2022/5/26(Thu)

Chapter 3.4 Orthogonal and Orthonormal Systems

Definition 3.4.1. (Orthogonal and orthonormal systems)

A family S of nonzero vectors in an inner product space E is called an orthogonal system if $x \perp y$ for any two distinct elements of S. If, in addition, $\|x\| = 1$ for all $x \in S$, then S is called an orthonormal system.

Every orthogonal set of nonzero vectors can be normalized: If S is an orthogonal system, then $S_1 = \{\frac{x}{\|x\|} : x \in S\}$ is an orthonormal system.

Note that if x is orthogonal to each of $y_1, \ldots, y_n$, then x is orthogonal to every linear combination of vectors $y_1, \ldots, y_n$. In fact, if $y = \sum_{k=1}^{n} \lambda_k y_k$, then we have

$$\langle x, y \rangle = \left\langle x, \sum_{k=1}^{n} \lambda_k y_k \right\rangle = \sum_{k=1}^{n} \overline{\lambda_k} \langle x, y_k \rangle = 0.$$

Theorem 3.4.2.

Orthogonal systems are linearly independent.

n 本のベクトル $v_1, \ldots, v_n$ が線形独立とは，$c_1, \ldots, c_n$ をスカラーとすると，

$$\sum_{i=1}^{n} c_i \mathbf{v_i} = 0 \Rightarrow c_1 = \cdots = c_n = 0$$

が成り立つことである．

証明

S を直交系とする．ある $x_1, \ldots, x_n \in S$ と $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ に対して $\sum_{k=1}^{n} \alpha_k x_k = 0$ と仮定すると，

$$0 = \sum_{m=1}^{n} \langle 0, \alpha_m x_m \rangle = \sum_{m=1}^{n} \left\langle \sum_{k=1}^{n} \alpha_k x_k, \alpha_m x_m \right\rangle = \sum_{m=1}^{n} |\alpha_m|^2 \|x_m\|^2$$

各 $m \in \mathbb{N}$ に対して $\alpha_m = 0$ となるので，$x_1, \ldots, x_n$ は線形独立となる．

Definition 3.4.3. (Orthonormal sequence)

A sequence of vectors which constitutes an orthonormal system is called an orthonormal sequence.

In applications, it is often convenient to use sequences indexed by the set of all integers, $\mathbb{Z}$. The condition of orthonormality of a sequence (x_n) can be expressed in terms of the Kronecker delta symbol (Leopold Kronecker (1823-1891)):

$$\langle x_m, x_n \rangle = \delta_{mn} = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}$$

Example 3.4.4.

For $e_n = (0, \ldots, 0, 1, 0, \ldots)$ with 1 in the nth position, the set $S = \{e_1, e_2, \ldots\}$ is an orthonormal system in l^2.

$1 \leq p < \infty$ として,

$$l^p := \{(x_n) \in l(\mathbb{N}) | (\sum_{n=1}^{\infty} x_n^p)^{\frac{1}{p}} < \infty\}$$

Example 3.4.5.

Let $\varphi_n(x) = \frac{e^{inx}}{\sqrt{2\pi}}, n \in \mathbb{Z}$. The set $\{\varphi_n : n \in \mathbb{Z}\}$ is an orthonormal system in $L^2([-\pi, \pi])$. Indeed, for $m \neq n$, we have

$$\langle \varphi_m, \varphi_n \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(m-n)x} dx = \frac{e^{\pi i(m-n)} - e^{-\pi i(m-n)}}{2\pi i(m-n)} = 0.$$

On the other hand,

$$\langle \varphi_n, \varphi_n \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(n-n)x} dx = 1.$$

Thus, $\langle \varphi_m, \varphi_n \rangle = \delta_{mn}$ for every pair of integers m and n.

確認用

$$\begin{aligned} \|f\|_{L^p} &:= \left(\int |f(x)|^p dx \right)^{\frac{1}{p}} \\ \langle f, g \rangle_{L^2} &:= \int f(x) g(x) dx \\ e^{i\theta} &= \cos\theta + i \sin\theta \end{aligned}$$

Example 3.4.6. & 3.4.7.

The Legendre polynomials and the Hermite polynomial. (省略)

Theorem 3.4.8 (Pythagorean formula)

If $x_1, \ldots, x_n$ are orthogonal vectors in an inner product space, then

$$\left\| \sum_{k=1}^{n} x_k \right\|^2 = \sum_{k=1}^{n} \|x_k\|^2 .$$

証明

$x_1 \perp x_2$ ならば, $\|x_1 + x_2\|^2 = \|x_1\|^2 + \|x_2\|^2$ が成り立つ (ピタゴラスの定理より). よって $n = 2$ の時, 定理 3.4.8 は正しい. $n-1$ で成り立つと仮定すると以下のようになる.

$$\left\| \sum_{k=1}^{n-1} x_k \right\|^2 = \sum_{k=1}^{n-1} \|x_k\|^2 .$$

ここで, $x = \sum_{k=1}^{n-1} x_k$ と $y = x_n$ とすると, $x \perp y$ より,

$$\left\| \sum_{k=1}^{n} x_k \right\|^2 = \|x + y\|^2 = \|x\|^2 + \|y\|^2 = \sum_{k=1}^{n-1} \|x_k\|^2 + \|x_n\|^2 = \sum_{k=1}^{n} \|x_k\|^2 .$$

Theorem 3.4.9. (Bessel's equality and inequality)

Let $x_1, \ldots, x_n$ be an orthonormal set of vectors in an inner product space E. Then, for every $x \in E$, we have

$$\left\| x - \sum_{k=1}^{n} \langle x, x_k \rangle x_k \right\|^2 = \|x\|^2 - \sum_{k=1}^{n} |\langle x, x_k \rangle|^2 \tag{3.23}$$

and

$$\sum_{k=1}^{n} |\langle x, x_k \rangle|^2 \leq \|x\|^2 \tag{3.24}$$

証明

Thorem 3.4.8.(Pythagorean formula) から，任意の $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ に対して，

$$\left\| \sum_{k=1}^{n} \alpha_k x_k \right\|^2 = \sum_{k=1}^{n} \|\alpha_k x_k\|^2 = \sum_{k=1}^{n} |\alpha_k|^2$$

よって，

$$\begin{aligned}
\left\| x - \sum_{k=1}^{n} \alpha_k x_k \right\|^2 &= \left\langle x - \sum_{k=1}^{n} \alpha_k x_k, x - \sum_{k=1}^{n} \alpha_k x_k \right\rangle \\
&= \|x\|^2 - \left\langle x, \sum_{k=1}^{n} \alpha_k x_k \right\rangle - \left\langle \sum_{k=1}^{n} \alpha_k x_k, x \right\rangle + \sum_{k=1}^{n} |\alpha_k|^2 \|x_k\|^2 \\
&= \|x\|^2 - \sum_{k=1}^{n} \overline{\alpha_k} \langle x, x_k \rangle - \sum_{k=1}^{n} \alpha_k \overline{\langle x, x_k \rangle} + \sum_{k=1}^{n} \alpha_k \overline{\alpha_k} \\
&= \|x\|^2 - \sum_{k=1}^{n} |\langle x, x_k \rangle|^2 + \sum_{k=1}^{n} |\langle x, x_k \rangle - \alpha_k|^2
\end{aligned}$$

ここで，$\alpha_k = \langle x, x_k \rangle$ とすれば，(3.23) 式を得る．また，(3.23) 式から

$$0 \leq \|x\|^2 - \sum_{k=1}^{n} |\langle x, x_k \rangle|^2$$

が成り立ち，(3.24) が示せた．

Theorem 3.4.10.

Let (x_n) be an orthonormal sequence in a Hilbert space H, and let (α_n) be a sequence of complex numbers. Then the series $\sum_{n=1}^{\infty} \alpha_n x_n$ converges if and only if $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$ and in that case

$$\left\| \sum_{n=1}^{\infty} \alpha_n x_n \right\|^2 = \sum_{n=1}^{\infty} |\alpha_n|^2 \tag{3.28}$$

証明

全ての $m > k > 0$ に対して，

$$\left\| \sum_{n=k}^{m} \alpha_n x_n \right\|^2 = \sum_{n=k}^{m} |\alpha_n|^2 \tag{3.29}$$

が， the Pythagorean formula (3.22) によって成り立つ．もし $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$ ならば，(3.29) より $s_m = \sum_{n=1}^{m} \alpha_n x_n$ と取ることができ，これはコーシー列となる．これは，H の完備性によって，$\sum_{n=1}^{\infty} \alpha_n x_n$ が収束列であることを意味している．逆に，もし $\sum_{n=1}^{\infty} \alpha_n x_n$ が収束するとしたら，$\sigma_m = \sum_{n=1}^{m} |\alpha_n|^2$ は $\mathbb{R}$ のコーシー列であるから，(3.29) より $\sum_{n=1}^{\infty} |\alpha_n|^2$ が収束すると言える．(3.28) を得るには，(3.29) を $k = 1$ と $m \to \infty$ とすれば良い．

Example 3.4.11.

Let $H = L^2([-\pi, \pi])$, and let $x_n(t) = \frac{1}{\sqrt{\pi}} \sin nt$ for $n = 1, 2, \ldots$. The sequence (x_n) is an orthonormal set in H. On the other hand, for $x(t) = \cos t$, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n(t) &= \sum_{n=1}^{\infty} \left[\frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} \cos t \sin nt dt \right] \frac{\sin nt}{\sqrt{\pi}} \\ &= \sum_{n=1}^{\infty} 0 \cdot \sin nt = 0 \neq \cos t \end{aligned}$$

Definition 3.4.12. (Complete orthonormal sequence)

An orthonormal sequence (x_n) in an inner product space E is said to be complete if for every $x \in E$ we have

$$x = \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n.$$

Definition 3.4.13. (Orthonormal basis)

An orthonormal system B in an inner product space E is called an orthonormal basis if every $x \in E$ has a unique representation

$$x = \sum_{n=1}^{\infty} \alpha_n x_n$$

where $\alpha_n \in \mathbb{C}$ and x_n 's are distinct elements of B.