Lecture 1

2.10 Find the inverse, if it exists, for each matrix.

(a) $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right]$$

$2x_{21} = 0, x_{21} = 0;\ x_{11} + 3x_{21} = 1, x_{11} = 1$

$2x_{22} = 1, x_{22} = \frac{1}{2};\ x_{12} + 3x_{22} = 0, x_{12} = -\frac{3}{2}$

Hence $A^{-1} = \begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$

(b) $B = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 6 & 0 & 1 \end{array}\right] \xrightarrow{R_2 = R_2 - 2R_1} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right]$$

The augmented matrix has a row $[0 \quad 0|d\,]$ with $d \neq 0$, hence the inverse of matrix B doesn't exist.

(c) $C = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 0 & 4 & 2 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 4 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 = R_3 - 4R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & -10 & 0 & -4 & 1 \end{array}\right]$$

$-10x_{31} = 0, x_{31} = 0; x_{21} + 3x_{31} = 0, x_{21} = 0;\ x_{11} - x_{21} + 2x_{31} = 1, x_{11} = 1$

$-10x_{32} = -4, x_{32} = \frac{2}{5};\ x_{22} + 3x_{32} = 1, x_{22} = \frac{-1}{5};\ x_{12} - x_{22} + 2x_{32} = 0, x_{12} = -1$

$-10x_{33} = 1, x_{33} = \frac{-1}{10};\ x_{23} + 3x_{33} = 0, x_{23} = \frac{3}{10};\ x_{13} - x_{23} + 2x_{33} = 0, x_{13} = \frac{1}{2}$

Hence $C^{-1} = \begin{bmatrix} 1 & -1 & \frac{1}{2} \\ 0 & \frac{-1}{5} & \frac{3}{10} \\ 0 & \frac{2}{5} & \frac{-1}{10} \end{bmatrix}$

1.16 Let A be an $n \times n$ matrix.
(a) If $A^2 = 0$, prove that A is singular.
(b) If $A^2 = A$ and $A \neq I$, prove that A is singular.

(a) Assume A is nonsingular then there exists an inverse matrix A^{-1}.

$$(A^2)A^{-1} = 0A^{-1}$$
$$A(AA^{-1}) = 0$$
$$AI = 0$$
$$A = 0$$

But $A = 0$ is singular, contradiction. A is singular. QED

(b) Assume A is nonsingular then there exists an inverse matrix A^{-1}.

$$(A^2)A^{-1} = AA^{-1}$$
$$A(AA^{-1}) = I$$
$$AI = I$$
$$A = I$$

But $A \neq I$, contradiction. A is singular. QED

Lecture 2

6.3 (b) Find the inner product of $[17 \quad 0 \quad -4 \quad 12 \quad 3]^{\mathrm{T}}, [1 \quad -1 \quad 5 \quad 9 \quad 2]^{\mathrm{T}}$

$\langle [17 \quad 0 \quad -4 \quad 12 \quad 3]^{\mathrm{T}}, [1 \quad -1 \quad 5 \quad 9 \quad 2]^{\mathrm{T}} \rangle$
$= 17 \times 1 + 0 + (-4) \times 5 + 12 \times 9 + 3 \times 2$
$= 111$

6.4 (b) Determine if $[1 \quad -2 \quad 5 \quad 7]^{\mathrm{T}}$ and $[-1 \quad 2 \quad 1 \quad 1]^{\mathrm{T}}$ is orthogonal.

$\langle [1 \quad -2 \quad 5 \quad 7]^{\mathrm{T}}, [-1 \quad 2 \quad 1 \quad 1]^{\mathrm{T}} \rangle$
$= 1 \times (-1) + (-2) \times 2 + 5 \times 1 + 7 \times 1$
$= 7 \neq 0$

This pair of vectors is not orthogonal.

6.8 Find the angle between the vectors $[-1 \quad 2 \quad 5]^{\mathrm{T}}$ and $[1 \quad -8 \quad 2]^{\mathrm{T}}$

Let θ be the angle between these two vectors, then

$$\cos(\theta) = \frac{\langle [-1 \quad 2 \quad 5]^{\mathrm{T}}, [1 \quad -8 \quad 2]^{\mathrm{T}} \rangle}{\|[-1 \quad 2 \quad 5]^{\mathrm{T}}\|_2 \|[1 \quad -8 \quad 2]^{\mathrm{T}}\|_2} = \frac{-1 - 16 + 10}{\sqrt{1+4+25}\sqrt{1+64+4}}$$

$$= \frac{-7}{\sqrt{30}\sqrt{69}} = -0.1539 \qquad (4.\mathrm{s.f})$$

$\theta = \cos^{-1}(-0.1539) = 98.85^{\circ} \qquad (4.\mathrm{s.f})$

Lecture 4

11.1 Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 5 & 4 \end{bmatrix}$, find the LU factorization of A step-by-step without pivoting.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 5 & 4 \end{bmatrix} \xrightarrow{R_2 = R_2 - (2)R_1} \begin{bmatrix} 1 & 2 & 3 \\ (2) & 1 & -2 \\ 3 & 5 & 4 \end{bmatrix}$$

$$\xrightarrow{R_3 = R_3 - (3)R_1} \begin{bmatrix} 1 & 2 & 3 \\ (2) & 1 & -2 \\ (3) & -1 & -5 \end{bmatrix} \xrightarrow{R_3 = R_3 - (-1)R_2} \begin{bmatrix} 1 & 2 & 3 \\ (2) & 1 & -2 \\ (3) & (-1) & -7 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & -7 \end{bmatrix} \text{ and } LU = A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 5 & 4 \end{bmatrix}$$

11.28 Let $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$.

(a) Find the decomposition $PA = LU$ for the matrix.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$

Pivot row $= 1$.

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix} \xrightarrow[R_3 = R_3 - (-1)R_1]{R_2 = R_2 - (-1)R_1} \begin{bmatrix} 1 & 0 & 1 \\ (-1) & 1 & 2 \\ (-1) & -1 & 2 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 2 \end{bmatrix}$$

Pivot row $= 2$.

$$\begin{bmatrix} 1 & 0 & 1 \\ (-1) & 1 & 2 \\ (-1) & -1 & 2 \end{bmatrix} \xrightarrow{R_3 = R_3 - (-1)R_2} \begin{bmatrix} 1 & 0 & 1 \\ (-1) & 1 & 2 \\ (-1) & (-1) & 4 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}, P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}, P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, PA = LU = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$

(b) Solve the system $Ax = [1 \quad 1 \quad 1]^{\mathrm{T}}$ using the results of part (c).

$$PAx = LUx = P[1 \quad 1 \quad 1]^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Let $y = Ux$, then $Ly = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$y_1 = 1; -y_1 + y_2 = 1, y_2 = 2; -y_1 - y_2 + y_3 = 1, y_3 = 4 \Rightarrow y = [1 \quad 2 \quad 4]^{\mathrm{T}}$

$$Ux = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$4x_3 = 4, x_3 = 1; 2x_3 + x_2 = 2, x_2 = 0; x_1 + x_3 = 1, x_1 = 0 \Rightarrow x = [0 \quad 0 \quad 1]^{\mathrm{T}}$

(c) Find the decomposition $PB = LU$ for the matrix $B = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$$

Pivot row $= 1$.

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix} \xrightarrow[R_3 = R_3 - (1/3)R_1]{R_2 = R_2 - (1/2)R_1} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ (\frac{1}{2}) & \frac{1}{12} & \frac{1}{12} \\ (\frac{1}{3}) & \frac{1}{12} & \frac{4}{45} \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 0 & 1 \end{bmatrix}, P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{12} & \frac{1}{12} \\ 0 & \frac{1}{12} & \frac{4}{45} \end{bmatrix}$$

Pivot row $= 2$.

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ (\frac{1}{2}) & \frac{1}{12} & \frac{1}{12} \\ (\frac{1}{3}) & \frac{1}{12} & \frac{-1}{90} \end{bmatrix} \xrightarrow{R_3 = R_3 - (1)R_2} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ (\frac{1}{2}) & \frac{1}{12} & \frac{1}{12} \\ (\frac{1}{3}) & (1) & \frac{1}{180} \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 1 & 1 \end{bmatrix}, P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{12} & \frac{1}{12} \\ 0 & 0 & \frac{1}{180} \end{bmatrix}$$

$$L=\begin{bmatrix}1&0&0\\ \frac{1}{2}&1&0\\ \frac{1}{3}&1&1\end{bmatrix}, U=\begin{bmatrix}1&\frac{1}{2}&\frac{1}{3}\\ 0&\frac{1}{12}&\frac{1}{12}\\ 0&0&\frac{1}{180}\end{bmatrix}, P=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, PB=LU=\begin{bmatrix}1&\frac{1}{2}&\frac{1}{3}\\ \frac{1}{2}&\frac{1}{3}&\frac{1}{4}\\ \frac{1}{3}&\frac{1}{4}&\frac{1}{5}\end{bmatrix}$$

(d) Solve the system $Bx=[1\quad 1\quad 1]^{\mathrm{T}}$ using the results of part (c).

$$PBx=LUx=P[1\quad 1\quad 1]^{\mathrm{T}}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}$$

Let $y=Ux$, then $Ly=\begin{bmatrix}1&0&0\\ \frac{1}{2}&1&0\\ \frac{1}{3}&1&1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}$

$$y_1=1;\frac{1}{2}y_1+y_2=1, y_2=\frac{1}{2};\frac{1}{3}y_1+y_2+y_3=1, y_3=\frac{1}{6}\Rightarrow y=[1\quad \frac{1}{2}\quad \frac{1}{6}]^{\mathrm{T}}$$

$$Ux=\begin{bmatrix}1&\frac{1}{2}&\frac{1}{3}\\ 0&\frac{1}{12}&\frac{1}{12}\\ 0&0&\frac{1}{180}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1\\ \frac{1}{2}\\ \frac{1}{6}\end{bmatrix}$$

$$\frac{1}{180}x_3=\frac{1}{6}, x_3=30;\frac{1}{12}x_3+\frac{1}{12}x_2=\frac{1}{2}, x_2=-24; x_1+\frac{1}{2}x_2+\frac{1}{3}x_3=1, x_1=3$$

$$\Rightarrow x=[3\quad -24\quad 30]^{\mathrm{T}}$$

Lecture 5

14.13

(a) Find an orthonormal basis for the subspace spanned by the columns of

$$A = \begin{bmatrix} 1 & 4 & 7 \\ -1 & 2 & 3 \\ 9 & 1 & 0 \\ 4 & 1 & 8 \end{bmatrix}$$

Let $v_1 = [1 \quad -1 \quad 9 \quad 4]^{\mathrm{T}}, v_2 = [4 \quad 2 \quad 1 \quad 1]^{\mathrm{T}}, v_3 = [7 \quad 3 \quad 0 \quad 8]^{\mathrm{T}}$

Then by the Gram-Schmidt process:

$$e_1 = \frac{v_1}{\|v_1\|_2} = \frac{[1 \quad -1 \quad 9 \quad 4]^{\mathrm{T}}}{\sqrt{1+1+81+16}} = \begin{bmatrix} \frac{1}{\sqrt{99}} & \frac{-1}{\sqrt{99}} & \frac{3}{\sqrt{11}} & \frac{4}{\sqrt{99}} \end{bmatrix}^{\mathrm{T}}$$

$$\approx [0.1005 \quad -0.1005 \quad 0.9045 \quad 0.4020]^{\mathrm{T}};$$

$$\langle v_2, e_1 \rangle = \left(4 \times \frac{1}{\sqrt{99}} + 2 \times \frac{-1}{\sqrt{99}} + \frac{3}{\sqrt{11}} + \frac{4}{\sqrt{99}}\right) = \frac{5}{\sqrt{11}}$$

$$u_2 = v_2 - \langle v_2, e_1 \rangle e_1 = [4 \quad 2 \quad 1 \quad 1]^{\mathrm{T}} - \frac{5}{\sqrt{11}} \begin{bmatrix} \frac{1}{\sqrt{99}} & \frac{-1}{\sqrt{99}} & \frac{3}{\sqrt{11}} & \frac{4}{\sqrt{99}} \end{bmatrix}^{\mathrm{T}}$$

$$= \begin{bmatrix} \frac{127}{33} & \frac{71}{33} & \frac{-4}{11} & \frac{13}{33} \end{bmatrix}^{\mathrm{T}}$$

$$\|u_2\|_2 = \frac{1}{33}\sqrt{127^2 + 71^2 + 12^2 + 13^2} = \sqrt{\frac{217}{11}}$$

$$e_2 = \frac{u_2}{\|u_2\|_2} = \sqrt{\frac{11}{217}} \begin{bmatrix} \frac{127}{33} & \frac{71}{33} & \frac{-4}{11} & \frac{13}{33} \end{bmatrix}^{\mathrm{T}} \approx [0.8665 \quad 0.4844 \quad -0.0819 \quad 0.0887]^{\mathrm{T}}$$

$$\langle v_3, e_1 \rangle = \left(7 \times \frac{1}{\sqrt{99}} + 3 \times \frac{-1}{\sqrt{99}} + 8 \times \frac{4}{\sqrt{99}}\right) = \frac{12}{\sqrt{11}}$$

$$\langle v_3, e_2 \rangle = \sqrt{\frac{1}{217}}\left(7 \times \frac{127}{3\sqrt{11}} + 3 \times \frac{71}{3\sqrt{11}} + 8 \times \frac{13}{3\sqrt{11}}\right) = \sqrt{\frac{1}{217}}\frac{402}{\sqrt{11}}$$

$$u_3 = v_3 - \langle v_3, e_1 \rangle e_1 - \langle v_3, e_2 \rangle e_2$$

$$= [7 \quad 3 \quad 0 \quad 8]^{\mathrm{T}} - \begin{bmatrix} \frac{4}{11} & \frac{-4}{11} & \frac{36}{11} & \frac{16}{11} \end{bmatrix}^{\mathrm{T}} - \frac{134}{217} \begin{bmatrix} \frac{127}{11} & \frac{71}{11} & \frac{-12}{11} & \frac{13}{11} \end{bmatrix}^{\mathrm{T}}$$

$$= \frac{1}{217}[-107 \quad -135 \quad -564 \quad 1262]^{\mathrm{T}}$$

$$\|u_3\|_2 = \frac{1}{217}\sqrt{107^2 + 135^2 + 564^2 + 1262^2} = \sqrt{\frac{8942}{217}}$$

$$e_3 = \frac{u_3}{\|u_3\|_2} \approx [-0.0768 \quad -0.0969 \quad -0.4049 \quad 0.9060]^{\mathrm{T}}$$

An orthonormal basis for the subspace spanned by the columns of A is:

$$\{e_1, e_2, e_3\} = \left\{ \begin{bmatrix} 0.1005 \\ -0.1005 \\ 0.9045 \\ 0.4020 \end{bmatrix}, \begin{bmatrix} 0.8665 \\ 0.4844 \\ -0.0819 \\ 0.0887 \end{bmatrix}, \begin{bmatrix} -0.0768 \\ -0.0969 \\ -0.4049 \\ 0.9060 \end{bmatrix} \right\}$$

14.28 The QR decomposition can be used to solve a linear system. Let A be an 3×3 matrix, with $A = QR$. Then, the linear system $Ax = b$ can be written as $QRx = b$. Find the least-squares solution x to $Ax = b$ by step:
(a) Solve $R^{\mathrm{T}} y = A^{\mathrm{T}} b$ for y.
(b) Solve $Rx = y$ for x.

(a) Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ Then

$$A^{\mathrm{T}} b = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_{11}b_1 + a_{21}b_2 + a_{31}b_3 \\ a_{12}b_1 + a_{22}b_2 + a_{32}b_3 \\ a_{13}b_1 + a_{23}b_2 + a_{33}b_3 \end{bmatrix}, R^{\mathrm{T}} = \begin{bmatrix} r_{11} & 0 & 0 \\ r_{12} & r_{22} & 0 \\ r_{13} & r_{23} & r_{33} \end{bmatrix},$$

$$\begin{bmatrix} r_{11} & 0 & 0 \\ r_{12} & r_{22} & 0 \\ r_{13} & r_{23} & r_{33} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} a_{11}b_1 + a_{21}b_2 + a_{31}b_3 \\ a_{12}b_1 + a_{22}b_2 + a_{32}b_3 \\ a_{13}b_1 + a_{23}b_2 + a_{33}b_3 \end{bmatrix}$$

$r_{11}y_1 = c_1,\ y_1 = \dfrac{c_1}{r_{11}}$

$r_{12}y_1 + r_{22}y_2 = c_2,\ y_2 = \dfrac{r_{11}c_2 - r_{12}c_1}{r_{11}r_{22}}$

$r_{13}y_1 + r_{23}y_2 + r_{33}y_3 = c_3, y_3 = \dfrac{r_{22}r_{11}c_3 - (r_{22}r_{13} - r_{23}r_{12})c_1 - r_{23}r_{11}c_2}{r_{11}r_{22}r_{33}}$

With $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a_{11}b_1 + a_{21}b_2 + a_{31}b_3 \\ a_{12}b_1 + a_{22}b_2 + a_{32}b_3 \\ a_{13}b_1 + a_{23}b_2 + a_{33}b_3 \end{bmatrix}$

(b) $Rx = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$r_{33}x_3 = y_3,\ x_3 = \dfrac{y_3}{r_{33}}$

$r_{23}x_3 + r_{22}x_2 = y_2,\ x_2 = \dfrac{r_{33}y_2 - r_{23}y_3}{r_{22}r_{33}}$

$r_{13}x_3 + r_{12}x_2 + r_{11}x_1 = y_1, x_1 = \dfrac{r_{22}r_{33}y_1 - (r_{22}r_{13} - r_{23}r_{12})y_3 - r_{12}r_{33}y_2}{r_{11}r_{22}r_{33}}$

13.1 find the LU decomposition of the tridiagonal matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 5 \\ 0 & 3 & 4 \end{bmatrix}$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_1 & 1 & 0 \\ 0 & l_2 & 1 \end{bmatrix}, U = \begin{bmatrix} u_1 & c_1 & 0 \\ 0 & u_2 & c_2 \\ 0 & 0 & u_3 \end{bmatrix}, \text{with } c_1 = 1, c_2 = 5$$

$u_1 = 1;$

$l_1 = \frac{2}{u_1} = \frac{2}{1} = 2;\ c_1 l_1 + u_2 = 1, u_2 = 1 - 1 \times 2 = -1;$

$l_2 = \frac{3}{u_2} = \frac{3}{-1} = -3;\ c_2 l_2 + u_3 = 4, u_3 = 4 + 5 \times 3 = 19;$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 5 \\ 0 & 0 & 19 \end{bmatrix}$$

13.4 Show matrix $C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ is positive definite by computing all its principle minors and showing that they are all positive.

$i = 1$:

$$\det C(1{:}i, 1{:}i) = \det C(1{:}1, 1{:}1) = c_{11} = 1 \geq 0$$

$i = 2$:

$$\det C(1{:}i, 1{:}i) = \det C(1{:}2, 1{:}2)$$
$$= \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 \times 2 - 1 \times 1 = 1 \geq 0$$

$i = 3$:

$$\det C(1{:}i, 1{:}i) = \det C$$
$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{vmatrix}$$
$$= 1 \times \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} - 1 \times \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} + 1 \times \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}$$
$$= 1 \times 5 - 1 \times 2 + 1 \times (-1) = 2 \geq 0$$

All leading principle minors of C are positive hence matrix C is positive definite.

QED

Lecture 6

15.16 Assume A is nonsingular with SVD $A = U\Sigma V^{\mathrm{T}}$.

(a) Prove that $\sigma_n\|x\|_2 \le \|Ax\|_2 \le \sigma_1\|x\|_2$.

(b) Show that $\frac{\|Ax\|_2}{\|x\|_2}$ attains its maximum value $\sigma_1$at $x = v_1$.

(c) Show that $\frac{\|A^{-1}x\|_2}{\|x\|_2}$ attains its maximum value $\frac{1}{\sigma_n}$ at $x = u_n$.

(a) Let $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \cdots, \sigma_n)$ with $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_n > 0$, and $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_3 \end{bmatrix}$, then:

$$A = U\Sigma V^{\mathrm{T}}, A^{\mathrm{T}} = (U\Sigma V^{\mathrm{T}})^{\mathrm{T}} = V\Sigma U^{\mathrm{T}}$$

$$\begin{aligned}
\|Ax\|_2^2 = \langle Ax, Ax\rangle &= x^{\mathrm{T}}A^{\mathrm{T}}Ax \\
&= x^{\mathrm{T}}(V\Sigma U^{\mathrm{T}})(U\Sigma V^{\mathrm{T}})x \\
&= x^{\mathrm{T}}(V\Sigma^2 V^{\mathrm{T}})x \\
&= x^{\mathrm{T}}\left(\sum\nolimits_{i=1}^{n} v_i\sigma_i^2 v_i^{\mathrm{T}}\right)x = x^{\mathrm{T}}\left(\sum\nolimits_{i=1}^{n} \sigma_i^2\right)x = \sum\nolimits_{i=1}^{n} \sigma_i^2\, x_i^2
\end{aligned}$$

And since $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_n > 0$:

$$\sigma_n^2\|x\|_2^2 = \sigma_n^2\sum\nolimits_{i=1}^{n} x_i^2 \le \sum\nolimits_{i=1}^{n} \sigma_i^2\, x_i^2 \le \sigma_1^2\sum\nolimits_{i=1}^{n} x_i^2 = \sigma_1^2\|x\|_2^2$$

$\|Ax\|_2^2 \le \sigma_1^2\|x\|_2^2 \Rightarrow \|Ax\|_2 \le \sigma_1\|x\|_2,\ \sigma_n^2\|x\|_2^2 \le \|Ax\|_2^2 \Rightarrow \sigma_n\|x\|_2 \le \|Ax\|_2$

$\Rightarrow \sigma_n\|x\|_2 \le \|Ax\|_2 \le \sigma_1\|x\|_2$

QED

(b) When $x = v_1, \|x\|_2 = \|v_1\|_2 = 1, Av_1$:

$$\begin{aligned}
Av_1 = \left(\sum\nolimits_{i=1}^{n} u_i\sigma_i\, v_i^{\mathrm{T}}\right)v_1 &= \sum\nolimits_{i=1}^{n} u_i\sigma_i\,\langle v_i, v_1\rangle \\
&= u_1\sigma_1
\end{aligned}$$

$$\|Av_1\|_2 = \|u_1\sigma_1\|_2 = \sigma_1\|u_1\|_2 = \sigma_1, \frac{\|Av_1\|_2}{\|v_1\|_2} = \frac{\sigma_1}{1} = \sigma_1, \text{and}$$

$$\|Ax\|_2 \le \sigma_1\|x\|_2 \Rightarrow \frac{\|Ax\|_2}{\|x\|_2} \le \frac{\sigma_1\|x\|_2}{\|x\|_2} = \sigma_1$$

$$\Rightarrow \frac{\|Ax\|_2}{\|x\|_2} \text{ attains its maximum } \sigma_1\text{at } x = v_1$$

QED

(c) $A^{-1} = (U\Sigma V^{\mathrm{T}})^{-1} = (V^{\mathrm{T}})^{-1}\Sigma^{-1}U^{-1} = V\Sigma^{-1}U^{\mathrm{T}}, \Sigma^{-1} = \mathrm{diag}\left(\frac{1}{\sigma_1}, \frac{1}{\sigma_2}, \cdots, \frac{1}{\sigma_n}\right)$

with $\frac{1}{\sigma_n} \geq \frac{1}{\sigma_{n-1}} \geq \cdots \geq \frac{1}{\sigma_1} > 0$. Hence the result from part(a) gives:

$$\frac{1}{\sigma_1}\|x\|_2 \leq \|A^{-1}x\|_2 \leq \frac{1}{\sigma_n}\|x\|_2 \Rightarrow \frac{\|Ax\|_2}{\|x\|_2} \leq \frac{\sigma_1\|x\|_2}{\|x\|_2} = \sigma_1$$

When $x = u_n, \|x\|_2 = \|u_n\|_2 = 1, A^{-1}u_n$:

$$A^{-1}u_n = \left(\sum_{i=1}^{n} \frac{1}{\sigma_i} v_i\, u_i^{\mathrm{T}}\right) u_n = \sum_{i=1}^{n} \frac{v_i}{\sigma_i}\langle u_i, u_n\rangle$$

$$= \frac{v_n}{\sigma_n}$$

$$\|A^{-1}u_n\|_2 = \left\|\frac{v_n}{\sigma_n}\right\|_2 = \frac{1}{\sigma_n}\|v_n\|_2 = \frac{1}{\sigma_n}, \frac{\|A^{-1}u_n\|_2}{\|u_n\|_2} = \frac{1}{\sigma_n}$$

$$\Rightarrow \frac{\|A^{-1}x\|_2}{\|x\|_2} \text{ attains its maximum } \frac{1}{\sigma_n} \text{ at } x = u_n$$

QED

15.20 (b) Find the SVD for the matrix $B = \begin{bmatrix} -1 & 1 & 2 & 3 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ -4 & -3 & -2 & -1 & 0 \end{bmatrix}$ and find an orthonormal basis for the range, null space, row space, and the null space of its transpose.

(b)
```
import numpy as np # Compute the SVD for B via Python
B = np.array([[-1, 1, 2, 3, 5], [1, 2, 3, 4, 5], [-4, -3, -2, -1, 0]])
U, D, V = np.linalg.svd(B)
```

Hence $B = U\Sigma V^{\mathrm{T}}$ with:

$$U = \begin{bmatrix} 0.6014 & 0.4114 & -0.6849 \\ 0.7500 & 0.0045 & 0.6614 \\ -0.2752 & 0.9114 & 0.3059 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 9.8803 & 0 & 0 & 0 & 0 \\ 0 & 5.2146 & 0 & 0 & 0 \\ 0 & 0 & 0.4340 & 0 & 0 \end{bmatrix}$$

$$V^{\mathrm{T}} = \begin{bmatrix} 0.1265 & 0.2963 & 0.4052 & 0.5141 & 0.6839 \\ -0.7772 & -0.4437 & -0.1892 & 0.0654 & 0.3988 \\ 0.2829 & -0.6445 & 0.0062 & 0.6568 & -0.2705 \\ 0.0305 & 0.3873 & -0.8356 & 0.3873 & 0.0305 \\ 0.5469 & -0.3873 & -0.3191 & -0.3873 & 0.5469 \end{bmatrix}$$

The orthonormal basis for

range (B): $\left\{\begin{bmatrix}0.6014\\0.7500\\-0.2752\end{bmatrix},\begin{bmatrix}0.4114\\0.0045\\0.9114\end{bmatrix},\begin{bmatrix}-0.6849\\0.6614\\0.3059\end{bmatrix}\right\}$

null (B): $\left\{\begin{bmatrix}0.0305\\0.3873\\-0.8356\\0.3873\\0.0305\end{bmatrix}\begin{bmatrix}0.5469\\-0.3873\\-0.3191\\-0.3873\\0.5469\end{bmatrix}\right\}$

row (B): $\left\{\begin{bmatrix}0.1265\\0.2963\\0.4052\\0.5141\\0.6839\end{bmatrix},\begin{bmatrix}-0.7772\\-0.4437\\-0.1892\\0.0654\\0.3988\end{bmatrix},\begin{bmatrix}0.2829\\-0.6445\\0.0062\\0.6568\\-0.2705\end{bmatrix}\right\}$

null (B^{T}): 0