MATH311 Homework 1

Woohyuk Choi (20210236)

Due : Feb 28 18:00

Problem 1

If r is rational ($r \neq 0$) and x is irrational, prove that $r+x$ and rx are irrational.

Proof. Since r is rational, we can denote $r = \frac{m}{n}$ where $m, n \in \mathbb{Z}$ are co-prime and $n \neq 0$. Here, I'm going to use 'contradiction' to prove.

Firstly, checking about $r + x$, we can assume $(r + x) \in \mathbb{Q}$ by using contradiction. Let $(r + x) = \frac{a}{b}$ where $a, b \in \mathbb{Z}$ and $b \neq 0$. We know $x = (r + x) - r$ by axiom. So, we can denote $x = \frac{a}{b} - \frac{m}{n} = \frac{an-bm}{bn}$ when $a, b, m, n \in \mathbb{Z}$. We know $(an-bm), bn \in \mathbb{Z}$ and $bn \neq 0$. So here, x is rational number. It is contradiction because we said x is irrational. Then, we can say $r + x$ is irrational.

And next, checking about rx , we can assume $rx \in \mathbb{Q}$ by using contradiction. Let $rx = \frac{c}{d}$ where $c, d \in F$ and $d \neq 0$. We know $x = rx \cdot \frac{1}{r}$ by axiom. So, we can denote $x = \frac{c}{d} \cdot \frac{n}{m} = \frac{cn}{dm}$, and $cn, dm \in \mathbb{Z}$ and $dm \neq 0$. So here, x is rational number. It is contradiction because we said x is irrational. Then we can say rx is irrational. $\square$

Problem 2

Prove that there is no rational number whose square is 12.

Proof. Let's say a rational number r whose square is 12 exists, by definition of rational number, we can denote $r = \frac{a}{b}$ where $a, b \in \mathbb{Z}$ are not both multiples of 3, and $b \neq 0$.

Then, $r^2 = \frac{a^2}{b^2} = 12$, so we get $a^2 = 12b^2$. By processing factorization of 12, 12 can be denoted $12 = 2^2 \cdot 3$, then $a^2 = 2^2 \cdot 3 \cdot b^2$.

Here, we can say a^2 is multiple of 3 , and a is a multiple of 2. Then a is a multiple of $3 \cdot 2 = 6$, so for $k \in \mathbb{Z}, a = 6k$. Here, we can say $36k^2 = 12b^2$ so $3k^2 = b^2$, hence b^2 is multiple of 3 , it means b be a multiple of 3, this contradicts with our assumption above.

So, there is no rational number whose square is 12. $\square$

Problem 3

Prove Proposition 1.15.

Proposition 1.15.
(a) If $x \neq 0$ and $xy = xz$ then $y = z$
(b) If $x \neq 0$ and $xy = x$ then $y = 1$
(c) If $x \neq 0$ and $xy = 1$ then $y = \frac{1}{x}$
(d) If $x \neq 0$ then $\frac{1}{\frac{1}{x}} = x$

Proof. (a) By using axioms for multiplication,

$$y = 1 \cdot y = (x \cdot \frac{1}{x}) \cdot y = \frac{1}{x} \cdot (xy) = \frac{1}{x} \cdot (xz) = (\frac{1}{x} \cdot x) \cdot z = 1 \cdot z = z$$

(b) By using axioms for multiplication again,

$$y = 1 \cdot y = (x \cdot \frac{1}{x}) \cdot y = \frac{1}{x} \cdot (xy) = \frac{1}{x} \cdot x = 1$$

(c) By using axioms for multiplication again,

$$y = 1 \cdot y = (x \cdot \frac{1}{x}) \cdot y = \frac{1}{x} \cdot (xy) = \frac{1}{x} \cdot 1 = \frac{1}{x}$$

(d) By the same way, we can use the result of (c). By changing x to $\frac{1}{x}$ and y to x

$$x = 1 \cdot x = (\frac{1}{x} \cdot x) \cdot x = \frac{1}{\frac{1}{x}} \cdot (\frac{1}{x} \cdot x) = \frac{1}{\frac{1}{x}} \cdot 1 = \frac{1}{\frac{1}{x}}$$

$\square$

Problem 4

Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E . Prove that $\alpha \leq \beta$.

Proof. Let E be a nonempty subset of an ordered set S. And $\alpha \in S$ is a lower bound of E , $\beta \in S$ is a upper bound of E.

Now, think about an element $x \in E$, we can say surely $\alpha \leq x$ and $x \leq \beta$ by definition of lower / upper bound.

So, we get $\alpha \leq x \leq \beta$ then $\alpha \leq \beta$ holds. $\square$

Problem 5

Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup(-A)$$

Proof. Now, we use the definition of inf and sup . And since A is a set of real numbers, we can say it is ordered set.

Without Loss of Generality, let $x_1, x_2 \in A$ satisfy $x_1 < x_2$, then we can say $-x_1, -x_2 \in -A$ satisfy $-x_1 > -x_2$. Let $\inf A = \alpha$ then $\forall x \in A$ satisfy $x > \alpha$ and any $c > \alpha$ is not a lower bound of A .

So, $\forall x_1, x_2 \in A, \alpha < x_1 < x_2$ holds. Here, $\forall - x_1, -x_2 \in -A, -x_2 < -x_1 < -\alpha$ holds. Also, any $-c < -\alpha$ is not a upper bound of A by multiplication of -1 each side.

So, $-\alpha = \sup(-A)$.

Then, $-(-\alpha) = -\sup(-A) = \alpha = \inf A$ holds. □