Research report of Core Adaptive Fourier Decomposition

Wu Yunni 2023

Abstract

There exist great interest and discussion to decomposition of signals in Hardy space $H^2(\mathbb{D})$, concerning paradox of negative instantaneous frequency. Of the positive frequency decomposition of signals, Core Adaptive Fourier decomposition (Core AFD) is a well-researched method. Core AFD decomposes functions into mono-components by using adaptive Takenaka-Malmquist systems. There are ample variant methods, including Cyclic AFD and n-best Blaschke form approximation, and generalization to non-Hardy spaces, including weighted Bergman and weighted Hardy spaces. However there are still several problems with regard to Core AFD and we propose some of the problems in this parper for future research.

Keywords: ***Instantaneous Frequency, Core Adaptive Fourier decomposition, Hardy space, Hyperbolic non-Separable Condition***

I. Introduction

This paper concerns some pending problems with regard to *Core Adaptive Fourier approximation*, or *Core AFD*, of a function in Hardy $H^2(\mathbb{D})$ space,

$$H^2(\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{C} : f = \sum_{k=0}^{\infty} c_k z^k, \sum_{k=0}^{\infty} |c_k|^2 < \infty\},$$

where $\mathbb{D}$ stands for the open unit disc on complex plane. For functions $f, g \in H^2(\mathbb{D})$, we have inner product $\langle f, g\rangle$ defined as

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\overline{g(e^{it})}dt. \tag{1}$$

Core AFD decomposes functions into mono-components by using adaptive *Takenaka-Malmquist systems* [9]. There are ample variant methods, including Cyclic AFD and n-best Blaschke form approximation, and generalization to non-Hardy spaces, including weighted Bergman and weighted Hardy spaces. However there are still several problems or possible improvement with regard to Core AFD and its variants. Below we introduce Core AFD to Hardy space functions and propose some of the problems in this parper for future research.

II. Core Adaptive Fourier Decomposition

The above defined $H^2(\mathbb{D})$ is a reproducing kernel Hilbert space (RKHS). For $a \in \mathbb{D}$, the function

$$k_a(z) = \frac{1}{1 - \overline{a}z}$$

is the *a-parameterized reproducing kernel* of the Hardy $H^2(\mathbb{D})$ space. Given a sequence $\{a_k \mid a_k \in \mathbb{D}\}_{k=1}^{\infty}$ with multiplicity allowed, we have a corresponding sequence of *multiple Szegö kernels* $\{\tilde{k}_{a_k}\}_{k=1}^{\infty}$, where

$$\tilde{k}_{a_k} = \left[\left(\frac{\partial}{\partial \overline{a}}\right)^{l(k)-1} k_a\right]_{a=a_k},$$

where $l(1) = 1$, and $l(k)$ is the multiple of a_k in $(a_1, \cdots, a_k)$. The Gram-Schmidt orthogonalization of the sequence $\{\tilde{k}_{a_k}\}$ forms the corresponding Takenaka-Malmquist (or T-M) system $\{B_k = B_{a_1,\cdots,a_k}\}_{k=1}^{\infty}$

$$B_k(z) = \frac{\sqrt{1-|a_k|^2}}{1-\bar{a}_k z} \prod_{l=1}^{k-1} \frac{z-a_l}{1-\bar{a}_l z}$$

(see [18]).

The L^2-normalization of k_a is

$$e_a = \frac{\sqrt{1-|a|^2}}{1-\bar{a}z}. \tag{2}$$

Hence,

$$\langle f, e_a \rangle = \sqrt{1-|a|^2} f(a). \tag{3}$$

Core AFD [12] adaptively select optimal parameters $a_1, \cdots, a_l, \cdots$, in the one-by-one manner such that,

$$|\langle f_l, e_{a_l} \rangle|^2 = \max\{|\langle f_l, e_z \rangle|^2 : z \in \mathbb{D}\} \tag{4}$$

and thus to result a fast convergence

$$f = \sum_{k=1}^{\infty} \langle f_k, e_{a_k} \rangle B_k, \tag{5}$$

and

$$\begin{aligned} \|f\|^2 &= \sum_{k=1}^{\infty} |\langle f, B_k \rangle|^2 \\ &= \sum_{k=1}^{\infty} |\langle f_k, e_{a_k} \rangle|^2 \\ &= \sum_{k=1}^{\infty} (1-|a_k|^2)|f_k(a_k)|^2, \end{aligned} \tag{6}$$

where $\{f_k \mid f_k \in H^2(\mathbb{D})\}_{k=1}^{\infty}$ are the *reduced reminders* generated by induction, with $f_1(z) \triangleq f(z)$, and

$$f_{k+1}(z) \triangleq \frac{1-\bar{a}_k z}{z-a_k} \left(f_k(z) - \langle f_k, e_{a_k} \rangle \right).$$

We say a_l is a maximal point to reduced reminder f_l if it satisfies equation (4), and existence of a maximal point a to $f \in H^2(\mathbb{D})$ has been repeatedly proved in [1]–[3], [20]. The corresponding series $\{a_l : a_l \in \mathbb{D}\}_{l=1}^{\infty}$ is called *maximal series* of f. f is said to be an *n-AFD form* if f can be explicitly written as

$$f = \sum_{k=1}^{n} \{f_k, e_{a_k}\} B_k, \tag{7}$$

where $a_1, \cdots, a_n$ is chosen one-by-one according to maximal selection principle. Maximal series of n-AFD form is $\{a_l : a_l \in \mathbb{D}\}_{l=1}^{n}$.

Furthermore, we note that

$$\langle f, B_k \rangle = \langle f_k, e_{a_k} \rangle = (1-|a_k|^2)|f_k(a_k)|^2.$$

The relation

$$f(z) = \sum_{k=1}^{n} \langle f_k, e_{a_k} \rangle B_k(z) + f_{n+1}(z) \prod_{k=1}^{n} \frac{z-a_k}{1-\bar{a}_k z}$$

exhibits the Beurling type decomposition: the H^2 space is the direct sum of the *backward shift-invariant* subspace as span of the $\{B_k\}_{k=1}^{n}$ and the *shift-invariant* subspace generated by the Blaschke product $\prod_{k=1}^{n} \frac{z-a_k}{1-\bar{a}_k z}$.

III. Property of Maximal Series

In this session, we introduce some properties of maximal series and presents some hypothesises that need further research.

A. Non-Equality

Let $n \geq 2$ be fixed, and $f \in H^2(\mathbb{D})$ be not identical with an m-AFD form for any $m < n$. Then maximal series $\{a_l\}_{l=1}^{\infty}$ satisfy

$$a_l \neq a_{l-1}, \qquad l = 2, \cdots, n. \tag{8}$$

Let $f_1, f_2, \cdots$, be the corresponding reduced reminders generated consecutively by $\{a_l\}_{l=1}^{\infty}$. We, in fact, have

$$f_l(a_{l-1}) = 0, \qquad l = 2, \cdots, n,$$

and this is proven in the lemma below.

Lemma 1. *Let $n \geq 2$ be fixed, and $f \in H^2(\mathbb{D})$ be not identical with an m-AFD form for any $m < n$. Let $\{a_l\}_{l=1}^{\infty}$ be a maximal series of f and $f_1, f_2, \cdots$,* *be the corresponding reduced reminders generated consecutively by $\{a_l\}_{l=1}^{\infty}$. Then we have*

$$f_l(a_{l-1}) = 0, \qquad l = 2, \cdots, n$$

Proof. Let $\{a_l\}_{l=1}^{\infty}$ be a maximal series to $f \in H^2(\mathbb{D})$ and $f_1, f_2, \cdots$, be the corresponding reduced reminders generated consecutively by $a_1, a_2, \cdots$. Then according to the maximal selection principle, for $l = 2, \cdots, n$, we have,

$$(1-|a_{l-1}|^2)|f_{l-1}(a_{l-1})|^2 = \max_{z\in\mathbb{D}}\{(1-|z|^2)|f_{l-1}(z)|^2\}.$$

And

$$\begin{aligned} 0 &= \frac{\partial}{\partial z}\left[(1-|z|^2)|f_{l-1}(z)|^2\right]_{z=a_{l-1}} \\ &= \left[\left(-\bar{z}f_{l-1}(z) + (1-|z|^2)f^{'}(z)\right)\overline{f_{l-1}(z)}\right]_{z=a_{l-1}}. \end{aligned}$$

The maximality implies that $f_{l-1}(a_{l-1}) \neq 0$. Hence

$$-\overline{a_{l-1}}f_{l-1}(a_{l-1}) + (1-|a_{l-1}|^2)f^{'}(a_{l-1}) = 0. \quad (9)$$

When $z \neq a_{l-1}$,

$$\begin{aligned} f_l(z) &= \frac{1-\overline{a_{l-1}}z}{z-a_{l-1}}\left(f_{l-1}(z) - \langle f_{l-1}, e_{a_{l-1}}\rangle e_{a_{l-1}}\right) \\ &= -\overline{a_{n-1}}f_{n-1}(a_{n-1}) \\ &\quad + (1-|a_{n-1}|^2)\frac{f_{n-1}(z) - f_{n-1}(a_{n-1})}{z-a_{n-1}}. \end{aligned}$$

Letting $z \to a_{n-1}$ while recalling (9), we have

$$\begin{aligned} f_l(a_{l-1}) &= -\overline{a_{l-1}}f_{l-1}(a_{l-1}) \\ &\quad + (1-|a_{l-1}|^2)f^{'}(a_{l-1}) \\ &= 0, \end{aligned}$$

$l = 2, \cdots, n$. □

B. Hyperbolic Non-Separable condition

We say series $\{z_l : z_l \in H^2(\mathbb{D})\}_{l=1}^{\infty}$ satisfy hyperbolic non-separable condition if

$$\sum_{l=1}^{\infty}(1-|z_l|) = \infty \quad (10)$$

and the corresponding T-M system, $\{B_l\}_{l=1}^{\infty}$, is a basis of Hardy space,

$$\overline{span}\{B_l\}_{l=1}^{\infty} = H^2(\mathbb{D}).$$

If $f \in H^2(\mathbb{D})$ is an n-AFD form, clearly maximal series $\{a_l\}_{l=1}^{\infty}$ of f do not satisfy hyperbolic non-separable condition, or condition (10). Given $f \in H^2(\mathbb{D})$ that is not an n-AFD form for any $n > 0$, whether its maximal series $\{a_l\}_{l=1}^{\infty}$ satisfy condition (10) requires further research.

So far, We have proven that for any polynomial $g \in H^2(\mathbb{D})$, its corresponding maximal series $\{a_l\}_{l=1}^{\infty}$ satisfy condition (10). The proof consists of three part:

1) Any polynomial $g \in H^2(\mathbb{D})$ is not an n-AFD form for any $n > 0$.

2) Let g be a polynomial of order $N \geq 1$. Then reduce reminders g_l of g is a polynomial of order $1 \leq n_l \leq N$.

3) Let g be a polynomial of order N and $a \in \mathbb{D}$ be a maximal point of g such that $|\langle g, e_a\rangle|^2 = \max\{|\langle g, e_z\rangle|^2 : z \in \mathbb{D}\}$. Then there exists a fixed $r_N \leq 1$ that for any polynomial of order N, its corresponding maximal point satisfies

$$|a| \leq r_N.$$

Particularly,

$$r_N = \sqrt{\frac{N}{N+1}},$$

with equality taken at $g = z^N$.

Below we present the proof of step (1) and (2).

Lemma 2. *Let $g = \alpha z + \beta \in H^2(\mathbb{D})$, then g is not an n-AFD form for any fixed $n > 0$ and reduced reminders $g_l \in H^2(\mathbb{D})$ is also a polynomial of order 1*

$$g_l = \alpha_l z + \beta_l, \qquad \alpha_l \neq 0.$$

Proof. Let $a_1, a_2, \cdots$, be any complex numbers $\in \mathbb{D}$, and $g_1, g_2, \cdots$ be the corresponding reduced reminders generated consecutively by $a_1, a_2, \cdots$,.

When $l = 1$, $g_1 \triangleq g$.

When $l > 1$,assume $g_{l-1} \in H^2(\mathbb{D})$ is a polynomial of order 1

$$g_{l-1} = \alpha_{l-1} z + \beta_{l-1}, \qquad \alpha_{l-1} \neq 0.$$

Then we have

$$\begin{aligned} g_l(z) &= -\bar{a}_{l-1} g_{l-1}(z) \\ &\quad + \frac{\left(1-|a_{l-1}|^2\right)\left(g_{l-1}(z) - g_{l-1}(a_{l-1})\right)}{z - a_{l-1}} \\ &= -\bar{a}_{l-1}\left(\alpha_{l-1} z + \beta_{l-1}\right) + \alpha_{l-1}\left(1-|a_{l-1}|^2\right), \end{aligned}$$

with

$$\alpha_l = -\bar{a}_{l-1}\alpha_{l-1},$$

$$\beta_l = \alpha_{l-1}\left(1-|a_{l-1}|^2\right) - \bar{a}_{l-1}\beta_{l-1}.$$

Assume $a_{l-1} = 0$, by **Lemma 1**,

$$g'_{l-1}(a_{l-1}) = \frac{\bar{a}_{l-1}}{1-|a_{l-1}|^2} g_{l-1}(a_{l-1})$$
$$\alpha_{l-1} = 0$$

By assumption, we have $\alpha_{l-1} \neq 0$. Contradiction. Hence we have

$$g_l = \alpha_l z + \beta_l, \qquad \alpha_l \neq 0.$$

□

Theorem 1. *Let $g = \sum_{k=0}^{N} c_k z^k \in H^2(\mathbb{D})$ be a polynomial of order N with $c_N \neq 0$. Then g is not an n-AFD form for any fixed $n > 0$ and reduced reminders $g_l \in H^2(\mathbb{D})$ is also a polynomial of order $1 \le N_l \le N$.*

Proof. Let $a_1, a_2, \cdots,$ be any complex numbers $\in \mathbb{D}$, and $g_1, g_2, \cdots$ be the corresponding reduced reminders generated consecutively by $a_1, a_2, \cdots,$.

When $l = 1$, $g_1 \triangleq g$.

When $l > 1$,assume $g_{l-1} = \sum_{k=0}^{N_{l-1}} c_{k,l-1} z^k \in H^2(\mathbb{D})$ is a polynomial of order $1 \le N_{l-1} \le N$ with $c_{N_{l-1},l-1} \neq 0$. If $N_{l-1} = 1$, by **lemma 2**, g_l is also a polynomial of order 1. For $2 \le N_{l-1} \le N$, we have

$$\begin{aligned} g_l(z) &= -\bar{a}_{l-1} g_{l-1}(z) \\ &\quad + \frac{\left(1-|a_{l-1}|^2\right)\left(g_{l-1}(z) - g_{l-1}(a_{l-1})\right)}{z - a_{l-1}}, \end{aligned}$$

where

$$\begin{aligned} \frac{g_{l-1}(z) - g_{l-1}(a_{l-1})}{z - a_{l-1}} &= \sum_{k=0}^{N_{l-1}} c_{k,l-1} \frac{z^k - a_{l-1}^k}{z - a_{l-1}} \\ &= \sum_{k=1}^{N_{l-1}} c_{k,l-1} \sum_{r=0}^{k-1} z^r a_{l-1}^{k-r-1}, \end{aligned}$$

is a polynomial of order $N_{l-1} - 1$ if $a_{l-1} \neq 0$, and $g_l(z)$ is a polynomial of order N_{l-1}.

When $a_{l-1} = 0$, we have

$$g_l(z) = \sum_{k=1}^{N_{l-1}} c_{k,l-1} z^{k-1},$$

is a polynomial of order $1 \le N_{l-1} - 1 < N$. □

We have shown that maximal series of polynomial in Hardy space satisfy hyperbolic non-separable condition and therefore the corresponding T-M system is a basis of H^2. For any $f \in H^2(\mathbb{D})$, whether T-M system generated consecutively by its maximal series still needs further investment.

IV. Variant AFD method

A. n-Best Rational Approximation

Below we introduce n-best rational approximation to Hardy space functions. Let p, q be a pair of co-prime polynomials, where all zeros of q are in the exterior of $\mathbb{D}$. The order of such formulated rational function p/q is defined as $\mathrm{ord}(p/q) = \max\{\deg(p), \deg(q)\}$ [5]. An n-best rational approximation to $f \in H^2(\mathbb{D})$ is a rational function p_1/q_1 with ord $(p_1/q_1) \le n$ such that

$$\|f - p_1/q_1\| \le \|f - p/q\|, \qquad \mathrm{ord}(p/q) \le n.$$

In [14] we show that n-best rational approximation is equivalent to *n-best Blaschke form approximation*.

f is said to be an *n-Blaschke form* if f can be explicitly written as $f = \sum_{k=1}^{n} c_k B_k$, where $k = 1, \cdots, n$ and c_k are complex numbers with $c_n \neq 0$, and $\{B_k\}_{k=1}^{n}$ is the T-M system induced by $\boldsymbol{A} = (a_1, \cdots, a_n) \in \mathbb{D}^n$. Let $E_f(\boldsymbol{Z}) = \sum_{k=1}^{n}(1 -$

$|z_k|^2)|f_k(z_k)|^2$ be the energy target function of f at $\mathbf{Z}$. We call $\boldsymbol{A}$ an n-best vector if it gives rise to

$$E_f(\boldsymbol{A}) = \sup\left\{\sum_{k=1}^{n}(1-|z_k|^2)|f_k(z_k)|^2 : \mathbf{Z}\in\mathbb{D}^n\right\}. \tag{11}$$

The corresponding $\sum_{k=1}^{n}\langle f, B_k\rangle B_k$ is called an *n-best Blachke form approximation to f*. Due to the orthogonality the same parameter vector $\mathbf{A}$ gives rise to the best approximation

$$\inf\left\{\|f-\sum_{k=1}^{n}\langle f,B_k\rangle B_k\|^2 : \mathbf{Z}\in\mathbb{D}^n\right\}.$$

Existence of an n-best Blaschke approximation to $f\in H^2(\mathbb{D})$ has been repeatedly proved in [1]–[3], [20]. Note that for a given n-vector $\mathbf{Z}\in\mathbb{D}^n$, the subspace of $H^2(\mathbb{D})$ spanned by the induced T-M system $\{B_k\}_{k=1}^{n}$ is invariant under permutation of the elements of the vector $\mathbf{Z}$, although the induced $\{B_k\}_{k=1}^{n}$ are different. Hence energy $E_f(\mathbf{Z})$ is invariant under permutation of $\mathbf{Z}$. In particular, if $\mathbf{Z}$ is an n-best vector to f, any permutation of $\mathbf{Z}$ is also an n-best vector.

B. Cyclic AFD

n-best Blaschke approximation can be viewed as a pole-tuple search algorithm. And in a pole-tuple search algorithm, a main issue is how to ensure accuracy of the best tuple within an acceptable search time. Assume that the values $f(e^{itj}), 1\le j\le J$, are known, and the search is taken over an n-dimensional tensor product net $\mathscr{M}^n$, where the one-dimensional rectangular ε-net $\mathscr{M}\subset\mathbb{D}$, has NM nodes. Then an exhaustive search on $\mathscr{M}^n$ needs $O((JNM)^n)$ times of operations, which is unpractical when the approximation degree n is large.

To reduce the computational cost, we introduce Cyclic AFD. The nature of Cyclic AFD is finding n-best parameters not simultaneously, but sequentially. Therefore, Cyclic AFD needs only $O(JNM)$ operations, dramatically reducing the time used in an exhaustive search.

C. Unsolved Problem

We have proved that Cyclic AFD is identical with n-best Blaschke approximation if the target function satisfies a certain condition [14]. However, whether Cyclic AFD is identical with n-best Blaschke approximation without this condition require future research.

We will investigate the case when $n=2$ and generalize the result to $n\ge 2$.

References

[1] L.Baratchart, M.Cardelli, M.Olivi, *Identification and rational L2 approximation-a gradient algorithm*, Automatica, 1991, 27: 413-418.

[2] L.Baratchart, E.B.Saff, F.Wielonsky, J. Analuse Mathematique, *A criterion for uniqueness of a critical point in H2 rational approximation*, 1996, 70: 225-266.

[3] L. Baratchart, *A remark on uniqueness of best rational approximation of degree 1 in L2 of the circle*, Electron. Trans.Numer. Anal., 2006, 25:54-66.

[4] S. Boyd, L. Vandenberghe, *Convex Optimization*, Cambridge University Press, 2004.

[5] J.B. Garnett, *Bounded Analytic Functions*, Academic Press, 1987.

[6] P. Fulcheri, M. Olivi, *Matrix rational H2 approximation: a gradient algorithm based on Schur analysis*, SIAM J. Control Optim., 1998, **36**(6):2103-2127

[7] J.L.Walsh, *The existence of rational functions of best approximation*, Ibid, 1931, 33:668-689.

[8] J.L.Walsh, *Interpolation and Approximation by Rational Functions in the Complex Plane*, AMS, 1965.

[9] T. Qian, I*ntrinsic mono-component decomposition of functions: An advance of Fourier theory*, Mathematical Methods in Applied Sciences, 2010, **33**(7): 880-891.

[10] T. Qian, L. M. Zhang, Z. X. Li, *Algorithm of Adaptive Fourier Decomposition* , IEEE Transa Signal Processing, 2011, **59**(2): 5899-5906.

[11] T. Qian, Y. B. Wang, *Adaptive Fourier series-a variation of greedy algorithm*, Advances in Computational Mathematics, 2011, **34**(3): 279-293.

[12] W. Mi, T. Qian, *Frequency Domain Identification: An Algorithm Based On Adaptive Rational Orthogonal System,* Automatica, 2012, **48**(6): 1154-1162.

[13] T. Qian, E. Wegert, *Optimal Approximation by Blaschke Forms*, Complex Variables and Elliptic Equations, 2013, **58**(1): 123-133.

[14] T. Qian, *Cyclic AFD algorithm for best approximation by rational functions of given order*, Mathematical Methods in the Applied Sciences, 2014, **37**(6):846-859

[15] T. Qian, J. Z. Wang, W. X. Mai, *An Enhancement Algorithm for Cyclic Adaptive Fourier Decomposition*, Applied and Computational Harmonic Analysis, 2019, **47**(2): 516-525.

[16] G. T. Deng, Y. Huang, T. Qian, *Reproducing Kernels of Some Weighted Bergman Spaces*, Journal of Geometric Analysis, 2021.

[17] Y. B. Wang, T. Qian, *Pseudohyperbolic distance and n-best rational approximation in H2 space*, Mathematical Methods in the Applied Sciences, 2021, **44**(11):8497-8504.

[18] T. Qian, *Sparse representation of randdom signals,* Mathematical Methods in the Applied Sciences, 2022, **45**(8): 4210-4230.

[19] Qian T, Dai L, Zhang L, Chen Z, *Granular sieving algorithm for selecting best n parameters*, Mathematical Methods in the Applied Sciences, 2022.

[20] Qu W, Qian T, Haichou Li, Kehe Z.*Best kernel approximation in Bergman spaces*[J]. Applied Mathematics and Computation, 2022, 416:126749.