Chapter 3 Hilbert Spaces and Orthonomal Systems

Definition 3.2.1. (Inner product space)

Let E be a complex vector space. A mapping $\langle\cdot,\cdot\rangle : E \times E \to \mathbb{C}$ is called an inner product in E if for any $x, y, z \in E$ and $\alpha, \beta \in \mathbb{C}$ the following conditions are satisfied:

(a) $\langle x, y\rangle = \overline{\langle y, x\rangle}$ (the bar denotes the complex conjugate);
(b) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$;
(c) $\langle x, x\rangle \geq 0$;
(d) $\langle x, x\rangle = 0$ implies $x = 0$.

A vector space with an inner product is called an inner product space.

According to the definition, the inner product of two vectors is a complex number. By (a), $\langle x, x\rangle = \overline{\langle x, x\rangle}$, which means that $\langle x, x\rangle$ is a real number for every $x \in E$. It follows from (b) that

$$\langle x, \alpha y + \beta z\rangle = \overline{\langle \alpha y + \beta z, x\rangle} = \overline{\alpha\langle y, x\rangle + \beta\langle z, x\rangle} = \bar{\alpha}\langle x, y\rangle + \bar{\beta}\langle x, z\rangle.$$

In particular

$$\langle \alpha x, y\rangle = \alpha\langle x, y\rangle \quad \text{and } \langle x, \alpha y\rangle = \bar{\alpha}\langle x, y\rangle.$$

Hence, if $\alpha = 0$, we have $\langle 0, y\rangle = \langle x, 0\rangle = 0$.

Example 3.2.2

The simplest, although important, example of an inner product space is the space of complex numbers $\mathbb{C}$. The inner product is defined by $\langle x, y\rangle = x\overline{y}$ □

Example 3.2.3

The space $\mathbb{C}^N$ of ordered N-tuples $x = (x_1, \ldots, x_N)$ of complex numbers, with the inner product defined by

$$\langle x, y\rangle = \sum_{k=1}^{N} x_k \overline{y}_k, \quad x = (x_1, \ldots, x_N),\ y = (y_1, \ldots, y_N)$$

is an inner product space.

Example 3.2.4

The space l^2 of all sequences $(x_1, x_2, x_3, \ldots)$ of complex numbers such that $\sum_{k=1}^{\infty} |x_k|^2 < \infty$, with the inner product defined by

$$\langle x, y\rangle = \sum_{k=1}^{\infty} x_k \overline{y}_k, \quad x = (x_1, x_2, x_3, \ldots),\ y = (y_1, y_2, y_3, \ldots)$$

is an infinite dimensional inner product space.

Example 3.2.6

The space $\mathcal{C}([a, b])$ of all continuous complex valued functions on the interval $[a, b]$, with the inner product

$$\langle x, y\rangle = \int_a^b f(x)\overline{g(x)}dx$$

is an inner product space.

これが上の $(a) \sim (b)$ を満たすのか？どのように確認すれば良いのか？ Example3.2.7 との違いは？

Example 3.2.7.

The space $L^2(\mathbb{R}\ )$ (see Section 2.13) with the inner product defined by

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}dx$$

and, more generally, the space $L^2\left(\mathbb{R}^N\right)$ with the inner product defined by

$$\int_{\mathbb{R}^N} f(x)\overline{g(x)}dx$$

are very important inner product spaces. In applications we often use a subset Ω of $\mathbb{R}^N$ and the space $L^2(\Omega)$ with the inner product defined by the integral over Ω. For example, the space $L^2([a,b])$ is the right setting in many cases. □

Example 3.2.8

Let E be the Cartesian product of inner product spaces E_1 and E_2, that is, $E = E_1 \times E_2 = \{(x,y) : x \in E_1, y \in E_2\}$. The space E is an inner product space with the inner product defined by

$$\langle (x_1, y_1), (x_2, y_2) \rangle = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle .$$

Note that E_1 and E_2 can be identified with the subspaces $E_1 \times \{0\}$ and $\{0\} \times E_2$, respectively.

Similarly, we can define the inner product on $E_1 \times \cdots \times E_n$. This method can be used to construct new examples of inner product spaces. □

An inner product space is a vector space with an inner product. It turns out that every inner product space is also a normed space with the norm defined by

$$\|x\| = \sqrt{\langle x, x \rangle}.$$

First notice that this functional is well defined because $\langle x, x \rangle$ is always a nonnegative (real) number and $\|x\| = 0$ if and only if $x = 0$. Moreover,

$$\|\lambda x\| = \sqrt{\langle \lambda x, \lambda x \rangle} = \sqrt{\lambda \bar{\lambda} \langle x, x \rangle} = |\lambda| \|x\|$$

It thus remains to prove the triangle inequality. This is not as simple as the first two conditions. We first prove the so-called Schwarz's inequality (Hermann Amandus Schwarz (1843-1921)), which will be used in the proof of the triangle inequality.

Theorem 3.2.9. (Schwarz's inequality)

For any two elements x and y of an inner product space, we have

$$|\langle x, y \rangle| \leq \|x\| \|y\|. \tag{3.1}$$

The equality $|\langle x, y \rangle| = \|x\| \|y\|$ holds if and only if x and y are linearly dependent.

Proof:

If $y = 0$, then (3.1) is satisfied because both sides are equal to zero. Assume then $y \neq 0$. We have

$$0 \leq \langle x + \alpha y, x + \alpha y \rangle = \langle x, x \rangle + \bar{\alpha}\langle x, y \rangle + \alpha \langle y, x \rangle + |\alpha|^2 \langle y, y \rangle \tag{3.2}$$

Now put $\alpha = -\langle x, y \rangle / \langle y, y \rangle$ in (3.2), and then multiply by $\langle y, y \rangle$ to obtain

$$0 \leq \langle x, x \rangle \langle y, y \rangle - |\langle x, y \rangle|^2.$$

This gives Schwarz's inequality. If x and y are linearly dependent, then $x = \alpha y$ for some $\alpha \in \mathbb{C}$. Hence,

$$|\langle x, y \rangle| = |\langle x, \alpha x \rangle| = |\bar{\alpha}| \langle x, x \rangle = |\alpha| \|x\| \|x\| = \|x\| \|\alpha x\| = \|x\| \|y\|.$$

Now, let x and y be vectors such that $|\langle x, y \rangle| = \|x\| \|y\|$, or, equivalently,

$$\langle x, y \rangle \langle y, x \rangle = \langle x, x \rangle \langle y, y \rangle \tag{3.3}$$

We will show that $\langle y, y \rangle x - \langle x, y \rangle y = 0$, which proves that x and y are linearly dependent. Indeed, by (3.3) we have

$$\begin{aligned} &\langle \langle y, y \rangle x - \langle x, y \rangle y, \langle y, y \rangle x - \langle x, y \rangle y \rangle \\ &\quad = \langle y, y \rangle^2 \langle x, x \rangle - \langle y, y \rangle \langle y, x \rangle \langle x, y \rangle - \langle x, y \rangle \langle y, y \rangle \langle y, x \rangle + \langle x, y \rangle \langle y, x \rangle \langle y, y \rangle \\ &= 0 \end{aligned}$$

completing the proof. □

Corollary 3.2.10. (Triangle inequality)

For any two elements x and y of an inner product space we have

$$\|x + y\| \leq \|x\| + \|y\|.$$

Proof: When $\alpha = 1$, Equation (3.2) can be written as

$$\begin{aligned} \|x + y\|^2 &= \langle x + y, x + y \rangle = \langle x, x \rangle + 2\,\mathrm{Re}\langle x, y \rangle + \langle y, y \rangle \\ &\leq \langle x, x \rangle + 2|\langle x, y \rangle| + \langle y, y \rangle \\ &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \quad \text{(by Schwarz's inequality)} \\ &= (\|x\| + \|y\|)^2 \end{aligned}$$

where $\mathrm{Re}\, z$ denotes the real part of $z \in \mathbb{C}$. □

The preceding discussion justifies the following definition:

Definition 3.2.11. (Norm in an inner product space)

By the norm in an inner product space E we mean the functional defined by $\|x\| = \sqrt{\langle x, x \rangle}$.

We have proved that every inner product space is a normed space. It is only natural to ask whether every normed space is an inner product space. More precisely: is it possible to define in a normed space $(E, \|\cdot\|)$ an inner product $\langle \cdot, \cdot \rangle$ such that $\|x\| = \sqrt{\langle x, x \rangle}$ for every $x \in E$? In general, the answer is negative. In the following theorem, we prove a property of the norm in an inner product space, which is a necessary and sufficient condition for a normed space to be an inner product space; see Exercise 12.

Theorem 3.2.12. (Parallelogram law)

For any two elements x and y of an inner product space, we have

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right).$$

Proof:
We have

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle$$

and hence

$$\|x+y\|^2 = \|x\|^2 + \langle x,y\rangle + \langle y,x\rangle + \|y\|^2.$$

Now replace y by $-y$ in (3.6) to obtain

$$\|x-y\|^2 = \|x\|^2 - \langle x,y\rangle - \langle y,x\rangle + \|y\|^2$$

By adding (3.6) and (3.7), we obtain the parallelogram law.

One of the most important consequences of having the inner product is the possibility of defining orthogonality of vectors. This makes the theory of Hilbert spaces very different from the general theory of Banach spaces.

Definition 3.2.13. (Orthogonal vectors)

Two vectors x and y in an inner product space are called orthogonal, denoted by $x \perp y$, if $\langle x,y\rangle = 0$.

If $x \perp y$, then $\langle y,x\rangle = \overline{\langle x,y\rangle} = 0$, and thus $y \perp x$. In other words, the relation $\perp$ is symmetric.
The next theorem is another example of the geometric character of the norm defined by an inner product.

Theorem 3.2.14. (Pythagorean formula)

For any pair of orthogonal vectors, we have

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2$$

Proof: If $x \perp y$, then $\langle x,y\rangle = \langle y,x\rangle = 0$, and thus the equality follows immediately from (3.6). □

In the definition of the inner product space we assume that E is a complex vector space. It is possible to define a real inner product space with the inner product of any two vectors being a real number. Then condition (b) in Definition 3.2.1 becomes $\langle x,y\rangle = \langle y,x\rangle$. All the preceding theorems hold in the real case. If, in Examples 3.2.2-3.2.7, the word complex is replaced by real and $\mathbb{C}$ by $\mathbb{R}$, we obtain a number of examples of real inner product spaces. A finite dimensional real inner product space is called a Euclidean space.

If $x = (x_1, \ldots, x_N)$ and $y = (y_1, \ldots, y_N)$ are vectors in $\mathbb{R}^N$, then the inner product $\langle x,y\rangle = \sum_{k=1}^{N} x_k y_k$ can be defined equivalently as $\langle x,y\rangle = \|x\|\|y\|\cos\theta$, where θ is the angle between vectors x and y. In this case, Schwarz's inequality follows from

$$\frac{|\langle x,y\rangle|}{\|x\|\|y\|} = |\cos\theta| \le 1, \quad x \neq 0, y \neq 0$$