JANUARY 2024 PRELIMINARY EXAM

Problem 1. *Suppose $\{a_n\}$ and $\{b_n\}$ are two complex sequences such that*

$$\lim_{n\to\infty} a_n b_n = 0.$$

Show that at least one of $\{a_n\}$ and $\{b_n\}$ has a subsequences that converges to zero.

Problem 2. *Suppose $f : [a,b] \to \mathbb{R}$ is bounded, and moreover f is Riemann integrable on $[a,c]$ for all $a < c < b$. Show that f is Riemann integrable on $[a,b]$.*

Problem 3. *Suppose* $f : \mathbb{R} \to \mathbb{R}$ *is differentiable and* $\lim_{x\to\infty} f'(x) = 0$. *Show that if the sequence* $\{f(n)\}_{n\in\mathbb{N}}$ *converges, then the limit* $\lim_{x\to\infty} f(x)$ *exists.*

Proof. Let $L := \lim_{n\in\mathbb{N}, n\to\infty} f(n)$. We will show $\lim_{x\to\infty} f(x) = L$.
Let $\epsilon > 0$. Then there exists an $N \in \mathbb{N}$ such that $|L - f(m)| < \epsilon/2$ for all $m \geq N$. Similarly, there exists an $M \in \mathbb{N}$ such that $|f'(p)| < \epsilon/2$ for all $p \geq M$. Set $P = \max\{N, M\}$. Then for all $x \geq P$ set $\tilde{x} = \lceil x \rceil$ so that $\tilde{x} \in \mathbb{N}$ and $\tilde{x} \geq P$ as well. Then

$$\begin{aligned} |L - f(x)| &= |L - f(x) - f(\tilde{x}) + f(\tilde{x})| \\ &\leq |L - f(\tilde{x})| + |f(\tilde{x}) - f(x)| \text{ by the triangle inequality,} \\ &< \epsilon/2 + |f(\tilde{x}) - f(x)| \text{ since } \tilde{x} \geq P. \end{aligned}$$

It remains to show that $|f(\tilde{x}) - f(x)| < \epsilon/2$. Since f is differentiable, by the Mean Value Theorem, there exists $c \in (x, \tilde{x})$ such that

$$\frac{|f(\tilde{x}) - f(x)|}{|\tilde{x} - x|} = |f'(c)| < \epsilon/2$$

where the inequality holds since $c \geq P$. Then,

$$|f(\tilde{x}) - f(x)| < |\tilde{x} - x|\epsilon/2.$$

Putting it altogether, we get the required bound

$$|L - f(x)| < \epsilon/2 + |f(\tilde{x}) - f(x)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

This shows that the limit $\lim_{x\to\infty} f(x)$ equals L, and hence, exists. □

Problem 4. *A function $f : \mathbb{R} \to \mathbb{R}$ is called Lipschitz if there exists $M > 0$ such that*

$$|f(x) - f(y)| \leq M|x - y|$$

for all $x, y \in \mathbb{R}$. Show that every Lipschitz function on $\mathbb{R}$ is uniformly continuous on $\mathbb{R}$, but not every uniformly continuous function on $\mathbb{R}$ is necessarily Lipschitz on $\mathbb{R}$.