



# Chapter 9

## Variational Methods: A Short Intro

*Multiple View Geometry*  
Summer 2016

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**Variational methods** are a class of optimization methods. They are popular because they allow to solve many problems in a mathematically transparent manner. Instead of implementing a heuristic sequence of processing steps (as was commonly done in the 1980's), one clarifies beforehand what properties an 'optimal' solution should have.

Variational methods are particularly popular for **infinite-dimensional** problems and **spatially continuous** representations.

Particular applications are:

- Image denoising and image restoration
- Image segmentation
- Motion estimation and optical flow
- Spatially dense multiple view reconstruction
- Tracking

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## Advantages of Variational Methods

Variational methods have **many advantages** over heuristic multi-step approaches (such as the Canny edge detector):

- A mathematical analysis of the considered cost function allows to make statements on the **existence** and **uniqueness** of solutions.
- Approaches with multiple processing steps are difficult to modify. All steps rely on the input from a previous step. Exchanging one module by another typically requires to re-engineer the entire processing pipeline.
- Variational methods make **all modeling assumptions transparent**, there are no hidden assumptions.
- Variational methods typically have **fewer tuning parameters**. In addition, the effect of respective parameters is clear.
- Variational methods are **easily fused** – one simply adds respective energies / cost functions.



## Example: Variational Image Smoothing



Let  $f : \Omega \rightarrow \mathbb{R}$  be a grayvalue input image on the domain  $\Omega \subset \mathbb{R}^2$ . We assume that the observed image arises by some 'true' image corrupted by additive noise. We are interested in a denoised version  $u$  of the input image  $f$ .

The approximation  $u$  should fulfill two properties:

- It should be as similar as possible to  $f$ .
- It should be spatially smooth (i.e. 'noise-free').

Both of these criteria can be entered in a cost function of the form

$$E(u) = E_{\text{data}}(u, f) + E_{\text{smoothness}}(u)$$

The first term measures the similarity of  $f$  and  $u$ . The second one measures the smoothness of the (hypothetical) function  $u$ .

Most variational approaches have the above form. They merely differ in the specific form of the data (similarity) term and the regularity (or smoothness) term.

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## Example: Variational Image Smoothing

For denoising a grayvalue image  $f : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ , specific examples of data and smoothness term are:

$$E_{\text{data}}(u, f) = \int_{\Omega} (u(x) - f(x))^2 dx,$$

and

$$E_{\text{smoothness}}(u) = \int_{\Omega} |\nabla u(x)|^2 dx,$$

where  $\nabla = (\partial/\partial x, \partial/\partial y)^\top$  denotes the **spatial gradient**.

Minimizing the weighted sum of data and smoothness term

$$E(u) = \int (u(x) - f(x))^2 dx + \lambda \int |\nabla u(x)|^2 dx, \quad \lambda > 0,$$

leads to a **smooth approximation**  $u : \Omega \rightarrow \mathbb{R}$  of the input image.

Such energies which assign a real value to a function are called a **functionals**. How does one minimize functionals where the argument is a function  $u(x)$  (rather than a finite number of parameters)?

# Functional Minimization & Euler-Lagrange Equation

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- As a necessary condition for minimizers of a functional the associated Euler-Lagrange equation must hold. For a functional of the form

$$E(u) = \int \mathcal{L}(u, u') dx,$$

it is given by

$$\frac{dE}{du} = \frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial u'} = 0$$

- The central idea of variational methods is therefore to determine solutions of the Euler-Lagrange equation of a given functional. For general non-convex functionals this is a difficult problem.
- Another solution is to start with an (appropriate) function  $u_0(x)$  and to modify it step by step such that in each iteration the value of the functional is decreased. Such methods are called descent methods.

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## Gradient Descent



One specific descent method is called **gradient descent** or **steepest descent**. The key idea is to start from an initialization  $u(x, t = 0)$  and iteratively march in direction of the negative energy gradient.

For the class of functionals considered above, the gradient descent is given by the following **partial differential equation**:

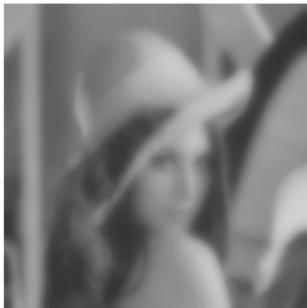
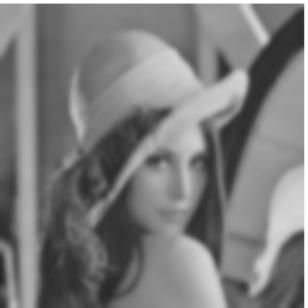
$$\begin{cases} u(x, 0) = u_0(x) \\ \frac{\partial u(x, t)}{\partial t} = -\frac{dE}{du} = -\frac{\partial \mathcal{L}}{\partial u} + \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial u'} \end{cases}$$

Specifically for  $\mathcal{L}(u, u') = \frac{1}{2}(u(x) - f(x))^2 + \frac{\lambda}{2}|u'(x)|^2$  this means:

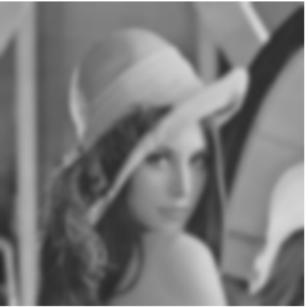
$$\frac{\partial u}{\partial t} = (f - u) + \lambda u''.$$

If the gradient descent evolution converges:  $\partial u / \partial t = -\frac{dE}{du} = 0$ , then we have found a solution for the Euler-Lagrange equation.

# Image Smoothing by Gradient Descent



$$E(u) = \int (f - u)^2 dx + \lambda \int |\nabla u|^2 dx \rightarrow \min.$$



$$E(u) = \int |\nabla u|^2 dx \rightarrow \min.$$

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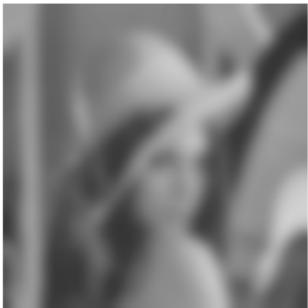
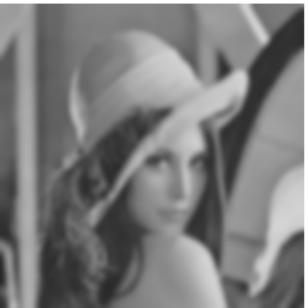
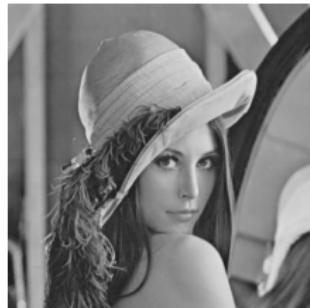
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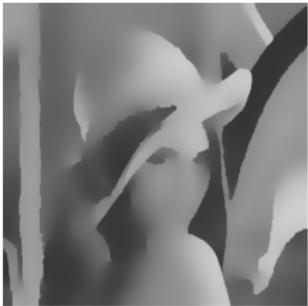
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# Discontinuity-preserving Smoothing



$$E(u) = \int |\nabla u|^2 dx \rightarrow \min.$$



$$E(u) = \int |\nabla u| dx \rightarrow \min.$$

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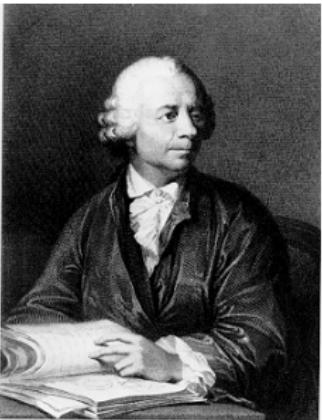
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Leonhard Euler (1707 – 1783)

- Published 886 papers and books, most of these in the last 20 years of his life. He is generally considered the most influential mathematician of the 18th century.
- Contributions: Euler number, Euler angle, Euler formula, Euler theorem, Euler equations (for liquids), Euler-Lagrange equations,...
- 13 children



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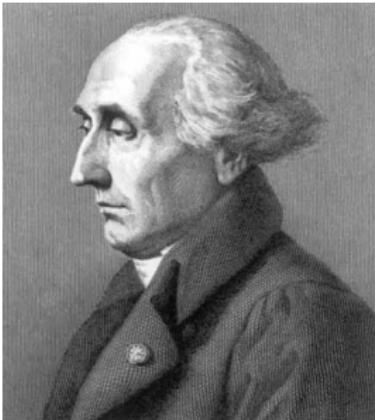
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# Joseph-Louis Lagrange

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Joseph-Louis Lagrange (1736 – 1813)

- born Giuseppe Lodovico Lagrangia (in Turin). Autodidact.
- At the age of 19: Chair for mathematics in Turin.
- Later worked in Berlin (1766-1787) and Paris (1787-1813).
- 1788: *La Méchanique Analytique*.
- 1800: *Leçons sur le calcul des fonctions*.



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