3.4 Orthogonal and Orthonormal Systems

Theorem 3.4.14.

An orthonormal sequence (x_n) in a Hilbert space H is complete if and only if $\langle x, x_n \rangle = 0$ for all $n \in \mathbb{N}$ implies $x = 0$.

Proof: (x_n) を H の完全正規直交系とすると，任意の $x \in H$ に対して，

$$x = \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n$$

よって，もし $\langle x, x_n \rangle = 0$ ならば $x = 0$.

一方で，任意の $n \in \mathbb{N}$ に対して $\langle x, x_n \rangle = 0$ ならば $x = 0$ とする．x を H の元とすると

$$y = \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n.$$

The sum y exists in H by (3.26) and Theorem 3.4.10. Since, for every $n \in \mathbb{N}$,

$$\begin{aligned}
\langle x - y, x_n \rangle &= \langle x, x_n \rangle - \left\langle \sum_{k=1}^{\infty} \langle x, x_k \rangle x_k, x_n \right\rangle \\
&= \langle x, x_n \rangle - \sum_{k=1}^{\infty} \langle x, x_k \rangle \langle x_k, x_n \rangle \\
&= \langle x, x_n \rangle - \langle x, x_n \rangle = 0
\end{aligned}$$

we have $x - y = 0$, and hence $x = \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n$.

Theorem 3.4.15. (Parseval's formula)

An orthonormal sequence (x_n) in a Hilbert space H is complete if and only if

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2 \tag{3.31}$$

for every $x \in H$.

Proof: $x \in H$ とする．ベッセルの不等式より，全ての $n \in \mathbb{N}$ に対して，

$$\left\| x - \sum_{k=1}^{n} \langle x, x_k \rangle x_k \right\|^2 = \|x\|^2 - \sum_{k=1}^{n} |\langle x, x_k \rangle|^2 \tag{3.32}$$

が成り立つ．

もし (x_n) が完全とすると，(3.32) の左辺は $n \to \infty$ としたとき 0 へ収束する．よって，

$$\lim_{n \to \infty} \left[\|x\|^2 - \sum_{k=1}^{n} |\langle x, x_k \rangle|^2 \right] = 0.$$

よって (3.31) が成り立つ．

次に (3.31) が成り立つとする．すると，(3.32) の右辺は $n \to \infty$ としたとき 0 へ収束し，

$$\lim_{n \to \infty} \left\| x - \sum_{k=1}^{n} \langle x, x_k \rangle x_k \right\|^2 = 0.$$

となる．よって (x_n) は完全である．

4.2 Example of Operators

We begin with some examples of operators. In each case, we are interested whether the operator is bounded. Recall that an operator A is called bounded if there is a number K such that $\|Ax\| \le K\|x\|$ for every x in the domain of A. The norm of A is defined as the infimum of all such numbers K, or equivalently, by

$$\|A\| = \sup_{\|x\|=1} \|Ax\|.$$

We will refer to this norm as the operator norm.

In Section 1.6, we proved that an operator A is bounded if and only if it is continuous. It is often much more difficult to find the norm of an operator than just to prove that it is bounded.

Example 4.2.1. (Identity operator and null operator)

The simplest examples of operators are the identity operator $\mathcal{I}$ and the null operator. The identity operator leaves every element unchanged, that is, $\mathcal{I}x = x$ for all $x \in E$. The null operator assigns the zero vector to every element of E. The null operator will be denoted by 0 . Obviously, the identity operator and the null operator are bounded and we have $\|\mathcal{I}\| = 1$ and $\|0\| = 0$. A scalar multiple $\alpha\mathcal{I}$ of the identity operator is the operator, which multiplies every element by the scalar α, that is, $(\alpha\mathcal{I})x = \alpha x$.

Example 4.2.2.

Let A be an operator on $\mathbb{C}^N$ and let $\{e_1, \dots, e_N\}$ be the standard orthonormal base in $\mathbb{C}^N$, that is,

$$\begin{aligned} e_1 &= (1, 0, 0, \dots, 0) \\ e_2 &= (0, 1, 0, \dots, 0) \\ &\vdots \\ e_N &= (0, 0, \dots, 0, 1) \end{aligned}$$

Define, for $i, j \in \{1, 2, \dots, N\}$,

$$\alpha_{ij} = \langle Ae_j, e_i \rangle .$$

Then, for $x = \sum_{j=1}^N \lambda_j e_j \in \mathbb{C}^N$, we have $Ax = \sum_{j=1}^N \lambda_j Ae_j$, and hence

$$\langle Ax, e_i \rangle = \sum_{j=1}^N \lambda_j \langle Ae_j, e_i \rangle = \sum_{j=1}^N \alpha_{ij}\lambda_j \tag{4.1}$$

Thus, every operator on the space $\mathbb{C}^N$ is defined by an $N \times N$ matrix. Conversely, for every $N \times N$ matrix (α_{ij}), formula (4.1) defines an operator on $\mathbb{C}^N$. We thus have a one-to-one correspondence between operators on N dimensional vector spaces and $N \times N$ matrices. If operator A is defined by the matrix (α_{ij}), then

$$\|A\| \le \sqrt{\sum_{i=1}^N \sum_{j=1}^N |\alpha_{ij}|^2}.$$

This implies that every operator on $\mathbb{C}^N$, and thus every operator on any finite dimensional Hilbert space, is bounded.

Example 4.2.3. (Differential operator)

One of the most important operators in applied mathematics is the differential operator

$$(Df)(x) = \frac{df}{dx}(x) = f'(x)$$

defined on a space of differentiable functions. Consider, for example, the differential operator on a subspace of $L^2([-\pi,\pi])$ defined as

$$\mathcal{D}(D) = \left\{ f \in L^2([-\pi,\pi]) : f' \in L^2([-\pi,\pi]) \right\}.$$

If $L^2([-\pi,\pi])$ is equipped with the standard norm $\|f\| = \sqrt{\int_{-\pi}^{\pi} |f(x)|^2 dx}$, then the differential operator is not bounded. Indeed, for $f_n(x) = \sin nx$, $n = 1, 2, 3, \ldots$, we have

$$\|f_n\| = \sqrt{\int_{-\pi}^{\pi} |\sin nx|^2 dx} = \sqrt{\pi}$$

and

$$\|Df_n\| = \sqrt{\int_{-\pi}^{\pi} |n \cos nx|^2 dx} = n\sqrt{\pi}.$$

This example can be easily generalized to an arbitrary interval $[a,b]$ or even $(-\infty,\infty)$

Example 4.2.4. (Integral operator)

Another important type of operators is an integral operator T defined by

$$(Tx)(s) = \int_a^b K(s,t)x(t)dt,$$

where a and b are finite or infinite, $a < b$, and K is a function defined on the square $(a,b) \times (a,b)$. The function K is called the kernel of the operator. The domain of an integral operator depends on K. If

$$\int_a^b \int_a^b |K(s,t)|^2 dt ds < \infty$$

then T is a bounded operator on $L^2([a,b])$ and

$$\|T\| \le \sqrt{\int_a^b \int_a^b |K(s,t)|^2 dt ds}$$

Indeed, for any $x \in L^2([a,b])$, we have

$$\begin{aligned}
\|Tx\|^2 &= \int_a^b \left| \int_a^b K(s,t)x(t)dt \right|^2 ds \\
&\le \int_a^b \left(\int_a^b |K(s,t)|^2 dt \int_a^b |x(t)|^2 dt \right) ds \qquad \text{(by Schwarz's inequality)} \\
&\le \int_a^b \int_a^b |K(s,t)|^2 dt ds \int_a^b |x(t)|^2 dt.
\end{aligned}$$

Thus,

$$\|Tx\| \le \sqrt{\int_a^b \int_a^b |K(s,t)|^2 dt ds \|x\|}$$