Historical Narrative of Maxwell's Equations

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Introduction

Maxwell's equations have taken many forms over the past century and a half. This paper will detail the historical development of Maxwell's equations, describing some of these forms and the people who contributed to them. I propose that through a historical look at the different forms of the equations and the reasoning used by various scientists, we can better understand the nature of physics and to some extent the meanings of Maxwell's equations. This also suggests some usefulness to integrating more teaching of physics history into physics classes.

Early Electromagnetism

In the early days of electromagnetism, many scientists were working on varied and still disjointed topics, including Oersted, Ampere, Biot, Savart, and Faraday (Arthur, 2013). For example, Gauss thought that electric actions propagate with finite velocity, but did not publish this work, because he couldn't prove it (Sengupta & Sarkar, 2003). Oersted discovered that a steady electric current will create a magnetic force, and Ampere described this in equation form (Israelson, 2014). Faraday's work included discovering (or continuing work on) rotational magnetic field around electric current, induction, diamagnetism, dielectrics, and more (Israelson, 2014). Maxwell brought together much of this work.

One of Faraday's contributions was the proposal that magnetic lines of force extended everywhere in space (Israelson, 2014). It is interesting to note the differences between how Faraday thought about these lines compared to how we think of field now. He thought there were many particles in the medium between charges that all exerted forces, thus not fully fixing the concerning theory of "action-at-a-distance," but limiting the distance to be very small. Though still in progress, Faraday's theory was "the first precise and quantitative concept of a field" (Israelson, 2014, p. 8), but he did not live to see his theory accepted.

When Faraday was about 65, Maxwell was about 25 and well positioned to expand on Faraday's work. Faraday had little formal education and liked to think about physics purely conceptually rather than through mathematical formalism. He saw capacity to explain experimental results as proof that concepts were physically real, thus thinking his "magnetic lines of force" had to physically exist (Israelson, 2014). On the other hand, Maxwell had more formal math training, allowing him to mathematically express Faraday's idea of field. He did so much more than just write 20 equations describing Faraday's idea of field, though, as is often characterized. Israelson says (2014, p. 11), "Maxwell defined a new kind of theoretical physics in which the classification of mathematical quantities, vector symbolism, and Lagrangian dynamics became major construction tools." Thus, through the production of a specific piece of physics, Maxwell changed all of physics. This one process – the development of the now-called Maxwell's equations – had far-reaching effects.

Maxwell

Most authors characterize Maxwell as having picked up from Faraday's work specifically (Israelson, 2014; Sengupta & Sarkar, 2003; Bork, 1963; Arthur, 2013). Many also agree though that Maxwell synthesized all prior work done by the lengthy list of scientists above (Sengupta & Sarkar, 2003; Arthur, 2013), but it is difficult to find explicit descriptions of the ways Maxwell interacted with work by earlier scientists besides Faraday, like Gauss, Ampere, and more. Perhaps, this is because there is no record of the explicit references, and Maxwell was instead influenced by these scientists' work indirectly and did not reference them. There is one implied link in the literature from Gauss's work to Maxwell's, however. Riemann was a student of Gauss's who continued trying after Gauss's death to prove Gauss's idea that electric actions propagate with finite velocity (Sengupta & Sarkar, 2003). Maxwell wrote a paper comparing his work to his contemporaries, like Riemann, Weber, and Lorenz (Bork, 1963), so it is likely he knew of Gauss's work through Riemann. Clearly, he was influenced in direct and indirect ways by prior scientific progress.

The first of Maxwell's three major papers was based directly on Faraday's idea of magnetic force lines, called "On Faraday's Lines of Force" (Bork, 1963). In his second and third (refined from the second) major papers, he writes 20 equations describing the electromagnetic field, first termed thus in the third paper (Bork, 1963; Israelson, 2014). According to Israelson (2014, p. 9),

> "The specific concepts that Maxwell adopted from Faraday included the field-based definitions of electric charge and current, the concept of conduction as the competition between polarization build-up and decay, and the reduction of all electric and magnetic actions to stresses in the field."

In the earliest forms of Maxwell's equations, he thought about electromagnetic actions in a mechanical analogy through the concept of "molecular vortices," but he ultimately switched from this theory to Faraday's theory of field (Arthur, 2013, p. 62). Maxwell's idea of field was that it pervades all of space and physical media alike, avoiding the problems of action-at-a-distance, although he seems to have thought about the field as being in the ether. Thus, he modified Faraday's idea of a field, bringing it slightly closer to the modern idea.

Maxwell's 20 equations in his second paper are shown in Figure 1 (Arthur, 2013, p. 65). He used a longhand notation of calling P, Q, and R the x, y, and z components of electric field respectively (not using E for electric field initially) (Arthur, 2013, p. 62). Maxwell's own writing is very thorough and organized – fortunately for historians of science. From "A Dynamical Theory of the Electromagnetic Field" (1865, p. 22), we find that f, g, and h are the electric displacements in the x, y, and z directions respectively; p, q, and r are electrical current components; F, G, and H are components of electromagnetic momentum; and α, β, and γ are magnetic forces. Clearly, we have much for which to thank Heaviside and Lorentz (Arthur, 2013, p. 62) for cleaning up these equations, as well as Maxwell for documenting his process. It would have been incredibly difficult for Maxwell's equations to have been clarified and condensed into the modern form without the clear descriptions Maxwell wrote of the meanings of his symbols.

$$\left.\begin{aligned} p' &= p + \frac{df}{dt}, \\ q' &= q + \frac{dg}{dt}, \\ r' &= r + \frac{dh}{dt}, \end{aligned}\right\} \quad . \quad (A)$$

$$\left.\begin{aligned} \mu\alpha &= \frac{dH}{dy} - \frac{dG}{dz}, \\ \mu\beta &= \frac{dF}{dz} - \frac{dH}{dx}, \\ \mu\gamma &= \frac{dG}{dx} - \frac{dF}{dy}. \end{aligned}\right\} \quad . \; . \; . \quad (B)$$

$$\left.\begin{aligned} \frac{d\gamma}{dy} - \frac{d\beta}{dz} &= 4\pi p', \\ \frac{d\alpha}{dz} - \frac{d\gamma}{dx} &= 4\pi q', \\ \frac{d\beta}{dx} - \frac{d\alpha}{dy} &= 4\pi r'. \end{aligned}\right\} \quad . \; . \quad (C)$$

$$\left.\begin{aligned} P &= \mu\left(\gamma\frac{dy}{dt} - \beta\frac{dz}{dt}\right) - \frac{dF}{dt} - \frac{d\Psi}{dx}, \\ Q &= \mu\left(\alpha\frac{dz}{dt} - \gamma\frac{dx}{dt}\right) - \frac{dG}{dt} - \frac{d\Psi}{dy}, \\ R &= \mu\left(\beta\frac{dx}{dt} - \alpha\frac{dy}{dt}\right) - \frac{dH}{dt} - \frac{d\Psi}{dz}. \end{aligned}\right\} \quad . \; . \quad (D)$$

$$\left.\begin{aligned} P &= kf, \\ Q &= kg, \\ R &= kh. \end{aligned}\right\} \quad . \; . \quad (E)$$

$$\left.\begin{aligned} P &= -\varrho p, \\ Q &= -\varrho q, \\ R &= -\varrho r. \end{aligned}\right\} \quad . \; . \quad (F)$$

$$e + \frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = 0 \quad . \; . \quad (G)$$

$$\frac{de}{dt} + \frac{dp}{dx} + \frac{dq}{dy} + \frac{dr}{dz} = 0 \quad . \; . \; . \quad (H)$$

Figure 1 Maxwell's 20 equations from 1865 "A Dynamical Theory of the Electromagnetic Field" mathematically describing Faraday's idea of field (Arthur, 2013, p. 65). Clearly, his notation was complicated and unwieldy.

The main concept he added to the work of those before him, besides simply furthering Faraday's field theory, was the idea of displacement current. The way that Maxwell wrote displacement current is as follows (Arthur, 2013, p. 63).

$$R = -4\pi E^2 h; r = \frac{dh}{dt}$$

Here, R is the electromotive force, E is a coefficient depending on the dielectric, h is the displacement, r is the value of the electric current due to displacement (Maxwell, 1861). Interestingly, he never made the explicit argument of the symmetry of the set of equations when displacement current is included, despite this being a common modern one (Bork, 1963). Such an argument only came later in the work of those who refined Maxwell's equations.

Electromagnetism after Maxwell

Maxwell's contemporaries, Boltzmann, Hertz, Kirchoff, Lorenz, and Weber did some work that helped cement Maxwell's equations (Arthur, 2013). For example, Hertz's discovery of electromagnetic waves helped Maxwell's theory gain traction in the scientific world, since his theory had suggested the existence of electromagnetic waves (Sengupta & Sarkar, 2003). Maxwell's work was poorly received at first, but clearly gained momentum.

Oliver Heaviside was a major player in bringing about the final form of Maxwell's equations. He was first to explicitly discuss the symmetry of the equations. In his *Electromagnetic Theory*, he "modifie[d] and extend[ed] Maxwell's equations" (Bork, 1963, p. 5) building on them with rationalized units, vector notation, and symmetry. Heaviside was among several to condense and clarify Maxwell's equations (Arthur, 2013), some of whom are called the Maxwellians (Sengupta & Sarkar, 2003). Heaviside converted Maxwell's wieldy quaternions to vector algebra notation, writing div and curl, while Gibbs introduced notation similar to modern notation. Heaviside did not condense the 20 equations to four, but his equations did include four that look similar to the currently recognized Maxwell's equations. Hertz further clarified the equations (sometimes known as Maxwell-Hertz equations), mostly by removing the potentials and giving the equations separate forms for free space, conductors, and more (Arthur, 2013).

Modern Electromagnetism

This process carries us to the modern way of writing Maxwell's equations and explains how Maxwell came to be a pivotal person in the formulation of electromagnetism by synthesizing the work done before him and influencing the work that followed. However, the equations have not remained static, untouched entities since the time of the Maxwellians. There is still a variety of ways to express, conceptualize, and prove Maxwell's equations. An example of one of the many modern ways of thinking about Maxwell's equations is Feynman's proof, using only Newton's laws of motion and the commutation relation for position and velocity of a nonrelativistic particle (Dyson, 1990). Feynman hoped from this proof to find physics models that were not describable through Lagrangians or Hamiltonians (Dyson, 1990, p. 2). Since it did not do this, Feynman considered the proof a failure, but Dyson considers the proof a successful historical relic to give context to historical physics. This view can inform the usefulness of studying the history of physics.

A more typical modern version of Maxwell's equations is the tensor form used to express the equations in consistent relativistic language (Griffiths, 2017, p. 539), credited to Minkowski (Arthur, 2013). That looks like the following.

$$\frac{\partial F^{\mu\nu}}{\partial x^{\nu}} = \mu_0 J^{\mu}, \frac{\partial G^{\mu\nu}}{\partial x^{\nu}} = 0$$

$F^{\mu\nu}$ is a component of the field tensor which consists of electric field components divided by the speed of light. $G^{\mu\nu}$ is a component of the dual tensor which contains components of magnetic field and electric field. Summing over ν for different values of μ gives Maxwell's equations. For example, $\mu = 0$ gives Gauss's Law from the first equation.

This relativistic form can be further simplified to a simple, elegant equation.

$$\Box^2 A^{\mu} = -\mu_0 J^{\mu}$$

Here, $\Box$ is the d'Alambertian, or $\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$. A^{μ} is a component of the potential four-vector, with the first component being the scalar potential divided by c, and the others the components of the vector potential. J^{μ} is a component of the current density four-vector, made up of charge density times the speed of light, and the components of current density.

The current relativistic form came into being partly due to various thought experiments about relativistic electromagnetism. One such thought experiment showed that the classical forms of the electromagnetic equations yield a dipole moving with constant proper acceleration and no external force – self-sustaining motion (Cornish, 1986). The energy for this motion comes from the electromagnetic field (Griffiths, 2017). This seems to suggest the intermixing of the electric and magnetic fields in different frames in relativity.

Discussion

This historical background of Maxwell's equations makes many things clear – some physics content and some tangential descriptors of the nature of science. The biggest truth that this history has pointed out is that science is not something done by any one person. This is already known, but it is eye opening to realize how scientists draw on other scientists' work without even always knowing and certainly without always directly referencing it. To miss referencing work is not a good scientific practice, but to a certain extent seems to have been unavoidable at times in history. Information on exactly how Maxwell was influenced by other scientists than Faraday is frustratingly difficult to find; it is just known that he drew from other scientists' work. There does not seem to be documentation of how Gauss's ideas reached Maxwell. Similarly, Lorentz presented a small set of electromagnetic equations for a microscopic model, never referencing Maxwell (Arthur, 2013). To some extent, it seems there is societal knowledge passed around in unclear ways, something of which we should certainly be wary.

This societal knowledge is passed around in ways that are messy, non-linear, and often undocumented. It is difficult to find a clear linear, chronological history of Maxwell's equations. This suggests many things: (a) the whole narrative is too lengthy and involved for the length of one paper (b) it's difficult to describe a clear linear historical story when history is not clear or linear, and (c) there is a lot of remaining work to be done in telling the history of physics. It was surprising how hard it was to find descriptions of the different forms of equations throughout history and how they changed as different scientists worked on them. This points to my flawed conceptual model of how equations come to be, expecting more linear passing on and clear influencing than actually happens.

It is also interesting that Maxwell seems to have been the center point of a great scientific movement that narrowed into his work and then broadened back out again. There was so much work prior to Maxwell that Maxwell synthesized that then also led to such a wide range of new physics. An open question remains if some scientists really do stand out as pivot points in history or if it only seems this way because, for example, we call the pivotal electromagnetic equations "Maxwell's equations" rather than "Gauss-Faraday-Maxwell-Heaviside-Hertz-etc. equations." Perhaps some scientists just get lucky to be working on a piece of science at the right time to get the most credit.

This historical research has also shown how much physics is done in collaboration, how changeable it is in time, that sometimes people try something that is not necessarily wrong but adds nothing new to physics, and many more such lessons. These all seem like they could be useful things for students to learn firsthand from more integration of history into physics education. I expected before doing this research to better understand Maxwell's equations from learning their history, which is only partly true, but there are many other lessons that history can teach.

Implications for Teaching

Stanley (2016) argues that more history should be taught in conjunction with physics for many reasons, some of which align with lessons discussed above. He suggests that physics education often neglects to teach aspects like the messy, social aspect of physics, and that learning the history of science can help fill this gap. The history of physics shows that physics is social and needs many kinds of people, showing a more human side of physics that may help student retention in physics. Learning history can also point to new science questions and foster curiosity, showing students that physics is never finished. Some of these points I learned from my historical analysis, suggesting there is usefulness to teaching Maxwell's equations alongside more of their history.

Conclusion

Prior to doing this research, I had thought I would understand Maxwell's equations significantly better from seeing their historical derivation and getting in the minds of those who worked on developing them. To some extent, this happened. However, most of my better understanding Maxwell's equations came simply from the sheer amount I read and thought about them. While Maxwell's papers are very thorough, his notation and ideas are so different from the modern ways of thinking that the historical derivations are unlikely to help a modern student see new meaning in the equations. Yet, there are so many other lessons I learned and that other students could gain from doing more research on the history of Maxwell's equations. Through encouraging students to learn more about the history of physics, students may better understand the nature of science as well as the physics concepts on which they focus.

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