STA 103 Lecture 3: Discrete Random Variables

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Announcement

Update on Assignment Due Dates

Hi everyone,

To give you time to review solutions before each exam, all assignments will now be due **one day earlier** than originally scheduled. This allows me to post solutions **24 hours before the exam**, so you can use them as an additional study resource.

To balance the shorter timeline, I've removed one problem from Assignment 1:

- **Before:** 6 problems in 7 days
- **Now:** 5 problems in 6 days (similar to having 12 days in a regular quarter as summer session moves twice as fast as a regular quarter)

The removed problem will **not** appear on the exam.

Revised Due Dates:

- Assignment 1: **Aug 12 (Tue) Noon**
- Assignment 2: **Aug 25 (Mon) Noon**
- Assignment 3: **Sep 8 (Mon) Noon**

The syllabus and assignment have been updated accordingly, **please check the latest versions**. Thanks for your understanding, and I hope this helps you better prepare for the exams!

Bernoulli Random Variables

- **Definition**: A Bernoulli random variable takes on only two values: 0 and 1, with probabilities $1 - p$ and p, respectively.

- PMF in table form:

x	$p(x)$
0	$1 - p$
1	p

- PMF in Function Form:

$$p(x) = \begin{cases} p^x (1-p)^{1-x} & \text{if } x = 0 \text{ or } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

- The Bernoulli random variable can be denoted by $X \sim \text{Bernoulli}(p)$.

Bernoulli Random Variables

For Bernoulli random variables:

- Expectation:

$$E(X) = \sum_x xp(x) = 0 \cdot (1-p) + 1 \cdot p = p.$$

- Variance:

$$E(X^2) = \sum_x x^2 p(x) = 0^2 \cdot (1-p) + 1^2 \cdot p = p,$$
$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = p - p^2 = p(1-p).$$

Binominal Random Variables

- **Definition**: A binominal random variable X counts the number of successes in n independent Bernoulli trials.
- Each trial has only two outcomes: "success" with probability p, and "failure" with probability $1 - p$.
- The total number of trials is fixed and denoted by n.
- The random variable X can take values $x = 0, 1, 2, \dots, n$.

Binomial Random Variables

- A binomial random variable is the sum of n independent and identically distributed **(i.i.d.)** Bernoulli random variables with the same success probability p:

$$Y = X_1 + X_2 + \cdots + X_n, \quad X_i \overset{\text{i.i.d.}}{\sim} \text{Bernoulli}(p).$$

- The PMF of Binomial RV, getting exactly x successes in n trials, is

$$P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}.$$

- The term $\binom{n}{x} = \frac{n!}{(n-x)!x!}$ counts the number of different ways to choose x successes out of n trials.

- The expression $p^x (1 - p)^{n-x}$ is the probability of any particular sequence with x successes and $n - x$ failures.

Binominal Random Variables

- How to calculate $E(X)$ and $\mathrm{Var}(X)$? Isn't it too complicated?

- Luckily here is the trick.
  - **Expectation**:

$$\begin{aligned} E(Y) &= E(X_1 + X_2 + \cdots + X_n) \\ &= E(X_1) + E(X_2) + \cdots + E(X_n). \end{aligned}$$

    The summation expansion of expectation always holds.

  - **Variance**:

$$\begin{aligned} \mathrm{Var}(Y) &= \mathrm{Var}(X_1 + X_2 + \cdots + X_n) \\ &= \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \cdots + \mathrm{Var}(X_n). \end{aligned}$$

    The summation expansion of variance holds only if $X_1, \ldots, X_n$ are independent. (We will discuss independence in the next lecture!)

Binominal Random Variables

Then,

- **Expectation**:

$$\begin{aligned} E(Y) &= E(X_1 + X_2 + \cdots + X_n) \\ &= E(X_1) + E(X_2) + \cdots + E(X_n) \\ &= n \times E(X_1) = np. \end{aligned}$$

- **Variance**:

$$\begin{aligned} \mathrm{Var}(Y) &= \mathrm{Var}(X_1 + X_2 + \cdots + X_n) \\ &= \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \cdots + \mathrm{Var}(X_n) \\ &= n \times \mathrm{Var}(X_1) = np(1-p). \end{aligned}$$

- We can denote it by $Y \sim$ Binomial(n, p). The Bernoulli(p) is a special case of Binomial(n, p) when $n = 1$.

Binominal Random Variables

- **Example** (Market Research Analyst): You are analyzing customer response behavior in an online survey conducted by a retail economics team. From past data, it is known that the probability a randomly selected customer responds to the survey is $p = 0.2$. You randomly contact $n = 50$ customers. Let the random variable X denote the number of customers who respond.

- **Question 1**: Identify the distribution of X. Justify your answer.

Binominal Random Variables

- **Question 2**: What is the expected number of responses and the standard deviation?
- **Question 3**: What is the probability that exactly 10 customers respond?
- **Question 4**: What is the probability that at most 2 customers respond?

Poisson Random Variable

- **Definition**: A Poisson random variable models the number of occurrences of an event in a fixed interval of time or space, under the assumptions that:
  - Events occur independently.
  - The average rate (events per interval) is constant.
  - Two events cannot occur at exactly the same time.

- Let X denote the number of events in a given interval and let $\lambda > 0$ be the expected number of events in that interval. Then X is said to follow a Poisson distribution with parameter λ, and its probability mass function is

$$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}, \text{ for } x = 0, 1, 2, \ldots$$

Poisson Random Variable

- We can denote it by $X \sim \text{Poisson}(\lambda)$.

- **Expectation and Variance**:

$$E(X) = \text{Var}(X) = \lambda.$$

Poisson Random Variable

Identifying a Poisson Random Variable: A random variable X can be modeled using a Poisson distribution if it satisfies the following conditions

- Counts of Events: X represents the number of times an event occurs in a fixed interval of time or space.
- Non-negative Integer Values: $X = 0, 1, 2, \ldots$
- Constant Rate: Events occur at a constant average rate λ over time or space.
- Rare Events: In a very small interval, the probability of more than one event occurring is negligible.

Poisson Random Variable

Examples that likely follow a Poisson distribution:

- Number of customer arrivals at a bank per hour.
- Number of website clicks or page views per second.
- Number of insurance claims filed by a policyholder per year.

Poisson Random Variable

- **Example**: A local bank observes that, on average, 6 customers arrive at the counter per hour. Assume customer arrivals follow a Poisson process. Let X be the number of customer arrivals in each hour. Then $X \sim \text{Poisson}(\lambda = 6)$.

- **Question 1**: What is the probability that exactly 4 customers arrive in one hour?

- **Question 2**: What is the probability that at most 2 customers arrive?

Poisson Random Variable

- **Example**: A local bank observes that, on average, 6 customers arrive at the counter per hour. Assume customer arrivals follow a Poisson process. Let X be the number of customer arrivals in each hour. Then $X \sim \text{Poisson}(\lambda = 6)$.

- **Question 3**: What is the probability that more than 1 customers arrive?

- **Question 4**: What are the mean, standard deviation and variance of X?

Conditional PMF

Suppose X is a random variable with PMF:

- PMF in a table form:

x	$p(x)$
1	0.5
2	0.1
3	0.1
4	0.2
5	0.1
	1

- Equivalently, we have the following box model:

5×	1×	1×	2×	1×
$\boxed{1}$	$\boxed{2}$	$\boxed{3}$	$\boxed{4}$	$\boxed{5}$

Box Model (10 tickets total)

Conditional PMF

- What if I sampled X but did not tell you which number it was but instead told you that $X \geq 3$.

- Then, the actual value of X is still random but now the possible values are $3, 4, 5$ instead of $1, 2, 3, 4, 5$.

- The conditional PMF given $X \geq 3$, denoted $p(x \mid X \geq 3)$, characterizes the randomness in X after observing $X \geq 3$.

- To find $p(x \mid X \geq 3)$, just zero out $p(x)$ for x's that are not observed and renormalize to 1.

Conditional PMF

- The original PMF in a table form:

x	$p(x)$
1	0.5
2	0.1
3	0.1
4	0.2
5	0.1
	1

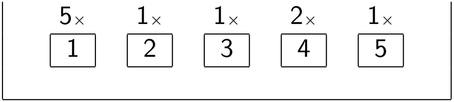
- After observing $X \geq 3$,

x	$p(x \mid X \geq 3)$
1	0
2	0
3	$0.1/0.4$
4	$0.2/0.4$
5	$0.1/0.4$
	$(0.1 + 0.2 + 0.1)/0.4$

Conditional PMF

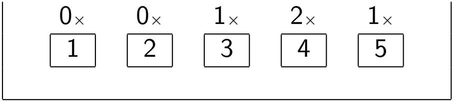
Equivalently,

- The original box model:



Box Model (10 tickets total)

- After observing $X \geq 3$,



Box Model (4 tickets total)

Conditional PMF

- **Question 1**: Find $P(X \geq 4)$?

x	$p(x)$
1	0.5
2	0.1
3	0.1
4	0.2
5	0.1
	1

- **Question 2**: Find $P(X \geq 4 \mid X \geq 3)$?

x	$p(x \mid X \geq 3)$
1	0
2	0
3	$0.1/$0.4
4	$0.2/$0.4
5	$0.1/$0.4
	$(0.1 + 0.2 + 0.1)/$0.4

Conditional PMF

- We can also compute the "new" expected value $E(X)$ after observing $X \geq 3$.

x	$p(x \mid X \geq 3)$
1	0
2	0
3	$0.1/0.4$
4	$0.2/0.4$
5	$0.1/0.4$
	$(0.1 + 0.2 + 0.1)/0.4$

$$
\begin{aligned}
E(X \mid X \geq 3) &= \sum_x x p(x \mid X \geq 3) \\
&= 3 \cdot \frac{1}{4} + 4 \cdot \frac{1}{2} + 5 \cdot \frac{1}{4} = \frac{3 + 8 + 5}{4} = 4.
\end{aligned}
$$

- **Interpretation** of $E(X \mid X \geq 3)$: Adjusted prediction after given new information that $X \geq 3$.

Conditional PMF

- Box average after observing $X \geq 3$.

$$\begin{array}{ccccc} 0\times & 0\times & 1\times & 2\times & 1\times \\ \boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} & \boxed{5} \end{array}$$

Box Model (4 tickets total)

$$\text{Box Average} = \frac{3+4+4+5}{4} = 4.$$

- **Interpretation**: Box Average after observing $X \geq 3$ is identical to $E(X \mid X \geq 3)$.

Conditional PMF

- Similarly, the $\mathrm{Var}(X)$, and $\mathrm{sd}(X)$ after observing $X \geq 3$ are:

x^2	x	$p(x \mid X \geq 3)$
1^2	1	0
2^2	2	0
3^2	3	$0.1/0.4 = 1/4$
4^2	4	$0.2/0.4 = 1/2$
5^2	5	$0.1/0.4 = 1/4$
		$(0.1 + 0.2 + 0.1)/0.4$

$$\begin{aligned}\mathrm{Var}(X \mid X \geq 3) &= E(X^2 \mid X \geq 3) - (E(X \mid X \geq 3))^2 \\ &= 3^2 \cdot \frac{1}{4} + 4^2 \cdot \frac{1}{2} + 5^2 \cdot \frac{1}{4} - (4)^2 \\ &= \frac{9 + 32 + 25}{4} - 16 = 1/2.\end{aligned}$$

$$\mathrm{sd}(X \mid X \geq 3) = \sqrt{\mathrm{Var}(X \mid X \geq 3)} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \approx 0.7071.$$