Chapter 3 Hilbert Spaces and Orthonomal Systems

Definition 3.2.13. (Orthogonal vectors)

Two vectors x and y in an inner product space are called orthogonal, denoted by $x \perp y$, if $\langle x, y\rangle = 0$.

If $x \perp y$, then $\langle y, x\rangle = \overline{\langle x, y\rangle} = 0$, and thus $y \perp x$. In other words, the relation $\perp$ is symmetric.

The next theorem is another example of the geometric character of the norm defined by an inner product.

Theorem 3.2.14. (Pythagorean formula)

For any pair of orthogonal vectors, we have

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2$$

Proof: If $x \perp y$, then $\langle x, y\rangle = \langle y, x\rangle = 0$, and thus the equality follows immediately from (3.6). □

In the definition of the inner product space we assume that E is a complex vector space. It is possible to define a real inner product space with the inner product of any two vectors being a real number. Then condition (b) in Definition 3.2.1 becomes $\langle x, y\rangle = \langle y, x\rangle$. All the preceding theorems hold in the real case. If, in Examples 3.2.2-3.2.7, the word complex is replaced by real and $\mathbb{C}$ by $\mathbb{R}$, we obtain a number of examples of real inner product spaces. A finite dimensional real inner product space is called a Euclidean space.

If $x = (x_1, \ldots, x_N)$ and $y = (y_1, \ldots, y_N)$ are vectors in $\mathbb{R}^N$, then the inner product $\langle x, y\rangle = \sum_{k=1}^{N} x_k y_k$ can be defined equivalently as $\langle x, y\rangle = \|x\|\|y\| \cos\theta$, where θ is the angle between vectors x and y. In this case, Schwarz's inequality follows from

$$\frac{|\langle x, y\rangle|}{\|x\|\|y\|} = |\cos\theta| \leq 1, \quad x \neq 0, y \neq 0$$

Definition 3.3.1. (Hilbert space)

A complete inner product space is called a *Hilbert space.*

Example 3.3.2

Since $\mathbb{C}$ is complete, is is a Hilbert space, and so is $\mathbb{C}^N$.

Example 3.3.3

l^2 is a Hilbert space. The completeness was proved in Section 1.4 (Example 1.4.6).

Example 3.3.4

The space E described in Example 3.2.5 is an inner product space, which is not a Hilbert space. It is not complete. The sequence

$$x_n = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, 0, \ldots\right)$$

is a Cauchy sequence:

$$\lim_{n,m\to\infty} \|x_n - x_m\| = \lim_{n,m\to\infty} \left[\sum_{k=\min[m,n]+1}^{\max\{m,n]} \frac{1}{k^2} \right]^{1/2} = 0.$$

However, the sequence does not converge in E, because its limit $\left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right)$ is not in E. (The sequence (x_n) converges in l^2.)

Example 3.3.5

The space discussed in Example 3.2.6 is another example of an incomplete inner product space. In fact, consider the following sequence of functions in $\mathcal{C}([0,1])$, (see Figure 3.1):

$$f_n(x) = \begin{cases} 1 & \text{if } 0 \le x \le \frac{1}{2}, \\ 1 - 2n\left(x - \frac{1}{2}\right) & \text{if } \frac{1}{2} \le x \le \frac{1}{2n} + \frac{1}{2}, \\ 0 & \text{if } \frac{1}{2n} + \frac{1}{2} \le x \le 1 \end{cases}$$

Evidently, f_n 's are continuous. Moreover,

$$\|f_n - f_m\| \le \left(\frac{1}{n} + \frac{1}{m}\right)^{1/2} \to 0, \qquad \text{as } m, n \to \infty.$$

Thus, (f_n) is a Cauchy sequence. It is easy to see that the sequence is pointwise convergent to the function

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x \le \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} < x \le 1 \end{cases}$$

The limit function is not continuous and thus not an element of $\mathcal{C}([0,1])$. Therefore the sequence (f_n) is not convergent in $\mathcal{C}([0,1])$. Consequently, $\mathcal{C}([0,1])$ is not a Hilbert space.