2022/6/20(Mon)

Separation of Two Convex Sets

Thus far we have discussed the separation of a convex set and a point not in the set and have also discussed the support of convex sets at boundary points. In addition, if we have two disjoint convex sets, they can be separated by a hyperplane H such that one of the sets belongs to H^+ and the other set belongs to H^-. In fact, this result holds true even if the two sets have some points in common, as long as their interiors are disjoint. This result is made precise by the following theorem.

2.4.8 Theorem

Let S_1 and S_2 be nonempty convex sets in R^n and suppose that $S_1 \cap S_2$ is empty. Then there exists a hyperplane that separates S_1 and S_2; that is, there exists a nonzero vector $\mathbf{p}$ in R^n such that

$$\inf\left\{\mathbf{p}^t\mathbf{x} : \mathbf{x} \in S_1\right\} \geq \sup\left\{\mathbf{p}^t\mathbf{x} : \mathbf{x} \in S_2\right\}$$

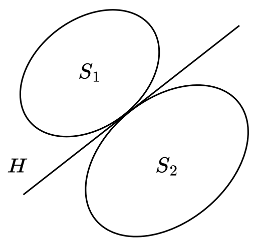

図 1　S_1 と S_2 を分離する H

$S = S_1 \ominus S_2 = \{\mathbf{x_1} - \mathbf{x_2} : \mathbf{x_1} \in S_1 \text{ and } \mathbf{x_2} \in S_2\}$ とおくと，S は凸集合となり，$S_1 \cap S_2 = \emptyset$ より，$\mathbf{0} \notin S$ となる．そうでなければ，$S_1 \cap S_2$ が空集合でないこととなり仮定に反する．よって，*Theorem 2.4.7* の *Corollary 1* において $\overline{\mathbf{x}} = \mathbf{0}$ とすれば，全ての $\mathbf{x} \in S$ に対して $\mathbf{p}^t\mathbf{x} \geq 0$ を満たす $\mathbf{p} \in \mathbb{R}^n$ が存在する．

すなわち，任意のベクトル $\mathbf{x_1} \in S_1$，$\mathbf{x_2} \in S_2$ に対して，$\mathbf{x_1} - \mathbf{x_2} \in S$ となるので，$\mathbf{p}^t\mathbf{x_1} \geq \mathbf{p}^t\mathbf{x_2}$ となる．

よって，任意の $\mathbf{x_1} \in S_1$ に対して

$$\mathbf{p}^t\mathbf{x_1} \geq \sup_{\mathbf{x} \in S_2} \mathbf{p}^t\mathbf{x}$$

が成り立ち，

$$\inf_{\mathbf{x} \in S_1} \mathbf{p}^t\mathbf{x} \geq \sup_{\mathbf{x} \in S_2} \mathbf{p}^t\mathbf{x}$$

となる．

Corollary 1

Let S_1 and S_2 be nonempty convex sets in R^n. Suppose that int S_2 is not empty and that $S_1 \cap \operatorname{int} S_2$ is empty. Then there exists a hyperplane that separates S_1 and S_2; that is, there exists a nonzero p such that

$$\inf\left\{\mathbf{p}^t\mathbf{x} : \mathbf{x} \in S_1\right\} \geq \sup\left\{\mathbf{p}^t\mathbf{x} : \mathbf{x} \in S_2\right\}$$

上限の性質と内積が連続であることより，

$$\sup_{\mathbf{x} \in S_2} \mathbf{p}^t\mathbf{x} \ = \sup_{\mathbf{x} \in \operatorname{int} S_2} \mathbf{p}^t\mathbf{x}$$

が成り立つ．よって，*Theorem 2.4.8* より

$$\inf_{\mathbf{x}\in S_1} \mathbf{p}^t\mathbf{x} \geq \sup_{\mathbf{x}\in S_2} \mathbf{p}^t\mathbf{x}$$

となる．

Corollary 2

Let S_1 and S_2 be nonempty sets in R^n such that $\operatorname{int}\operatorname{conv}(S_i) \neq \varnothing$, for $i = 1, 2$, but $\operatorname{int}\operatorname{conv}(S_1) \cap \operatorname{int}\operatorname{conv}(S_2) = \varnothing$. Then there exists a hyperplane that separates S_1 and S_2.

Note the importance of assuming nonempty interiors in Corollary 2. Otherwise, for example, two crossing lines in R^2 can be taken as S_1 and S_2 [or as $\operatorname{conv}(S_1)$ and $\operatorname{conv}(S_2)$], and we would have $\operatorname{int}\operatorname{conv}(S_1) \cap \operatorname{int}\operatorname{conv}(S_2) = \varnothing$. But there does not exist a hyperplane that separates S_1 and S_2.