JANUARY 2017 PRELIMINARY EXAM

Advanced Calculus, Mandatory Problems

(1) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and f' is continuous. Show that the restriction of f to any closed interval $[a, b]$ is Lipschitz continuous.

(2) Suppose that (X, d) is a metric space, fix a point a, and let $f(x) = d(a, x)$. Show that the function $f : X \to \mathbb{R}$ is uniformly continuous.

Advanced Calculus, Optional Problems

(3) Let $f : (a, b) \to \mathbb{R}$ be differentiable. Suppose that $f'(c) = 0$ for some $c \in (a, b)$. Show that if f has a local minimum at $x = c$ and $f''(c)$ exists, then $f''(c) \geq 0$.

(4) Suppose that $f \in C[0, 1]$ and

$$\int_a^b x^n f(x) dx = 0 \qquad \text{for all integer } n \geq 0.$$

Show that f is the zero function.

Real Analysis, Mandatory Problems

(1) For $f \in L^1(\mathbb{R})$, the Fourier transform of f is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx.$$

(a) Show that $\hat{f}$ is continuous in $\mathbb{R}$.
(b) Show that $\hat{f} \to 0$ as $|\xi| \to \infty$. For this part, you may assume that the set of all linear combinations of characteristic functions over bounded open intervals is dense in $L^1(\mathbb{R})$.

(2) Let f be an integrable function in $\mathbb{R}^d$. Show that

$$\lim_{\alpha \to \infty} \alpha m \left\{ x \in \mathbb{R}^d : |f(x)| > \alpha \right\} = 0.$$

Real Analysis, Optional Problems

(3) (a) State Egorov's Theorem.
(b) Use Egorov's Theorem to prove the Bounded Convergence Theorem: if f_k is a sequence of measurable functions on a measurable set E with $m(E) < \infty$, such that $f_k \to f$ a.e. in E and $|f_k| \le M$ a.e. in E for some finite constant M, then

$$\int_E |f_k - f| \, dx \to \qquad \text{as } k \to \infty.$$

(4) For a measurable function f on a measurable set $E \subset \mathbb{R}$, define

$$\|f\|_{L^\infty(E)} = \inf \left\{ \alpha : m \left\{ x \in E : |f(x)| > \alpha \right\} = 0 \right\}.$$

Show that if $\|f\|_{L^\infty(E)} < \infty$ and $0 < m(E) < \infty$, then

$$\lim_{p \to \infty} \left(\frac{1}{m(E)} \int_E |f|^p \, dx \right)^{\frac{1}{p}} = \|f\|_{L^\infty(E)}.$$