January 2017 Preliminary Exam

Committee Members: No record

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Advanced Calculus, Mandatory Problems

1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and f' is continuous. Show that the restriction of f to any closed interval $[a, b]$ is Lipschitz continuous.

2. Suppose that (X, d) is a metric space, fix a point a, and let $f(x) = d(a, x)$. Show that the function $f : X \to \mathbb{R}$ is uniformly continuous.

Advanced Calculus, Optional Problems

3. Let $f : (a, b) \to \mathbb{R}$ be differentiable. Suppose that $f'(c) = 0$ for some $c \in (a, b)$. Show that if f has a local minimum at $x = c$ and $f''(c)$ exists, then $f''(c) \geq 0$.

4. Suppose that $f \in C[0, 1]$ and

$$\int_a^b x^n f(x)dx = 0 \quad \text{for all integer } n \geq 0.$$

Show that f is the zero function.

Real Analysis, Mandatory Problems

1. For $f \in L^1(\mathbb{R})$, the Fourier transform of f is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx.$$

   (a) Show that $\hat{f}$ is continuous in $\mathbb{R}$.

   (b) Show that $\hat{f} \to 0$ as $|\xi| \to \infty$. For this part, you may assume that the set of all linear combinations of characteristic functions over bounded open intervals is dense in $L^1(\mathbb{R})$.

2. Let f be an integrable function in $\mathbb{R}^d$. Show that

$$\lim_{\alpha \to \infty} \alpha m \left\{ x \in \mathbb{R}^d : |f(x)| > \alpha \right\} = 0.$$

Real Analysis, Optional Problems

3. (a) State Egorov's Theorem.

   (b) Use Egorov's Theorem to prove the Bounded Convergence Theorem: if f_k is a sequence of measurable functions on a measurable set E with $m(E) < \infty$, such that $f_k \to f$ a.e. in E and $|f_k| \leq M$ a.e. in E for some finite constant M, then

$$\int_E |f_k - f| \, dx \to \qquad \text{as } k \to \infty.$$

4. For a measurable function f on a measurable set $E \subset \mathbb{R}$, define

$$\|f\|_{L^\infty(E)} = \inf \left\{ \alpha : m \left\{ x \in E : |f(x)| > \alpha \right\} = 0 \right\}.$$

   Show that if $\|f\|_{L^\infty(E)} < \infty$ and $0 < m(E) < \infty$, then

$$\lim_{p \to \infty} \left(\frac{1}{m(E)} \int_E |f|^p \, dx \right)^{\frac{1}{p}} = \|f\|_{L^\infty(E)}.$$

Solutions

Advanced Calculus, Mandatory Problems

1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and f' is continuous. Show that the restriction of f to any closed interval $[a, b]$ is Lipschitz continuous.

   Proof. For a function to be Lipschitz continuous, we must have that $|f(x) - f(y)| \leq M|x - y|$ for some constant M. Since f is continous and differentiable, in particular on $[a, b]$, the Mean Value Theorem implies that for any $x, y \in [a, b]$, there exists some $c \in [a, b]$ such that
   $$f'(c) = \frac{f(x) - f(y)}{x - y}.$$
   Applying the absolute value to each side, we obtain that $|f(x) - f(y)| = |f'(c)|\,|x - y|$.

   Now, since f' is continuous, in particular on $[a, b]$, we know that f' must attain some maximum, call it M, on $[a, b]$. Thus we have
   $$|f(x) - f(y)| = |f'(c)|\,|x - y| \leq M\,|x - y|\,.$$
   Therefore, f is Lipschitz continuous on any closed interval $[a, b]$. $\square$

2. Suppose that (X, d) is a metric space, fix a point a, and let $f(x) = d(a, x)$. Show that the function $f : X \to \mathbb{R}$ is uniformly continuous.

   Proof. A function f is uniformly continuous if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$, $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$. Since X is a metric space, we know by the triangle inequality and symmetry that $d(a, x) \leq d(a, y) + d(x, y)$. This implies that $d(a, x) - d(a, y) \leq d(x, y)$.

   Choose $x, y \in X$ so that $d(x, y) < \delta$ and set $\varepsilon = \delta$. Then we have

   $$|f(x) - f(y)| = |d(a, x) - d(a, y)| \leq d(x, y) < \delta = \varepsilon.$$

   Thus, f is uniformly continuous. $\square$

Advanced Calculus, Optional Problems

3. Let $f : (a, b) \to \mathbb{R}$ be differentiable. Suppose that $f'(c) = 0$ for some $c \in (a, b)$. Show that if f has a local minimum at $x = c$ and $f''(c)$ exists, then $f''(c) \geq 0$.

   Proof. Since f has a local minimum at $x = c$, we know that c is a critical point of f. Furthermore, since we know that f'' exists, we may apply the second derivative test which states that for any critical point x_0, if $f''(x_0) > 0$, then f has a local minimum at x_0 and if $f''(x_0) < 0$ then f has a local maximum at x_0. The test is inconclusive if $f''(x_0) = 0$.

   Now, since c is a critical point of f, if $f''(c) < 0$, the second derivative test would imply that f has a local maximum at $x = c$, which we know is false. Note that if $f''(c) = 0$, the test is inconclusive and so doesn't directly contradict that f has a local minimum at c. Thus, it must be that $f(c) \geq 0$. $\square$

4. Suppose that $f \in C[0,1]$ and

$$\int_a^b x^n f(x) dx = 0 \quad \text{for all integer } n \geq 0.$$

Show that f is the zero function.

Proof. $\square$

Real Analysis, Mandatory Problems

1. For $f \in L^1(\mathbb{R})$, the Fourier transform of f is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i x\xi}dx.$$

   (a) Show that $\hat{f}$ is continuous in $\mathbb{R}$.

   Proof. Since $f \in L^1(\mathbb{R})$, we know that f is integrable. That is, $\int_{\mathbb{R}} f(x) < \infty$. Furthermore, since f is integrable, $|f|$ is as well.

   Consider some sequence $\xi_n \to \xi$ and let $g_n(x) = f(x)e^{-2\pi i x\xi_n}$. Observe that this is a measurable function for all n as it is the product of two measurable functions. Furthermore, $g_n(x) \to f(x)e^{-2\pi i x\xi} =: g(x)$ for all $x \in \mathbb{R}$. Next, note that $\left|f(x)e^{-2piix\xi_n}\right| = |f(x)|$. Thus, we have that $g_n(x) \leq |f(x)|$ where f is an integrable function and g_n is a sequence of measurable functions with $g_n(x) \to g(x)$. So by the Dominated Convergence Theorem, we know that

$$\int_{\mathbb{R}} f(x)e^{-2\pi i x\xi_n}dx \to \int_{\mathbb{R}} f(x)e^{-2\pi i x\xi}dx.$$

   That is, $\hat{f}(\xi_n) \to \hat{f}(\xi)$. Recall that a function h is continuous if and only if it takes convergent sequences to convergent sequences. In other words, if $x_n \to x$, then $h(x_n) \to h(x)$.

   Therefore, since $\xi_n \to \xi$ and we have that $\hat{f}(\xi_n) \to \hat{f}(\xi)$, it follows that $\hat{f}$ is continuous. $\square$

   (b) Show that $\hat{f} \to 0$ as $|\xi| \to \infty$. For this part, you may assume that the set of all linear combinations of characteristic functions over bounded open intervals is dense in $L^1(\mathbb{R})$.