# The indexed adjoint functor theorem

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## Contents



## 1 Background

We only assume familiarity with the most fundamental definitions about fibrations and indexed categories: what they are, and how they are equivalent. A reading of the first few chapters of [Str18] suffices.

| | Fibrations over $B$ | $B$-indexed categories |
|---|---|---|
| 0-cells | Grothendieck fibration $X \to B$ | Pseudofunctor $B^{op} \to \mathrm{Cat}$ |
| 1-cells | Cartesian functor | Pseudonatural transformation |
| 2-cells | Vertical transformation | Modification |

The direction $\implies$ takes a cleavage inducing functors $x^*$ which assemble naturally into a pseudofunctor. The opposite direction takes the total category / Grothendieck construction of the pseudofunctor.

For notation, we always use blackboard bold $\mathbb{C}$ for indexed categories induced by a fibration with domain $\mathcal{C}$.

## 2 Locally small and well-powered categories

To give some motivation, let $i : \mathbf{Set} \to \mathbf{Cat}$ be the discrete category functor, so for any category $\mathcal{C}$ we have an indexing induced by $X \to \mathcal{C}^{iX}$. We actually write $\mathcal{C}^I$.

Let's assume $\mathcal{C}$ is locally small. Then given families $X, Y \in \mathcal{C}^I$, there's an obvious candidate hom-set $[X, Y]$ indexed by $I$ with $[X, Y]_i = [X_i, Y_i]$. Letting $p : [X, Y] \to I$, we take $p^*X$. We then have a canonical indexed map

$\phi : p^*X \to p^*Y$ whose $f$-component is $f$. Then $[X, Y]$ being a hom-set can be expressed by the following universal property: for any $h : J \to I$, $\psi : h^*X \to p^*Y$ we have a unique factoring depicted below

$$\begin{tikzcd} h^*X \arrow[rr, dashed] \arrow[dr, "\psi"'] & & p^*X \arrow[dl, "\phi"] \\ & Y & \end{tikzcd} \qquad \begin{tikzcd} J \arrow[rr, dashed] \arrow[dr, "h"'] & & {[X,Y]} \arrow[dl, "p"] \\ & I & \end{tikzcd}$$

It's pretty straightforward to see this: $\psi$ consists of $h^*X \to h^*Y$, that is, a $J$-indexed collection of maps $[X, Y]$, but this already gives us the map on the right, and consequently the map on the left.

With the above motivation in mind, consider:

**Definition 2.1.** *Let $P : X \to B$ be a fibration.*

*i) Given $X, Y \in X^I$,* $\mathrm{Hom}_I(X, Y)$ *is the category whose objects are pairs*

$$\begin{tikzcd} & U \arrow[dl, "\varphi"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

*s.t. $P(\varphi) = P(f)$ where $\varphi$ is cartesian and whose arrows are s.t.*

$$\begin{tikzcd} & X & \\ U \arrow[ur, "\varphi"] \arrow[rr, "\theta"] \arrow[dr, "f"'] & & V \arrow[ul, "\varphi'"'] \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

*We call $P$ locally small if the fibration* $\mathrm{Hom}_I(X, Y) \to B/I$ *is representable, that is,* $\mathrm{Hom}_I(X, Y)$ *admits a terminal object.*

Have the reader read [Joh02], they'll probably be familiar with a different definition:

**Definition 2.2.**

*i) If $I$ is any category,* $\mathrm{Rect}(I, \mathbb{C})$ *is the category of vertical diagrams (in the non-indexed sense, on the fibration domain) in $\mathbb{C}$ and transformations with cartesian components.*

*ii) We say $\mathbb{C}$ is locally small if the forgetful functor* $\mathrm{Rect}(\mathbb{2}, \mathbb{C}) \to \mathrm{Rect}(\mathbf{2}, \mathbb{C})$ *has a right adjoint.*

*iii) More generally, we say $\mathbb{C}$* ***satisfies the comprehension scheme*** *for $F : I \to J$ if $F^*$ :* $\mathrm{Rect}(J, \mathbb{C}) \to \mathrm{Rect}(I, \mathbb{C})$ *has a right adjoint.*

Expanding the second definition in terms of comma categories, we have a straightforward equivalence.

**Definition 2.3.**

*i) We write* $\mathrm{Rect}_{mono}(I, \mathbb{C})$ *for the full subcategory of* $\mathrm{Rect}(I, \mathbb{C})$ *whose diagrams have all arrows mono.*

*ii) We say $\mathbb{C}$ is well-powered if each $x^*$ preserves monos and the codomain functor* $\mathrm{Rect}_{mono}(\mathbb{2}, \mathbb{C}) \to \mathrm{Rect}(\mathbf{1}, \mathbb{C})$ *has a right adjoint.*

# 3 Unpacking the properties

We now consider what does it mean for an indexed category to be locally-small, well-powered, etc. Recall that in the Grothendieck fibration obtained from $\mathbb{C}$, cartesian arrows are precisely $(f, u)$ with $u$ an isomorphism.

**Lemma 3.1.** *An indexed category $\mathbb{C}$ is locally small iff for $X, Y$, we have $d : \mathrm{hom}(X, Y) \to I \times J, g : d_0^*X \to d_1^*Y$ s.t. for every $u : K \to I \times J$ and $f : u_0^*X \to u_1^*Y$ we have unique $v : K \to \mathrm{hom}(X, Y)$ such that $u = d \circ v$ and*

$$f = \psi_{d,v} \circ v^*g \circ \psi_{d,v}^{-1}$$

*Proof.*

$$\begin{tikzcd} V \arrow[rr, "(g,\alpha)"] \arrow[dr, "(1,\psi_{g,1}^{-1}\alpha)"'] & & X \\ & g^*X \arrow[ur, "(g,1)"'] & \end{tikzcd} \qquad \begin{tikzcd} & u^*X \arrow[dl, "(u,1)"'] \arrow[dr, "(u,f)"] \arrow[dd, dashed, "(v,k)"] & \\ X & & Y \\ & d^*X \arrow[ul, "(d,1)"] \arrow[ur, "(d,g)"'] & \end{tikzcd}$$

□

In the same way,

**Theorem 3.2.** *An indexed category whose transition functors preserve monos is well-powered if given $A \in C^I$ we have $d : \mathrm{sub}(A) \to I$ and $m : X \rightarrowtail d^*A$ generic in the sense that for any other $m' : B \to u^*A$ we have $d \circ v = u$ and $v^*m = m'$ as subobjects.*

Let us consider some other properties.

**Definition 3.3.**

*i) We say $\mathbb{C}$ has definable equality if it satisfies the comprehension scheme w.r.t. the functor collapsing two parallel arrows.*

*ii) We say $\mathbb{C}$ has definable invertibility if it satisfies the comprehension scheme w.r.t. the functor including $\mathbb{2}$ inside its grupoid closure.*

The following are easily proved by unpacking the right adjoint definition with comma categories:

**Lemma 3.4.**

*i) $\mathbb{C}$ has definable equality iff for every $f, g \in \mathcal{C}^I$ we have a subobject $I' \rightarrowtail I$ s.t. every $x : J \to I$ s.t. $x^*f = x^*g$ factors through $I'$.*

*ii) $\mathbb{C}$ has definable invertibility iff for every $f \in \mathcal{C}^I$ we have a subobject $I' \rightarrowtail I$ s.t. every $x : J \to I$ s.t. $x^*f$ is iso factors through $I'$.*

**Theorem 3.5.** *Let $\mathbb{C}$ be a $\mathcal{S}$-indexed locally small category:*

*i) If $\mathcal{S}$ has equalizers, $\mathbb{C}$ has definable equality*

*ii) If $\mathcal{S}$ also has pullbacks, $\mathbb{C}$ has definable invertibility*

*Proof.* i): Let $d : J \to I \times I$ be the arrow indexing morphisms given by $\mathbb{C}$ being locally small. Then both $f$ and $g$ correspond to arrows $I \to J$, and taking their equalizer we obtain the desired subobject.

ii): Let $f : X \to Y \in \mathcal{C}^I$ be given. Let

$$\begin{tikzcd} J \arrow[r, tail, "v"] \arrow[d, "x"'] \arrow[dr, phantom, "\lrcorner", very near start] & \hom(Y,X) \arrow[d] \\ I \arrow[r, tail] & I \times I \end{tikzcd}$$

Letting $h$ be the generic such morphism, we put $g = x^*h$, where $h$ is the generic morphism. Then we take $K \rightarrowtail J$ to be the intersection of arrows measuring equality of $x^*(f) \circ g = 1, g \circ x^*(f)$, respectively. Then one can see that factorizations of some $y$ through $K \rightarrowtail J \to I$ correspond to inverses of $y^*(f)$, but by uniqueness the latter arrow is mono. □

We will later use these to transfer properties from an indexed category to its comma construction.

# 4 Indexed (co)completeness

Through this entire section, we fix a base category $\mathcal{S}$ which we'll assume for our ends to be a topos (though it needn't be in a lot of results we'll state).

**Definition 4.1.** *We say $\mathbb{C}$ has $\mathcal{S}$-indexed products if every $x^*$ has a right adjoint $\Pi_X$, satisfying the following Beck-Chevalley condition: for every pullback square*

$$\begin{tikzcd} I \arrow[r, "w"] \arrow[d, "x"'] \arrow[dr, phantom, "\lrcorner", very near start] & K \arrow[d, "y"] \\ J \arrow[r, "z"'] & L \end{tikzcd}$$

*we have a 2-cell isomorphism*

$$\begin{tikzcd} C^J \arrow[r, "w^*"] \arrow[d, "\Pi_y"'] \arrow[dr, phantom, "\overset{\Phi}{\nearrow}\cong"] & C^I \arrow[d, "\Pi_x"] \\ C^L \arrow[r, "z^*"'] & C^K \end{tikzcd}$$

*where $\Phi_A$ is the $x^*$-transpose of*

$$x^*z^*\Pi_y A \xrightarrow{\cong} w^*y^*\Pi_y A \xrightarrow{w^*\epsilon^y_A} w^*A$$

*We also have the dual definition of indexed coproducts.*

**Remark 4.2.** *Note that due do uniqueness of adjoints we can get coherence isos $\Theta, \psi$ for $\Pi_x$, for instance, given by*

$$\Theta^{-1} = \epsilon^1 \circ \theta_{\Pi_1}$$

$$\psi_{y,x} = \Pi_{yx}\epsilon^x \circ \Pi_{yx}x^*\epsilon^y_{\Pi_x} \circ \Pi_{yx}\phi^{-1}_{y,x\,\Pi_y\Pi_x} \circ \eta^{yx}_{\Pi_y\Pi_x}$$

*Which gives us*

$$(\Theta^{-1} * \theta^{-1}) \circ \eta^1 = \theta^{-1} \circ \epsilon_{1^*} \circ \theta_{\Pi_1 1^*} \circ \eta^1 = 1$$

*Also,*

$$\psi_{y,x} = \Pi_{yx}\epsilon^x \circ \Pi_{yx}x^*\epsilon^y_{\Pi_x} \circ \Pi_{yx}\phi^{-1}_{y,x\,\Pi_y\Pi_x} \circ \eta^{yx}_{\Pi_y\Pi_x}$$

*so a straightforward but long and tedious calculation shows:*

$$\begin{aligned}
&(\psi_{y,x} * \phi_{y,x}) \circ \Pi_y\eta^x_{y^*} \circ \eta^y \\
&= \Pi_{yx}\phi_{y,x} \circ [(\Pi_{yx}\epsilon^x \circ \Pi_{yx}x^*\epsilon^y_{\Pi_x} \circ \Pi_{yx}\phi^{-1}_{y,x\,\Pi_y\Pi_x} \circ \eta^{yx}_{\Pi_y\Pi_x}) * (x^*y^*)] \circ \Pi_y\eta^x_{y^*} \circ \eta^y \\
&= \Pi_{yx}\phi_{y,x} \circ [(\Pi_{yx}\epsilon^x \circ \Pi_{yx}x^*\epsilon^y_{\Pi_x} \circ \Pi_{yx}\phi^{-1}_{y,x\,\Pi_y\Pi_x}) * (x^*y^*)] \circ \Pi_{yx}(yx)^*\Pi_y\eta^x_{y^*} \circ \eta^{yx}_{\Pi_y y^*} \circ \eta^y && \textit{(Naturality of } \eta^{yx}\textit{)} \\
&= \Pi_{yx}\phi_{y,x} \circ [(\Pi_{yx}\epsilon^x \circ \Pi_{yx}x^*\epsilon^y_{\Pi_x}) * (x^*y^*)] \circ \Pi_{yx}x^*y^*\Pi_y\eta^x_{y^*} \circ \Pi_{yx}\phi^{-1}_{y,x\,\Pi_y y^*} \circ \eta^{yx}_{\Pi_y y^*} \circ \eta^y && \textit{(Naturality of } \phi^{-1}_{y,x}\textit{)} \\
&= \Pi_{yx}\phi_{y,x} \circ [(\Pi_{yx}\epsilon^x) * (x^*y^*)] \circ \Pi_{yx}x^*\eta^x_{y^*} \circ \Pi_{yx}x^*\epsilon^y_{y^*} \circ \Pi_{yx}\phi^{-1}_{y,x\,\Pi_y y^*} \circ \eta^{yx}_{\Pi_y y^*} \circ \eta^y && \textit{(Naturality of } \epsilon^y\textit{)} \\
&= \Pi_{yx}\phi_{y,x} \circ \Pi_{yx}x^*\epsilon^y_{y^*} \circ \Pi_{yx}\phi^{-1}_{y,x\,\Pi_y y^*} \circ \eta^{yx}_{\Pi_y y^*} \circ \eta^y && \textit{(Triangle identity for } x\textit{)} \\
&= \Pi_{yx}\phi_{y,x} \circ \Pi_{yx}x^*\epsilon^y_{y^*} \circ \Pi_{yx}\phi^{-1}_{y,x\,\Pi_y y^*} \circ \Pi_{yx}(yx)^*\eta^y \circ \eta^{yx} && \textit{(Naturality of } \eta^{yx}\textit{)} \\
&= \Pi_{yx}\phi_{y,x} \circ \Pi_{yx}x^*\epsilon^y_{y^*} \circ \Pi_{yx}x^*y^*\eta^y \circ \Pi_{yx}\phi^{-1}_{y,x} \circ \eta^{yx} && \textit{(Naturality of } \phi^{-1}_{y,x}\textit{)} \\
&= \eta^{yx} && \textit{(Triangle identity for } y\textit{)}
\end{aligned}$$

A nice characterization of the existence of internal coproducts comes from the following result:

**Theorem 4.3.** *Let $P : \mathcal{C} \to \mathcal{S}$ be a Grothendieck fibration.*

*i) Each $x^*$ has a left adjoint iff $P$ is an opfibration as well.*

*ii) $\mathbb{C}$ has $\mathcal{S}$-indexed coproducts iff $P$ is an opfibration and the pullback of any cocartesian arrow along a cartesian arrow exists and is again cocartesian.*

*Proof.* [Joh02, Lemma B.1.4.5] □

**Definition 4.4.** *An indexed category $\mathbb{C}$ is $\mathcal{S}$-complete if it has $\mathcal{S}$-indexed products and $\mathcal{S}$-indexed finite limits (that is, each $C^I$ has finite limits preserved by the reindexing functors).*

**Definition 4.5.** *We call an indexed functor between $\mathcal{S}$-complete categories $F : \mathbb{C} \to \mathbb{D}$ $\mathcal{S}$-continuous if it is componentwise complete and the mate in the right is an isomorphism as well for every $x : I \to J$:*

$$\begin{tikzcd}
C^J \arrow[r, "x^*"] \arrow[d, "F^J"'] \arrow[dr, phantom, "\overset{\cong}{\Nearrow}"] & C^I \arrow[d, "F^I"] \\
D^J \arrow[r, "x^*"'] & D^I
\end{tikzcd}
\quad \iff \quad
\begin{tikzcd}
C^I \arrow[r, "F^I"] \arrow[d, "\Pi_x"'] \arrow[dr, phantom, "\overset{\cong}{\Nearrow}"] & D^I \arrow[d, "\Pi_x"] \\
C^J \arrow[r, "F^I"'] & D^J
\end{tikzcd}$$

# 5 Interlude: Beck-Chevalley

Recall we can define adjointness using the 2-cells, which are transformations $\alpha : F \implies G$ with, for every $x : J \to I$,

$$\alpha_{Jx^*} \circ F^x = G^x \circ x^* \alpha_I$$

if we unpack what it means to be a modification in this context,

Thus we can define a notion of adjunction in this context, and if we do we'll end up with the following: $F \dashv G$ implies there we have componentise units and counits satisfying (recall the 2-cells are modifications)

$$\begin{tikzcd}
x^* \arrow[r, "x^*\eta^I"] \arrow[d, "\eta^J_{x^*}"'] & x^* G^I F^I \arrow[d, "G^x F^I"] \\
G^J F^J x^* \arrow[r, "(G^J F^x)^{-1}"'] & G^J x^* F^I
\end{tikzcd}
\qquad
\begin{tikzcd}
F^J x^* G^I \arrow[r, "(F^x{}_{G^J})^{-1}"] \arrow[d, "F^I G^x"'] & x^* F^I G^I \arrow[d, "x^*\epsilon^J"] \\
F^J G^J x^* \arrow[r, "\epsilon^I x^*"'] & x^*
\end{tikzcd}$$

Let us take the mate of $(F^x)^{-1}$:

$$\begin{tikzcd}
D^I \arrow[r] \arrow[dr, no head] & C^I \arrow[r, "x^*"] \arrow[d, "F^I"] \arrow[dl, phantom, "\epsilon^I \Swarrow"] \arrow[dr, phantom, "(F^x)^{-1} \Swarrow"] & C^J \arrow[d, "F^J"] \arrow[dr] \arrow[dr, phantom, "\Swarrow \eta^J"'] & \\
& D^I \arrow[r, "x^*"'] & D^J \arrow[r] & C^J
\end{tikzcd}$$

We get

$$\begin{aligned}
&G^J x^* \epsilon^I \circ G^J (F^x)^{-1} G^I \circ \eta^J x^* G^I \\
&= G^J x^* \epsilon^I \circ G^x F^I G^I \circ x^* \eta^I G^I && \text{(by using the identities above)} \\
&= G^x \circ x^* G^I \epsilon^I \circ x^* \eta^I G^I && \text{(naturality of } G^x\text{)} \\
&= G^I && \text{(functoriality and triangle identities)}
\end{aligned}$$

Does converse hold? Yes, as long as $(F^x)^{-1}$ is invertible.

$$
\begin{aligned}
& G^x F^I \circ x^* \eta^I \\
&= G^J x^* \epsilon^I_{F^I} \circ G^J (F^x_{F^I})^{-1} \circ \eta^J_{x^* G^J} \circ x^* \eta^I && ((F^x)^{-1} \text{ and } G^x \text{ are mates}) \\
&= G^J x^* \epsilon^I_{F^I} \circ G^J (F^x_{F^I})^{-1} \circ G^J F^J x^* \eta^I \circ \eta^J_{x^*} && (\text{naturality of } \eta^J) \\
&= G^J x^* \epsilon^I_{F^I} \circ G^J x^* F^I \eta^I \circ G^J (F^x_{F^I})^{-1} \circ \eta^J_{x^*} && (\text{naturality of } G^J (F^x)^{-1}) \\
&= G^J (F^x_{F^I})^{-1} \circ \eta^J_{x^*} && (\text{functoriality and triangle identities})
\end{aligned}
$$

Now, what does Beck-Chevalley has to do with all this?
Firstly, recall tha for

$$
\begin{tikzcd}
I \arrow[r, "w"] \arrow[d, "x"'] \arrow[dr, phantom, "\lrcorner", very near start] & K \arrow[d, "y"] \\
J \arrow[r, "z"'] & L
\end{tikzcd}
$$

we have an isomorphism given by the $x^*$-transpose of

$$
x^* z^* \Pi_y A \xrightarrow{\cong} w^* y^* \Pi_y A \xrightarrow{w^* \epsilon^y_A} w^* A
$$

which is

$$
\Pi_x w^* \epsilon^y \circ \Pi_x \phi^{-1}_{w,y} \phi_{x,z_{\Pi_y}} \circ \eta^x_{z^* \Pi_y}
$$

that is,

$$
\begin{tikzcd}
C^K \arrow[r, "\Pi_y"] \arrow[dr] & C^L \arrow[r, "z^*"] \arrow[d, "y^*"'] \arrow[l, phantom, "\Swarrow \epsilon", near start, shift right=2] \arrow[dr, phantom, "\Swarrow \phi^{-1}\phi"] & C^J \arrow[d, "x^*"] \arrow[dr, no head] \arrow[dr, phantom, "\Swarrow \eta", shift right=3] & \\
& C^K \arrow[r, "w^*"'] & C^I \arrow[r, "\Pi_x"'] & C^J
\end{tikzcd}
$$

the mate of $\phi^{-1}\phi$.

**Definition 5.1.** *A comprehension category is a "cartesian" functor $Q$*

$$
\begin{tikzcd}
\mathcal{E} \arrow[rr, "Q"] \arrow[dr, "P"'] & & \mathcal{S}^{\to} \arrow[dl, "\mathrm{cod}"] \\
& \mathcal{S} &
\end{tikzcd}
$$

*except that we don't assume $\mathcal{S}$ has pullbacks, hence* cod *is not necessarily a Grothendieck fibration, but $P$ is.*

**Definition 5.2.** *Given a comprehension category $Q : \mathcal{E} \to \mathcal{S}^{\to}$ and an $\mathcal{S}$-indexed category $\mathbb{C}$ given by $F : \mathcal{S}^{op} \to \mathrm{Cat}$, $[Q]_{\mathbb{C}}$ is the indexed functor given by*

$$
\begin{tikzcd}
\mathcal{E}^{op} \arrow[r, bend left=40, "(\mathrm{cod} \circ Q)^{op}", ""{name=U, below}] \arrow[r, bend right=40, "(\mathrm{dom} \circ Q)^{op}"', ""{name=D, above}] & \mathcal{S}^{op} \arrow[r, "F"] & \mathrm{Cat}
\arrow[from=U, to=D, Rightarrow, "(\mathrm{arr} * Q)^{op}"]
\end{tikzcd}
$$

Given a functor $P : \mathcal{E} \to \mathcal{S}$, we write $\mathrm{cart}(P)$ for the (possibly non-full) subcategory of $\mathcal{E}$ whose arrows are cartesian w.r.t. $P$. Consider thus the comprehension category $Q : \mathrm{cart}(\mathrm{cod}) \to \mathcal{S}^{\to}$ and the induced indexed functor $F = [Q]_{\mathbb{C}}$.

**Theorem 5.3.** *The category $\mathbb{C}$ has $\mathcal{S}$-indexed coproducts iff $F$ has an indexed left adjoint.*

*Proof.* We've seen above that as long as the mate of each $G^x$ satisfies invertibility and we have componentwise adjoints, we can construct an indexed left adjoint.

Let us look at the indexed functor $[Q]_{\mathbb{C}} = F : \mathbb{D} \to \mathbb{E}$. Let

$$\begin{tikzcd} I \arrow[r, "w"] \arrow[d, "x"'] \arrow[dr, phantom, very near start, "\lrcorner"] & K \arrow[d, "y"] \\ J \arrow[r, "z"'] & L \end{tikzcd}$$

be an arrow $x \to y$ in cart(cod).

- Its $x$-component is $x^* : C^J \to C^I$
- $F^{(}w, z)$ is given by $\phi^{-1}_{w,y}\phi_{x,z}$

It thus follows from what has been discussed previously. ☐

As a last thing, we are going to relate the Beck-Chevalley condition to spans, so that it'll be applied in the next section

**Remark 5.4.** *Let $\mathbb{D}$ be an $\mathcal{S}$-complete category. Consider the category $\langle X, Y$ of spans*

$$\begin{tikzcd} S \arrow[r, "y"] \arrow[d, "x"'] & Y \\ X & \end{tikzcd}$$

*with fixed endpoints, and the obvious morphism.*

*Then we can associate with the above span the functor $\Pi_y x^* : D^X \to D^Y$. In fact, we can functorially associate these by taking*

$$\begin{tikzcd} S \arrow[rr, "y"] \arrow[dr, "u"] \arrow[dd, "x"'] & & Y \\ & S' \arrow[ur, "y'"'] \arrow[dl, "x'"] & \\ X & & \end{tikzcd}$$

*to*

$$\Pi_{y'} x'^* \xrightarrow{\Pi_{y'}\eta^u x'^*} \Pi_{y'}\Pi_u u^* x'^* \xrightarrow{\psi_{y',u} * \phi_{x',u}} \Pi_y x^*$$

*thus yielding a contravariant action.*

*Furthermore, given two spans in $< X, Y >, < Y, Z >$ respectively, we can form their composition in $< X, Z >$ by taking pullbacks*

$$\begin{tikzcd} U \arrow[r, "t"] \arrow[d, "s"'] \arrow[dr, phantom, very near start, "\lrcorner"] & T \arrow[r, "z"] \arrow[d, "w"] & Z \\ S \arrow[r, "y"'] \arrow[d, "x"'] & Y & \\ X & & \end{tikzcd}$$

*provided such pullback exists. Then the Beck-Chevalley condition under the action above identifies the expression with $\Pi_z w^* \Pi_y x^*$ (that is, both paths are canonically isomorphic).*

We actually need something more precise, and the following result will be crucial.

**Lemma 5.5.** *Consider the following situation:*

$$
\begin{tikzcd}
D \arrow[dr, "f"] \arrow[rrr, bend left, "v"] \arrow[dddr, bend right, "u"'] & & & \\
 & C \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, "d"] & B \arrow[r, "z"] \arrow[d, "w"] & Z \\
 & A \arrow[r, "y"'] \arrow[d, "x"'] & Y & \\
 & X & &
\end{tikzcd}
\qquad
\begin{tikzcd}
D \arrow[dr, "bf"] \arrow[drr, bend left, "v"] \arrow[ddr, bend right, "df"'] & & \\
 & B \arrow[r, "z"] \arrow[d, "w"'] & Z \\
 & Y &
\end{tikzcd}
\qquad
\begin{tikzcd}
D \arrow[dr, "af"] \arrow[drr, bend left, "df"] \arrow[ddr, bend right, "u"'] & & \\
 & A \arrow[r, "y"] \arrow[d, "x"'] & Y \\
 & X &
\end{tikzcd}
$$

*Let the maps induced by the contravariant action above be denoted by $\gamma, \beta, \alpha$, respectively, and the Beck-Chevalley isomorphism by $\tau : w^*\Pi_y \to \Pi_b a^*$.*

*Then*

$$\Pi_v \epsilon^{df} u^* \circ (\beta * \alpha) = \gamma \circ (\psi_{z,b} * \phi_{x,a}) \circ \tau$$

*Proof.* We have

$$\tau = \mathrm{Mate}(\phi_{y,a}^{-1}\phi_{w,b}) = \Pi_b a^* \epsilon^y \circ \Pi_b \phi_{y,a}^{-1} \phi_{w,b_{\Pi_y}} \circ \eta^b_{w^*\Pi_y}$$

$$
\begin{aligned}
&\Pi_v \epsilon^{df} u^* \circ (\beta * \alpha) \\
&= \Pi_v \epsilon^{df} u^* \circ ((\psi_{z,bf} * \phi_{w,bf}) \circ \Pi_z \eta^{bf}_{w^*}) * ((\psi_{y,af} * \phi_{x,af}) \circ \Pi_y \eta^{af}_{x^*})) \\
&= \Pi_v \epsilon^{df}_{u^*} \circ ((\psi_{zb,f} \circ \psi_{z,b_{\Pi_f}} * \phi_{wb,f} \circ f^* \phi_{w,b}) \circ \Pi_z \Pi_b \eta^f_{b^*w^*} \circ \Pi_z \eta^b_{w^*}) \\
&\quad *((\psi_{ya,f} \circ \psi_{y,a_{\Pi_f}} * \phi_{xa,f} \circ f^* \phi_{x,a}) \circ \Pi_y \Pi_a \eta^f_{a^*x^*} \circ \Pi_y \eta^a_{x^*})) && (1) \\
&= \Pi_v \epsilon^f_{u^*} \circ \Pi_v f^* \epsilon^d_{\Pi_f u^*} \circ ((\psi_{zb,f} \circ \psi_{z,b_{\Pi_f}} * f^* \phi_{w,b}) \circ \Pi_z \Pi_b \eta^f_{b^*w^*} \circ \Pi_z \eta^b_{w^*}) \\
&\quad *((\psi_{y,a_{\Pi_f}} * \phi_{xa,f} \circ f^* \phi_{x,a}) \circ \Pi_y \Pi_a \eta^f_{a^*x^*} \circ \Pi_y \eta^a_{x^*})) && (2) \\
&= \Pi_v \epsilon^f_{u^*} \circ \Pi_v f^* \epsilon^a_{\Pi_f u^*} \circ \Pi_v f^* a^* \epsilon^y_{\Pi_a \Pi_f u^*} \circ (\psi_{zb,f} \circ \psi_{z,b_{\Pi_f}} * f^* \phi^{-1}_{y,a} \circ f^* \phi_{w,b}) \circ \Pi_z \Pi_b \eta^f_{b^*w^*} \circ \Pi_z \eta^b_{w^*}) \\
&\quad *(\Pi_y \Pi_a \Pi_f (\phi_{xa,f} \circ f^* \phi_{x,a}) \circ \Pi_y \Pi_a \eta^f_{a^*x^*} \circ \Pi_y \eta^a_{x^*})) && (3) \\
&= (\psi_{zb,f} * \phi_{xa,f}) \circ \psi_{z,b_{\Pi_f f^*(xa)^*}} \circ \Pi_z \Pi_b \Pi_f \epsilon^f_{f^*(xa)^*} \circ \Pi_z \Pi_b \Pi_f f^* \epsilon^a_{\Pi_f f^*(xa)^*} \circ \Pi_z \Pi_b \Pi_f f^* a^* \epsilon^y_{\Pi_a \Pi_f f^*(xa)^*} \\
&\quad \circ(\Pi_z \Pi_b \Pi_f (f^* \phi^{-1}_{y,a} \circ f^* \phi_{w,b})) \circ \Pi_z \Pi_b \eta^f_{b^*w^*} \circ \Pi_z \eta^b_{w^*}) \\
&\quad *(\Pi_y \Pi_a \eta^f_{(xa)^*} \circ \Pi_y \Pi_a \phi_{x,a} \circ \Pi_y \eta^a_{x^*})) && (4) \\
&= (\psi_{zb,f} * \phi_{xa,f}) \circ \psi_{z,b_{\Pi_f f^*(xa)^*}} \circ \Pi_z \Pi_b \Pi_f f^* \epsilon^a_{(xa)^*} \circ \Pi_z \Pi_b \Pi_f f^* a^* \epsilon^y_{\Pi_a (xa)^*} \\
&\quad \circ(\Pi_z \Pi_b \Pi_f (f^* \phi^{-1}_{y,a} \circ f^* \phi_{w,b})) \circ \Pi_z \Pi_b \eta^f_{b^*w^*} \circ \Pi_z \eta^b_{w^*}) * (\Pi_y \Pi_a \phi_{x,a} \circ \Pi_y \eta^a_{x^*})) && (5) \\
&= [(\psi_{zb,f} * \phi_{xa,f}) \circ \Pi_{zb} \eta^f_{(xa)^*}] \circ \psi_{z,b_{(xa)^*}} \circ \Pi_z \Pi_b \epsilon^a_{(xa)^*} \circ \Pi_z \Pi_b a^* \epsilon^y_{\Pi_a (xa)^*} \\
&\quad \circ(\Pi_z \Pi_b (\phi^{-1}_{y,a} \circ \phi_{w,b})) \circ \Pi_z \eta^b_{w^*}) * (\Pi_y \Pi_a \phi_{x,a} \circ \Pi_y \eta^a_{x^*})) && (6) \\
&= \gamma \circ \psi_{z,b_{(xa)^*}} \circ \Pi_z \Pi_b \epsilon^a_{(xa)^*} \circ \Pi_z \Pi_b a^* \epsilon^y_{\Pi_a (xa)^*} \circ \Pi_z \Pi_b a^* y^* (\Pi_y \Pi_a \phi_{x,a} \circ \Pi_y \eta^a_{x^*}) \circ \Pi_z \tau_{x^*} \\
&= \gamma \circ (\psi_{z,b} * \phi_{x,a}) \circ \tau && (7)
\end{aligned}
$$

(1): Expansion of $\eta^{af}, \eta^{bf}$ and iso coherence
(2): Expression for $\epsilon^{df}$
(3): Expression for $\epsilon^{ya}$
(4): Naturality of the left epsilons and naturality of $\eta^f$ in right side
(5): Naturality of left epsilons and triangle identity for $f$
(6): naturality of $\eta^f$
(7): Naturality of the $\eta$s, triangle identity for $a$ ∎

We shall prove a result that will come very handy to prove some comonad identities.
First, we need a little technical lemma

**Lemma 5.6.** *The Beck-Chevalley morphism is compatible with gluing, that is*

$$
\begin{tikzcd}
A \arrow[r, "x"] \arrow[d, "w"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[r, "y"] \arrow[d, "h"] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "z"] \\
C \arrow[r, "v"'] & D \arrow[r, "u"'] & F
\end{tikzcd}
\qquad
\begin{tikzcd}
{[E]} \arrow[r, "y^*"] \arrow[rr, bend left, "(yx)^*"] \arrow[d, "\Pi_z"'] \arrow[dr, phantom, "\Nearrow"] & {[B]} \arrow[r, "x^*"] \arrow[d, "\Pi_h"] \arrow[dr, phantom, "\Nearrow"] \arrow[u, phantom, "\Uparrow"] & {[A]} \arrow[d, "\Pi_w"] \\
{[F]} \arrow[r, "u^*"'] \arrow[rr, bend right, "(uv)^*"'] & {[D]} \arrow[r, "v^*"'] \arrow[d, phantom, "\Uparrow"] & {[C]}
\end{tikzcd}
$$

*Proof.* Immediate from it being the described mate and the coherence of the composition isomorphisms ☐

We will not put out the whole definition of a bicategory, which the reader can get from [JY21, Chapter 2].

**Definition 5.7.** *Let $\mathcal{S}$ be a category with pullbacks. Then we have a bicategory* $\mathrm{Span}(\mathcal{S})$ *of spans with composition being "span composition" as previously defined, and horizontal composition being precisely taking the pullback-induced arrow. We still write $< X, Y >$ for the corresponding hom-categories of spans.*

We now refine our previous construction using bicategories.

**Lemma 5.8.** *Let $\mathbb{D}$ be an $\mathcal{S}$-complete category. We define a functor $\mathcal{G}$ from* $\mathrm{Span}(\mathcal{S})$*:*

- *$\mathcal{G}$ takes $X$ to $D^X$*
- *$\mathcal{G}$ takes a span*

$$
\begin{tikzcd}
S \arrow[r, "y"] \arrow[d, "x"'] & Y \\
X &
\end{tikzcd}
$$

*to $\Pi_y x^*$*

- *$\mathcal{G}$ takes*

$$
\begin{tikzcd}
S \arrow[rr, "y"] \arrow[dr, "f"] \arrow[dd, "x"'] & & Y \\
& T \arrow[ur, "w"'] \arrow[dl, "z"] & \\
X & &
\end{tikzcd}
$$

*to $(\psi_{w,f} * \psi_{z,f}) \circ \Pi_w \eta^f_{z^*}$.*

*Let us write $D^\square$ for the 2-category with objects $D^X$ whose hom-categories are inverted, that is, $[D^X, D^Y]^{op}$. Then $\mathcal{G}$ defines a pseudofunctor* $\mathrm{Span}(\mathcal{S}) \to D^\square$.

*Proof.* The coherence isos for composition are given by Beck-Chevalley and the coherence isos for $\mathbb{D}$. The associativity commutativity conditions follow from the previous lemma.

For instance, let us consider the condition

$$\begin{tikzcd}
 & <X,Y> \times <Y,Z> \arrow[dl, "\times"'] \arrow[dr, "\mathcal{G}\times\mathcal{G}"] & \\
<X,Z> \arrow[dr, "\xi^{-1}_{T,S}\mathcal{G}\xi_{V,U}"'] & & {[D^X,D^Y]^{op} \times [D^Y,D^Z]^{op}} \arrow[dl, "*"] \\
 & {[D^X,D^Z]^{op}} &
\end{tikzcd}$$

where $\xi_{S,T} : \mathcal{G}(S) \circ \mathcal{T} \to \mathcal{G}(S \circ T)$ is the coherence iso given by the obvious composition of the Beck-Chevalley isomorphism and $\phi, \psi$.

Let us consider the above situation with the spans

$$\begin{tikzcd}
T \circ S \arrow[rr] \arrow[dr, "u\times v"] \arrow[dd] & & T \arrow[rr, "z"] \arrow[dr, "v"] \arrow[dd, "w"] & & Z \\
 & V \circ U \arrow[rr] \arrow[dd] & & V \arrow[ur] \arrow[dl] & \\
S \arrow[rr, "y"] \arrow[dr, "u"] \arrow[dd, "x"'] & & Y & & \\
 & U \arrow[ur] \arrow[dl] & & & \\
X & & & &
\end{tikzcd}$$

Now, we have

$$\begin{aligned}
&\Pi_{zp_T}\epsilon^{d_{S,T}}_{(xp_S)^*} \circ (\mathcal{G}(p_T) * \mathcal{G}(p_S)) \\
&= \Pi_{zp_T}\epsilon^{d_{S,T}}_{(xp_S)^*} \circ (((\psi_{z,p_T} * \phi_{w,P_T}) \circ \Pi_z\eta^{p_T}_{w^*}) * ((\psi_{y,p_S} * \phi_{x,p_S}) \circ \Pi_y\eta^{p_S}_{x^*})) \\
&= \Pi_{zp_T}\epsilon^{p_S}_{(xp_S)^*} \circ \Pi_{zp_T}p_S^*\epsilon^{y}_{\Pi_{p_S}(xp_S)^*} \circ \Pi_{zp_T}(\phi^{-1}_{y,p_S} * \psi^{-1}_{x,p_S})_{(xp_S)^*} \circ (((\psi_{z,p_T} * \phi_{w,P_T}) \circ \Pi_z\eta^{p_T}_{w^*}) * ((\psi_{y,p_S} * \phi_{x,p_S}) \circ \Pi_y\eta^{p_S}_{x^*})) \\
&= \psi_{z,p_T} \circ \Pi_z\Pi_{p_T}\epsilon^{p_S}_{(xp_S)^*} \circ \Pi_z\Pi_{p_T}p_S^*\epsilon^{y}_{\Pi_{p_S}(xp_S)^*} \circ ((\Pi_z\Pi_{p_T}(\phi^{-1}_{y,P_S}\phi_{w,P_T}) \circ \Pi_z\eta^{p_T}_{w^*}) * (\Pi_y\Pi_{P_S}(\phi_{x,p_S}) \circ \Pi_y\eta^{p_S}_{x^*})) \quad (1) \\
&= (\psi_{z,p_T} * \phi_{x,P_s}) \circ \Pi_z\Pi_{p_T}p_S^*\epsilon^{y}_{\Pi_{p_S}(xp_S)^*} \circ (\Pi_z\Pi_{p_T}(\phi^{-1}_{y,P_S}\phi_{w,P_T}) \circ \Pi_z\eta^{p_T}_{w^*}) \quad (2) \\
&= \xi_{S,T}
\end{aligned}$$

(1): Naturality of $\psi_{z,p_T}$, cancelled $\psi_{z,p_S}$
(2): Naturality of the epsilons, triangle identity for $p_S$

Now it suffices to invoke Lemma 5.5 and we're done.

Also, note that we have the expected identities for different pullbacks in Beck-Chevalley, that is, if we have weo candidate pullbacks

$$\begin{tikzcd}
 & D \arrow[dl, "\cong\ \alpha"'] \arrow[dr, "b"] \arrow[dd, "a"] & \\
E \arrow[rr, "b'"] \arrow[dr, "a'"'] & & B \arrow[d, "v"] \\
 & A \arrow[r, "u"'] & C
\end{tikzcd}$$

then (where $\tau, \tau'$ are the B-C isos for $D, E$, respectively),

$$
\begin{aligned}
\mathcal{G}(\alpha) \circ \tau' &= (\psi_{b',\alpha} * \phi_{a',\alpha}) \circ \Pi_{b'} \eta^{\alpha}_{a'^*} \circ \tau' \\
&= (\psi_{b',\alpha} * \phi_{a',\alpha}) \circ \Pi_{b'} \eta^{\alpha}_{a'^*} \circ \Pi_{b'} a'^* \epsilon^u \circ \Pi_{b'} {\phi^{-1}_{u,a'} \phi_{v,b'}}_{\Pi_u} \circ \eta^{b'}_{v^* \Pi_u} \\
&= \Pi_b a^* \epsilon^u \circ \Pi_b {\phi_{a',\alpha}}_{u^* \Pi_u} \circ \Pi_b \alpha^* {\phi^{-1}_{u,a'} \phi_{v,b'}}_{\Pi_u} \circ {\psi_{b',\alpha}}_{\alpha^* b'^* v^* \Pi_u} \circ \Pi_{b'} \eta^{\alpha}_{b'' v^* \Pi_u} \circ \eta^{b'}_{v^* \Pi_u} && (1) \\
&= \Pi_b a^* \epsilon^u \circ \circ \Pi_b \alpha^* \phi^{-1}_{u,a} {\phi_{v,b}}_{\Pi_u} \circ \Pi_b {\phi_{b',\alpha}}_{v^* \Pi_u} \circ {\psi_{b',\alpha}}_{\alpha^* b'^* v^* \Pi_u} \circ \Pi_{b'} \eta^{\alpha}_{b'' v^* \Pi_u} \circ \eta^{b'}_{v^* \Pi_u} && (2) \\
&= \tau && (3)
\end{aligned}
$$

(1): Naturality of $\eta^{\alpha}, \psi_{\alpha,b'}, \phi_{a',\alpha}$
(2): Associativity condition for isos $\phi$
(3): Identity for $\eta^{yx}$

□

# 6 (Co) monadicity of diagrams

**Definition 6.1.** *Let $\mathbb{E}$ be a $\mathcal{S}$-indexed category. A $\mathbb{C}$-shaped diagram in $\mathbb{E}$ consists of the following data:*

- *An object $P \in \mathbb{E}^{C_0}$*
- *An arrow $\Phi : d_0^* P \to d_1^* P$*

*such that $i^* \Phi = 1_P$ and $c^* \Phi = p_1^* \Phi \circ p_0^* \Phi$ modulo the coherence isos, that is,*

- $\phi_{d_0,i} i^* \Phi \phi^{-1}_{d_1,i} = 1_{1^* P}$
- $c^* \Phi = \phi^{-1}_{d_1,c} \phi_{d_1,p_1} p_1^* \Phi \phi^{-1}_{d_0,p_1} \phi_{d_1,p_0} p_0^* \Phi \phi^{-1}_{d_0,p_0} \phi_{d_0,c}$

*Morphisms between such diagrams are defined in the obvious way.*
*We write $\mathbb{E}^{\mathbb{C}}$ for the corresponding category.*

It is not hard to see that under externalization of small categories we have an equivalence of indexed functors $\underline{\mathbb{C}} \to \mathbb{E}$ and $\mathbb{C}$-shaped diagrams in $\mathbb{E}$. A nice intuition for diagrams is considering the notion of in-universe functor as an action in the slice indexing of $\mathcal{S}$ over itself. In fact, the reader might check that under the standard Set-indexing of categories, diagrams correspond to the usual notion, yielding a mapping whose functoriality is expressed by the last two conditions. Internal diagrams give rise to diagram categories, with $\left(\mathbb{E}^{\mathbb{C}}\right)^I = \mathbb{E}^{\mathbb{C} \times I}$, with the obvious transition functors.

Now, if we wish to talk about the indexed version of the limit and colimit functors, it seems reasonable to look for adjoints of the pre-composition indexed functor $F^* : \mathbb{E}^{\mathbb{D}} \to \mathbb{E}^{\mathbb{C}}$ (where $F$ is an internal functor). In fact, we look at adjoints of the indexing functors, by endowing $\mathbb{E}$ with a $\mathrm{Cat}(\mathcal{S})$-indexed structure in the obvious way.

**Theorem 6.2.** *Let $\mathbb{E}$ have the canonical* Cat($\mathcal{S}$)*-indexed category structure. Then if $\mathbb{E}$ is (co) complete (in the indexed sense), its transition functors have right (left) adjoints. Moreover, $\mathbb{D}^{\mathbb{C}}$ has $\mathcal{S}$-indexed (co)products.*

First, we need to prove

**Theorem 6.3.** *Let $\mathbb{D}$ have $\mathcal{S}$-indexed products. Then the category $\mathcal{D}^{\mathbb{C}}$ (we use this notation for the non-indexed version) is comonadic over $\mathcal{D}^{\mathbb{C}}$.*

*Proof.* Thanks to the results laid out in the previous section, we shall be able to easily prove this result.

Consider the following map

$$
\Pi_{d_0} d_1^* \xrightarrow[\Pi_{d_0} \eta_{d_1^*}]{} \Pi_{d_0} \Pi_i i^* d_1^* \xrightarrow[\psi_{d_0,i} * \phi_{d_1,i}]{\cong} 1
$$

which we denote by $\eta$.

Now, if $(P, \Phi)$ is an internal diagram, consider the $d_0^*$-transpose $\Phi' : P \to \Pi_{d_0} d_1^* P$.

Then we have the unit property

$$
\begin{aligned}
&(\theta'^{-1}\psi_{d_1,i} * \theta^{-1}\phi_{d_0,i}) \circ \Pi_{d_0}\eta^i_{d_1^*} \circ \Pi_{d_0}\Phi \circ \eta^{d_0}_P \\
&= (\theta'^{-1}\psi_{d_1,i} * \theta^{-1}\phi_{d_0,i}) \circ \circ\Pi_{d_0}\Pi_i i^*\Phi \circ \Pi_{d_0}\eta^i_{d_0^*} \circ \eta^{d_0}_P && \text{(naturality of } \eta^i) \\
&= 1 && \text{(unit identities proved in Remark 4.2)}
\end{aligned}
$$

Note that by the selfsame identities (that is, right and left side are isos) and $id_0 = 1$, we can recover such equality so converse holds.

Consider also

$$\Pi_{d_0}d_1^* \xrightarrow{\Pi_{d_0}\eta_{d_1^*}} \Pi_{d_0}\Pi_c c^* d_1^* \xrightarrow{\cong} \Pi_{d_0}\Pi_{p_0}p_1^* d_1^* \longrightarrow \Pi_{d_0}d_1^*\Pi_{d_0}d_1^*$$

which we denote by $\mu$, where the last arrow (which we denote by $\alpha$) is induced by the Beck-Chevalley condition on the pullback

$$
\begin{tikzcd}
C_2 \arrow[r, "p_1"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_1 \arrow[d, "d_0"] \\
C_1 \arrow[r, "d_1"'] & C_0
\end{tikzcd}
$$

Then we have the composition property for the transpose of our diagram $\Phi$:

$$
\begin{aligned}
&\Pi_{d_0}\alpha^{-1}_{d_1^*} \circ (\psi^{-1}_{d_0,p_0}\psi_{d_0,c} * \phi^{-1}_{d_1,p_1}\phi_{d_1,c}) \circ \Pi_{d_0}\eta^c_{d_1^*} \circ \Pi_{d_0}\Phi \circ \eta^{d_0} \\
&= \Pi_{d_0}d_1^*\Pi_{d_0}\Phi \circ \Pi_{d_0}\alpha^{-1}_{d_0^*} \circ \Pi_{d_0}\Pi_{p_0}\left(\phi^{-1}_{d_0,p_1} \circ \phi_{d_1,p_0} \circ p_0^*\Phi \circ \phi^{-1}_{d_0,p_0} \circ \phi_{d_0,c}\right) \circ (\psi^{-1}_{d_0,p_0}\psi_{d_0,c})_{c^*d_0^*} \circ \Pi_{d_0}\eta^c_{d_0^*} \circ \eta^{d_0} && (1) \\
&= \Pi_{d_0}d_1^*\Pi_{d_0}\Phi \circ \Pi_{d_0}\alpha^{-1}_{d_0^*} \circ \Pi_{d_0}\Pi_{p_0}\left(\phi^{-1}_{d_0,p_1} \circ \phi_{d_1,p_0} \circ p_0^*\Phi\right) \circ \Pi_{d_0}\eta^{p_0}_{d_0^*} \circ \eta^{d_0} && (2) \\
&= \Pi_{d_0}d_1^*\Pi_{d_0}\Phi \circ \Pi_{d_0}\alpha^{-1}_{d_0^*} \circ \Pi_{d_0}\Pi_{p_0}p_1^*(\epsilon^{d_0}_{d_0^*} \circ d_0^*\eta^{d_0}) \circ \Pi_{d_0}\Pi_{p_0}\left(\phi^{-1}_{d_0,p_1} \circ \phi_{d_1,p_0}\right) \circ \Pi_{d_0}\eta^{p_0}_{d_1^*} \circ \Pi_{d_0}\Phi \circ \eta^{d_0} && (3) \\
&= \Pi_{d_0}d_1^*\Pi_{d_0}\Phi \circ \Pi_{d_0}d_1^*\eta^{d_0} \circ \Pi_{d_0}\Phi \circ \eta^{d_0} && (4)
\end{aligned}
$$

(1): Naturality of $\eta^c$, expansion of $c^*\Phi$, naturality of $\psi, \alpha^{-1}$
(2): Identity for $\eta^{yx}$ for $d_0c, p_0c$
(3): Swap $\eta^{p_0}$ and $\Phi$, then apply the triangle identity in reverse for $d_0$
(4): Pass new $\eta^{d_0}$ rightward using naturality, then we get the exact expression for $\alpha$

Note for converse, we use step 0 to 1, cutting off all isos, and noting that the expression we're left is nothing more than two transposes.

Now we verify that $(\Pi_{d_0}d_1^*, \eta, \mu)$ has comonad structure. Recall Remark 5.4.

In the language of spans, an internal category is given by spans

$$
\begin{tikzcd}
C_1 \arrow[r, "d_0"] \arrow[d, "d_1"'] & C_0 \\
C_0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
C_2 \arrow[r, "p_0"] \arrow[d, "p_1"'] & C_1 \\
C_1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
C_0 \arrow[r, no head] \arrow[d, no head] & C_0 \\
C_0 &
\end{tikzcd}
$$

(where the second is the obvious composite) and the appropriate arrows $c, i$ between. Thanks to Lemma 5.8, we can easily prove commutativity as desired. Recall the notation with the functor $\mathcal{G}$ and transition isos $\xi_{T,S}$. Then for $c(1 \times i)$, for instance, it follows immediately that

$$1 = \mathcal{G}(c(1\times i)) = \mathcal{G}(1 \times i) \circ \mathcal{G}(c) = \xi^{-1}_{C_1,C_0}(\mathcal{G}(i)*)\xi^{-1}_{C_1,C_1}\mathcal{G}(c) = \ \eta' * \Pi_{d_0}d_1^* \circ \mu'$$

by choosing $C_1 \circ C_0 = C_1$ (noting that the Beck-Chevalley coherence isos are entirely compatible with the pullback isomorhisms, as proved in the end of the previous section).

Also,

$$\mathcal{G}(c(1\times c)) = \xi_{C_2,C_1}(\mathcal{G} * \mathcal{G}(1)\xi^{-1}_{C_1,C_1}\mathcal{G}(c)$$

Now note by coherence

$$\begin{tikzcd}
\mathcal{G}(C_2)\circ\mathcal{G}(C_1) \arrow[r, "\xi_{C_1,C_1}{}^{-1}\Pi_{d_0}d_1^*"] \arrow[d, "\xi_{C_2,C_1}"'] & \mathcal{G}(C_1)\circ\mathcal{G}(C_1)\circ\mathcal{G}(C_1) \\
\mathcal{G}(C_3) \arrow[r, "\xi^{-1}_{C_2,C_1}"'] & \mathcal{G}(C_1)\circ\mathcal{G}(C_2) \arrow[u, "\Pi_{d_0}d_1^*\xi^{-1}_{C_1,C_1}"']
\end{tikzcd}$$

so postcomposing we get the desired equality. □

We've just proved a crucial theorem towards our goal, but we still need another to constructed the desired colimits, which we'll prove in the next section

# 7 Interlude: morphisms of monads

Just like in the previous section, we lay out results which'll be useful later on the next section. Let $T, S$ be two monads. We write $\eta^T, \eta^S, \mu^T, \mu^S$ for the respective unit/multiplication arrows. Let

$$\begin{tikzcd}
C^T \arrow[r, "Q"'] \arrow[d, shift left, "U^T"] & D^S \arrow[d, shift left, "U^S"] \\
C \arrow[r, "R"'] \arrow[u, shift left, "F^T"] & D \arrow[u, shift left, "F^S"]
\end{tikzcd}$$

be a square s.t. $RU^T = U^S Q$.

Let

$$\delta = \mathrm{Mate}(RU^T = U^S Q) = \epsilon^S_{QF^T}\circ F^S R\eta^T$$

and $\lambda = U^S\delta$. Then clearly by the triangle inequality

$$\begin{tikzcd}
R \arrow[r, "\eta^S_R"] \arrow[dr, "R\eta^T"'] & SR \arrow[d, "\lambda"] \\
 & RT
\end{tikzcd}$$

Furthermore,

$$\begin{aligned}
& R\mu^T \circ \lambda_T \circ S\lambda && (1)\\
&= U^S Q\epsilon^T_{F^T}\circ(U^S\epsilon^S_{QF^T T}\circ SR\eta^T_T)\circ(SU^S\epsilon^S_{QF^T}\circ SSR\eta^T) && (2)\\
&= U^S\epsilon^S_{QF^T}\circ(SU^S\epsilon^S_{QF^T}\circ SSR\eta^T) \quad \text{(Naturality of } \epsilon^S \text{ and triangle identity)} && (3)\\
&= (U^S\epsilon^S_{QF^T}\circ SR\eta^T)\circ U^S\epsilon^S_{F^S R} \quad \text{(Naturality of the left } \epsilon^S) && (4)\\
& \lambda\circ\mu^R && (5)
\end{aligned}$$

so we have

$$\begin{tikzcd}
S^2R \arrow[rr, "\mu^S_R"] \arrow[d, "S\lambda"'] & & SR \arrow[dd, "\lambda"] \\
SRT \arrow[d, "\lambda_T"'] & & \\
RT^2 \arrow[rr, "R\mu^T"'] & & RT
\end{tikzcd}$$

In fact, the reader might know that monads correspond to lax functors $1 \to \mathrm{Cat}$, and in fact the transformation $\lambda$ yields a lax transformation between such functors.

**Lemma 7.1.** *We have $Q(A, a) = (RA, Ra \circ \lambda_A)$*

*Proof.*

$$RU^T\epsilon^T_{(A,a)} \circ \lambda_{U_T} = U^S\epsilon^S_Q$$

by naturality of $\epsilon^S$ and the triangle identity. □

Thus there is an equivalence between the data of $(R, \lambda)$ and the data given by the square at the start of the section, as we can reconstruct $Q : C^S \to D^T$ from $R$. We call the pair $(R, \lambda)$ satisfying the aforementioned identities a morphism of monads. In fact, we can compose these 1-cells. Let $(R, \lambda) : T \to S, (L, \delta) : S \to K$ be given. Then the obvious composition is the desired composition: $L\lambda \circ \delta_R$.

Of course, if one unravels the definition of modification (2-cells for lax functors) then we have that a 2-cell $(R, \lambda) \to (L, \delta)$ consists of $\gamma : R \implies L$ s.t. $\gamma_T \circ \lambda = \delta \circ S\gamma$. Furthermore, we have horizontal composition by taking the horizontal composition of the underlying transformations.

We define the notion of comorphism of monads $(L, \delta) : S \to T$ as being induced by an oplax transformation instead, which gives us the dual identities:

$$\begin{tikzcd}
L \arrow[r, "L\eta^S"] \arrow[dr, "\eta^T_L"'] & LS \arrow[d, "\delta"] \\
& TL
\end{tikzcd}$$

$$\begin{tikzcd}
LS^2 \arrow[rr, "L\mu^S"] \arrow[d, "\delta_S"'] & & LS \arrow[dd, "\delta"] \\
TLS \arrow[d, "T\delta"'] & & \\
T^2L \arrow[rr, "\mu^T_L"'] & & TL
\end{tikzcd}$$

We shall write $T \rightharpoonup S$ and $T \rightharpoondown S$ for morphisms and comorphisms of moands respectively, from now own.

**Lemma 7.2.** *Suppose $(R, \lambda) : T \rightharpoonup S$ and $R$ has a left adjoint $L$. Then the mate of $\lambda$ yields $(L, \delta) : S \rightharpoondown T$.*

*Proof.* Recall that by mate we mean

$$\delta = \epsilon^L_{TL} \circ L\lambda_L \circ LS\eta^L$$

Verification of the unit identity is straightforward. To verify the composition identity:

$$\begin{aligned}
&\mu^T_L \circ T\delta \circ \delta_S \\
&= \epsilon^S_{TL} \circ LR\mu^T \circ L\lambda_{TLS} \circ LSR\delta \circ LS\eta^L_S && \text{(Naturality of } \epsilon^S \text{ and } \lambda) \\
&= \epsilon^S_{TL} \circ LR\mu^T_L \circ L\lambda_{TLS} \circ LSR(\epsilon^L_{TL} \circ L\lambda_L \circ LS\eta^L) \circ LS\eta^L && \text{(Expanded expression)} \\
&= \epsilon^S_{TL} \circ LR\mu^T_L \circ L\lambda_{TL} \circ LS\lambda_L \circ LSS\eta^L && \text{(Naturality of rightmost } \eta^L \text{ and triangle identity)} \\
&= \epsilon^S_{TL} \circ L(\lambda \circ \mu^S_R)_L \circ LSS\eta^L && \text{(Composition identity)} \\
&= \delta \circ L\mu^S
\end{aligned}$$

□

Just like for morphisms of monads, if we are given

$$\begin{tikzcd}
C^T \arrow[d, shift left, "U^T"] & D^S \arrow[l, "P"'] \arrow[d, shift left, "U^S"] \\
C \arrow[u, shift left, "F^T"] & D \arrow[l, "L"] \arrow[u, shift left, "F^S"]
\end{tikzcd}$$

s.t. $F^T L = PF^S$, we can get a comorphism $(L, \delta) : S \rightharpoondown T$ by taking the mate $\pi$ of the equality

$$\begin{tikzcd}
D \arrow[r, "Q"] \arrow[d, "F^S"'] \arrow[dr, phantom, "\overset{1}{\Swarrow}"] & C \arrow[d, "F^T"] \\
D^S \arrow[r, "P"'] & C^T
\end{tikzcd}$$

then setting $\delta = \pi_{F^S}$. We leave it to the interested reader to verify the identities are satisfied.

**Lemma 7.3.** *Let*

$$\begin{tikzcd}
C^T \arrow[d, shift left, "U^T"] \arrow[r, shift right, "Q"'] & D^S \arrow[l, shift right, "P"'] \arrow[d, shift left, "U^S"] \\
C \arrow[u, shift left, "F^T"] \arrow[r, shift right, "R"'] & D \arrow[l, shift right, "L"'] \arrow[u, shift left, "F^S"]
\end{tikzcd}$$

*such that*

$$\begin{tikzcd}
(RU^T, _) \arrow[r, "\cong"] \arrow[d, equal, "="'] & (_, F^T L) \arrow[d, equal, "="] \\
(U^S Q, _) \arrow[r, "\cong"] & (_, PF^S)
\end{tikzcd}$$

*Then the two ways of obtaining a comorphism $(L, \delta) : S \rightharpoondown T$ give the same arrow.*

*Proof.* We talk about horizontal and vertical mates, in the obvious sense. Note that by the condition on the transposes, the identity transformation representing $PF^S = F^T L$ is given by the horizontal transpose followed by the vertical transpose of $RU^T = U^S Q$. Now, if we write $\circ_V, \circ_H$ for vertical and horizontal 2-cell square composition, respectively, we have:

$$\begin{aligned}
&\mathrm{VMate}((RU^T = U^S Q) \circ_V \mathrm{HMate}(RU^T = U^S Q)) \\
&= \mathrm{VMate}(RU^T = U^S Q) \circ_V (RU^T = U^S Q)
\end{aligned}$$

□

We thus obtain the following results which are crucial for the next section in terms of motivation:

**Corollary 7.4.** *With the same conditions as the previous lemma, one has*

$$P\epsilon^{F^S}_{F^S} = \epsilon^{F^T}_{F^T L} \circ F^T \delta = \epsilon^{F^T}_{F^T L} \circ F^T \epsilon^L_{TL} \circ F^T L U^S \epsilon^{F^s}_L \circ F^T LSR\eta^T_L \circ F^T LS\eta^L$$

**Lemma 7.5.** *If we write*

$$\pi = \mathrm{Mate}(RU^T = U^S Q) = \epsilon^S_{QF^T} \circ F^S R\eta^T$$

*we have*

$$\pi_L \circ F^S \eta^S = \eta^Q_{F^S}$$

*Proof.*

$$\begin{aligned}
&\pi_L \circ F^S \eta^S \\
&= \epsilon^S_{QF^T L} \circ F^S R\eta^T_L \circ F^S \eta^R \\
&= \epsilon^S_{QF^T L} \circ F^S U^S \eta^Q_{F^S} \circ F^S \eta^S && \text{(Commutativity condition on transposes)} \\
&= \eta^Q_{F^S}
\end{aligned}$$

□

Before we move on, we still need the following result crucially:

**Lemma 7.6.** *Let*

$$\pi = \mathrm{Mate}(RU^T = U^S Q) = \epsilon^S_{QF^T} \circ F^S R\eta^T$$

$$\varphi = \pi_L \circ F^S \eta^L$$

$$\omega = \epsilon^{F^T}_{F^T L} \circ F^T \delta$$

*Without any assumption on the existence of a left adjoint for $Q$, we have:*

$$\begin{tikzcd}
F^S S \arrow[r, "\epsilon^S_{F^S}"] \arrow[d, "\varphi_S"'] & F^S \arrow[d, "\varphi"] \\
QF^T LS \arrow[r, "Q\omega"'] & QF^T L
\end{tikzcd}$$

*Proof.*

$$\begin{aligned}
&U^S Q\omega \circ U^S \varphi_S \\
&= RU^T \epsilon^{F^T}_{F^T L} \circ RT\delta \circ \lambda_{LS} \circ S\eta^L_S \\
&= RU^T \epsilon^{F^T}_{F^T L} \circ \lambda_{TL} \circ SR\epsilon^L_{TL} \circ S\eta^L_{RTL} \circ S(\lambda_L \circ S\eta^L) && \text{(Naturality of the right } \lambda \text{ and } \eta\text{)} \\
&= RU^T \epsilon^{F^T}_{F^T L} \circ \lambda_{TL} \circ S(\lambda_L \circ S\eta^L) && \text{(Triangle identity)} \\
&= \lambda_L \circ S\eta^L \circ \mu^S && \text{(Composition identity and naturality of } \epsilon\text{)} \\
&= U^S(\varphi \circ \epsilon^S_{F^S})
\end{aligned}$$

□

# 8 The adjoint lifting theorem

We follow [Bor94, Chapter 4.5]. Consider the following situation, where $T, S$ are monads:

$$\begin{tikzcd}
C^T \arrow[r, "Q"] \arrow[d, "U^T"'] & D^S \arrow[d, "U^S"] \\
C \arrow[r, "R"'] & D
\end{tikzcd}$$

We shall prove that if $C^T$ has coequalizers, $R$ being a right adjoint implies $Q$ is a right adjoint. To start, suppose $R$ and $Q$ have right adjoints. Then without loss of generality we can suppose both adjoint squares commute:

$$\begin{tikzcd}
C^T \arrow[d, shift left, "U^T"] \arrow[r, shift right, "Q"'] & D^S \arrow[l, shift right, "P"'] \arrow[d, shift left, "U^S"] \\
C \arrow[u, shift left, "F^T"] \arrow[r, shift right, "R"'] & D \arrow[l, shift right, "L"'] \arrow[u, shift left, "F^S"]
\end{tikzcd}$$

We fix some notation:

- $\eta^F, \epsilon^F$ are the unit/counit for $F \dashv G$
- $\eta^T, \mu^T$ are the unit and multiplication for the monad $T$

We'll always omit when there is no ambiguity.

Recall that we have the following coequalizer (generalizing a common algebraic construction):

$$\begin{tikzcd}
F^S S(D) \arrow[r, shift left, "\mu_{F^S(D)}"] \arrow[r, shift right, "F^S(d)"'] & F^S(D) \arrow[r] & (D, d)
\end{tikzcd}$$

Then since $P$ preserves all colimits, passing the image we get an expression of $P(D, d)$ as a coequalizer. Now, since $PF_T = F^S L$, we start by retrieving $P\mu^S_{F^S}$. Thanks to Corollary 7.4 in the previous section, we already know hot to do it, by setting

$$\omega = \epsilon^{F^T}_{F^T L} \circ F^T \delta$$

We also set

$$\varphi = \pi_L \circ F^S \eta^L$$

Now, taking the coequalizer

$$\begin{tikzcd}
F^T LS(D) \arrow[r, shift left, "\omega_D"] \arrow[r, shift right, "F^T L(d)"'] & F^T L(D) \arrow[r, "x"] & X
\end{tikzcd}$$

we have, as proved in the previous section,

$$\begin{tikzcd}
F^S S(D) \arrow[r, shift left, "\epsilon^S_{F^S(D)}"] \arrow[r, shift right, "F^S(d)"'] \arrow[d, "\varphi_{S(D)}"'] & F^S(D) \arrow[r, "d"] \arrow[d, "\varphi_D"'] & (D, d) \arrow[d, dashed, "\xi"] \\
QF^T LS(D) \arrow[r, shift left, "Q\omega_D"] \arrow[r, shift right, "QF^T L(d)"'] & QF^T L(D) \arrow[r, "Qx"'] & QX
\end{tikzcd}$$

We prove $\xi$ does in fact satisfies the property of the unit arrow (recall in the previous section we had $\varphi = \eta^Q_{F^S}$).

$$(D, d) \xrightarrow{\quad y \quad} Q(A, a) = (Ra, Ra \circ \lambda_A)$$

be given, so that

$$Ra \circ \lambda_A \circ Sy = y \circ d$$

For the sake of intuition, let us again suppose we have a suitable left adjoint $P$ for $Q$. Then one has

$$\begin{aligned}
&\epsilon^{F^T} \circ F^T \epsilon^L_{U^T} \circ F^T LU^S(y) \\
&= \epsilon^Q \circ P\epsilon^{F^S}_Q \circ F^T LU^S(y) \\
&= \epsilon^Q \circ P(y) \circ P\epsilon^S \\
&= (y)^\flat \circ P(d)
\end{aligned}$$

Thus we set

$$\chi = \epsilon^{F^T} \circ F^T \epsilon^L_{U^T} \circ F^T LU^S(y) = a \circ F^T \epsilon^L_A \circ F^T LU^S(y)$$

and verify it equalizes the arrows $\omega_D, F^T L(d)$:

$$\begin{aligned} & U^S \left( a \circ F^T \epsilon^L_{U^T} \circ F^T LU^S(y) \circ \epsilon^{F^T}_{F^T L(D)} \circ F^T \delta_D \right) \\ = & U^S \left( a \circ \epsilon^{F^T}_{F^T U^T} \circ F^T \epsilon^L_{TU^T} \circ F^T L \lambda_{U^T} \circ F^T LSU^S(y) \right) \quad \text{(First naturality, then expand } \delta \text{ and triangle identity)} \\ = & U^S(a) \circ \mu_A \circ T \epsilon^L_{TA} \circ TL \lambda_A \circ TLSU^S(y) \end{aligned}$$

On the other hand

$$\begin{aligned} & U^S \left( a \circ F^T \epsilon^L_{U^T} \circ F^T LU^S(y) \circ F^T L(d) \right) \\ = & U^S \left( a \circ F^T \epsilon^L_{U^T} \circ F^T LR(a) \circ F^T L \lambda_{U^T} \circ F^T LSU^S(y) \right) \\ = & U^S \left( a \circ \mu^T_A \circ F^T \epsilon^L_{TA} \circ F^T L \lambda_A \circ TLSU^S(y) \right) \end{aligned}$$

Hence we have a factorization

$$\begin{tikzcd} F^T LS(D) \arrow[r, shift left, "\omega_D"] \arrow[r, shift right, "F^T L(d)"'] & F^T L(D) \arrow[r, "x"] \arrow[dr, "\chi"'] & X \arrow[d, dashed, "y^\flat"] \\ & & (A, a) \end{tikzcd}$$

We now verify $Qy^\beta \circ \xi \circ d = y \circ d$ (since $d$ is epi):

$$U^S (y \circ d) = RU^T a \circ \lambda_A \circ SU^S(y)$$

$$\begin{aligned} & U^S \left( Qy^\flat \circ \xi \circ d \right) \\ = & U^S \left( Q\chi \circ \varphi_D \right) \\ = & RU^T (a \circ F^T \epsilon^L_{U^T} \circ F^T LU^S(y)) \circ \lambda_{L(D)} \circ S\eta^L_D \\ = & RU^T a \circ \lambda_A \circ SR\epsilon^T_A \circ S\eta^L_A \circ SU^S(y) \\ = & RU^T a \circ \lambda_A \circ SU^S(y) \end{aligned}$$

Furthermore $Qf \circ \xi = Qg \circ \xi \iff Q(fx) \circ \varphi = Q(gx) \circ \varphi$
but upon expansion this becomes (in transpose notation)

$$U^S \flat^S \#^R \#^T (fx)$$

hence $fx = gx$ but $x$ is epi.

# 9 Indexed monads

In the 2-category of $\mathcal{S}$-indexed categories, as in any, one can define monads. In fact, one can speak of algebras (since the monad is a functor):

**Definition 9.1.** *Let $\mathbb{T}$ be a monad on $\mathbb{C}$. Then we have an indexed category of $\mathbb{T}$-algebras $\mathbb{C}^{\mathbb{T}}$ whose $I$-component is $(C^I)^{T^I}$ and whose transition functors $x^*$ act in the obvious way. This functor has domain in $T^J$-algebras, which follows from the fact $\eta, \mu$ are indexed transformations:*

$$
\begin{tikzcd}
x^*A \arrow[r, "\eta^J_{x^*A}"] \arrow[dr, "x^*\eta^I_A"'] \arrow[ddr, no head] & T^J x^*A \arrow[d, "T^{x}{}^{-1}_A"] \\
& x^*T^I A \arrow[d, "x^*a"] \\
& x^*A
\end{tikzcd}
\qquad
\begin{tikzcd}
{T^J}^2 x^*A \arrow[r, "T^{x2}{}^{-1}_A"] \arrow[dd, "\mu^J_{x^*A}"'] & x^*{T^I}^2 A \arrow[r] \arrow[dd, "x^*\mu^I_A"] & T^J x^*A \arrow[d, "T^{x}{}^{-1}_A"] \\
& & x^*T^I A \arrow[d, "x^*a"] \\
T^J x^*A \arrow[r, "T^{x}{}^{-1}_A"'] & x^*T^I A \arrow[r, "x^*a"'] & x^*A
\end{tikzcd}
$$

*Furthermore, we have a forgetful indexed functor and a forgetful free functor $F$, which is an indexed functor since multiplication is an indexed transformation:*

$$
\begin{tikzcd}
T^J x^*T^I A \arrow[r, "T^J T^x_A"] \arrow[d, "T^{x}{}^{-1}_{TA}"] & {T^J}^2 x^*A \arrow[dd, "\mu^J_{x^*A}"] \\
x^*{T^I}^2 A \arrow[ur, "T^{2x}_A"'] \arrow[d, "x^*\mu^I_A"] & \\
x^*T^I A \arrow[r, "T^x_A"'] & T^J x^*A
\end{tikzcd}
$$

*We have (noting $F^x = T^x$)*

$$
\begin{aligned}
&\epsilon^J_{x^*(A,a)} \circ (FU)^x \\
&= [x^*a \circ T^{x}{}^{-1}_A] \circ F^J U^x \circ F^x_{U^I} \\
&= [x^*a] = x^*\epsilon^I_{(A,a)}
\end{aligned}
$$

*hence $F \dashv U$ in the indexed sense.*

Analogously to the non-indexed case, we have:

**Theorem 9.2.** *Let $\mathbb{T}$ be an indexed monad on $\mathbb{C}$.*

*i) If $\mathbb{C}$ has $\mathcal{S}$-indexed products, so does $\mathbb{C}^{\mathbb{T}}$, and those are preserved by the forgetful functor $U$*

*ii)*

*iii) If $\mathbb{C}$ has $\mathcal{S}$-indexed coproducts which are preserved by $\mathbb{T}$, so does $\mathbb{C}^{\mathbb{T}}$, and those are preserved by the forgetful functor $U$*

*Proof.* i): We define $\Pi_x$ as taking $T^I A \to A$ to $T^J \Pi_x A \to \Pi_x T^I A \to \Pi_x A$, where the first arrow is obtained as the mate of $x^*T^J \cong T^I x^*$. It's easy to see by taking mates in the modification diagrams that we can imitate the same argument to show the resulting arrow has algebra structure. The unity and counit are lifted from the base category and so is the Beck-Chevalley condition.

ii): Analogous to the previous argument, but this time we need that $T$ commutes with $\Sigma_x$ since it cannot simply be obtained by mates due to the direction. $\square$

Packing together everything we've proved thus far, we have:

**Theorem 9.3.** *Let $\mathbb{D}$ be a $\mathcal{S}$-complete (resp. cocomplete) category. Then in the canonical* $\mathrm{Cat}(\mathcal{S})$*-indexing of $\mathbb{D}$, the category has indexed finite (co) limits, and the transition functors have right (resp.left) adjoints. In particular, we have limit (colimit) functors* $\lim_{\mathbb{C}} : \mathbb{D}^{\mathbb{C}} \to \mathbb{D}$*. Furthermore, $\mathbb{D}^{\mathbb{C}}$ ahs $\mathcal{S}$-indexed (co) products.*

*Proof.* The last statement follows directly from the previous result. Furthermore, the comonad constructed in section 5 preserves all finite limits, hence the co-forgetful functor creates them, hence $D^T \simeq D^{\mathbb{C}}$ has all finite limits, hence we can apply the adjoint lifting theorem proved in the previous section.

Finally, due to the Beck-Chevalley conditio, each monad of $D^{\mathbb{C}\times I}$ over $D^{C_0\times I}$ assembles into an indexed comonad. Let $T^I = \Pi_{d_0\times I}(d_1\times I)^*$. Firstly, note that we have

$$\begin{tikzcd}
(C_0\times x)^*\Pi_{d_0\times I}(d_1\times I)^* \arrow[r] \arrow[dr, "T^x"'] & \Pi_{d_0\times J}(C_1\times x)^*(d_1\times I)^* \arrow[d, "\Pi_{d_0\times J}\phi"] \\
& \Pi_{d_0\times J}(d_1\times J)^*(C_0\times x)^*
\end{tikzcd}$$

where the last arrow is induced by the usual coherence isos and the first by applying Beck-Chevalley to

$$\begin{tikzcd}
C_1\times J \arrow[r, "1\times x"] \arrow[d, "d_0\times 1"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_1\times I \arrow[d, "d_0\times 1"] \\
C_0\times J \arrow[r, "1\times x"'] & C_0\times I
\end{tikzcd}$$

Thus we have an indexed comonad $\mathbb{T}$. Now, we verify the unit $\eta$ is an indexed transformation:

$$\begin{tikzcd}
(C_0\times x)^*\Pi_{d_0\times I}(d_1\times I)^* \arrow[rr, "(C_0\times x)^*\Pi_{d_0\times I}\eta^{i\times I}_{(d_1\times J)^*}"] \arrow[d, "\text{B-C}"] & & (C_0\times x)^*\Pi_{d_0\times I}\Pi_{i\times I}(i\times I)^*(d_1\times I)^* \arrow[dl] \arrow[ddd, "(C_0\times x)^*(\Theta^{-1}\psi*\theta^{-1}\phi)"] \\
\Pi_{d_0\times J}(C_1\times x)^*(d_1\times I)^* \arrow[r] \arrow[ddr] \arrow[ddd, "\Pi_{d_0\times J}\phi"'] & \Pi_{d_0\times J}(C_1\times x)^*\Pi_{i\times I}(i\times I)^*(d_1\times I)^* \arrow[dd, "\phi\circ\text{B-C}"] \arrow[d, phantom, "\mathbf{A}"'] & \\
& {} & {} \\
{} \arrow[r, phantom, "\mathbf{B}"] & \Pi_{d_0\times J}\Pi_{i\times J}(i\times J)^*(C_1\times x)^*(d_1\times I)^* \arrow[d] \arrow[ur, phantom, "\mathbf{C}"] & \\
\Pi_{d_0\times J}(d_1\times J)^*(C_0\times x)^* \arrow[r, "\Pi_{d_0\times J}\eta^{i\times J}_{(d_1\times J)^*(C_0\times x)^*}"'] & \Pi_{d_0\times J}\Pi_{i\times J}(i\times J)^*(d_1\times J)^*(C_0\times x)^* \arrow[r, "(\Theta^{-1}\psi*\theta^{-1}\phi)_{(C_0\times x)^*}"'] & (C_0\times x)^*
\end{tikzcd}$$

Starting by triangle **A**, we have

$$\begin{tikzcd}
C^K \arrow[r, "y^*"] \arrow[d, "\Pi_w"'] & C^I \arrow[d, "\Pi_x"] & & \Pi_a z^* b^* \arrow[r] \arrow[d] & \Pi_a z^*\Pi_w w^* b^* \arrow[d] \\
C^L \arrow[r, "z^*"'] & C^J & \rightarrow & \Pi_a\Pi_x x^* z^* b^* & \Pi_a\Pi_x y^* w^* b^* \arrow[l]
\end{tikzcd}$$

which is really easy and short to verify.The square **B** is even easier, just straightforward naturality. Finally, for **C**, we apply the argument of **A**, using glued pullbacks and considering the case of $\eta^1$, which is invertible and whose inverse is precisely the iso arrows at the bottom right.

Now, for $\mu$, we have a diagram whose left side is

$$
\begin{tikzcd}
(C_0 \times x)^*\Pi_{d_0\times J}(d_1 \times J)^* \arrow[r] \arrow[dd, "\text{B-C}_{(d_1\times J)^*}"'] & (C_0 \times x)^*\Pi_{d_0\times J}\Pi_{c\times J}(c \times J)^*(d_1 \times J)^* \arrow[r] \arrow[dd] & (C_0 \times x)^*\Pi_{d_0\times J}\Pi_{p_0\times J}(p_1 \times J)^*(d_1 \times J)^* \arrow[d, "\text{B-C}_{\Pi_{p_0\times J}(p_1\times J)^*(d_1\times J)^*}"] \\
& & \Pi_{d_0\times I}(C_1 \times x)^*\Pi_{p_0\times J}(p_1 \times J)^*(d_1 \times J)^* \arrow[dd] \\
\Pi_{d_0\times I}(C_1 \times x)^*(d_1 \times J)^* \arrow[r] \arrow[dd, "\Pi_{d_0\times I}\phi"'] & \Pi_{d_0\times I}(C_1 \times x)^*\Pi_{c\times J}(c \times J)^*(d_1 \times J)^* \arrow[dd, "\text{B-C}"] \arrow[r, phantom, "\mathbf{B}"] & \\
\arrow[r, phantom, "\mathbf{A}"] & & \Pi_{d_0\times I}\Pi_{p_0\times I}(p_1 \times I)^*(C_1 \times x)^*(d_1 \times J)^* \arrow[d] \\
\Pi_{d_0\times I}(d_1 \times I)^*(C_0 \times x)^* \arrow[r] & \Pi_{d_0\times I}\Pi_{c\times I}(c \times I)^*(d_1 \times I)^*(C_0 \times x)^* \arrow[r] & \Pi_{d_0\times I}\Pi_{p_0\times I}(p_1 \times I)^*(d_1 \times I)^*(C_0 \times x)^*
\end{tikzcd}
$$

and right side is

$$
\begin{tikzcd}
(C_0 \times x)^*\Pi_{d_0\times J}\Pi_{p_0\times J}(p_1 \times J)^*(d_1 \times J)^* \arrow[r] \arrow[d, "\text{B-C}_{\Pi_{p_0\times J}(p_1\times J)^*(d_1\times J)^*}"] & (C_0 \times x)^*\Pi_{d_0\times J}(d_1 \times J)^*\Pi_{d_0\times J}(d_1 \times J)^* \arrow[d, "\text{B-C}_{(d_1\times J)^*\Pi_{d_0\times J}(d_1\times J)^*}"] \\
\Pi_{d_0\times I}(C_1 \times x)^*\Pi_{p_0\times J}(p_1 \times J)^*(d_1 \times J)^* \arrow[r] \arrow[dd] & \Pi_{d_0\times I}(C_1 \times x)^*(d_1 \times J)^*\Pi_{d_0\times J}(d_1 \times J)^* \arrow[d, "\Pi_{d_0\times I}\phi_{\Pi_{d_0\times J}(d_1\times J)^*}"] \\
\arrow[r, phantom, "\mathbf{C}"] & \Pi_{d_0\times I}(d_1 \times I)^*(C_0 \times x)^*\Pi_{d_0\times J}(d_1 \times J)^* \arrow[d, "\Pi_{d_0\times I}(d_1\times I)^*\text{B-C}_{(d_1\times J)^*}"] \\
\Pi_{d_0\times I}\Pi_{p_0\times I}(p_1 \times I)^*(C_1 \times x)^*(d_1 \times J)^* \arrow[r] \arrow[d] & \Pi_{d_0\times I}(d_1 \times I)^*\Pi_{d_0\times I}(C_1 \times x)^*(d_1 \times J)^* \arrow[d, "\Pi_{d_0\times I}(d_1\times I)^*\Pi_{d_0\times I}\phi"] \\
\Pi_{d_0\times I}\Pi_{p_0\times I}(p_1 \times I)^*(d_1 \times I)^*(C_0 \times x)^* \arrow[r] & \Pi_{d_0\times I}(d_1 \times I)^*\Pi_{d_0\times I}(d_1 \times I)^*(C_0 \times x)^*
\end{tikzcd}
$$

For **A**, it is the same as for **A** in the previous diagram. For **B**, it is easily seen that the Beck-Chevalley iso satisfies a vertical gluing coherence condition:

$$
\begin{tikzcd}
& C^C \arrow[r, "v^*"] \arrow[dl, "\Pi_j"'] \arrow[d, "\Pi_k"] \arrow[dr, phantom, "\Nearrow"] & C^A \arrow[d, "\Pi_u"] \arrow[dr, "\Pi_e"] & \\
C^D \arrow[r, Rightarrow] \arrow[dr, "\Pi_l"'] & C^D \arrow[r, "w^*"] \arrow[d, "\Pi_y"] \arrow[dr, phantom, "\Nearrow"] & C^B \arrow[r, Rightarrow] \arrow[d, "\Pi_x"] & C^B \arrow[dl, "\Pi_f"] \\
& C^F \arrow[r, "z^*"'] & C^E &
\end{tikzcd}
\qquad
\begin{tikzcd}
z^*\Pi_y\Pi_k \arrow[r] \arrow[d] & \Pi_x w^*\Pi_k \arrow[r] & \Pi_x\Pi_u v^* \arrow[d] \\
z^*\Pi_l\Pi_j \arrow[r] & \Pi_f w^*\Pi_j \arrow[r] & \Pi_f\Pi_e z^*
\end{tikzcd}
$$

Finally, for **C**, we want to show

$$\begin{tikzcd}
& & & \bullet \arrow[d, "(p_1\times I)^*"] \arrow[dr, "\Pi_{d_0\times I}"] \\
& \bullet \arrow[urr, "(C_1\times X)^*"] \arrow[r, "(p_1\times J)^*"] \arrow[dl, "(C_1\times X)^*"'] \arrow[d, "\Pi_{d_0\times J}"] \arrow[dr, phantom, "\Nearrow"] & \bullet \arrow[r, "(C_2\times X)^*"] \arrow[d, "\Pi_{p_0\times J}"] \arrow[dr, phantom, "\Nearrow"] \arrow[ur, phantom, "\Nearrow"] & \bullet \arrow[d, "\Pi_{p_0\times I}"] \arrow[r, phantom, "\Longrightarrow"] & \bullet \arrow[dl, "(d_1\times I)^*"] \\
\bullet \arrow[r, phantom, "\Longrightarrow"] \arrow[dr, "\Pi_{d_0\times I}"'] & \bullet \arrow[r, "(d_1\times J)^*"] \arrow[d, "(C_0\times X)^*"] \arrow[dr, phantom, "\Nearrow"] & \bullet \arrow[r, "(C_1\times X)^*"] & \bullet \\
& \bullet \arrow[urr, "(d_1\times I)^*"']
\end{tikzcd}$$

that is:

$$\begin{tikzcd}
& \bullet \arrow[r, "w^*"] \arrow[dl, "q^*"'] \arrow[d, "\Pi_y"] \arrow[dr, phantom, "\Swarrow"] & \bullet \arrow[d, "\Pi_x"] \\
\bullet \arrow[r, phantom, "\Longleftarrow"] \arrow[dr, "\Pi_p"'] & \bullet \arrow[r, "z^*"] \arrow[d, "u^*"] \arrow[dr, phantom, "\Swarrow"] & \bullet \arrow[d, "v^*"] \\
& \bullet \arrow[r, "k^*"] & \bullet
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r, "w^*"] \arrow[d, "q^*"] \arrow[dr, phantom, "\Swarrow"] & \bullet \arrow[d, "m^*"] \arrow[dr, "\Pi_x"] \\
\bullet \arrow[r, "l^*"] \arrow[d, "\Pi_p"] \arrow[dr, phantom, "\Swarrow"] & \bullet \arrow[d, "\Pi_h"] \arrow[r, phantom, "\Longleftarrow"] & \bullet \arrow[dl, "v^*"] \\
\bullet \arrow[r, "k^*"] & \bullet
\end{tikzcd}$$

or

$$\begin{tikzcd}
\bullet \arrow[r, "q^*"] \arrow[d, "\Pi_y"] \arrow[dr, phantom, "\Nearrow"] & \bullet \arrow[r, "l^*"] \arrow[d, "\Pi_p"] \arrow[dr, phantom, "\Nearrow"] & \bullet \\
\bullet \arrow[r, "u^*"] & \bullet \arrow[r, "k^*"] & \bullet \arrow[u, "\Pi_h"']
\end{tikzcd}
=
\begin{tikzcd}
& \bullet \arrow[dr, "l^*"] \\
\bullet \arrow[ur, "q^*"] \arrow[r, "w^*"'] \arrow[d, "\Pi_y"] \arrow[dr, phantom, "\Nearrow"] & \bullet \arrow[u, phantom, "\Uparrow"] \arrow[r, "m^*"'] \arrow[d, "\Pi_x"] \arrow[dr, phantom, "\Nearrow"] & \bullet \arrow[d, "\Pi_h"] \\
\bullet \arrow[r, "z^*"] \arrow[dr, "u^*"'] & \bullet \arrow[r, "v^*"] & \bullet \\
& \bullet \arrow[u, phantom, "\Uparrow"] \arrow[ur, "k^*"']
\end{tikzcd}$$

However, looking at the pullback digrams, we see equality holds, so it is immediate by coherence of the Beck-Chevalley iso with horizontal gluing.

We just proved the $\eta^I, \mu^I$ assemble into modifications, so we have an indexed monad. Therefore, we can apply the previous result to conclude $\mathbb{D}^{\mathbb{C}}$ has indexed products. $\square$

# 10 Proving the theorem

The indexed version of generator is pretty intuitive.

**Definition 10.1.** *An object $G \in C^I$ is called a generator if for $A, B \in C^J$, $f \neq g : A \to B$ we have a span*

$$\begin{tikzcd}
K \arrow[r, "y"] \arrow[d, "x"] & J \\
I
\end{tikzcd}$$

*and $h : x^*(G) \to y^*(A)$ s.t. $y^*(f)h \neq y^*(g)h$. We also have the dual notion called cogenerator.*

**Lemma 10.2.** *Suppose $\mathbb{C}$ is $\mathcal{S}$-cocomplete an locally small. Then $G \in \mathbb{C}^I$ is a separating family iff for every $A \in \mathbb{C}^J$ there's a span*

$$\begin{tikzcd} K \arrow[r, "y"] \arrow[d, "x"'] & J \\ I & \end{tikzcd}$$

*and an epi $e : \Sigma_y x^*(G) \twoheadrightarrow A$.*

*Proof.* One direction is obvious. Suppose $G$ is a separating family, and let $(v, w) : K \to I \times J$ index morphisms $G \to A$ with universal arrow $h$, Now, if $f \neq g$, we have $h'$ s.t. $y^*(f)h' \neq y^*(g)h'$, but $h' = z^*(h)$, where $z$ is s.t. $(x, y) \circ z = (v, w)$, hence $w^*(f)h \neq w^*(g)h$. But since this holds for any $f, g$, clearly the transpose is epi. □

**Definition 10.3** (Indexed comma)**.** *Let $F : \mathbb{C} \to \mathbb{E}, G : \mathbb{D} \to \mathbb{E}$ be $\mathcal{S}$-indexed functors. Then $F \downarrow G$ is an $\mathcal{S}$-indexed category with*

*i)* $(F \downarrow G)^I = F^I \downarrow G^I$

*ii) For $h : F^I A \to G^I B$, $x : J \to I$,*

$$\begin{tikzcd} F^I A \arrow[r, "h"] & G^I B \end{tikzcd} \qquad \begin{tikzcd} F^I A \arrow[r, "F^I f"] & F^I B \end{tikzcd}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\begin{tikzcd} x^* F^I A \arrow[r, "x^* h"] & x^* G^I B \arrow[d, "\phi_G^x"] \\ F^J x^* A \arrow[u, "\phi_F^{x\,-1}"] & G^J x^* B \end{tikzcd} \qquad \begin{tikzcd} F^J x^* A \arrow[r, "F^J x^* f"] \arrow[d, "\cong"'] & F^J x^* B \\ x^* F^I A \arrow[r, "x^* F^I f"] & x^* F^I B \arrow[u, "\cong"'] \end{tikzcd}$$

We assume $\mathcal{S}$ is finitely complete for the next results.

The following is tedious but straightforward from the definitions:

**Lemma 10.4.** *If $F : \mathbb{C} \to \mathbb{D}$ is $\mathcal{S}$-continuous, the indexed comma category $B \downarrow F$ (where $B \in \mathbb{C}^1$) is $\mathcal{S}$-complete.*

**Lemma 10.5.** *If $\mathbb{C}$ is locally small, $B \downarrow F$ is also locally small.*

*Proof.* The prime candidate for a generic morphism between the objects comprising the span

$$\begin{tikzcd} I^* B \arrow[r, "g"] \arrow[d, "f"'] & F^I Y \\ F^I X & \end{tikzcd}$$

would be $F^I h$, where $h$ is generic for morphisms $X \to Y$ and we have $d : J \to I \times I$. The problem is that the corresponding triangle needn't commute. Therefore we use 3.5 i), since we are assuming $\mathcal{S}$ is cartesian, we take $z : K \rightarrowtail J$ generic w.rt. making the triangle commute. Then $d \circ z$ is the desired indexing morphism. □

**Lemma 10.6.** *If $\mathbb{C}$ is well-powered and $F$ is continuous, $B \downarrow F$ is also well-powered.*

*Proof.* We want a morphism generic for subobjects of $X$ for which $f : I^* B \to F^I X$ factors through $F^I A \rightarrowtail F^I X$. Let $m : A \to m^* X$ be the generic suboject with $x : J \to I$ indexing. We take the pullback

$$\begin{tikzcd} D \arrow[r] \arrow[d, "h"'] \arrow[dr, phantom, "\lrcorner", very near start] & F^I A \arrow[d, tail] \\ J^* B \arrow[r] & F^J x^* X \end{tikzcd}$$

and then use the fact invertibility is definable in $\mathbb{D}$ bt 3.5 ii), to get a morphism $z : K \rightarrowtail J$ universal for morphisms which take $h$ to an iso. Then $x \circ z : K \to I$ is the desired indexing map. $\square$

We also have:

**Theorem 10.7.** *If $\mathbb{C}$ admits a cogenerator object, so does $F \downarrow B$.*

*Proof.* The proof is quite analogous to the non-indexed case. Let $G$ be the generator and let $k : J^*B \to x^*F^IG$ be a generic arrow for morphisms $B \to F^IG$. We wish to prove $J^*B \to x^*F^IX \cong F^Jx^*X$ is the desired generator object. Let us have two different arrows

$$\begin{tikzcd} & F^K X \arrow[dd, shift right, "F^K f"'] \arrow[dd, shift left, "F^K g"] \\ K^* B \arrow[ur, "a"] \arrow[dr, "b"'] & \\ & F^K Y \end{tikzcd}$$

Then we have

$$\begin{tikzcd} L \arrow[r, "y"] \arrow[d, "x"'] & K \\ J & \end{tikzcd}$$

and $h : y^*Y \to x^*G$ s.t. $hy^*(f) \neq hy^*(g)$. Thus we have a morphism

$$\begin{tikzcd} & F^K y^* Y \arrow[dd, "F^K h"] \\ L^* B \arrow[ur, "y^* b"] \arrow[dr, "u^* k"'] & \\ & F^K x^* G \end{tikzcd}$$

(modulo coherence isos).

$\square$

Before moving on to the first big proof, we need some lemmas:

**Lemma 10.8.** *Let $\mathbb{D}$ be an internal category with a weakly terminal object (non-uniqueness). Then $\mathbb{D}$ is connected, that is,*

$$\begin{tikzcd} D_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & D_0 \arrow[r, two heads] & \pi_0(\mathbb{D}) = 1 \end{tikzcd}$$

*Proof.* Having a weakly terminal object means

$$\begin{tikzcd} C_1 \arrow[r] & C_0 \times C_0 \\ & X \arrow[u, "(x, \top)"'] \arrow[ul, dashed, "\exists"] \end{tikzcd}$$

But then the coequalizer arrow equalizes $x$ and $\top$, meaning this is the unique such arrow, which always exists, so $\pi_0(\mathbb{D})$ is terminal. $\square$

**Theorem 10.9.** *Let $\mathbb{D}$ be a connected internal category in $\mathcal{S}$ which admits an arrow $1 \to D_0$, and $\mathbb{C}$ an $\mathcal{S}$-indexed locally small category. Then the constant diagram functor $\mathbb{D}^* : \mathbb{C} \to \mathbb{C}^{\mathbb{D}}$ is fully faithful.*

*Proof.* Faithfulness is straightforward.

Let $f : D_0^* A \to D_0^* B$. Let $g : J^* A \to J^* B$ classify morphisms $A \to B$. Then we have $f \cong u^*(g)$, but since $d_0^*(f) = d_1^*(f)$ (as the constant diagrams have identities for maps), we have that $ud_0 = ud_1$, inducing, by connectedness, a factorization $u = x \circ \top_{D_0}$, but then we just proved $f = Q_0^*(x^*(g))$. We leave it to the reader to parse this statement in light of the existence of coherence isos, but we promise everything works out. $\square$

**Theorem 10.10** (The special indexed initial object theorem)**.** *Let $\mathcal{S}$ be cartesian. Let $\mathbb{C}$ be complete, locally small, well-powered and with a cogenerator. Then $\mathbb{C}$ has an indexed initial object.*

*Proof.* Let us first recall the non-indexed version: the desired initial object is the intersection of all the subobjects of the product of such cogenerating family.

It suffices to prove it for $\mathcal{C}^1$. Let $m : Y \to U^*(X)$ be the generic subobject of $X = \Pi_I(G)$. Now, we take the intersection

$$\begin{tikzcd} \pi_1^*(Y) \cap \pi_2^*(Y) \arrow[r] \arrow[d] \arrow[dr, "a\cap b = z^*(m)"] & \pi_1^*(Y) \arrow[d, "\pi_1^*(m)=a"] \\ \pi_2^*(Y) \arrow[r, "\pi_2^*(m)=b"'] & U \times U^*(X) \end{tikzcd}$$

which is again classified by some map $z : U \times U \to U$. Then we take the equalizer

$$\begin{tikzcd} U_1 \arrow[r, tail, "h"] & U \times U \arrow[r, shift left, "z"] \arrow[r, shift right, "\pi_1"'] & U \end{tikzcd}$$

If the reader is looking for intuition, consider again the naive Set-indexing: $z$ is the intersection map in $U = \mathrm{Sub}(A)$, and $U_1$ is precisely the subset representing the relation $\leq$. We thus have a category $\mathbb{U}$ with $d_0 = \pi_1 h, d_1 = p_2 h$. In fact, it is an internal poset since $h$ is mono.

Now, consider the following diagram structure $(Y, \Phi)$ on $\mathbb{C}^{\mathbb{U}}$: $\Phi = (h\pi_1)^*(Y) \to (h\pi^2)^*(Y)$ is precisely the image under $h^*$ of the left leg on the intersection pullback, which $h$ collapses to a triangle. Furthermore, consider the constant diagram on $X$, $\mathbb{U}^* X$. Then the universal mono $m : Y \to U^*(X)$ induces a morphism of diagrams $(Y, \Phi) \rightarrowtail \mathbb{U}^* X$, which is again mono because the forgetful functor on diagrams is faithful. Then we pass the limit functor $\lim_{\mathbb{U}}$ on this arrow, which is a right adjoint and gives us a mono arrow $B \rightarrowtail A$.

Now, by the results above, we have $A \cong X$, so we actually have an arrow $k : B \rightarrowtail X$. Furthermore, consider the unit $\eta : 1 \implies \Pi_{U \to \mathbb{U}}(U \to \mathbb{U})^*$. It is clearly mono since $(U \to \mathbb{U})^*$ is clearly faithful, which means we obtain a a mono arrow $h = \lim_{\mathbb{U}} \eta_{(Y,\Phi)} : B = \lim_{\mathbb{U}}(Y, \Phi) \to \Pi_U(Y)$ (intuitively this expresses the limit as a subobject of the product).

Now, if we take the transpose $h' : U^*(B) \to Y$, we have the following commutative diagram

$$\begin{tikzcd} U^*(B) \arrow[r, "h'"] \arrow[dr, "U^*(k)"'] & Y \arrow[d, "m"] \\ & U^*(X) \end{tikzcd}$$

since (we omit coherence isos and write $[m]$ for the induced arrow in $\mathbb{C}^{\mathbb{U}}$, $I$ for the functor $U_0 \to \mathbb{U}$):

$$k = {\eta^{\mathbb{U}}}_X^{-1} \circ \lim_{\mathbb{U}}([m])$$

$$
\begin{aligned}
& m \circ h' \\
&= I^*([m]) \circ \epsilon^U \circ U^* \lim_{\mathbb{U}} \eta^I_Y \\
&= \epsilon^U_{U^*} \circ U^* \lim_{\mathbb{U}} \Pi_I I^*([m]) \circ U^* \lim_{\mathbb{U}} \eta^I_Y \\
&= \epsilon^U_{U^*} \circ U^* \lim_{\mathbb{U}} \eta^I_{\mathbb{U}^* X} \circ {\eta^{\mathbb{U}}}_X \circ {\eta^{\mathbb{U}}}^{-1}_X \circ \lim_{\mathbb{U}}([m]) \\
&= {\eta^{\mathbb{U}}}^{-1}_X \circ \lim_{\mathbb{U}}([m]) = k
\end{aligned}
$$

Therefore we conclude $k : B \rightarrowtail X$ is in fact the minimal subobject. We now prove that $B$ is indeed the desired initial object of $\mathbb{C}^1$. Let $C \in \mathbb{C}^1$. Then by the dual of Lemma 10.2, we have $x : J \to I$ and a mono $C \rightarrowtail \Pi_J x^*(G)$. Since, $\Pi_J \cong \Pi_I \Pi_x$, we form the pullback

$$
\begin{tikzcd}
B \arrow[d, tail] \arrow[dr] & \\
U \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Pi_I(G) = X \arrow[d, "\Pi_I \eta^x_G"] \\
C \arrow[r, tail] & \Pi_J x^*(G)
\end{tikzcd}
$$

Furthermore, clearly any coequalizer from $B$ must be an isomorphism by minimality, hence $B$ is in fact an initial object. $\square$

Finally we can prove the desired theorem, but first we need another useful lemma.

**Lemma 10.11.** *Let $\mathcal{S}$ have finite products. Then the change of base functor $\Sigma_I^* : \mathrm{Cat}_{\mathcal{S}} \to \mathrm{Cat}_{\mathcal{S}/\mathcal{I}}$ preserves the property of having a (co) generator.*

*Proof.* Let $G \in \mathbb{C}^J$. We prove that $\pi_J^*(G)$ is the desired generator in the new category. Let $f \neq g$ in $\mathbb{C}^U, u : U \to I$. Then we have

$$
\begin{tikzcd}
K \arrow[r, "y"] \arrow[d, "x"'] & U \\
J &
\end{tikzcd}
\quad \to \quad
\begin{tikzcd}
K \arrow[r, "{(x,uy)}"] \arrow[d] \arrow[dr, "uy"] & J \times I \arrow[d, "\pi_I"] \\
U \arrow[r, "u"'] & I
\end{tikzcd}
$$

$\square$

Furthermore, such operation clearly preserves the other properties we need.

**Theorem 10.12** (The special indexed adjoint functor theorem)**.** *Let $\mathbb{S}$ be cartesian, and $\mathbb{C}, \mathbb{D}$ $\mathcal{S}$-indexed categories which are locally small and complete, and $\mathbb{C}$ $\mathcal{S}$-complete. Then $F : \mathbb{C} \to \mathbb{D}$ has a left adjoint iff it is continuous.*

*Proof.* We apply the previous theorem on each instance $B \downarrow F$, so that we may construct $H^1 \dashv F^1$. Furthermore, we can apply chance-of-base on the previous lemma to consider categories $B \downarrow F$ where $B \in \mathbb{D}^I$ originally. Now, we have $H^I \dashv F^I$ componentwise, but we need an **indexed** adjunction. To this end, we invoke the observation on section 5 that it suffices for the mate of $F^x$ to be invertible, but this follows easily from $\mathcal{S}$-continuity if we take all left adjoints of the mate (see Definition 4.5). $\square$

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