2022/5/12(Thu)

Chapter 3 Hilbert Spaces and Orthonomal Systems

Example 3.3.6.

The space $L^2(\mathbb{R})$ and $L^2([a,b])$ are Hilbert spaces.

内積空間が完備性を持つときにヒルベルト空間となる．L^p 空間が完備性を持つことは Theorem 2.13.4 に示されている．

■L^p **空間**　$\Omega \subset \mathbb{R}^d$ を開集合，$p \geq 1$ $(p \in \mathbb{R})$ とする．$L^p(\Omega)$ を可測関数 $f : \Omega \to \mathbb{R} \cup \{\pm\infty\}$ で，$|f|^p$ が Ω 上可積分であるものの同値類の集合とする．$f \in L^p(\Omega)$ のとき

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f(x)|^p dx\right)^{\frac{1}{p}}$$

とおく．

Example 3.3.7.

Let ρ be a measurable function defined on the interval $[a,b]$ such that $\rho(x) > 0$ almost everywhere in $[a,b]$. Denote by $L^{2,p}([a,b])$ the space of all complex-valued measurable functions on $[a,b]$ such that

$$\int_a^b |f(x)|^2 \rho(x) dx < \infty$$

This is a Hilbert space with the inner product

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}\rho(x) dx$$

To prove completeness, consider a Cauchy sequence (f_n) in $L^{2,\rho}([a,b])$. Then

$$\|f_m - f_n\|_{L^{2,\rho}([a,b])} = \int_a^b |f_m(x) - f_n(x)|^2 \rho(x) dx \to 0 \quad \text{as } n \to \infty$$

Define

$$F_n = f_n\sqrt{\rho}, \quad n \in \mathbb{N}$$

Since

$$\begin{aligned}
\|F_m - F_n\|^2_{L^2([a,b])} &= \int_a^b |F_m(x) - F_n(x)|^2 dx \\
&= \int_a^b \left| f_m(x)\sqrt{\rho(x)} - f_n(x)\sqrt{\rho(x)} \right|^2 dx \\
&= \int_a^b |f_m(x) - f_m(x)|^2 \rho(x) dx \\
&= \|f_m - f_n\|^2_{L^2\rho([a,b])}
\end{aligned}$$

(F_n) is a Cauchy sequence in $L^2([a,b])$. Thus, there exists $F \in L^2([a,b])$ such that

$$\|F_n - F\|^2_{L^2([a,b])} = \int_a^b |F_n(x) - F(x)|^2 dx \to 0.$$

We can show that $\frac{F}{\sqrt{\rho}} \in L^{2,\rho}([a,b])$ and $f_n \to \frac{F}{\sqrt{\rho}}$ in $L^{2,\rho}([a,b])$, proving completeness of $L^{2,\rho}([a,b])$.

Example 3.3.8. (Sobolev spaces)

Let Ω be an open set in $\mathbb{R}^N$. Denote by $\tilde{H}^m(\Omega), m = 1, 2, \ldots$, the space of all complex-valued functions $f \in \mathcal{C}^m(\Omega)$ such that $D^\alpha f \in L^2(\Omega)$ for all $|\alpha| \le m$, where $\alpha = (\alpha_1, \ldots, \alpha_N), \alpha_1, \ldots, \alpha_N$ are non-negative integers, $|\alpha| = \alpha_1 + \cdots + \alpha_N$, and

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_N^{\alpha_N}}.$$

For example, if $N = 2, \alpha = (2, 1)$, we have

$$D^\alpha f = \frac{\partial^3 f}{\partial x_1^2 \partial x_2}.$$

For $f \in \tilde{H}^m(\Omega)$, we thus have

$$\int_\Omega \left| \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_N^{\alpha_N}} \right|^2 < \infty$$

for every multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ such that $|\alpha| \le m$. The inner product in $\tilde{H}^m(\Omega)$ is defined by

$$\langle f, g \rangle = \int_\Omega \sum_{|\alpha| \le m} D^\alpha f \overline{D^\alpha g}.$$

If $\Omega \subseteq \mathbb{R}^2$, then the inner product in $\tilde{H}^2(\Omega)$ is

$$\langle f, g \rangle = \int_\Omega \left(f\bar{g} + f_x \overline{g_x} + f_y \overline{g_y} + f_{xx} \overline{g_{xx}} + f_{yy} \overline{g_{yy}} + f_{xy} \overline{g_{xy}} \right).$$

If $\Omega = (a, b) \subset \mathbb{R}$, the inner product in $\tilde{H}^m(a, b)$ is

$$\langle f, g \rangle = \int_a^b \sum_{n=0}^m \frac{d^n f}{dx^n} \overline{\frac{d^n g}{dx^n}}.$$

$\tilde{H}^m(\Omega)$ is an inner product space, but it is not a Hilbert space because it is not complete. The completion of $\tilde{H}^m(\Omega)$, denoted by $H^m(\Omega)$, is a Hilbert space. The space $H^m(\Omega)$ is a particular case of a general class of spaces denoted by $W^{m,p}(\Omega)$ (see Section 6.3), introduced by Sergei Lvovich Sobolev (19081989). We have $H^m(\Omega) = W^{m,2}(\Omega)$. Because of the applications to partial differential equations, spaces $H^m(\Omega)$ belong to the most important examples of Hilbert spaces.

Since every inner product space is a normed space, it is equipped with a convergence, namely the convergence defined by the norm. This convergence will be called strong convergence.

現状，なぜソボレフ空間を導入する必要があるのかは理解していない．

Definition 3.3.9. (Strong convergence)

A sequence (x_n) of vectors in an inner product space E is called *strongly convergent* to a vector x in E if $\|x_n - x\| \to 0$ as $n \to \infty$.

Definition 3.3.9. (Weak convergence)

A sequence (x_n) of vectors in an inner product space E is called *weakly convergent* to a vector x in E if $\langle x_n, y \rangle \to \langle x, y \rangle$ as $n \to \infty$, for every $y \in E$.

Theorem 3.3.11

A strongly convergent sequence is weakly convergent (to the same limit), that is, $x_n \to x$ implies $x_n \xrightarrow{w} x$.

Theorem 3.3.12.

If $x_n \to x$ and $y_n \to y$, then $\langle x_n, y_n \rangle \to \langle x, y \rangle$.

Theorem 3.3.13.

if $x_n \xrightarrow{w} x$ and $\|x_n\| \to \|x\|$, then $x_n \to x$.

Theorem 3.3.14.

Let S be a subset of an inner product space E such that span S is dense in E. If (x_n) is a bounded sequence in E and

$$\langle x_n, y \rangle \to \langle x, y \rangle \quad \text{for every } y \in S,$$

then $x_n \xrightarrow{w} x$. Proof: Clearly, if $\langle x_n, y \rangle \to \langle x, y \rangle$ for every $y \in S$, then $\langle x_n, y \rangle \to \langle x, y \rangle$ for every $y \in \operatorname{span} S$. Let $z \in E$ and let ε be an arbitrary positive number. Since span S is dense in E, there exists $y_0 \in \operatorname{span} S$ such that

$$\|z - y_0\| < \frac{\varepsilon}{3M},$$

where M is a positive constant such that $\|x\| \le M$ and $\|x_n\| \le M$ for all $n \in \mathbb{N}$. Since $\langle x_n, y \rangle \to \langle x, y \rangle$ for every $y \in \operatorname{span} S$, there exists $n_0 \in \mathbb{N}$ such that

$$|\langle x_n, y_0 \rangle - \langle x, y_0 \rangle| < \frac{\varepsilon}{3} \quad \text{for all } n > n_0.$$

Now, for any $n > n_0$, we have

$$\begin{aligned} |\langle x_n, z \rangle - \langle x, z \rangle| &\le |\langle x_n, z \rangle - \langle x_n, y_0 \rangle| + |\langle x_n, y_0 \rangle - \langle x, y_0 \rangle| \\ &\quad + |\langle x, y_0 \rangle - \langle x, z \rangle| \\ &< \|x_n\| \, \|z - y_0\| + \frac{\varepsilon}{3} + \|x\| \, \|y_0 - z\| \\ &< M \frac{\varepsilon}{3M} + \frac{\varepsilon}{3} + M \frac{\varepsilon}{3M} = \varepsilon \end{aligned}$$

Since z and ε are arbitrary, we conclude that $x_n \xrightarrow{w} x$.

The following theorem describes an important property of weakly convergent sequences in Hilbert spaces. The proof is not elementary; it is based on the Banach-Steinhaus theorem.

Theorem 3.3.15.

Weakly convergent sequences in a Hilbert space are bounded, that is, if (x_n) is a weakly convergent sequence, then there exists a number M such that $\|x_n\| \le M$ for all $n \in \mathbb{N}$.

Proof: Let (x_n) be a weakly convergent sequence in a Hilbert space H. Define

$$f_n(x) = \langle x, x_n \rangle, \quad n \in \mathbb{N}.$$

Then $f_n : H \to \mathbb{C}$ is a bounded linear functional for every $n \in \mathbb{N}$, by Theorem 3.3.11. Since, for every $x \in H$, the sequence $(\langle x, x_n \rangle)$ converges, it is bounded, that is, there exists a constant M_x such that

$$|f_n(x)| = |\langle x, x_n \rangle| \le M_x \quad \text{for all } n \in \mathbb{N}.$$

By the Banach-Steinhaus theorem (Theorem 1.5.13), there exists a constant M such that

$$\|f_n\| \le M \quad \text{for all } n \in \mathbb{N}.$$

Since

$$|f_n(x)| = |\langle x, x_n \rangle| \leq \|x\| \, \|x_n\|$$

for all $x \in H$, we have $\|f_n\| \leq \|x_n\|$. On the other hand,

$$|f_n\,(x_n)| = |\langle x_n, x_n \rangle| = \|x_n\|^2 \,.$$

Consequently, $\|f_n\| = \|x_n\|$, and thus

$$\|x_n\| \leq M \qquad \text{for all } n \in \mathbb{N}.$$