JKO scheme for Fokker-Planck equation

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1 Introduction

In this file, I will briefly present the derivation of the scheme we use.

2 Fokker-Planck and Equilibrium

Consider following Fokker-Planck equation:

$$\partial_t f = \nabla \cdot (vf + \nabla f)$$

It's well-known that it has global equilibrium

$$M = \frac{C_m}{(2\pi)^{d/2}} e^{-\frac{v^2}{2}}$$

where C_m is the mass of f.

3 Gradient Flow

According to ..., we know

$$\begin{aligned}\partial_t f &= \frac{1}{\varepsilon} \nabla_v \cdot (vf + \nabla_v f) \\ &= -\nabla_{d_{\mathcal{W}}} E(f)\end{aligned}$$

The desired energy functional defined as:

$$E(f|M) = \int_{\mathbb{R}^d} f \ln(\frac{f}{M}) \mathrm{d}v.$$

4 JKO scheme

JKO scheme:

$$f^{n+1} \in \arg\min_f \frac{1}{2} \|f^n - f\|^2_{d_{\mathcal{W}}} + \tau E(f|M)$$

5 Benamou-Brenier formula

$$\|f^n - f\|_{d_{\mathcal{W}}}^2 = \min_{f,m} \int_0^1 \int_\Omega \Phi(f(t,v), m(t,v)) \mathrm{d}v\mathrm{d}t$$

Where

$$\Phi(f,m) = \begin{cases} \frac{\|m\|^2}{f} & \text{if } f > 0, \\ 0 & \text{if } (f,m) = (0,0), \\ +\infty & \text{otherwise .} \end{cases}$$

With following linear constraints

$$\begin{aligned} \partial_t f + \nabla \cdot m = 0 &\text{ on } \Omega \times [0,1] \\ m \cdot \nu = 0 &\text{ on } \partial\Omega \times [0,1] \\ f(\cdot,0) = f_0, f(\cdot,1) = f_1 &\text{ on } \Omega \end{aligned}$$

6 Fully discretized JKO scheme

$$f_{\boldsymbol{j}}^{n+1} \in \arg\min_{f,m} \sum_{\boldsymbol{j}} (\varepsilon \Phi(f_{\boldsymbol{j}}, m_{\boldsymbol{j}}) + 2\tau f_{\boldsymbol{j}} \ln(\frac{f_{\boldsymbol{j}}}{M_{\boldsymbol{j}}})) := F(f_{\boldsymbol{j}}, m_{\boldsymbol{j}})$$

$$\text{s.t.} \quad f_{\boldsymbol{j}} - f_{\boldsymbol{j}}^* + D_{\boldsymbol{v}} m_{\boldsymbol{j}} = 0$$
$$m_{\boldsymbol{j}}|_{\partial\Omega} = 0$$

7 Douglas-Rachford Solver

Introduce indicator function for constraints set C and Rewrite JKO as following:

$$\min_u F(u) + \mathsf{I}_C(u)$$

where $u = [f; m]$. Then Douglas-Rachford scheme reads:

$$\begin{aligned} y_{k+1} &= \mathrm{prox}_F\,(u_k) \\ u_{k+1} &= u_k + \mathrm{prox}_{\mathsf{I}}\,(2y_{k+1} - u_k) - y_{k+1} \end{aligned}$$