MATH351 Homework 1

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1 Problem 1 (1.1.1)

Show that the following equations have at least one solution in the given intervals.

- General Idea : Use I.V.T ! (Especially, in this case, we can use Bolzano's Theorem, which is I.V.T with $k = 0$.)

1.1 Problem 1-(a)

$x \cos x - 2x^2 + 3x - 1 = 0$, $[0.2, 0.3]$ and $[1.2, 1.3]$

Proof. By using theorem mentioned above, if f is continuous on an interval $[a, b]$ with $f(a)f(b) < 0$, then there is a value $c \in (x_1, x_2)$ such that $f(c) = 0$.

For $f(x) = x \cos x - 2x^2 + 3x - 1 = 0$, $g(x) = x \cos x$ is continuous since it is product of two continuous functions. $h(x) = -2x^2 + 3x - 1$ is continuous since it is polynomial. So, $f(x) = g(x) + h(x)$ is also continuous. Then we can use theorem I introduced above.

At first, let's think about the interval $[0.2, 0.3]$. $f(0.2) \simeq -0.28 < 0$, and $f(0.3) \simeq 6.6 \times 10^{-3} > 0$. So, we get $f(0.2)f(0.3) < 0$.

Then we can say that there is a value $r_1 \in (0.2, 0.3)$, which means first root of this equation, such that $f(r_1) = 0$. So, this equation have at least one solution in $[0.2, 0.3]$.

And next, for interval $[1.2, 1.3]$, $f(1.2) \simeq 0.15 > 0$, and $f(1.3) \simeq -0.13 < 0$ So we can get $f(1.2)f(1.3) < 0$.

Then, we can say that there is a value $r_2 \in (1.2, 1.3)$, which means second root of this equation, such that $f(r_2) = 0$. So, this equation have at least one solution in $[1.2, 1.3]$.

$\square$

1.2 Problem 1-(c)

$2x \cos(2x) - (x - 2)^2 = 0$, $[2, 3]$ and $[3, 4]$

Proof. By the same logic with 1-(a) ,

Let's think about the interval $[2,3]$ first, $f(2) \simeq -2.61 < 0$, and $f(3) \simeq 4.76 > 0$. So, we get $f(2)f(3) < 0$.

Then we can say that there is a value $r_1 \in (2,3)$ such that $f(r_1) = 0$. So, this equation have at least one solution in $[2,3]$.

Next, let's think about the interval $[3,4]$. $f(3) \simeq 4.76 > 0$, $f(4) \simeq -5.16 < 0$. So, we get $f(3)f(4) < 0$.

Then we can say that there is a value $r_2 \in (3,4)$ such that $f(r_2) = 0$. So, this equation have at least one solution in $[3,4]$.

□

2 Problem 2 (1.1.4)

Find intervals containing solutions to the following solutions.

2.1 Problem 2-(a)

$x - 3^{-x} = 0$

Solution. Here, we need to check a property of this function, $f(x) = x - 3^{-x}$. By differentiation, we get $f'(x) = 1 + 3^{-x} \ln 3 > 0$, $\forall x \in \mathbb{R}$.

Then, $f(x)$ is an increasing function, so this function has 1 root at maximum.

By taking two real numbers, 0 and 1 , we get $f(0) = -1$, $f(1) = \frac{2}{3}$. Then, $f(0)f(1) < 0$, by Intermediate value theorem, there is one root in interval $(0,1)$.

■

2.2 Problem 2-(c)

$x^3 - 2x^2 - 4x + 2 = 0$

Solution. By differentiation of function $f(x) = x^3 - 2x^2 - 4x + 2$, we get $f'(x) = 3x^2 - 4x - 4 = (3x+2)(x-2)$.

Thus, if $x \in (-\frac{2}{3}, 2)$, $f'(x) < 0$ then $f(x)$ decreases. And if $x \in (-\infty, -\frac{2}{3}) \cup (2, \infty)$, $f'(x) > 0$ then $f(x)$ increases.

Here, for convenience, we check $f(-2)$, $f(-\frac{2}{3})$, $f(2)$, $f(4)$.

$$f(-2) = -6, f(-\frac{2}{3}) = \frac{94}{27}, f(2) = -6, f(4) = 18$$

We get

$$f(-2)f(-\frac{2}{3}) < 0, f(-\frac{2}{3})f(2) < 0, f(2)f(4) < 0$$

By Intermediate value theorem, $f(x)$ has a single root in each of these three intervals : $(-2, -\frac{2}{3})$, $(-\frac{2}{3}, 2)$,$(2,4)$.

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3 Problem 3 (1.1.6)

Find $\max_{a \le x \le b} |f(x)|$ for the following functions and intervals.

3.1 Problem 3-(a)

$f(x) = \frac{2x}{x^2+1}$, $[0, 2]$

Solution. We can get $f'(x) = \frac{-2x^2+2}{(x^2+1)^2}$. So in interval $[0, 2]$, we have three critical points, when $x = 0, 1, 2$.

$$f(0) = 0, f(1) = 1, f(2) = \frac{4}{5}$$

Then,

$$\max_{0 \le x \le 2} |f(x)| = |f(1)| = 1$$

■

3.2 Problem 3-(c)

$f(x) = x^3 - 4x + 2$, $[1, 2]$

Solution. We can get $f'(x) = 3x^2 - 4$. So in interval $[1, 2]$, we have three critical points, when $x = 1, \frac{2}{\sqrt{3}}, 2$.

$$f(1) = -1, f(\frac{2}{\sqrt{3}}) = 2 - \frac{16\sqrt{3}}{9}, f(2) = 2$$

Then,

$$\max_{1 \le x \le 2} |f(x)| = |f(2)| = 2$$

■

4 Problem 4 (1.1.14)

Let $f(x) = 2x\cos(2x) - (x-2)^2$ and $x_0 = 0$

4.1 Problem 4-(a)

Find the third Taylor polynomial $P_3(x)$ and use it to approximate $f(0.4)$.

Solution. Third Taylor polynomial can be denoted

$$P_3(x) = f(x_0) \cdot 1 + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3$$

Since

$$f'(x) = 2\cos(2x) - 4x\sin(2x) - 2(x - 2)$$

$$f''(x) = -8\sin(2x) - 8x\cos(2x) - 2$$
$$f'''(x) = -24\cos(2x) + 16x\sin(2x)$$

Then, here

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$
$$= -4 + 6x - x^2 - 4x^3$$
$$P_3(0.4) = -4 + 2.4 - 0.16 - 4 \cdot 0.064 = -2.016 \approx f(0.4)$$

■

4.2 Problem 4-(b)

Use the error formula in Taylor's Theorem to find an upper bound for the error $|f(0.4) - P_3(0.4)|$. Compute the actual error.

Solution. Let E_3 be an upper bound for the error of the third Taylor polynomial. Then, we can denote

$$E_3 = |f(0.4) - P_3(0.4)|_{\max} = \frac{f^{(4)}(\xi(x))}{4!}x^4$$

for $\xi \in (0, 0.4)$.

Since

$$f^{(4)}(x) = 32x\cos(2x) + 64\sin(2x)$$

For $f^{(5)}(x) = 160\cos(2x) - 64x\sin(2x)$, its solution is not in $(0, 0.4)$. Actually, the nearest one is $x = 0.657$ (by using wolfram-alpha.) Then here, we need to check two critical points, $x = 0, 0.4$,

$$f^{(4)}(0) = 0, f^{(4)}(0.4) = 54.829$$

So,

$$E_3 = 54.829 \cdot \frac{0.4^4}{4!} \simeq 0.058$$

Actual Error is

$$E = |f(0.4) - P_3(0.4)| \simeq 0.013$$

This error is less than $E_3 = 0.058$. ■

4.3 Problem 4-(c)

Find the fourth Taylor polynomial $P_4(x)$ and use it to approximate $f(0.4)$.

Solution.

$$P_4(x) = f(x_0) \cdot 1 + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \frac{f^{(4)}(x_0)}{4!}(x - x_0)^4$$
$$= -4 + 6x - x^2 - 4x^3$$

Then,

$$P_4(0.4) \simeq -2.016 \approx f(0.4)$$

■

4.4 Problem 4-(d)

Use the error formula in Taylor's Theorem to find an upper bound for the error$|f(0.4) - P_4(0.4)|$. Compute the actual error.

Solution. Let E_4 be an upper bound for the error of the fourth Taylor polynomial. Then, we can denote

$$E_4 = |f(0.4) - P_4(0.4)|_{\max} = \frac{f^{(5)}(\xi(x))}{5!}x^5$$

for $\xi \in (0, 0.4)$.

Since

$$f^{(5)}(x) = 160\cos(2x) - 64x\sin(2x)$$

For $f^{(6)}(x) = -384\sin(2x) - 128x\cos(2x)$, its solution is not in $(0, 0.4)$. Actually, the nearest one is $x = 1.358$ (by using wolfram-alpha.) Then here, we need to check two critical points, $x = 0, 0.4$,

$$f^{(5)}(0) = 160, f^{(5)}(0.4) = 93.109$$

So,

$$E_3 = 160 \cdot \frac{0.4^5}{5!} \simeq 0.013653$$

Actual Error is

$$E \simeq 0.013365$$

This error is less than E_4. ∎

5 Problem 5 (1.3.3)

The Maclaurin series for the arctangent function converges for $-1 < x \leq 1$ and is given by

$$\arctan x = \lim_{n\to\infty} P_n(x) = \lim_{n\to\infty} \sum_{i=1}^{n} (-1)^{i+1} \frac{x^{2i-1}}{2i-1}$$

5.1 Problem 5-(a)

Use the fact that $\tan(\frac{\pi}{4}) = 1$ to determine the number of n terms of the series that need to be summed to ensure that $|4P_n(1) - \pi| < 10^{-3}$.

Solution. Consider about error, by Alternating series estimating theorem,

$$E = |\arctan x - P_n(x)|$$

$$= |\sum_{i=0}^{\infty} (-1)^{i+1} \frac{x^{2i-1}}{2i-1} - \sum_{i=0}^{n} (-1)^{i+1} \frac{x^{2i-1}}{2i-1}| \leq \frac{x^{2n+1}}{2n+1}$$

By applying $x = 1$,

$$|\arctan 1 - P_n(1)| = |\frac{\pi}{4} - P_n(1)| \leq \frac{1}{2n+1}$$

$$|P_n(1) - \frac{\pi}{4}| \leq \frac{1}{2n+1}$$

$$\Rightarrow |4P_n(1) - \pi| \leq \frac{4}{2n+1} < 10^{-3}$$

$\therefore 4 < 10^{-3}(2n+1)$, then the smallest n satisfy the condition is $n = 2000$. ■

5.2 Problem 5-(b)

The C++ programming language requires the value of π to be within 10^{-10} . How many terms of the series would we need to sum to obtain this degree of accuracy?

Solution. As an extension of the solution of 5-(a) , now we check for 10^{-10} .

$$|4P_n(1) - \pi| \leq \frac{4}{2n+1} < 10^{-10}$$

$$4 \times 10^{10} < (2n+1)$$

, then the smallest n satisfy the condition is $n = 2 \times 10^{10}$. ■

6 Problem 6 (1.3.9)

Suppose that $0 < q < p$ and that $F(h) = L + O(h^p)$.

6.1 Problem 6-(a)

Show that $F(h) = L + O(h^q)$.

Solution. Since we have $F(h) = L + O(h^p)$, $\exists K > 0$ s.t.

$$|F(h) - L| < Kh^p$$

We assumed that h is sufficiently small already, it is good to say $h < 1$. Then, $h^p < h^q$ holds. Hence...

$$|F(h) - L| < Kh^p < Kh^q$$

$$F(h) = L + O(h^q)$$

■

6.2 Problem 6-(b)

Make a table listing h, h^2, h^3, h^4 for $h = 0.5, 0.1, 0.01, 0.001$ and discuss the varying rates of convergence of these powers of h as h approaches zero.

h	h^2	h^4
0.5	0.25	0.0625
0.1	10^{-2}	10^{-4}
0.01	10^{-4}	10^{-8}
0.001	10^{-6}	10^{-12}

Solution. As powers of h increase, convergence will be faster.

The table is located above this solution. ∎

7 Problem 7 (1.3.15)

7.1 Problem 7-(a)

How many multiplications and additions are required to determine a sum of the form

$$\sum_{i=1}^{n}\sum_{j=1}^{i} a_i b_j$$

Solution. Consider for each summation, at first, $\sum_{j=1}^{i} a_i b_j$. In this (inner) summation, addition will be processed $(i-1)$ times for each i , and multiplication will be processed i times. Then,

$$\sum_{i=1}^{n}(i-1) = \frac{n(n-1)}{2}, \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

And, in outer summation, n additions will be processed by adding the result for $i = 1 \sim n$.

Then, $\frac{n(n-1)}{2} + (n-1) = \frac{n^2+n-2}{2}$ additions and $\frac{n(n+1)}{2}$ multiplications are required to determine this summation. ∎

7.2 Problem 7-(b)

Modify the sum in part (a) to an equivalent form that reduces the number of computations.

Solution. Consider about another form,

$$\sum_{i=1}^{n}(a_i(\sum_{j=1}^{i} b_j))$$

Since a_i is independent with j , we can change the form like this.

The number of additions will be same with 7-(a), but the number of multiplications will be changed since we eliminated unnecessary multiplications by changing the form.

Multiplications will be processed only n times ; which is smaller than $\frac{n(n+1)}{2}$. ■