Lecture 7

16.1 Three variables *x*, *y*, and *z* are required to satisfy the equations

$$3x - y + 7z = 0$$
$$2x - y + 4z = 1/2$$
$$x - y + z = 1$$
$$6x - 4y + 10z = 3$$

Is there a solution in the normal sense, not a least-squares solution? If so, is it unique, or are there infinitely many solutions?

$$3x - y + 7z = 0 \Rightarrow y = 3x + 7z$$

Substitute y back to the equations above:

$$2x - y + 4z = \frac{1}{2} \Rightarrow x + 3z = -\frac{1}{2}$$
$$x - y + z = 1 \Rightarrow 2x + 6z = -1$$
$$6x - 4y + 10z = 3 \Rightarrow 6x + 18z = -3$$

$$\Rightarrow x + 3z = -\frac{1}{2}, x = -\frac{1}{2} - 3z, y = -\frac{3}{2} - 7z$$

Hence the equations have a solution in normal sense and there are infinitely many of those with $x = -\frac{1}{2} - 3z, y = -\frac{3}{2} - 7z$ and $z \in \mathbf{R}$.

16.4 If A is an $m \times n$ full-rank matrix, $m \geq n$, show that

$$\kappa\ (A) = \|A^{\dagger}\|_2 \|A\|_2 = \frac{\sigma_1}{\sigma_n}$$

where σ_1 and σ_n are the largest and smallest singular values of A, respectively.

Proof:

By SVD decomposition, there exists orthogonal matrix $U, V$and a diagonal $m \times n$ matrix $\tilde{\Sigma}$ such that

$$A = U\tilde{\Sigma}V^{\mathrm{T}}$$
$$\tilde{\Sigma} = \begin{bmatrix}\Sigma \\ 0\end{bmatrix}$$

Where $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \cdots, \sigma_n)$

By spectral theorem, $\|A\|_2 = \sigma_1$as σ_1 is the largest singular value of A.

$$A^{\mathrm{T}} = \left(U\tilde{\Sigma}V^{\mathrm{T}}\right)^{\mathrm{T}} = V\tilde{\Sigma}^{\mathrm{T}}U^{\mathrm{T}}, A^{\mathrm{T}}A = V\tilde{\Sigma}^{\mathrm{T}}(U^{\mathrm{T}}U)\tilde{\Sigma}V^{\mathrm{T}} = V\tilde{\Sigma}^{\mathrm{T}}\tilde{\Sigma}V^{\mathrm{T}}$$

With $\tilde{\Sigma}^{\mathrm{T}} = [\Sigma \quad 0], \tilde{\Sigma}^{\mathrm{T}}\tilde{\Sigma} = \mathrm{diag}({\sigma_1}^2, {\sigma_2}^2, \cdots, {\sigma_n}^2)$

$$(A^{\mathrm{T}}A)^{-1} = V\left(\tilde{\Sigma}^{\mathrm{T}}\tilde{\Sigma}\right)^{-1}V^{\mathrm{T}}$$

With $\left(\tilde{\Sigma}^{\mathrm{T}}\tilde{\Sigma}\right)^{-1} = \mathrm{diag}({\sigma_1}^{-2}, {\sigma_2}^{-2}, \cdots, {\sigma_n}^{-2})$

$$A^{\dagger} = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$$
$$= V\left(\tilde{\Sigma}^{\mathrm{T}}\tilde{\Sigma}\right)^{-1}(V^{\mathrm{T}}V)\tilde{\Sigma}^{\mathrm{T}}U^{\mathrm{T}}$$

$$= V\left(\tilde{\Sigma}^{\mathrm{T}}\tilde{\Sigma}\right)^{-1}\tilde{\Sigma}^{\mathrm{T}}U^{\mathrm{T}} = V\widetilde{D}U^{\mathrm{T}}$$

With $\widetilde{D} = \left(\tilde{\Sigma}^{\mathrm{T}}\tilde{\Sigma}\right)^{-1}\tilde{\Sigma}^{\mathrm{T}}$ a diagonal $n \times m$ matrix equals $[D \quad 0]$ and

$$D = \mathrm{diag}\left(\frac{1}{\sigma_1}, \frac{1}{\sigma_2}, \cdots, \frac{1}{\sigma_n}\right), \text{with} \frac{1}{\sigma_n} \geq \cdots \geq \frac{1}{\sigma_2} \geq \frac{1}{\sigma_1}.$$

Then by spectral theorem, $\|A^{\dagger}\|_2 = \left\|V\widetilde{D}U^{\mathrm{T}}\right\|_2 = \left\|\widetilde{D}\right\|_2 = \frac{1}{\sigma_n}$ as $\frac{1}{\sigma_n}$ is the largest singular value of $\widetilde{D}$.

$$\kappa\ (A) = \|A^{\dagger}\|_2\|A\|_2 = \frac{\sigma_1}{\sigma_n}$$

QED

16.18 Let

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 0 & 1 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix}$$

In parts **(a)–(d)**, find the unique least-squares solution *x* using

(a) $x = A^{\dagger}b$

(b) The normal equations

(c) The QR method

(d) The SVD method

(e) Find$\kappa\ (A)$

(a)

$$A^{\dagger} = \begin{bmatrix} 0.6667 & 0.1667 & -1.1667 \\ -0.3333 & 0.1667 & 0.8333 \end{bmatrix}, x = A^{\dagger}b = \begin{bmatrix} -0.3333 \\ 1.6667 \end{bmatrix}$$

(b)

% Solve the overdetermined least-squares problem using the normal equations

$$A^{\mathrm{T}}Ax = A^{\mathrm{T}}b, A^{\mathrm{T}}b = \begin{bmatrix} 10 \\ 16 \end{bmatrix}$$

%Use the Cholesky decomposition

$$A^{\mathrm{T}}A = R^{\mathrm{T}}R$$

$$R = \begin{bmatrix} 2.2360 & 3.1305 \\ 0 & 1.0954 \end{bmatrix}$$

$$R^{\mathrm{T}}y = A^{\mathrm{T}}b = \begin{bmatrix} 2.2361 & 0 \\ 3.1305 & 1.0954 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 16 \end{bmatrix}$$

$$y_1 = \frac{10}{2.2361} = 4.4721, y_1 = \frac{16 - 3.1305y_2}{1.0954} = 1.8527;\ x = \begin{bmatrix} 4.4721 \\ 1.8527 \end{bmatrix}$$

$$Rx = y = \begin{bmatrix} 2.2361 & 3.1305 \\ 0 & 1.0954 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4.4721 \\ 1.8257 \end{bmatrix}$$

$$x_2 = \frac{1.8257}{1.0954} = 1.6667, x_1 = \frac{4.4721 - 3.1305x_2}{2.2361} = -0.3333;\ x = \begin{bmatrix} -0.3333 \\ 1.6667 \end{bmatrix}$$

(c)

$$A = QR, Ax = QRx = b \Rightarrow y = Q^{\mathrm{T}}b, Rx = y$$

%Compute the reduced QR decomposition of A

$$Q = \begin{bmatrix} -0.4472 & 0.3651 \\ -0.8944 & -0.1826 \\ 0 & -0.9129 \end{bmatrix}, R = \begin{bmatrix} -2.2361 & -3.1305 \\ 0 & -1.0954 \end{bmatrix}$$

$$Q^{\mathrm{T}}b = \begin{bmatrix} -4.4721 \\ -1.8257 \end{bmatrix},$$

$$Rx = \begin{bmatrix} -2.2361 & -3.1305 \\ 0 & -1.0954 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4.4721 \\ -1.8257 \end{bmatrix}$$

$$x_2 = \frac{-1.8257}{-1.0954} = 1.6667, x_1 = \frac{-4.4721 + 3.1305x_2}{-2.2361} = -0.3333;\ x = \begin{bmatrix} -0.3333 \\ 1.6667 \end{bmatrix}$$

(d)

$$A = UDV^{\mathrm{T}}, Ax = UDV^{\mathrm{T}}x = b \Rightarrow Dy = U^{\mathrm{T}}b, x = Vy$$

%Compute the reduced SVD of A

$$U = \begin{bmatrix} -0.3506 & 0.4587 & 0.8165 \\ -0.9124 & 0.0294 & -0.4082 \\ -0.2113 & -0.8881 & 0.4082 \end{bmatrix},$$

$$D = \begin{bmatrix} 3.9517 & 0 \\ 0 & 0.6199 \\ 0 & 0 \end{bmatrix},$$

$$V^{\mathrm{T}} = \begin{bmatrix} -0.5505 & -0.8348 \\ 0.8348 & -0.5505 \end{bmatrix}$$

$$U^{\mathrm{T}}b = \begin{bmatrix} -4.7733 \\ -0.7412 \\ -1.6330 \end{bmatrix}, y = \begin{bmatrix} \frac{-4.7733}{3.9517} \\ \frac{-0.7412}{0.6199} \end{bmatrix} = \begin{bmatrix} -1.2079 \\ -1.1958 \end{bmatrix} \Rightarrow x = Vy = \begin{bmatrix} -0.3333 \\ 1.6667 \end{bmatrix}$$

(e)

$$\kappa\ (A) = \frac{\sigma_1}{\sigma_2} = \frac{3.9517}{0.6199} = 6.3751$$

Lecture 8

20.7 Show that

(a) The Jacobi iteration converges for $A_1 = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$ but the Gauss-Seidel iteration does not converge.

(b) The Gauss-Seidel iteration for $A_2 = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ but the Jacobi iteration converges does not converge.

(a)

$$D_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, L_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 3 & 2 & 0 \end{bmatrix}, U_1 = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B_J = -D_1^{-1}(L_1 + U_1) = \begin{bmatrix} 0 & -\frac{1}{2} & 1 \\ -1 & 0 & -1 \\ -3 & -2 & 0 \end{bmatrix}$$

Compute the eigenvalues λ of B_J:

$$\det(B_J - \lambda I) = \begin{vmatrix} -\lambda & -\frac{1}{2} & 1 \\ -1 & -\lambda & -1 \\ -3 & -2 & -\lambda \end{vmatrix}$$

$$= -\lambda \begin{vmatrix} -\lambda & -1 \\ -2 & -\lambda \end{vmatrix} + \frac{1}{2} \begin{vmatrix} -1 & -1 \\ -3 & -\lambda \end{vmatrix} + \begin{vmatrix} -1 & -\lambda \\ -3 & -2 \end{vmatrix}$$

$$= -\lambda(\lambda^2 - 2) + \frac{1}{2}(\lambda - 3) + (2 - 3\lambda) = -\left(\lambda^3 + \frac{1}{2}\lambda - \frac{1}{2}\right) = 0$$

When $\lambda \geq 1, -\left(\lambda^3 + \frac{1}{2}\lambda - \frac{1}{2}\right) \leq -1 < 0$. When $\lambda \leq -1, -\left(\lambda^3 + \frac{1}{2}\lambda - \frac{1}{2}\right) \geq 2 > 0$

$$\det(B_J - \lambda I) = -\left(\lambda^3 + \frac{1}{2}\lambda - \frac{1}{2}\right) = 0 \Rightarrow |\lambda| < 1, \rho(B_J) = \max|\lambda| < 1$$

The Jacobi iteration converges for $A_1 = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$.

QED

$$B_{GS} = -(L_1 + D_1)^{-1}U_1 = -\begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{vmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} - 0 + 0 = 2 \times 1 = 2$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \\ -\begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ -1 & -4 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & -2 & 1 \end{bmatrix}$$

$$B_{GS} = -\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2} & 1 \\ 0 & \frac{1}{2} & -2 \\ 0 & \frac{1}{2} & 1 \end{bmatrix}$$

Compute the eigenvalues λ of B_{GS}:

$$\det(B_{GS} - \lambda I) = \begin{bmatrix} -\lambda & -\frac{1}{2} & 1 \\ 0 & \frac{1}{2} - \lambda & -2 \\ 0 & \frac{1}{2} & 1 - \lambda \end{bmatrix}$$

$$= -\lambda \begin{vmatrix} \frac{1}{2} - \lambda & -2 \\ \frac{1}{2} & 1 - \lambda \end{vmatrix} + \frac{1}{2}\begin{vmatrix} 0 & -2 \\ 0 & 1 - \lambda \end{vmatrix} + \begin{vmatrix} 0 & \frac{1}{2} - \lambda \\ 0 & \frac{1}{2} \end{vmatrix}$$

$$= -\lambda\left(\left(\frac{1}{2} - \lambda\right)(1 - \lambda) + 1\right) = -\lambda(\lambda^2 - \frac{3}{2}\lambda + \frac{3}{2}) = 0$$

$$\Rightarrow \lambda_1 = 0, \lambda_2 = \frac{3}{4} + i\frac{\sqrt{15}}{4}, \lambda_3 = \frac{3}{4} - i\frac{\sqrt{15}}{4}$$

$$\Rightarrow \rho(B_{GS}) = \max|\lambda| = \sqrt{\frac{9}{16} + \frac{15}{16}} = \sqrt{\frac{3}{2}} > 1$$

The Gauss-Seidel iteration does not converge for $A_1 = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$.

QED

(b)

$$D_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, L_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}, U_1 = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B_J = -D_1^{-1}(L_1 + U_1) = \begin{bmatrix} 0 & -\frac{1}{2} & -\frac{3}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{3} & -\frac{1}{3} & 0 \end{bmatrix}$$

Compute the eigenvalues λ of B_J:

$$\det(B_J - \lambda I) = \begin{vmatrix} -\lambda & -\frac{1}{2} & -\frac{3}{2} \\ -\frac{1}{2} & -\lambda & -\frac{1}{2} \\ -\frac{1}{3} & -\frac{1}{3} & -\lambda \end{vmatrix}$$

$$= -\lambda \begin{vmatrix} -\lambda & -\frac{1}{2} \\ -\frac{1}{3} & -\lambda \end{vmatrix} + \frac{1}{2} \begin{vmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{3} & -\lambda \end{vmatrix} - \frac{3}{2} \begin{vmatrix} -\frac{1}{2} & -\lambda \\ -\frac{1}{3} & -\frac{1}{3} \end{vmatrix}$$

$$= -\lambda\left(\lambda^2 - \frac{1}{6}\right) + \frac{1}{2}\left(\frac{\lambda}{2} - \frac{1}{6}\right) - \frac{3}{2}\left(\frac{1}{6} - \frac{\lambda}{3}\right) = -\left(\lambda^3 - \frac{11}{12}\lambda + \frac{1}{3}\right) = 0$$

When $\lambda = -1, -\left(\lambda^3 - \frac{11}{12}\lambda + \frac{1}{3}\right) = -\frac{1}{4} < 0$. When $\lambda = -2, -\left(\lambda^3 - \frac{11}{12}\lambda + \frac{1}{3}\right) = \frac{35}{6} > 0$.

Since $\left(\lambda^3 - \frac{11}{12}\lambda + \frac{1}{3}\right)$ is continuous, B_J has an eigenvalue $-2 < \lambda_i < -1$.

$\Rightarrow \rho(B_{GS}) = \max|\lambda| \geq |\lambda_i| > 1$

The Jacobi iteration does not converge for $A_2 = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$.

QED

$$B_{GS} = -(L_1 + D_1)^{-1} U_1 = -\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{vmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 3 \end{vmatrix} = 2\begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix} - 0 + 0 = 2 \times 6 = 12$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 3 \end{bmatrix}^{-1} = \frac{1}{12}\begin{bmatrix} \begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 0 & 0 \\ 2 & 0 \end{vmatrix} \\ -\begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} \end{bmatrix}$$

$$= \frac{1}{12}\begin{bmatrix} 6 & 0 & 0 \\ 3 & 6 & 0 \\ -1 & -2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{4} & \frac{1}{2} & 0 \\ -\frac{1}{12} & -\frac{1}{6} & \frac{1}{3} \end{bmatrix}$$

$$B_{GS} = -\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{4} & \frac{1}{2} & 0 \\ -\frac{1}{12} & -\frac{1}{6} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2} & -\frac{3}{2} \\ 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & -\frac{1}{12} & -\frac{5}{12} \end{bmatrix}$$

Compute the eigenvalues λ of B_{GS}:

$$\det(B_{GS} - \lambda I) = \begin{bmatrix} -\lambda & -\frac{1}{2} & -\frac{3}{2} \\ 0 & \frac{1}{4} - \lambda & \frac{1}{4} \\ 0 & -\frac{1}{12} & -\frac{5}{12} - \lambda \end{bmatrix}$$

$$= -\lambda\begin{vmatrix} \frac{1}{4} - \lambda & \frac{1}{4} \\ -\frac{1}{12} & -\frac{5}{12} - \lambda \end{vmatrix} + \frac{1}{2}\begin{vmatrix} 0 & \frac{1}{4} \\ 0 & -\frac{5}{12} - \lambda \end{vmatrix} - \frac{3}{2}\begin{vmatrix} 0 & \frac{1}{4} - \lambda \\ 0 & -\frac{1}{12} \end{vmatrix}$$

$$= -\lambda\left(\left(\frac{1}{4} - \lambda\right)\left(-\frac{5}{12} - \lambda\right) + \frac{1}{48}\right) = -\lambda\left(\lambda^2 + \frac{\lambda}{6} - \frac{1}{12}\right) = 0$$

When $\lambda \geq 1, -\lambda\left(\lambda^2 + \frac{\lambda}{6} - \frac{1}{12}\right) \leq -\frac{13}{12} < 0$.

When $\lambda \leq -1, -\lambda\left(\lambda^2 + \frac{\lambda}{6} - \frac{1}{12}\right) \geq \frac{3}{4} > 0$

$\det(B_J - \lambda I) = -\left(\lambda^3 + \frac{1}{2}\lambda - \frac{1}{2}\right) = 0 \;\Rightarrow\; |\lambda| < 1, \rho(B_J) = \max|\lambda| < 1$

The Gauss-Seidel iteration does not converge for $A_2 = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$.

QED

20.17 The matrix $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ is positive definite.

(a) Is matrix A strictly row diagonally dominant?

(b) Is the Jacobi method guaranteed to converge? What about the Gauss-Seidel and SOR iterations?

(c) If the Gauss-Seidel method converges, solve $Ax = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$

(d) If the SOR iteration converges, use optomega to estimate an optimal ω, and demonstrate that sor improves the convergence rate relative to Gauss-Seidel using $\text{tol} = 1.0 \times 10^{-10}$ and $\text{maxiter} = 100$.

(a) Matrix $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ is not strictly row diagonally dominant.

(b)

Jacobi method isn't guaranteed to converge for matrix $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ since A is not strictly row diagonally dominant.

Gauss-Seidel and SOR iterations with $0 < \omega < 2$ are guaranteed to converge since A is a symmetric positive definite matrix.

(c)

```
# Gauss Seidel Iteration

# Defining equations to be solved
f1 = lambda x,y,z: (1+y)/2
f2 = lambda x,y,z: (3+x+z)/2
f3 = lambda x,y,z: (7+y)/2

# Initial setup
x0 = 0
y0 = 0
z0 = 0
count = 1

# Reading tolerable error
e = float(10**(-10))

# Implementation of Gauss Seidel Iteration
print('\nCount\tx\ty\tz\n')
```

```
condition = True

while condition:
	x1 = f1(x0,y0,z0)
	y1 = f2(x1,y0,z0)
	z1 = f3(x1,y1,z0)
	print('%d\t%0.4f\t%0.4f\t%0.4f\n' %(count, x1,y1,z1))
	e1 = abs(x0-x1);
	e2 = abs(y0-y1);
	e3 = abs(z0-z1);

	count += 1
	x0 = x1
	y0 = y1
	z0 = z1

	condition = e1>e and e2>e and e3>e

print('\nSolution: x=%0.3f, y=%0.3f and z = %0.3f\n'% (x1,y1,z1))
```

```
22        4.0000   7.0000   7.0000

23        4.0000   7.0000   7.0000

24        4.0000   7.0000   7.0000

25        4.0000   7.0000   7.0000

26        4.0000   7.0000   7.0000

27        4.0000   7.0000   7.0000

28        4.0000   7.0000   7.0000

29        4.0000   7.0000   7.0000

30        4.0000   7.0000   7.0000

31        4.0000   7.0000   7.0000

32        4.0000   7.0000   7.0000

33        4.0000   7.0000   7.0000

34        4.0000   7.0000   7.0000

35        4.0000   7.0000   7.0000

36        4.0000   7.0000   7.0000

Solution: x=4.000,  y=7.000 and z = 7.000
```

$$x = \begin{bmatrix} 4.000 \\ 7.000 \\ 7.000 \end{bmatrix}$$

(d)

```
# SOR iteration

# Defining equations to be solved
f1 = lambda x,y,z: (1+y)/2
f2 = lambda x,y,z: (3+x+z)/2
f3 = lambda x,y,z: (7+y)/2

# Initial setup
```

```
x0 = 0
y0 = 0
z0 = 0
count = 1

# Reading tolerable error
e = float(10**(-10))

# Implementation of SOR iteration
print('\nCount\tx\ty\tz\n')

# Estimate an optimal w
w = 2/(1+0.5**0.5)

condition = True

while condition:
    if count == 1:
        x1 = f1(x0,y0,z0)
        y1 = f2(x1,y0,z0)
        z1 = f3(x1,y1,z0)
        print('%d\t%0.4f\t%0.4f\t%0.4f\n' %(count, x1,y1,z1))
        e1 = abs(x0-x1);
        e2 = abs(y0-y1);
        e3 = abs(z0-z1);

        count += 1
        x0 = x1
        y0 = y1
        z0 = z1
    else:
        x1 = w*f1(x0,y0,z0)+(1-w)*x0
        y1 = w*f2(x1,y0,z0)+(1-w)*y0
        z1 = w*f3(x1,y1,z0)+(1-w)*z0
        print('%d\t%0.4f\t%0.4f\t%0.4f\n' %(count, x1,y1,z1))
        e1 = abs(x0-x1);
        e2 = abs(y0-y1);
        e3 = abs(z0-z1);

        count += 1
        x0 = x1
        y0 = y1
        z0 = z1
        condition = e1>e and e2>e and e3>e and count<=100
```

```
print('\nSolution: x=%0.3f, y=%0.3f and z = %0.3f\n'% (x1,y1,z1))
print(w)
```

```
3         3.2023   6.4385   6.8035

4         3.8080   6.8688   6.9568

5         3.9561   6.9715   6.9907

6         3.9908   6.9941   6.9981

7         3.9981   6.9988   6.9996

8         3.9996   6.9998   6.9999

9         3.9999   7.0000   7.0000

10        4.0000   7.0000   7.0000

11        4.0000   7.0000   7.0000

12        4.0000   7.0000   7.0000

13        4.0000   7.0000   7.0000

14        4.0000   7.0000   7.0000

15        4.0000   7.0000   7.0000

16        4.0000   7.0000   7.0000

17        4.0000   7.0000   7.0000


Solution: x=4.000, y=7.000 and z = 7.000, w=1.172
```

$$x = \begin{bmatrix} 4.000 \\ 7.000 \\ 7.000 \end{bmatrix}$$

The SOR iteration gives same solution as Gauss-Seidel method but takes less iteration (17) to get to the same tol $= 1.0 \times 10^{-10}$ than Gauss-Seidel method (36).

Lecture 9

17.8 Prove that if u is chosen to be parallel to vector $x - y$, where $\mathrm{x} \neq \mathrm{y}$ but $\|x\|_2 = \|y\|_2$, then $H_u x = y$.

Proof:

Let $u = c(x - y)$, where c is a scalar. Then

$$\begin{aligned}\|u\|_2^2 = u^T u &= c^2(x-y)^T(x-y)\\ &= c^2(x^T - y^T)(x-y)\\ &= c^2(x^Tx - x^Ty - y^Tx + y^Ty)\\ &= c^2\|x\|_2^2 + \|y\|_2^2 - \langle x,y\rangle - \langle y,x\rangle\\ &= 2c^2(\|x\|_2^2 - \langle y,x\rangle)\end{aligned}$$

$$\begin{aligned}uu^Tx &= c^2(x-y)(x-y)^Tx\\ &= c^2(x-y)(x^T - y^T)x\\ &= c^2(x-y)(x^Tx - y^Tx)\\ &= c^2(x-y)(\|x\|_2^2 - \langle y,x\rangle)\end{aligned}$$

Hence $\left(\frac{2}{\|u\|_2^2}\right)uu^Tx = \frac{2c^2(x-y)(\|x\|_2^2-\langle y,x\rangle)}{2c^2(\|x\|_2^2-\langle y,x\rangle)} = x - y$, and

$$H_u x = \left(I - \left(\frac{2}{\|u\|_2^2}\right)uu^T\right)x$$

$$= x - \left(\frac{2}{\|u\|_2^2}\right)uu^Tx = x - (x - y) = y$$

QED

17.10 If u and v are $n \times 1$ vectors, the $n \times n$ matrix uv^T has rank 1 (Problem 10.3). If A is an $n \times n$ matrix, we say that the matrix $B = A + uv^T$ is a *rank 1 update* of A. Let $A = QR$ be the QR decomposition of A. Show that

$$A + uv^T = Q(R + wv^T)$$

Where $w = Q^T u$.

Proof:

$$w = Q^Tu \;\Rightarrow\; Qwv^T = Q(Q^Tu)v^T = (QQ^T)uv^T$$

Since Q is an orthogonal, $QQ^T = I$.

$$Qwv^T = uv^T$$

$$Q(R + wv^T) = QR + Qwv^T = A + uv^T$$

QED

17.11 Problem 17.10 defines a rank 1 update of a matrix. A Householder reflection,

$$H_u = I - (\frac{2}{\|u\|_2^2})uu^T$$

is a rank 1 update of the identity matrix. We know a Householder matrix is symmetric, orthogonal, and is its own inverse ($H_u^2 = I$). This problem investigates a more general

rank 1 update of the identity, $R_1 = I - uv^{\mathrm{T}}$.
(a) Prove that $R_1$is nonsingular if and only if $\langle v, u\rangle \neq 1$.
(b) If $R_1$is nonsingular, show that $R_1^{-1} = I - \beta uv^{\mathrm{T}}$. Do this by finding a formula for β.

(a)
R_1 is nonsingular $\Rightarrow$ $\langle v, u\rangle \neq 1$
Suppose $R_1$is nonsingular and $\langle v, u\rangle = 1$, then

$$\begin{aligned} R_1 u &= (I - uv^{\mathrm{T}})u \\ &= u - u(v^{\mathrm{T}}u) \\ &= u - u\langle v, u\rangle = u - u = 0 \end{aligned}$$

Hence $u \in N(R_1)$, R_1 isn't full rank and therefore is singular. Contradiction.
If $R_1$is nonsingular, then $\langle v, u\rangle \neq 1$.

$\langle v, u\rangle \neq 1$ $\Rightarrow R_1$ is nonsingular
Suppose $R_1$is singular and $\langle v, u\rangle \neq 1$, then there exists a vector $x \neq 0$ such that

$$R_1 x = x - uv^{\mathrm{T}}x = x - u\langle v, x\rangle = 0$$

Since $\langle v, x\rangle$ is a scalar, let $\langle v, x\rangle = c$, c is a nonzero constant. $x = cu$.
Then $\langle v, x\rangle = \langle v, cu\rangle = c\langle v, u\rangle$

$$\begin{aligned} x - u\langle v, x\rangle &= cu - cu\langle v, u\rangle \\ &= cu(1 - \langle v, u\rangle) = 0 \end{aligned}$$

Since $\langle v, u\rangle \neq 1, 1 - \langle v, u\rangle \neq 0$ and $u \neq 0, c$ must equal to 0. Contradiction.
If $\langle v, u\rangle \neq 1,$ then $R_1$is nonsingular
$R_1$is nonsingular if and only if $\langle v, u\rangle \neq 1$.

QED

If $R_1$is nonsingular, then by part(a) $\langle v, u\rangle \neq 1$. Let $A = I - \beta uv^{\mathrm{T}}$.

$$\begin{aligned} AR_1 &= (I - \beta uv^{\mathrm{T}})(I - uv^{\mathrm{T}}) \\ &= I - (\beta + 1)uv^{\mathrm{T}} + \beta(uv^{\mathrm{T}})(uv^{\mathrm{T}}) \\ &= I - (\beta + 1)uv^{\mathrm{T}} + \beta u(v^{\mathrm{T}}u)v^{\mathrm{T}} \\ &= I - (\beta + 1)uv^{\mathrm{T}} + \beta\langle v, u\rangle uv^{\mathrm{T}} \\ &= I + (\beta\langle v, u\rangle - \beta - 1)uv^{\mathrm{T}} \end{aligned}$$

When $\beta = \frac{1}{\langle v,u\rangle - 1}, AR_1 = I + (1-1)uv^{\mathrm{T}} = I$

$$\Rightarrow A = R_1^{-1} = I - \frac{uv^{\mathrm{T}}}{\langle v, u\rangle - 1}$$