JANUARY 2024 PRELIMINARY EXAM

Problem 1. *Suppose* $\{a_n\}$ *and* $\{b_n\}$ *are two complex sequences such that*

$$\lim_{n\to\infty} a_n b_n = 0.$$

Show that at least one of $\{a_n\}$ *and* $\{b_n\}$ *has a subsequences that converges to zero.*

Problem 2. *Suppose* $f : [a,b] \to \mathbb{R}$ *is bounded, and moreover* f *is Riemann integrable on* $[a,c]$ *for all* $a < c < b$*. Show that* f *is Riemann integrable on* $[a,b]$*.*

Problem 3. *Suppose* $f : \mathbb{R} \to \mathbb{R}$ *is differentiable and* $\lim_{x\to\infty} f'(x) = 0$*. Show that if the sequence* $\{f(n)\}_{n\in\mathbb{N}}$ *converges, then the limit* $\lim_{x\to\infty} f(x)$ *exists.*

Problem 4. *A function* $f : \mathbb{R} \to \mathbb{R}$ *is called Lipschitz if there exists* $M > 0$ *such that*

$$|f(x) - f(y)| \le M|x - y|$$

for all $x, y \in \mathbb{R}$*. Show that every Lipschitz function on* $\mathbb{R}$ *is uniformly continuous on* $\mathbb{R}$*, but not every uniformly continuous function on* $\mathbb{R}$ *is necessarily Lipschitz on* $\mathbb{R}$*.*