## 4.3 Bilinear Functionals and Quadratic Forms

### Definition 4.3.1. (Bilinear functional)

By a bilinear functional $\varphi$ on a complex vector space $E$, we mean a mapping $\varphi : E\times\ E \to \mathbb{C}$ satisfying the following two conditions:

(a) $\varphi(\alpha x_1 + \beta x_2, y) = \alpha\varphi(x_1, y) + \beta\varphi(x_2, y)$,
(b) $\varphi(x, \alpha y_1 + \beta y_2) = \bar{\alpha}\varphi(x, y_1) + \bar{\beta}\varphi(x, y_2)$,

for any scalars $\alpha$ and $\beta$ and any $x, x_1, x_2, y, y_1, y_2 \in E$.

Bilinear functionals are often called sesquilinear. Note that a bilinear functional is linear with respect to the first variable and antilinear with respect to the second variable. Clearly, all bilinear functionals on $E$ constitute a vector space.

### Example 4.3.2.

Inner product is a bilinear functional.

$f(x, y) = x \cdot y$ とすれば良い.

### Example 4.3.3.

Let $A$ and $B$ be operators on an inner product space $E$. Then $\varphi_1(x, y) = \langle Ax, y\rangle, \varphi_2(x, y) = \langle x, By\rangle$, and $\varphi_3(x, y) = \langle Ax, By\rangle$ are bilinear functionals on $E$.

### Example 4.3.4.

Let $f$ and $g$ be linear functionals on a vector space $E$. Then $\varphi(x, y) = f(x)\overline{g(y)}$ is a bilinear functional on $E$.

### Definition 4.3.5. (Symmetric, positive, strictly positive, and bounded bilinear functionals)

Let $\varphi$ be a bilinear functional on $E$.

(a) $\varphi$ is called symmetric if $\varphi(x, y) = \overline{\varphi(y, x)}$ for all $x, y \in E$.
(b) $\varphi$ is called positive if $\varphi(x, x) \geq 0$ for every $x \in E$.
(c) $\varphi$ is called strictly positive if it is positive and $\varphi(x, x) > 0$ for all $x \neq 0$.
(d) If $E$ is a normed space, then $\varphi$ is called bounded if $|\varphi(x, y)| \leq K\|x\|\|y\|$ for some $K > 0$ and all $x, y \in E$. The norm of a bounded bilinear functional is defined by

$$\|\varphi\| = \sup_{\|x\|=\|y\|=1} |\varphi(x, y)|.$$

If $f = g$ in Example 4.3.4, then $\varphi$ is symmetric and positive. Inner product is strictly positive. If operators $A$ and $B$ in Example 4.3.3 are bounded, then $\varphi_1$, $\varphi_2$, and $\varphi_3$ are bounded. Similarly, if $f$ and $g$ in Example 4.3.4 are bounded, then the defined bilinear functional is also bounded. Note that for a bounded bilinear functional $\varphi$ on $E$ we have

$$|\varphi(x, y)| \leq \|\varphi\|\|x\|\|y\| \quad \text{for all } x, y \in E.$$

## Definition 4.3.6. (Quadratic form)

Let $\varphi$ be a bilinear functional on a vector space $E$. The function $\Phi : E \to \mathbb{C}$ defined by $\Phi(x) = \varphi(x, x)$ is called the quadratic form associated with $\varphi$. A quadratic form $\Phi$ on a normed space $E$ is called bounded if there exists a constant $K > 0$ such that $|\Phi(x)| \leq K\|x\|^2$ for all $x \in E$. The norm of a bounded quadratic form is defined by

$$\|\Phi\| = \sup_{\|x\|=1} |\Phi(x)|.$$

Note that for a bounded quadratic form $\Phi$ on a normed space we have $|\Phi(x)| \leq \|\Phi\|\|x\|^2$. A bilinear functional and the associated quadratic form have properties similar to an inner product $\langle x, y\rangle$ and the square of the norm defined by that inner product $\|x\|^2 = \langle x, x\rangle$, respectively.

## Theorem 4.3.7. (Polarization identity)

Let $\varphi$ be a bilinear functional on $E$, and let $\Phi$ be the quadratic form associated with $\varphi$. Then

$$4\varphi(x, y) = \Phi(x + y) - \Phi(x - y) + i\Phi(x + iy) - i\Phi(x - iy)$$

for all $x, y \in E$.

■Proof: For any $\alpha, \beta \in \mathbb{C}$, we have

$$\begin{aligned}\Phi(\alpha x + \beta y) &= \varphi(\alpha x + \beta y, \alpha x + \beta y)\\ &= |\alpha|^2\Phi(x) + \alpha\bar{\beta}\varphi(x, y) + \bar{\alpha}\beta\varphi(y, x) + |\beta|^2\Phi(y).\end{aligned}$$

Using this equality subsequently for $\alpha = \beta = 1; \alpha = 1$ and $\beta = -1; \alpha = 1$ and $\beta = i; \alpha = 1$ and $\beta = -i$; we get

$$\begin{aligned}\Phi(x + y) &= \Phi(x) + \varphi(x, y) + \varphi(y, x) + \Phi(y)\\ -\Phi(x - y) &= -\Phi(x) + \varphi(x, y) + \varphi(y, x) - \Phi(y)\\ i\Phi(x + iy) &= i\Phi(x) + \varphi(x, y) - \varphi(y, x) + i\Phi(y)\\ -i\Phi(x - iy) &= -i\Phi(x) + \varphi(x, y) - \varphi(y, x) - i\Phi(y).\end{aligned}$$

By adding these equalities we obtain (4.2). The following simple, but somewhat surprising, result is often useful.

## Corollary 4.3.8.

Let $\varphi_1$ and $\varphi_2$ be bilinear functionals on E. If $\varphi_1(x, x) = \varphi_2(x, x)$ for all $x \in E$, then $\varphi_1 = \varphi_2$, that is, $\varphi_1(x, y) = \varphi_2(x, y)$ for all $x, y \in E$. Similarly, if $A$ and $B$ are operators on $E$ such that $\langle Ax, x\rangle = \langle Bx, x\rangle$ for all $x \in E$, then $A = B$.

Proof: If $\varphi_1(x, x) = \varphi_2(x, x)$ for all $x \in E$, then the quadratic forms $\Phi_1$ and $\Phi_2$ associated with $\varphi_1$ and $\varphi_2$, respectively, are equal, and hence, by (4.2), the functionals $\varphi_1$ and $\varphi_2$ are equal. The proof for operators is obtained by letting $\varphi_1(x, y) = \langle Ax, y\rangle$ and $\varphi_2(x, y) = \langle Bx, y\rangle$.

## Theorem 4.3.9.

A bilinear functional $\varphi$ on $E$ is symmetric if and only if the associated quadratic form $\Phi$ is real.

Proof: If $\varphi(x, y) = \overline{\varphi(y, x)}$ for all $x, y \in E$, then

$$\Phi(x) = \varphi(x, x) = \overline{\varphi(x, x)} = \overline{\Phi(x)}$$

for every $x \in E$, and thus $\Phi$ is real. Assume now $\Phi(x) = \overline{\Phi(x)}$ for all $x \in E$. Define a bilinear functional $\psi$ on $E$ by

$$\psi(x, y) = \overline{\varphi(y, x)}.$$

Then, for the associated quadratic form $\Psi$ we have

$$\Psi(x) = \overline{\varphi(x,x)} = \overline{\Phi(x)} = \Phi(x).$$

Thus, by Corollary 4.3.8, $\varphi(x,y) = \psi(x,y)$ for all $x, y \in E$. Clearly, this means that $\varphi(x,y) = \overline{\varphi(y,x)}$ for all $x, y \in E$.

### Theorem 4.3.10.

A bilinear functional $\varphi$ on a normed space E is bounded if and only if the associated quadratic form $\Phi$ is bounded. Moreover, we have

$$\|\Phi\| \leq \|\varphi\| \leq 2\|\Phi\|.$$

Proof: Since

$$\|\Phi\| = \sup_{\|x\|=1} |\Phi(x)| = \sup_{\|x\|=1} |\varphi(x,x)| \leq \sup_{\|x\|=\|y\|=1} |\varphi(x,y)| = \|\varphi\|,$$

if $\varphi$ is bounded, then $\Phi$ is bounded and the first inequality follows. Suppose now that $\Phi$ is bounded. In view of (4.2), we have

$$\begin{aligned} |\varphi(x,y)| &= \frac{1}{4}|\Phi(x+y) - \Phi(x-y) + i\Phi(x+iy) - i\Phi(x-iy)| \\ &\leq \frac{1}{4}\|\Phi\| \left(\|x+y\|^2 + \|x-y\|^2 + \|x+iy\|^2 + \|x-iy\|^2\right). \end{aligned}$$

Hence, by the parallelogram law,

$$|\varphi(x,y)| \leq \|\Phi\| \left(\|x\|^2 + \|y\|^2\right).$$

Consequently,

$$\sup_{\|x\|=\|y\|=1} |\varphi(x,y)| \leq \sup_{\|x\|=\|y\|=1} \|\Phi\| \left(\|x\|^2 + \|y\|^2\right) = 2\|\Phi\|$$

Thus, if $\Phi$ is bounded, then $\varphi$ is bounded and the second inequality in (4.3) follows.

### Theorem 4.3.11.

Let $\varphi$ be a bilinear functional on a normed space E and let $\Phi$ be the associated quadratic form. If $\varphi$ is symmetric and bounded, then $\|\varphi\| = \|\Phi\|$. Proof: By Theorem 4.3.10, $\|\Phi\| \leq \|\varphi\|$. We need to show that the opposite inequality holds as well. Since $\varphi$ is symmetric, $\Phi$ is real, by Theorem 4.3.9. Then, by the polarization identity, we obtain

$$\operatorname{Re} \varphi(x,y) = \frac{1}{4}[\Phi(x+y) - \Phi(x-y)],$$

and hence

$$\begin{aligned} |\operatorname{Re} \varphi(x,y)| &\leq \frac{1}{4}\|\Phi\| \left(\|x+y\|^2 + \|x-y\|^2\right) \\ &= \frac{1}{2}\|\Phi\| \left(\|x\|^2 + \|y\|^2\right) \end{aligned}$$

by the parallelogram law. Let $x$ and $y$ be arbitrary fixed elements of $E$ such that $\|x\| = \|y\| = 1$, and let $\theta$ be a complex number such that $|\theta| = 1$ and