Appendices

A Proofs of Main Theorems

We provide the remaining part of the proof for Theorem 3 (2).

Proof of Theorem 3 (2). S_T^{GLW*} and S_T^* converge in distribution conditional on $\mathcal{S}$ to the distribution of $\Lambda_p\big(1\,(d_{m,1} \geq 0) \cdot S_{a,1}^{GLW}, 1\,(d_{m,2} \geq 0) \cdot S_{0,2}, \ldots, 1\,(d_{m,m} \geq 0) \cdot S_{0,m}\big)$ and $\Lambda_p\big(1\,(d_{m,1} \geq 0) \cdot S_{a,1}, 1\,(d_{m,2} \geq 0) \cdot S_{0,2}, \ldots, 1\,(d_{m,m} \geq 0) \cdot S_{0,m}\big)$, respectively. Since $S_{a,1}^{GLW} > S_{a,1} > 0$ almost everywhere under the local alternatives,

$$
\begin{aligned}
&\Lambda_p\left(1\,(d_{m,1} \geq 0) \cdot S_{a,1}^{GLW}, 1\,(d_{m,2} \geq 0) \cdot S_{0,2}, \ldots, 1\,(d_{m,m} \geq 0) \cdot S_{0,m}\right) \\
&\; > \Lambda_p\left(1\,(d_{m,1} \geq 0) \cdot S_{a,1}, 1\,(d_{m,2} \geq 0) \cdot S_{0,2}, \ldots, 1\,(d_{m,m} \geq 0) \cdot S_{0,m}\right)
\end{aligned}
$$

almost everywhere and so $c_{T,\alpha,\eta}^{GLW*} > c_{T,\alpha,\eta}^{*} + o_p(1)$ under the local alternatives.

Then, under the local alternatives $H_a^{(m)}$, we also have

$$
\begin{aligned}
&\lim_{T_1,T_2\to\infty} P_T\{S_T > c_{T,\alpha,\eta}^*\} \\
&= \lim_{T_1,T_2\to\infty} P_T\left\{\Lambda_p\left(S_{T,1}, S_{T,2}, \ldots, S_{T,m}\right) > c_{T,\alpha,\eta}^*\right\} \\
&= \lim_{T_1,T_2\to\infty} P_T\left\{\Lambda_p\left(S_{T,1}, \nu_T^{(2)}(\overline{x}) + \sqrt{T}\mu_2(\overline{x}), \ldots, \nu_T^{(m)}(\overline{x}) + \sqrt{T}\mu_m(\overline{x})\right) > c_{T,\alpha,\eta}^*\right\} \\
&= P\Bigg\{\Lambda_p\Bigg(\int_{\mathcal{C}_a^0}\left\{(1-\epsilon)\left[\nu_{1,2}^{(m)}(x)+\delta(x)\right]_+ + \epsilon\left[\nu_{1,2}^{(m)}(x)+\delta(x)\right]_-\right\}dx \\
&\qquad\qquad + (1-\epsilon)\int_{\mathcal{C}_a^+}\left\{\nu_{1,2}^{(m)}(x)+\delta(x)\right\}dx + \epsilon\int_{\mathcal{C}_a^-}\left\{\nu_{1,2}^{(m)}(x)+\delta(x)\right\}dx, \\
&\qquad\qquad 1\,(\mu_2(\overline{x})=0)\cdot\nu_{1,2}^{(2)}(\overline{x}), \ldots, 1\,(\mu_m(\overline{x})=0)\cdot\nu_{1,2}^{(m)}(\overline{x})\Bigg) > c_{T,\alpha,\eta}^*\Bigg\} \\
&> P\Bigg\{\Lambda_p\Bigg(\int_{\mathcal{C}_a^0}\left\{(1-\epsilon)\left[\nu_{1,2}^{(m)}(x)+\delta(x)\right]_+ + \epsilon\left[\nu_{1,2}^{(m)}(x)+\delta(x)\right]_-\right\}dx \\
&\qquad\qquad + (1-\epsilon)\int_{\mathcal{C}_a^+}\left\{\nu_{1,2}^{(m)}(x)+\delta(x)\right\}dx + \epsilon\int_{\mathcal{C}_a^-}\left\{\nu_{1,2}^{(m)}(x)+\delta(x)\right\}dx, \\
&\qquad\qquad 1\,(\mu_2(\overline{x})=0)\cdot\nu_{1,2}^{(2)}(\overline{x}), \ldots, 1\,(\mu_m(\overline{x})=0)\cdot\nu_{1,2}^{(m)}(\overline{x})\Bigg) > c_{T,\alpha,\eta}^{GLW*}\Bigg\}.
\end{aligned} \tag{A.1}
$$

Thus, we obtain the desired result in Theorem 3 (2). □

B Auxiliary Lemmas and Proofs of Lemmas

We provide the proofs of Lemma B.1, Lemma 1, and Lemma 2.

Proof of Lemma B.1. We prove this lemma under Assumption 2. The proof of the lemma under Assumption 1 is straightforward.

(1) Since the total boundedness of pseudometric space $(\mathcal{X}, \rho)$ is clear from boundedness of $\mathcal{X}$, we only need to verify (a) finite dimensional convergence and (b) the stochastic equicontinuity result: that is, for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$\overline{\lim_{T\to\infty}} \left\| \sup_{\rho(x_1,x_2)<\delta} \left| \nu_T^{(m)}(x_1) - \nu_T^{(m)}(x_2) \right| \right\|_q < \epsilon, \tag{B.1}$$

where the pseudo-metric on $\mathcal{X}$ is given by

$$\begin{aligned}\rho(x_1, x_2) = \Big\{ & E[((x_1 - X_{1,t})^{m-1}1(X_{1,t} \le x_1) - (x_1 - X_{2,t})^{m-1}1(X_{2,t} \le x_1)) \\ & - ((x_2 - X_{1,t})^{m-1}1(X_{1,t} \le x_2) - (x_2 - X_{2,t})^{m-1}1(X_{2,t} \le x_2))]^2 \Big\}^{1/2}.\end{aligned}$$

The finite dimensional convergence result holds by the Cramer-Wold device and a CLT for bounded random variables of Corollary 5.1. of Hall and Heyde (1980) since $\{(X_{1,t}, X_{2,t})^T : t = 1, \dots, T\}$ is strictly stationary and α-mixing with $\sum_{m=1}^{\infty} \alpha(m) < \infty$ by Assumption 2 (a).

To show the stochastic equicontinuity condition, let

$$\mathcal{F} = \{f_t(x) : x \in \mathcal{X}\},$$

where

$$f_t(x) = (x - X_{1,t})^{m-1}1(X_{1,t} \le x) - (x - X_{2,t})^{m-1}1(X_{2,t} \le x).$$

Then, $\mathcal{F}$ is a class of uniformly bounded functions that satisfy the L^2-continuity condition since, for some $C_1, C_2 > 0$,

$$\begin{aligned}
E \sup_{\substack{x_1\in\mathcal{X}\\|x_1-x|\le r}} [f_t(x_1) - f_t(x)]^2 &\le 2E \sup_{\substack{x_1\in\mathcal{X}\\|x_1-x|\le r}} [((x_1 - X_{1,t})^{m-1}1(X_{1,t} \le x_1) - (x - X_{1,t})^{m-1}1(X_{1,t} \le x))^2 \\
&\quad + ((x_1 - X_{2,t})^{m-1}1(X_{2,t} \le x_1) - (x - X_{2,t})^{m-1}1(X_{2,t} \le x))^2] \\
\le 4E \sup_{\substack{x_1\in\mathcal{X}\\|x_1-x|\le r}} & [((x_1 - X_{1,t})^{m-1} - (x - X_{1,t})^{m-1})^2 + (1(X_{1,t} \le x_1) - 1(X_{1,t} \le x))^2 \\
&\quad + ((x_1 - X_{2,t})^{m-1} - (x - X_{2,t})^{m-1})^2 + (1(X_{2,t} \le x_1) - 1(X_{2,t} \le x))^2] \\
\le 4E \sup_{\substack{x_1\in\mathcal{X}\\|x_1-x|\le r}} & [C_1|x_1 - x| + 1(x < X_{1,t} \le x_1) + 1(x < X_{2,t} \le x_1)] \\
\le 4E[C_1 r + 1(|X_{1,t} &- x| \le r) + 1(|X_{2,t} - x| \le r)] \le C_2 r,
\end{aligned}$$

where the first two inequalities hold by $(a + b)^2 \le 2(a^2 + b^2)$, the third inequality holds by $a^n - b^n \le (a - b) \sum_{k=0}^{n-1} a^{n-k-1} b^k$ and boundedness of random variables, and the last inequality holds by absolute continuity with respect to Lebesgue measure of Assumption 2 (b). Then, the bracketing number satisfies $N(\epsilon, \mathcal{F}) \leq C \cdot (1/\epsilon)^2$ for some $C > 0$ and so $\int_0^1 \epsilon^{-1/2} N(\epsilon, \mathcal{F})^{1/q} dx < \infty$. Furthermore, Assumption 2 (a) implies that $\sum_{m=1}^{\infty} m^{q-2} \alpha(m)^{2/(q+2)} = \sum_{m=1}^{\infty} O(m^{q-2-A\cdot 2/(q+2)}) < \infty$. Thus, the stochastic equicontinuity condition holds by Theorem 2.2. of Andrews and Pollard (1994) with $Q = q$ and $\gamma = 2$. This yields Lemma B.1. (i). $\square$

(2) Lemma B.1 (ii) follows from weak convergence results for Hilbert space valued random variables. Specifically, we can apply Theorem 3.1 of Politis and Romano (1994). First, $\{Z_t \equiv (\cdot - X_{1,t})^{m-1} 1(X_{1,t} \leq \cdot) - (\cdot - X_{2,t})^{m-1} 1(X_{2,t} \leq \cdot) : t = 1, \ldots, T\}$ is a stationary sequence of Hilbert space valued random variables which are bounded and satisfy the mixing condition $\sum_j \alpha_Z(j) < \infty$ by Assumption 2 (a). Second, Assumption 3 satisfies the condition related to the stationary resampling scheme. Thus, we have the desired result by applying the bootstrap central limit theorem.

Proof of Lemma 1. It suffices to show that for $1 \leq j \leq m$,

$$P\left\{[S_{T,j}]_+ = \left[\psi_j\left(\sqrt{T} d_{m,j}\right) \cdot S_{T,j}\right]_+\right\} \to 1 \tag{B.2}$$

since we consider a function Λ_p of the form (2.2) or (2.3) which satisfies $\Lambda_p(a_1, \ldots, a_j, \ldots, a_m) = \Lambda_p(a_1, \ldots, [a_j]_+, \ldots, a_m)$ for $a_j \in \mathbb{R}$, $1 \leq j \leq m$. Note that

$$\begin{aligned} S_{T,j} &= \psi_j\left(\sqrt{T} d_{m,j}\right) \cdot S_{T,j} + \left(1 - \psi_j\left(\sqrt{T} d_{m,j}\right)\right) \cdot S_{T,j} \\ &= \begin{cases} S_{T,j}, & \text{if } \sqrt{T} d_{m,j} \geq -\kappa_{T,j} \\ S_{T,j}, & \text{otherwise} \end{cases} \end{aligned}$$

and

$$\psi_j\left(\sqrt{T} d_{m,j}\right) \cdot S_{T,j} = \begin{cases} S_{T,j}, & \text{if } \sqrt{T} d_{m,j} \geq -\kappa_{T,j} \\ 0, & \text{otherwise.} \end{cases}$$

Suppose $\psi_j\left(\sqrt{T} d_{m,j}\right) = 0$, i.e., $\sqrt{T} d_{m,j} < -\kappa_{T,j}$. Then, we have

$$\begin{aligned} [S_{T,j}]_+ &= \max\left\{S_{T,j} - \sqrt{T} d_{m,j} + \sqrt{T} d_{m,j}, 0\right\} \\ &\leq \max\left\{S_{T,j} - \sqrt{T} d_{m,j} - \kappa_{T,j}, 0\right\} \\ &\leq \max\left\{\left|S_{T,j} - \sqrt{T} d_{m,j}\right| - \kappa_{T,j}, 0\right\}. \end{aligned} \tag{B.3}$$

Since $\left|S_{T,j} - \sqrt{T} d_{m,j}\right| = O_p(1)$ by Lemma B.1 (i) and $\kappa_{T,j}$ goes to infinity, the upper bound of $[S_{T,j}]_+$ is $o_p(1)$ when $1 - \psi_j\left(\sqrt{T}d_{m,j}\right) = 1$. Thus, we obtain

$$
\begin{aligned}
[S_{T,j}]_+ &= \psi_j\left(\sqrt{T}d_{m,j}\right) \cdot [S_{T,j}]_+ + \left(1 - \psi_j\left(\sqrt{T}d_{m,j}\right)\right) \cdot [S_{T,j}]_+ \\
&= \left[\psi_j\left(\sqrt{T}d_{m,j}\right) \cdot S_{T,j}\right]_+ + 1\left(\sqrt{T}d_{m,j} < -\kappa_{T,j}\right)[S_{T,j}]_+ \\
&= \left[\psi_j\left(\sqrt{T}d_{m,j}\right) \cdot S_{T,j}\right]_+,
\end{aligned}
$$

with probability approaching 1.

It only remains to show $\left|S_{T,j} - \sqrt{T}d_{m,j}\right| = O_p(1)$ for $1 \le j \le m$. Since $S_{T,1}$ takes a different from the other $S_{T,j}$'s, we consider two cases: $j = 1$ and $2 \le j \le m$. When $j = 1$, we have

$$
\begin{aligned}
\left|S_{T,1} - \sqrt{T}d_{m,1}\right| = &\left|\int_{\mathcal{X}} \sqrt{T}\left\{\left[\bar{F}_1^{(m)}(x) - \bar{F}_2^{(m)}(x)\right]_+ - \left[F_1^{(m)}(x) - F_2^{(m)}(x)\right]_+\right\} dx\right. \\
&\left. - \epsilon \int_{\mathcal{X}} \sqrt{T}\left\{\left|\bar{F}_1^{(m)}(x) - \bar{F}_2^{(m)}(x)\right| - \left|F_1^{(m)}(x) - F_2^{(m)}(x)\right|\right\} dx\right| \\
\le &\int_{\mathcal{X}} \left|\sqrt{T}\left[\left(\bar{F}_1^{(m)}(x) - \bar{F}_2^{(m)}(x)\right) - \left(F_1^{(m)}(x) - F_2^{(m)}(x)\right)\right]_+\right| \\
&+ \epsilon \int_{\mathcal{X}} \left|\sqrt{T}\left|\left(\bar{F}_1^{(m)}(x) - \bar{F}_2^{(m)}(x)\right) - \left(F_1^{(m)}(x) - F_2^{(m)}(x)\right)\right|\right| \\
\le &(1 + \epsilon) \cdot Q(\mathcal{X}) \cdot \sup_{x \in \mathcal{X}} \left|\nu_T^{(m)}(x)\right| = O_p(1),
\end{aligned} \tag{B.4}
$$

where the inequality holds by $[a]_+ + [b]_+ \le [a - b]_+$ and $|a| - |b| \le |a - b|$ for $a, b \in \mathbb{R}$, and the last equality holds by Lemma B.1 (i).

When $2 \le j \le m$, we have

$$
\begin{aligned}
\left|S_{T,j} - \sqrt{T}d_{m,j}\right| &= \left|\sqrt{T}\left[\bar{F}_1^{(j)}(\overline{x}) - \bar{F}_2^{(j)}(\overline{x})\right] - \left[F_1^{(j)}(\overline{x}) - F_2^{(j)}(\overline{x})\right]\right| \\
&\le \left|\nu_T^{(j)}(\overline{x})\right| = O_p(1),
\end{aligned} \tag{B.5}
$$

where the last equality holds by Lemma B.1 (i). Thus, we have the desired result. $\square$

Proof of Lemma 2. Since the empirical processes $\nu_T^{(m)}(\cdot)$ is asymptotically tight by Lemma B.1 (i), we have

$$
\begin{aligned}
&P\left(\sqrt{T}\sup_{x \in \mathcal{X}}\left|\left(\bar{F}_1^{(m)}(x) - \bar{F}_2^{(m)}(x)\right) - \left(F_1^{(m)}(x) - F_2^{(m)}(x)\right)\right| > \widehat{c}_T - c_{T,L}\right) \to 0 \\
&P\left(\sqrt{T}\sup_{x \in \mathcal{X}}\left|\left(\bar{F}_1^{(m)}(x) - \bar{F}_2^{(m)}(x)\right) - \left(F_1^{(m)}(x) - F_2^{(m)}(x)\right)\right| > c_{T,U} - \widehat{c}_T\right) \to 0
\end{aligned}
$$

by Assumption 4. Equivalently, we have

$$P\left(\sqrt{T}\sup_{x\in\mathcal{X}}\left|\left(\bar{F}_1^{(m)}(x)-\bar{F}_2^{(m)}(x)\right)-\left(F_1^{(m)}(x)-F_2^{(m)}(x)\right)\right|\leq\widehat{c}_T-c_{T,L}\right)\to 1$$
$$P\left(\sqrt{T}\sup_{x\in\mathcal{X}}\left|\left(\bar{F}_1^{(m)}(x)-\bar{F}_2^{(m)}(x)\right)-\left(F_1^{(m)}(x)-F_2^{(m)}(x)\right)\right|\leq c_{T,U}-\widehat{c}_T\right)\to 1.$$

(1) Let $x\in\mathcal{C}_0(c_{T,L})$. Then, by the triangular inequality,

$$\begin{aligned}\sqrt{T}\left|\bar{F}_1^{(m)}(x)-\bar{F}_2^{(m)}(x)\right| &\leq \sqrt{T}\left|\left(\bar{F}_1^{(m)}(x)-\bar{F}_2^{(m)}(x)\right)-\left(F_1^{(m)}(x)-F_2^{(m)}(x)\right)\right| \\ &\quad+\sqrt{T}\left|F_1^{(m)}(x)-F_2^{(m)}(x)\right| \\ &\leq (\widehat{c}_T-c_{T,L})+c_{T,L}=\widehat{c}_T\end{aligned}$$

with probability approaching 1. Thus, we have $P\left(\mathcal{C}_0(c_{T,L})\subset\widehat{\mathcal{C}}_0(\widehat{c}_T)\right)\to 1$. Now, let $x\in\widehat{\mathcal{C}}_0(\widehat{c}_T)$. The triangular inequality implies

$$\begin{aligned}\sqrt{T}\left|F_1^{(m)}(x)-F_2^{(m)}(x)\right| &\leq \sqrt{T}\left|\bar{F}_1^{(m)}(x)-\bar{F}_2^{(m)}(x)\right| \\ &\quad+\sqrt{T}\left|\left(\bar{F}_1^{(m)}(x)-\bar{F}_2^{(m)}(x)\right)-\left(F_1^{(m)}(x)-F_2^{(m)}(x)\right)\right| \\ &\leq \widehat{c}_T+(c_{T,U}-\widehat{c}_T)=c_{T,U}\end{aligned}$$

with probability approaching 1. Thus, we have $P\left(\widehat{\mathcal{C}}_0(\widehat{c}_T)\subset\mathcal{C}_0(c_{T,U})\right)\to 1$.

(2) Let $x\in\mathcal{C}_+(c_{T,U})$. Then, we have

$$\begin{aligned}\sqrt{T}\left(\bar{F}_1^{(m)}(x)-\bar{F}_2^{(m)}(x)\right) &= \sqrt{T}\left[\left(\bar{F}_1^{(m)}(x)-\bar{F}_2^{(m)}(x)\right)-\left(F_1^{(m)}(x)-F_2^{(m)}(x)\right)\right] \\ &\quad+\sqrt{T}\left(F_1^{(m)}(x)-F_2^{(m)}(x)\right) \\ &> (\widehat{c}_T-c_{T,U})+c_{T,U}=\widehat{c}_T\end{aligned}$$

with probability approaching 1. Thus, we have $P\left(\mathcal{C}_+(c_{T,U})\subset\widehat{\mathcal{C}}_+(\widehat{c}_T)\right)\to 1$. Now, let $x\in\widehat{\mathcal{C}}_+(\widehat{c}_T)$. The triangular inequality implies

$$\begin{aligned}\sqrt{T}\left(F_1^{(m)}(x)-F_2^{(m)}(x)\right) &= \sqrt{T}\left(\bar{F}_1^{(m)}(x)-\bar{F}_2^{(m)}(x)\right) \\ &\quad+\sqrt{T}\left[\left(\bar{F}_1^{(m)}(x)-\bar{F}_2^{(m)}(x)\right)-\left(F_1^{(m)}(x)-F_2^{(m)}(x)\right)\right] \\ &> \widehat{c}_T-(\widehat{c}_T-c_{T,L})=c_{T,L}\end{aligned}$$

with probability approaching 1. Thus, we have $P\left(\widehat{\mathcal{C}}_+(\widehat{c}_T)\subset\mathcal{C}_+(c_{T,L})\right)\to 1$.

(3) Let $x\in\mathcal{C}_-(c_{T,U})$. Then, we have

$$\sqrt{T}\left(\bar{F}_1^{(m)}(x)-\bar{F}_2^{(m)}(x)\right) = \sqrt{T}\left[\left(\bar{F}_1^{(m)}(x)-\bar{F}_2^{(m)}(x)\right)-\left(F_1^{(m)}(x)-F_2^{(m)}(x)\right)\right]$$

$$
\begin{aligned}
&+ \sqrt{T}\left(F_1^{(m)}(x) - F_2^{(m)}(x)\right) \\
&< (c_{T,U} - \widehat{c}_T) - c_{T,U} = -\widehat{c}_T
\end{aligned}
$$

with probability approaching 1. Thus, we have $P\left(\mathcal{C}_-(c_{T,U}) \subset \widehat{\mathcal{C}}_-(\widehat{c}_T)\right) \to 1$. Now, let $x \in \widehat{\mathcal{C}}_-(\widehat{c}_T)$. The triangular inequality implies

$$
\begin{aligned}
\sqrt{T}\left(F_1^{(m)}(x) - F_2^{(m)}(x)\right) &= \sqrt{T}\left(\bar{F}_1^{(m)}(x) - \bar{F}_2^{(m)}(x)\right) \\
&\quad + \sqrt{T}\left[\left(\bar{F}_1^{(m)}(x) - \bar{F}_2^{(m)}(x)\right) - \left(F_1^{(m)}(x) - F_2^{(m)}(x)\right)\right] \\
&< -\widehat{c}_T + (\widehat{c}_T - c_{T,L}) = -c_{T,L}
\end{aligned}
$$

with probability approaching 1. Thus, we have $P\left(\widehat{\mathcal{C}}_-(\widehat{c}_T) \subset \mathcal{C}_-(c_{T,L})\right) \to 1$. Therefore, we obtain

$$
\begin{aligned}
&P\left\{\mathcal{C}_0(c_{T,L}) \subset \widehat{\mathcal{C}}_0(\widehat{c}_T) \subset \mathcal{C}_0(c_{T,U})\right\} \to 1 \\
&P\left\{\mathcal{C}_+(c_{T,U}) \subset \widehat{\mathcal{C}}_+(\widehat{c}_T) \subset \mathcal{C}_+(c_{T,L})\right\} \to 1 \qquad \text{(B.6)} \\
&P\left\{\mathcal{C}_-(c_{T,U}) \subset \widehat{\mathcal{C}}_-(\widehat{c}_T) \subset \mathcal{C}_-(c_{T,L})\right\} \to 1.
\end{aligned}
$$

Next, we claim

$$
\begin{aligned}
&P\left\{\mathcal{C}_0(c_{T,U}) \subset \mathcal{C}_0\right\} \to 1 \\
&P\left\{\mathcal{C}_+ \subset \mathcal{C}_+(c_{T,U})\right\} \to 1 \qquad \text{(B.7)} \\
&P\left\{\mathcal{C}_- \subset \mathcal{C}_-(c_{T,U})\right\} \to 1.
\end{aligned}
$$

(1) Suppose $x \in \mathcal{C}_0(c_{T,U})$. Then, for large enough T, we have $|F_1(x) - F_2(x)| \leq \frac{c_{T,U}}{\sqrt{T}} < \epsilon$ for arbitrary $\epsilon > 0$ by definition of $c_{T,U}$. Thus, $F_1(x) = F_2(x)$ for large enough T, which implies that $x \in \mathcal{C}_0$ with probability approaching 1. We have the first part of the claim.

(2) Suppose $x \in \mathcal{C}_+$. Then, $F_1(x) - F_2(x) = c > 0$. By definition of $c_{T,U}$, we have $\frac{c_{T,U}}{\sqrt{T}} < c$ for large enough T. Thus, $F_1(x) - F_2(x) = c > \frac{c_{T,U}}{\sqrt{T}}$ for large enough T, which implies that $x \in \mathcal{C}_+(c_{T,U})$ with probability approaching 1. We have the second part of the claim.

(2) Suppose $x \in \mathcal{C}_-$. Then, $F_1(x) - F_2(x) = c < 0$. By definition of $c_{T,U}$, we have $-\frac{c_{T,U}}{\sqrt{T}} > -c$ for large enough T. Thus, $F_1(x) - F_2(x) = c < -\frac{c_{T,U}}{\sqrt{T}}$ for large enough T, which implies that $x \in \mathcal{C}_-(c_{T,U})$ with probability approaching 1. We have the last part of the claim.

Since we trivially have $\mathcal{C}_0 \subset \mathcal{C}_0(c_{T,L})$, $\mathcal{C}_+(c_{T,L}) \subset \mathcal{C}_+$, and $\mathcal{C}_-(c_{T,L}) \subset \mathcal{C}_-$ almost everywhere, combining (B.6) and (B.7) yields the desired result of the lemma. □

C Simulation Results for ASSD

We investigate the following data generating processes:

$$X \sim F$$
$$Y \sim G$$

where

$$F(x) = \frac{-x^2 + 2mx}{m^2} 1(0 \le x \le m) + 1(x > m)$$
$$G(x) = \frac{x-a}{b-a} 1(a \le x \le b) + 1(x > b)$$

with $m = 30$ and varying $a, b \in \mathbb{R}$. The data generating processes are from Osuna (2012).

To investigate the size property of the ASSD test, we consider the interior and binding cases of the null hypothesis $H_0^{(2)}$. There are four cases: $d_{2,1} < 0$ and $d_{2,2} < 0$, $d_{2,1} = 0$ and $d_{2,2} < 0$, $d_{2,1} < 0$ and $d_{2,2} = 0$, and $d_{2,1} = 0$ and $d_{2,2} = 0$. We can further divide these cases depending on whether there are crossings between two distributions. When $d_{2,1} < 0$ and $d_{2,2} < 0$ without crossings, X dominates Y by SSD and so by ASSD, which we call "Dominance 1" case. When $d_{2,1} < 0$ and $d_{2,2} < 0$ with crossings, X dominates Y by ASSD but not by SSD, which we call "Crossing (Interior)" case. When $d_{2,1} = 0$ and $d_{2,2} < 0$, there are crossings between two distributions, so we call this "Crossing (Boundary 1)" case. When $d_{2,1} < 0$ and $d_{2,2} = 0$ without crossings, X dominates Y by SSD and so by ASSD, which we call "Dominance 2" case. When $d_{2,1} < 0$ and $d_{2,2} = 0$ with crossings, X dominates Y by ASSD but not by SSD, which we call "Crossing (Boundary 2)" case. When $d_{2,1} = 0$ and $d_{2,2} = 0$ without crossings, X trivially dominates Y by SSD and so by ASSD, which we call "Same Distribution" case. When $d_{2,1} = 0$ and $d_{2,2} = 0$ with crossings, X dominates Y by ASSD but not by SSD, which we call "Crossing (Boundary 3)" case.

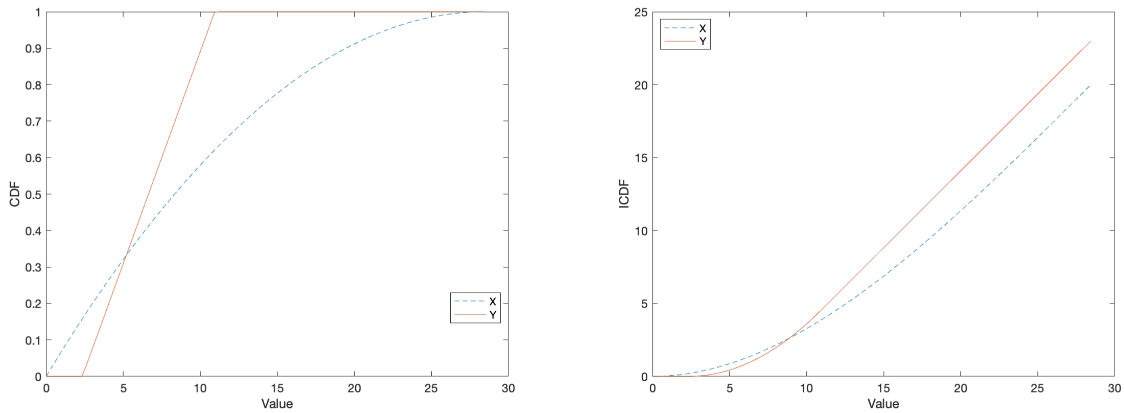
Likewise, we consider the the exterior cases of the null hypothesis $H_0^{(2)}$ to investigate the power property of the ASSD test. There are five cases: $d_{2,1} > 0$ and $d_{2,2} < 0$, $d_{2,1} > 0$ and $d_{2,2} = 0$, $d_{2,1} < 0$ and $d_{2,2} > 0$, $d_{2,1} = 0$ and $d_{2,2} > 0$, and $d_{2,1} > 0$ and $d_{2,2} > 0$. When $d_{2,1} > 0$ and $d_{2,2} < 0$, there are crossings between two distributions, so we call this "Crossing (Exterior 1)" case. When $d_{2,1} > 0$ and $d_{2,2} = 0$ without crossings, Y dominates X by SSD and so by ASSD, which we call "Reverse Dominance 1" case. When $d_{2,1} > 0$ and $d_{2,2} = 0$ with crossings, X does not dominate Y by ASSD and Y does not dominate X by SSD, which we call "Crossing (Exterior 2)" case. When $d_{2,1} < 0$ and $d_{2,2} > 0$, there are crossings between two distributions, so we call this "Crossing (Exterior 3)" case. When $d_{2,1} = 0$ and $d_{2,2} > 0$, there are crossings between two distributions, so we call this "Crossing (Exterior 4)" case. When $d_{2,1} > 0$ and $d_{2,2} > 0$

without crossings, Y dominates X by SSD and so by ASSD, which we call "Reverse Dominance 2" case. When $d_{2,1} > 0$ and $d_{2,2} > 0$ with crossings, X does not dominate Y by ASSD and Y does not dominate X by SSD, which we call "Crossing (Exterior 5)" case. We summarize our parameter choice for each of these cases in Table C.1. As an illustration, Figure C.1 shows the (integrated) cumulative distribution functions of X and Y for "Crossing (Boundary 1)" case.

Table C.1: Simulation Design for ASSD

	Case	a	b	$(d_{2,1}, d_{2,2})$
Size	Dominance 1	-12	26	$(-4.9833, -3)$
	Crossing (Interior)	1	13	$(-1.7449, -3)$
	Crossing (Boundary 1)	2.4705	11.5295	$(0, -3)$
	Dominance 2	-20	40	$(-6.25, 0)$
	Crossing (Boundary 2)	-7	27	$(-1.139, 0)$
	Same Distribution	-	-	$(0, 0)$
	Crossing (Boundary 3)	-4.2767	24.2767	$(0, 0)$
Power	Crossing (Exterior 1)	4	10	$(1.7181, -3)$
	Crossing (Exterior 1)	5	9	$(2.5097, -3)$
	Crossing (Exterior 1)	6	8	$(2.9847, -3)$
	Reverse Dominance 1	0	20	$(7.9167, 0)$
	Reverse Dominance 1	2	18	$(13.6167, 0)$
	Reverse Dominance 1	5	15	$(19.7917, 0)$
	Crossing (Exterior 2)	-1	21	$(4.8787, 0)$
	Crossing (Exterior 2)	-0.4	20.4	$(6.6476, 0)$
	Crossing (Exterior 2)	-0.1	20.1	$(7.5988, 0)$
	Crossing (Exterior 3)	-40	62	$(-9.2803, 1)$
	Crossing (Exterior 3)	-130	156	$(-72.919, 3)$
	Crossing (Exterior 3)	-230	260	$(-227.0833, 5)$
	Crossing (Exterior 4)	-23.8046	45.8046	$(0, 1)$
	Crossing (Exterior 4)	-83.1492	109.1492	$(0, 3)$
	Crossing (Exterior 4)	-144.067	174.0666	$(0, 5)$
	Reverse Dominance 2	0	22	$(23.1167, 1)$
	Reverse Dominance 2	0	26	$(49.7167, 3)$
	Reverse Dominance 2	0	30	$(71.25, 5)$
	Crossing (Exterior 5)	-13	35	$(3.6021, 1)$
	Crossing (Exterior 5)	-18	44	$(32.9038, 3)$
	Crossing (Exterior 5)	-23	53	$(82.084, 5)$

Figure C.1: CDFs and ICDFs



(a) CDFs

(b) ICDFs

For ASSD, we only report the results using the max type statistic for $p = 1$ since the sum type statistic or $p = 2$ variations show similar results. We relegate sensitivity analysis result for C_{cs} to the appendix. Table C.2 to C.5 show the rejection ratios under the null hypothesis, while Table C.6 to C.10 show that under the alternative hypothesis. In the interior case of the null hypothesis, the ASSD test shows proper size control. When the null hypothesis binds, the ASSD test has an exact size under every case of the LFC. Consistent with the theoretical prediction, the GLW test under "Crossing (Boundary)" case show conservative size control. Under the alternative hypothesis, the ASSD test is more powerful than the GLW test under "Reverse Dominance" case. As the local power analysis shows, the ASSD test is even more powerful under "Crossing (Exterior)" case.

Table C.2: Size Property $(d_{2,1} < 0, d_{2,2} < 0)$

		$H_{0,1}^{(2)}: d_{2,1} \leq 0$		$H_{0,2}^{(2)}: d_{2,2} \leq 0$	$H_0^{(2)}$		
					C_{cs}		
$(d_{2,1}, d_{2,2})$	T	$H_{0,1}^{GLW}$	$H_{0,1}^{(2)}$	$H_{0,2}^{(2)}$	0.1	0.2	0.3
(-4.9833,-3)	100	0.000	0.000	0.000	0.000	0.000	0.000
Dominance 1	200	0.000	0.000	0.000	0.000	0.000	0.000
	300	0.000	0.000	0.000	0.000	0.000	0.000
	400	0.000	0.000	0.000	0.000	0.000	0.000
	500	0.000	0.000	0.000	0.000	0.000	0.000
(-1.7449,-3)	100	0.000	0.001	0.000	0.000	0.000	0.000
Crossing (Interior)	200	0.000	0.001	0.000	0.001	0.001	0.001
	300	0.000	0.000	0.000	0.000	0.000	0.000
	400	0.000	0.000	0.000	0.000	0.000	0.000
	500	0.000	0.000	0.000	0.000	0.000	0.000

Table C.3: Size Property $(d_{2,1} = 0, d_{2,2} < 0)$

		$H_{0,1}^{(2)}: d_{2,1} \leq 0$		$H_{0,2}^{(2)}: d_{2,2} \leq 0$	$H_0^{(2)}$		
					C_{cs}		
$(d_{2,1}, d_{2,2})$	T	$H_{0,1}^{GLW}$	$H_{0,1}^{(2)}$	$H_{0,2}^{(2)}$	0.1	0.2	0.3
(-0.0000,-3)	100	0.000	0.001	0.000	0.006	0.000	0.000
Crossing (Boundary 1)	200	0.000	0.008	0.000	0.019	0.008	0.003
	300	0.000	0.018	0.000	0.025	0.018	0.009
	400	0.000	0.019	0.000	0.033	0.019	0.010
	500	0.000	0.021	0.000	0.029	0.021	0.017

Table C.4: Size Property ($d_{2,1} < 0, d_{2,2} = 0$)

		$H_{0,1}^{(2)} : d_{2,1} \leq 0$		$H_{0,2}^{(2)} : d_{2,2} \leq 0$	$H_0^{(2)}$		
						C_{cs}	
$(d_{2,1}, d_{2,2})$	T	$H_{0,1}^{GLW}$	$H_{0,1}^{(2)}$	$H_{0,2}^{(2)}$	0.1	0.2	0.3
(-6.2500,0)	100	0.002	0.002	0.051	0.051	0.051	0.051
Dominance 2	200	0.001	0.002	0.046	0.046	0.046	0.046
	300	0.000	0.002	0.049	0.049	0.049	0.049
	400	0.000	0.000	0.053	0.053	0.053	0.053
	500	0.000	0.003	0.048	0.048	0.048	0.048
(-1.1390,0)	100	0.007	0.006	0.053	0.053	0.047	0.044
Crossing (Boundary 2)	200	0.003	0.005	0.050	0.050	0.046	0.043
	300	0.002	0.003	0.054	0.054	0.054	0.049
	400	0.000	0.000	0.051	0.051	0.051	0.050
	500	0.003	0.007	0.053	0.053	0.053	0.052

Table C.5: Size Property ($d_{2,1} = 0, d_{2,2} = 0$)

		$H_{0,1}^{(2)} : d_{2,1} \leq 0$		$H_{0,2}^{(2)} : d_{2,2} \leq 0$	$H_0^{(2)}$		
						C_{cs}	
$(d_{2,1}, d_{2,2})$	T	$H_{0,1}^{GLW}$	$H_{0,1}^{(2)}$	$H_{0,2}^{(2)}$	0.1	0.2	0.3
(0,0)	100	0.047	0.052	0.056	0.053	0.053	0.053
Same Distribution	200	0.043	0.048	0.045	0.043	0.043	0.043
	300	0.048	0.052	0.049	0.050	0.050	0.050
	400	0.040	0.049	0.054	0.050	0.050	0.050
	500	0.041	0.047	0.058	0.052	0.052	0.052
(-0.0000,0)	100	0.019	0.019	0.057	0.053	0.049	0.049
Crossing (Boundary 3)	200	0.015	0.018	0.049	0.048	0.046	0.046
	300	0.019	0.022	0.055	0.054	0.051	0.050
	400	0.008	0.013	0.051	0.051	0.049	0.048
	500	0.017	0.019	0.050	0.050	0.049	0.046

Table C.6: Power Property ($d_{2,1} > 0, d_{2,2} < 0$)

		$H_{0,1}^{(2)} : d_{2,1} \leq 0$		$H_{0,2}^{(2)} : d_{2,2} \leq 0$	$H_0^{(2)}$		
						C_{cs}	
$(d_{2,1}, d_{2,2})$	T	$H_{0,1}^{GLW}$	$H_{0,1}^{(2)}$	$H_{0,2}^{(2)}$	0.1	0.2	0.3
(1.7181,-3)	100	0.000	0.073	0.000	0.135	0.063	0.016
Crossing (Exterior 1)	200	0.000	0.255	0.000	0.337	0.255	0.174
	300	0.000	0.419	0.000	0.472	0.419	0.348
	400	0.000	0.533	0.000	0.597	0.533	0.489
	500	0.000	0.648	0.000	0.700	0.648	0.599
(2.5097,-3)	100	0.000	0.231	0.000	0.347	0.221	0.091
Crossing (Exterior 1)	200	0.000	0.576	0.000	0.658	0.576	0.472
	300	0.000	0.777	0.000	0.821	0.777	0.709
	400	0.000	0.882	0.000	0.917	0.882	0.856
	500	0.000	0.947	0.000	0.964	0.947	0.937
(2.9847,-3)	100	0.000	0.379	0.000	0.523	0.369	0.200
Crossing (Exterior 1)	200	0.000	0.781	0.000	0.825	0.781	0.692
	300	0.000	0.931	0.000	0.944	0.931	0.905
	400	0.000	0.975	0.000	0.979	0.975	0.967
	500	0.000	0.994	0.000	0.996	0.994	0.992

Table C.7: Power Property ($d_{2,1} > 0, d_{2,2} = 0$)

		$H_{0,1}^{(2)} : d_{2,1} \leq 0$		$H_{0,2}^{(2)} : d_{2,2} \leq 0$	$H_0^{(2)}$		
						C_{cs}	
$(d_{2,1}, d_{2,2})$	T	$H_{0,1}^{GLW}$	$H_{0,1}^{(2)}$	$H_{0,2}^{(2)}$	0.1	0.2	0.3
(7.9167,0)	100	0.138	0.147	0.050	0.113	0.113	0.113
Reverse Dominance 1	200	0.162	0.180	0.054	0.150	0.150	0.150
	300	0.216	0.236	0.053	0.196	0.196	0.196
	400	0.260	0.287	0.049	0.236	0.236	0.236
	500	0.282	0.314	0.047	0.269	0.269	0.269
(13.6167,0)	100	0.256	0.279	0.047	0.231	0.231	0.231
Reverse Dominance 1	200	0.373	0.399	0.051	0.343	0.343	0.343
	300	0.488	0.533	0.052	0.465	0.462	0.462
	400	0.611	0.712	0.048	0.590	0.590	0.589
	500	0.714	0.860	0.045	0.710	0.702	0.697
(19.7917,0)	100	0.515	0.616	0.044	0.483	0.479	0.475
Reverse Dominance 1	200	0.772	0.952	0.054	0.757	0.754	0.753
	300	0.950	1.000	0.055	0.948	0.948	0.948
	400	0.985	1.000	0.048	0.989	0.989	0.989
	500	0.999	1.000	0.044	1.000	1.000	1.000
(4.8787,0)	100	0.093	0.097	0.050	0.080	0.080	0.080
Crossing (Exterior 2)	200	0.101	0.117	0.054	0.083	0.083	0.083
	300	0.109	0.124	0.056	0.099	0.099	0.099
	400	0.139	0.150	0.050	0.123	0.123	0.123
	500	0.136	0.151	0.050	0.127	0.127	0.127
(6.6476,0)	100	0.117	0.126	0.048	0.099	0.099	0.099
Crossing (Exterior 2)	200	0.140	0.153	0.054	0.123	0.123	0.123
	300	0.163	0.191	0.052	0.155	0.155	0.155
	400	0.198	0.218	0.049	0.182	0.182	0.182
	500	0.210	0.240	0.048	0.198	0.198	0.198
(7.5988,0)	100	0.128	0.139	0.048	0.110	0.110	0.110
Crossing (Exterior 2)	200	0.156	0.170	0.055	0.146	0.146	0.146
	300	0.204	0.220	0.052	0.187	0.187	0.187
	400	0.245	0.269	0.049	0.213	0.213	0.213
	500	0.269	0.294	0.047	0.246	0.246	0.246

Table C.8: Power Property ($d_{2,1} < 0, d_{2,2} > 0$)

		$H_{0,1}^{(2)} : d_{2,1} \leq 0$		$H_{0,2}^{(2)} : d_{2,2} \leq 0$	$H_0^{(2)}$		
						C_{cs}	
$(d_{2,1}, d_{2,2})$	T	$H_{0,1}^{GLW}$	$H_{0,1}^{(2)}$	$H_{0,2}^{(2)}$	0.1	0.2	0.3
(-9.2803,1)	100	0.002	0.010	0.097	0.097	0.097	0.097
Crossing (Exterior 3)	200	0.001	0.006	0.115	0.115	0.115	0.115
	300	0.001	0.005	0.131	0.131	0.131	0.131
	400	0.000	0.002	0.150	0.150	0.150	0.150
	500	0.000	0.006	0.165	0.165	0.165	0.165
(-72.9190,3)	100	0.003	0.011	0.092	0.092	0.092	0.092
Crossing (Exterior 3)	200	0.001	0.007	0.123	0.123	0.123	0.123
	300	0.001	0.007	0.152	0.152	0.152	0.152
	400	0.000	0.005	0.164	0.164	0.164	0.164
	500	0.000	0.007	0.180	0.180	0.180	0.180
(-227.0833,5)	100	0.004	0.011	0.089	0.089	0.089	0.089
Crossing (Exterior 3)	200	0.001	0.007	0.124	0.124	0.124	0.124
	300	0.001	0.007	0.149	0.149	0.149	0.149
	400	0.000	0.005	0.166	0.166	0.166	0.166
	500	0.000	0.007	0.175	0.175	0.175	0.175

Table C.9: Power Property ($d_{2,1} = 0, d_{2,2} > 0$)

		$H_{0,1}^{(2)} : d_{2,1} \leq 0$		$H_{0,2}^{(2)} : d_{2,2} \leq 0$	$H_0^{(2)}$		
						C_{cs}	
$(d_{2,1}, d_{2,2})$	T	$H_{0,1}^{GLW}$	$H_{0,1}^{(2)}$	$H_{0,2}^{(2)}$	0.1	0.2	0.3
(0.0000,1)	100	0.003	0.011	0.128	0.128	0.128	0.127
Crossing (Exterior 4)	200	0.002	0.011	0.149	0.149	0.149	0.149
	300	0.002	0.010	0.201	0.201	0.201	0.201
	400	0.000	0.010	0.235	0.235	0.235	0.235
	500	0.003	0.015	0.260	0.260	0.260	0.260
(0.0000,3)	100	0.006	0.016	0.131	0.131	0.131	0.131
Crossing (Exterior 4)	200	0.001	0.010	0.168	0.168	0.168	0.168
	300	0.001	0.009	0.243	0.243	0.243	0.243
	400	0.000	0.011	0.273	0.273	0.273	0.273
	500	0.003	0.017	0.307	0.307	0.307	0.307
(0.0000,5)	100	0.008	0.016	0.138	0.138	0.138	0.138
Crossing (Exterior 4)	200	0.001	0.013	0.168	0.168	0.168	0.168
	300	0.001	0.008	0.245	0.245	0.245	0.245
	400	0.000	0.011	0.271	0.271	0.271	0.271
	500	0.002	0.017	0.309	0.309	0.309	0.309

Table C.10: Power Property ($d_{2,1} > 0, d_{2,2} > 0$)

		$H_{0,1}^{(2)}: d_{2,1} \leq 0$		$H_{0,2}^{(2)}: d_{2,2} \leq 0$	$H_0^{(2)}$		
						C_{cs}	
$(d_{2,1}, d_{2,2})$	T	$H_{0,1}^{GLW}$	$H_{0,1}^{(2)}$	$H_{0,2}^{(2)}$	0.1	0.2	0.3
(23.1167,1)	100	0.397	0.420	0.282	0.372	0.372	0.372
Reverse Dominance 2	200	0.589	0.612	0.430	0.567	0.567	0.567
	300	0.721	0.747	0.559	0.699	0.699	0.699
	400	0.830	0.857	0.658	0.819	0.819	0.819
	500	0.910	0.924	0.757	0.900	0.900	0.900
(49.7167,3)	100	0.857	0.869	0.877	0.876	0.876	0.876
Reverse Dominance 2	200	0.988	0.989	0.995	0.994	0.994	0.994
	300	1.000	1.000	1.000	1.000	1.000	1.000
	400	1.000	1.000	1.000	1.000	1.000	1.000
	500	1.000	1.000	1.000	1.000	1.000	1.000
(71.2500,5)	100	0.987	0.988	0.997	0.997	0.997	0.997
Reverse Dominance 2	200	1.000	1.000	1.000	1.000	1.000	1.000
	300	1.000	1.000	1.000	1.000	1.000	1.000
	400	1.000	1.000	1.000	1.000	1.000	1.000
	500	1.000	1.000	1.000	1.000	1.000	1.000
(3.6021,1)	100	0.012	0.013	0.161	0.161	0.160	0.155
Crossing (Exterior 5)	200	0.005	0.021	0.212	0.212	0.212	0.211
	300	0.005	0.026	0.306	0.306	0.306	0.306
	400	0.001	0.032	0.352	0.352	0.352	0.352
	500	0.007	0.032	0.416	0.416	0.416	0.416
(32.9038,3)	100	0.059	0.096	0.474	0.474	0.474	0.474
Crossing (Exterior 5)	200	0.086	0.208	0.686	0.686	0.686	0.686
	300	0.126	0.370	0.829	0.829	0.829	0.829
	400	0.171	0.484	0.913	0.913	0.913	0.913
	500	0.223	0.595	0.964	0.964	0.964	0.964
(82.0840,5)	100	0.156	0.253	0.694	0.694	0.694	0.693
Crossing (Exterior 5)	200	0.274	0.541	0.911	0.911	0.911	0.911
	300	0.452	0.736	0.985	0.985	0.985	0.985
	400	0.591	0.873	0.998	0.998	0.998	0.998
	500	0.695	0.952	1.000	1.000	1.000	1.000

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