1.5 Linear Mappings (P27 $\sim$)

Theorem 1.5.7.

A linear mapping is continuous if and only if it is bounded.

Proof:

If L is bounded and $x_n \to 0$, then $\|Lx_n\| \leq \alpha \|x_n\| \to 0$. Thus, L is continuous at 0 and hence, by Theorem 1.5.5, L is continuous.

If L is not bounded, then for every $n \in \mathbb{N}$ there exists an $x_n \in E_1$ such that $\|Lx_n\| > n \|x_n\|$. Define

$$y_n = \frac{x_n}{n \|x_n\|}, \quad n = 1, 2, \ldots$$

Then $y_n \to 0$. Since $\|Ly_n\| > 1$ for all $n \in \mathbb{N}, L$ is not continuous. □

Since every linear mapping between finite dimensional spaces is bounded, it is continuous. The above theorem implies also that, for linear mappings, continuity and uniform continuity are equivalent.

The space of all linear mappings from a vector space E_1 into a vector space E_2 becomes a vector space if the addition and multiplication by scalars are defined as follows:

$$(L_1 + L_2)\, x = L_1 x + L_2 x \text{ and } (\lambda L)x = \lambda(Lx)$$

If E_1 and E_2 are normed spaces, then the set of all bounded linear mappings from E_1 into E_2, denoted by $\mathcal{B}(E_1, E_2)$, is a vector subspace of the space defined above.

Theorem 1.5.8.

If E_1 and E_2 are normed spaces, then $\mathcal{B}(E_1, E_2)$ is a normed space with the normed difined by

$$\|L\| = \sup_{\|x\|=1} \|Lx\|.$$

Proof:

We will only show that the norm (1.14) satisfies the triangle inequality. If $L_1, L_2 \in \mathcal{B}(E_1, E_2)$ and $x \in E_1$ is such that $\|x\| = 1$, then

$$\|L_1 x + L_2 x\| \leq \|L_1 x\| + \|L_2 x\|.$$

This implies

$$\|L_1 x + L_2 x\| \leq \sup_{\|x\|=1} \|L_1 x\| + \sup_{\|x\|=1} \|L_2 x\| = \|L_1\| + \|L_2\|,$$

and hence

$$\|L_1 + L_2\| = \sup_{\|x\|=1} \|L_1 x + L_2 x\| \leq \|L_1\| + \|L_2\|.$$

□

It follows from (1.14) that $\|Lx\| \leq \|L\|\|x\|$ for all $x \in E_1$. In fact, $\|L\|$ is the least number α such that $\|Lx\| \leq \alpha\|x\|$ for all $x \in E_1$.

The norm defined in (1.14) is the standard norm on $\mathcal{B}(E_1, E_2)$ and this is the norm we mean when we say "the normed space $\mathcal{B}(E_1, E_2)$." Convergence with respect to this norm is called uniform convergence.