Convergence Analysis of Optimizers

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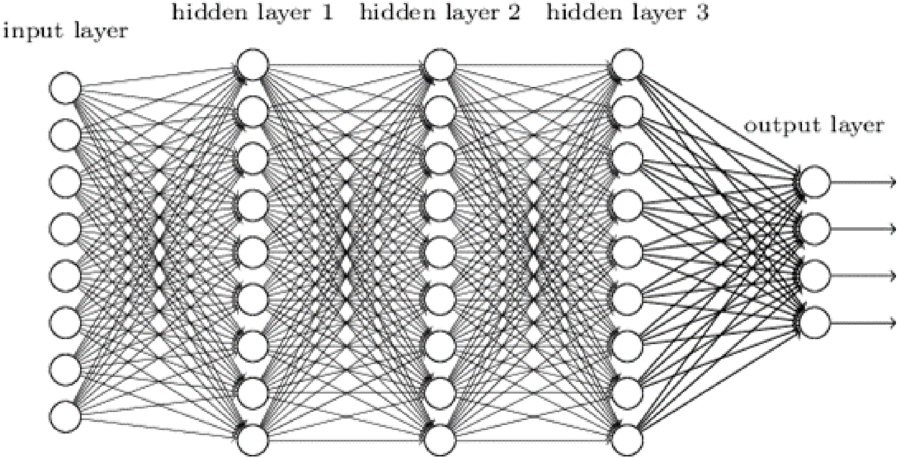
1. Deep Learning Optimization Overview

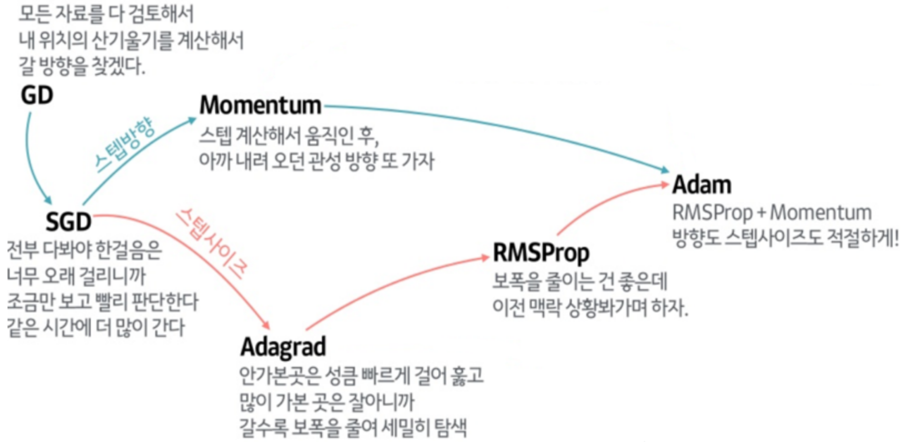
- The optimization problem in deep learning is non-convex.
- Through initialization, (dynamic) learning rate, and stochastic elements, we can make gradient descent-based methods converge to the global minimum.



2. Unified Perspective for Optimizers (GD)

$$\mathbf{x_{t+1}} = \mathbf{x_t} - \alpha \nabla \mathbf{f}(\mathbf{x})$$

- Step size (learning rate) : α
- Step direction : $\nabla f(x)$



2. Unified Perspective for Optimizers (SGD)

Stochastic Gradient Descent (Mini-batch)

- $\mathbf{g_t = \nabla \tilde{f}_t(x) = \frac{1}{t}\sum_{i=1}^{t} \nabla f_i(x_t)}$
- $\mathbf{x_{t+1} = x_t - \alpha g_t}$

Mini-batch: A training method dividing the entire dataset into small subsets

- Noise Injection $\rightarrow$ escape local minima, find global minima
- Stochastic Learning
- Computational Efficiency
- Memory Efficiency

2. Unified Perspective for Optimizers (SGDM)

Stochastic gradient descent with momentum

- $\mathbf{g_t} = \frac{\mathbf{1}}{\mathbf{t}} \sum_{\mathbf{i=1}}^{\mathbf{t}} \nabla \mathbf{f_i}(\mathbf{x_t})$
- $\mathbf{m_t} = \beta \mathbf{m_{t-1}} + (\mathbf{1} - \beta)\mathbf{g_t}$
- $\mathbf{x_{t+1}} = \mathbf{x_t} - \alpha \mathbf{m_t}$

Momentum: Inertia from previous gradients + current gradient

- $m_t = (0.081 g_{t-2} + 0.09 g_{t-1}) + 0.1 g_t$, $(\beta = 0.9)$
- SGDM is SGD with inertia (SGD is special case of SGDM in $\beta = 1$)
- Reduction of oscillations

2. Unified Perspective for Optimizers (Adagrad)

Adagrad : dynamic learning rate based on the accumulated square sum of gradients

- $\mathbf{g_i^{(k)} = \frac{1}{t}\sum_{j=1}^{k} \nabla f_i(x_j)}$
- $\mathbf{v_i^{(k)} = \sum_{j=1}^{k} (g_i^{(j)})^2}$
- $\mathbf{x_i^{(k+1)} = x_i^{(k)}} - \frac{\alpha}{\sigma + \sqrt{\mathbf{v_i^{(k)}}}}\mathbf{g_i^{(k)}}$

  (σ is a small value to prevent division by zero)

Dynamic learning rate: less sensitive to learning rate

- Adagrad is SGD with dynamic learning rate
  (SGD is special case of Adagrad in $v_i^{(k)} = 0$)
- large step size initially, small step size near optimum

2. Unified Perspective for Optimizers (RMSProp)

RMSProp : exponentially weighted moving average

- $\mathbf{g_i^{(k)}} = \mathbf{\frac{1}{t} \sum_{j=1}^{k} \nabla f_i(x_j)}$
- $\mathbf{v_i^{(k+1)}} = \beta_\mathbf{2} \mathbf{v_i^{(k)}} + (\mathbf{1} - \beta_\mathbf{2})(\mathbf{g_i^{(k)}})^\mathbf{2}$
- $\mathbf{\hat{v}_i}^{\mathbf{(k+1)}} = \dfrac{\mathbf{v_i^{(k+1)}}}{\mathbf{1} - \beta_\mathbf{2}^{\mathbf{k+1}}}$
- $\mathbf{x_i^{(k+1)}} = \mathbf{x_i^{(k)}} - \dfrac{\alpha}{\sigma + \sqrt{\mathbf{\hat{v}_i^{(k+1)}}}} \mathbf{g_i^{(k)}} = \mathbf{x_i^{(k)}} - \dfrac{\alpha}{\sigma + \mathrm{RMS}(\mathbf{g_i})} \mathbf{g_i^{(k)}}$

Exponentially weighted moving average: not monotonically decreasing

- to prevent the learning rate from decreasing too rapidly in Adagrad

2. Unified Perspective for Optimizers (Adam)

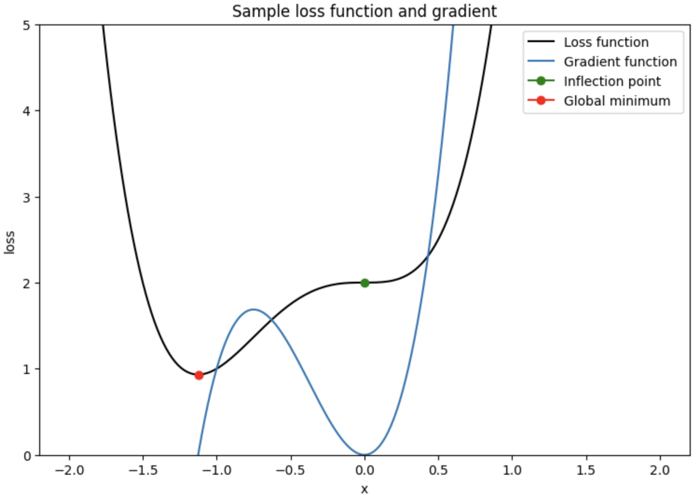
Adam(Momentum + RMSProp)

- $\mathbf{g_i^{(k)} = \frac{1}{t} \sum_{j=1}^{k} \nabla f_i(x_j)}$
- $\mathbf{m^{(k+1)} = \beta_1 m^{(k)} + (1 - \beta_1) g_i^{(k)}}$
- $\mathbf{v_i^{(k+1)} = \beta_2 v_i^{(k)} + (1 - \beta_2)(g_i^{(k)})^2}$
- $\mathbf{\hat{m}^{(k+1)} = \frac{m^{(k+1)}}{1 - \beta_1^{k+1}}}$
- $\mathbf{\hat{v}_i^{(k+1)} = \frac{v_i^{(k+1)}}{1 - \beta_2^{k+1}}}$
- $\mathbf{x_i^{(k+1)} = x_i^{(k)}} - \frac{\alpha}{\sigma + \sqrt{\mathbf{\hat{v}_i^{(k+1)}}}} \mathbf{m^{(k+1)}} = x_i^{(k)} - \frac{\alpha}{\sigma + \mathsf{RMS}(g_i)} \mathbf{m^{(k+1)}}$

Momentum + RMSProp:

- Adam is the most generalized form among all optimizers (SGDM, Adagrad, RMSProp are special cases of Adam)

3. Polynomial Example



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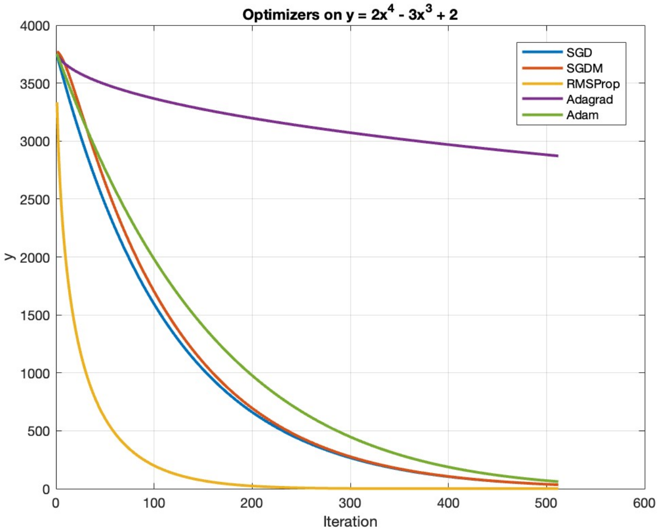

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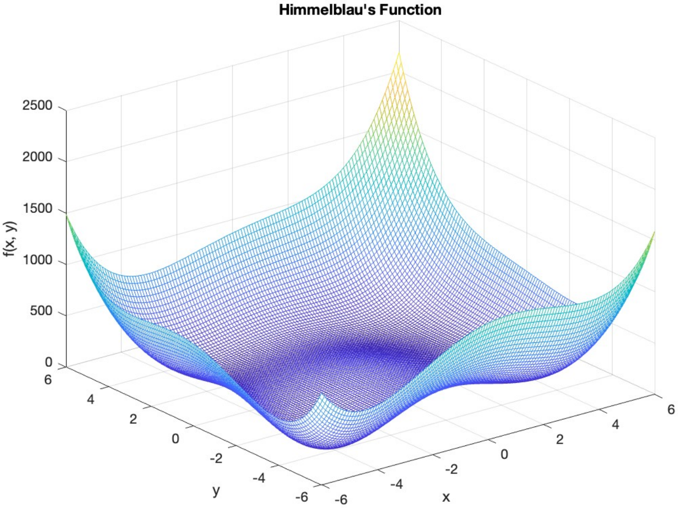

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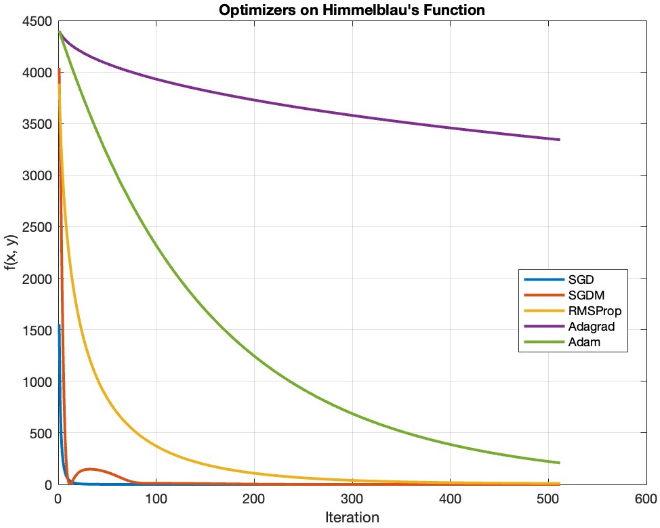
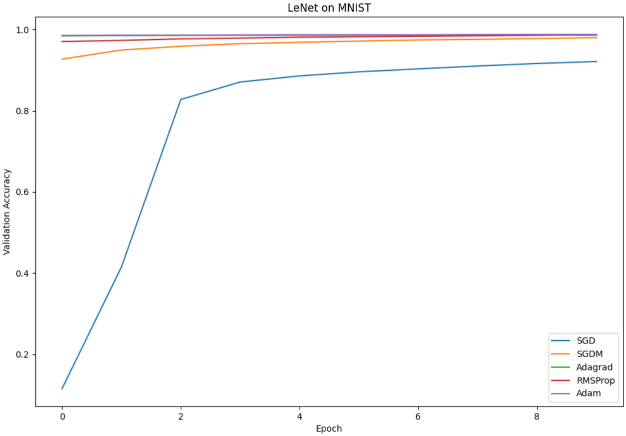


4. Broad Range of Experiments

Multiple Datasets, Different Model Architectures, and various optimizer

Dataset	**Model Architecture**
Mnist	LeNet-5
Fashion-mnist	AlexNet
Cifar-10	VGGNet

4. Broad Range of Experiments



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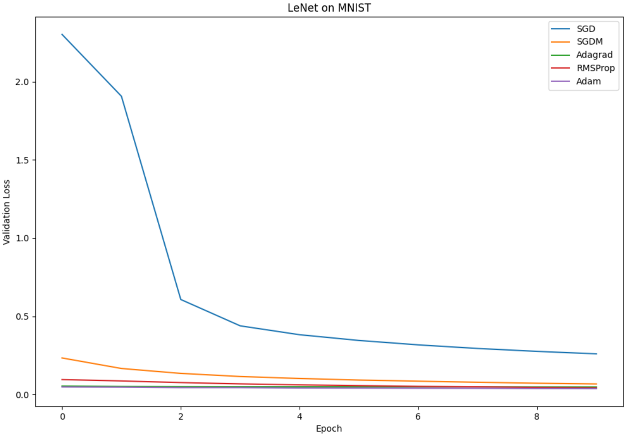

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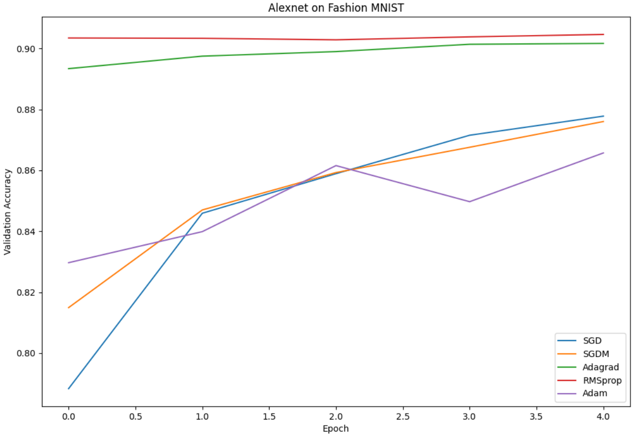

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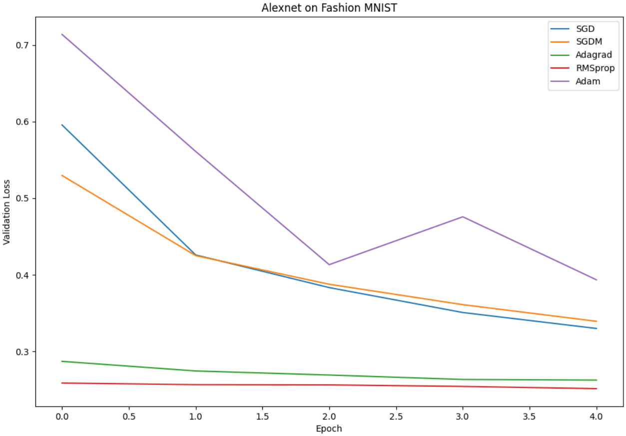

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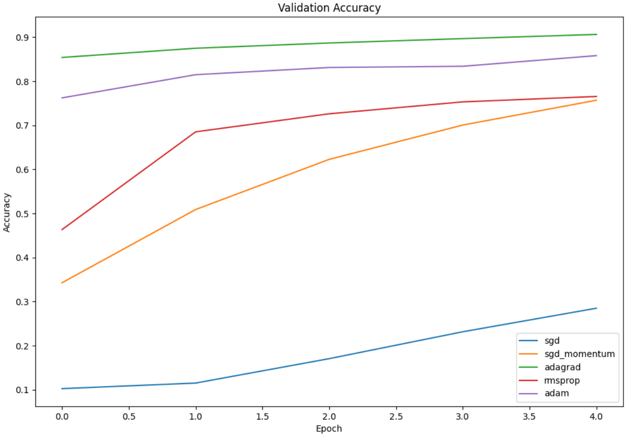

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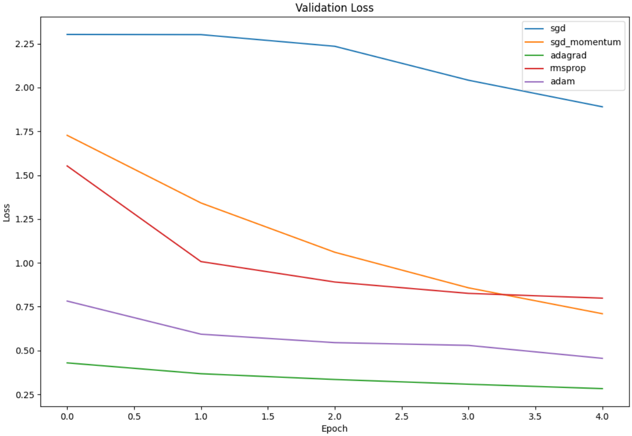

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