STA 103 Lecture 2: Random Variable, Probability Distribution

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Announcement

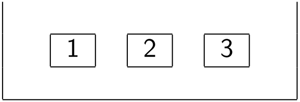
- Assignment 1 (Due Aug 13th, Wed) will be posted on Canvas today.
- Problem set 1, covering topics in Midterm 1, will be posted on Canvas today.
- Office hour location will be MSB 1143 (Thu 2-3 pm).

Random Variables

- **Definition**: A random variable (RV for short) is a numerical description of the outcome of a random experiment. (i.e, daily revenue, daily visit of website).

- It usually denoted with capital letters X, Y, Z. The values of a random variable can vary with each repetition of an experiment.

Select sample X



out of box with 3 tickets

- Random variables can take on values with specific probabilities.
  - **Example**: I will flip two fair coins. Let Y be the number of heads I get. Then, Y is random variable.

Probability Distribution

- **Definition**: A probability distribution is a function of the random variable that determines which values are more likely than others. It assigns the probabilities to the possible values of the random variable.
- Discrete Variable: Random variables that can assume a countable number of values. Discrete RV has probability mass function (PMF) as probability distribution. (Midterm 1)
- Continuous Variable: Random variables that can assume values corresponding to any of the points contained in an interval. Continuous RV has probability density function (PDF) as probability distribution. (Midterm 2)

Examples: Discrete vs Continuous

Scenario 1: A retailer tracks the number of customers who make a purchase each day.

- Discrete RV: Let X be number of purchasing customers in a day. Assume X follows Poisson distribution with mean $\lambda = 20$.
- Probability Mass Function (PMF):

$$P(X = x) = p(x) = \frac{e^{-20} \cdot 20^x}{x!}, \quad x = 0, 1, 2, \ldots$$

Scenario 2: An economist analyzes the monthly household income in a city.

- Continuous RV: Let Y be monthly household income. Assume Y follows normal distribution with mean $\mu = 5$, standard deviation $\sigma = 1$.
- Probability Density Function (PDF):

$$P(Y = y) = p(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-5)^2}{2}}, \quad y \in \mathbb{R}.$$

Probability Mass Function (PMF)

- Every discrete RV X has a unique PMF usually denoted $p(x)$. Discrete means the set of possible values can be put in a list.

- Everything can be computed from $p(x)$.
  - Use $p(x)$ to check if X has the same randomness as another random variable.
  - Use $p(x)$ to visualize X.

- Random quantities are hard but functions are easy $\rightarrow$ X is hard, $p(x)$ is easier.

- We will see **2** different ways to define the PMF $p(x)$ for a random variable X.

Definition of PMF

Let $p(x) = P(X = x)$: Input is a possible X value x, output is the chance that $X = x$.

- **Definition 1 (PMF in Table Form)**: The probability mass function $p(x)$ for a random variable X, given in table form is the following:

x	$p(x)$
1	1/3
2	2/3

- **Definition 2 (PMF in Function Form)**: The probability mass function $p(x)$ for a random variable X, given in function form is the following:

$$p(x) = \begin{cases} \frac{1}{3} & \text{if } x = 1, \\ \frac{2}{3} & \text{if } x = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Properties of PMF

- $p(x) \geq 0$ for every x.
- $\sum_x p(x) = 1$.
- No repeat on the input value x.

Examples and Non-Examples

- Example 1:

x	$p(x)$
-1	1/4
0	2/4
1	1/4

- Example 2:

x	$p(x)$
1	2^{-1}
2	2^{-2}
3	2^{-3}
4	2^{-4}
$\vdots$	$\vdots$

Examples and Non-Examples

- Non-Example 1:

x	$p(x)$
0	1/4
1	2/4
0	1/4

- Non-Example 2:

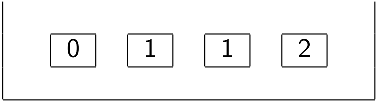
x	$p(x)$
2	3/4
1	-1/2
0	3/4

- Non-Example 3:

x	$p(x)$
2	0
1	1/2
0	3/4

Two random variables with the same PMF are identically distributed

- RV X: The number of heads obtained when flipping **2** fair coins.
- RV Y: The selected ticket when one ticket is drawn at random from the following box.



box with 4 tickets

- **Question**: Are X and Y identical (in terms of randomness)?
- **Answer**: The way to tell is to compare the PMFs.

Use the PMF to find any probability

- Illustrate with previous PMF example:

$$p(x) = \begin{cases} 2^{-x} & \text{for } x \in \{1, 2, 3, \cdots\}, \\ 0 & \text{otherwise.} \end{cases}$$

- Think of X as the outcome of a game you can bet on. For example, if you bet $X = 3$ on next play then

$$p(3) = 2^{-3} = 1/8$$

  is the chance you win that bet.

- But you can bet on other things e.g., bet $X \leq 3$, we have

$$P(X \leq 3) = p(1) + p(2) + p(3) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8}.$$

- This X can also be interpreted as the number of games played until a streak of consecutive wins ends (including the final losing game), in a game where the probability of winning each round is $\frac{1}{2}$.

Use the PMF to find any probability

- Illustrate with previous PMF example:

$$p(x) = \begin{cases} 2^{-x} & \text{if } x \in \{1, 2, 3, \cdots\}, \\ 0 & \text{otherwise.} \end{cases}$$

- **Question**: What is the probability of winning when you bet $X \leq 2$ or $X \geq 4$?

- **Question**: What is the probability of winning when you bet X is even?

- In general, for any set of possible outcomes A for X,

$$P(X \in A) = \sum_{x \in A} p(x).$$

  The left hand side is the chance you win a bet that X is in A. The right hand side is to compute $p(x)$ for each x in A and add them up.

Finding the PMF to match an X

- Start by listing the possible values of X in the left column of a PMF table.
- Then go back and figure out $p(x) = P(X = x)$ for each x in the right column.
- **Example**: Flip a fair coin until you see the first heads. Let X denote the number of flips it takes.
- For example, if we have T, T, H, then $X = 3$ (took 3 flips).

Finding the PMF to match an X

- First think of the possible values for X.

x	
1	
2	
3	
4	
$\vdots$	$\vdots$

- Now start with $x = 1$. Only way is first flip is heads. Then, $P(X = 1) = P(\text{first flip heads}) = 1/2$.

x	$p(x)$
1	1/2
2	
3	
4	
$\vdots$	$\vdots$

Finding the PMF to match an X

- Now think about $x = 2$.
  $P(X = 2) = P(\text{first flip tails, then heads}) = 1/2 \cdot 1/2 = 1/4$.

x	$p(x)$
1	1/2
2	1/4
3	
4	
$\vdots$	$\vdots$

- In a similar argument,

x	$p(x)$
1	$1/2$
2	$1/4$
3	$1/8$
4	$1/16$
$\vdots$	$\vdots$

We always need to check the second column sums to 1.

Finding X to match a PMF

- Gives this PMF table

x	$p(x)$
-1	0.51
0	0.15
1	0.34

- Make a new column (large number)$\times p(x)$, so all elements are integers.

x	$100 \times p(x)$
-1	51
0	15
1	34

- Now make 100 tickets with x column values written on them duplicated by the frequency $100 \times p(x)$. The number showing on a random draw is a X.

51×	15×	34×
$\boxed{-1}$	$\boxed{0}$	$\boxed{1}$

Expected Value

- If X is a discrete random variable with PMF $p(x)$, the expected value $E(X)$ is defined as

$$E(X) = \sum_x xp(x).$$

- The expected value $E(X)$ is a fundamental quantity useful for all sort of things. $E(X)$ is analogous to sample mean (average of a list $x_1, x_2, \ldots, x_n$.)

$$\frac{1}{n}\sum_{i=1}^{n} x_i.$$

- **Example**: The expected value of a random variable is its weighted average value and can be viewed as a central value of the random variable. $E(X)$ is the best prediction of the future RV X.

Expected Value

- **Example 1**: A bookstore owner tracks the number of books sold daily. Let the random variable X represent the number of books sold in a day. The probability mass function (PMF) is:

x (Books Sold)	0	1	2	3	4
$p(x)$	0.10	0.25	0.30	0.20	0.15

- **Question**: What is the expectation number of books sold in a day?

- **Answer**:

$$
\begin{aligned}
E(X) = \sum_x xp(x) &= 0 \times 0.1 + 1 \times 0.25 + 2 \times 0.3 + 3 \times 0.2 + 4 \times 0.15 \\
&= 0 + 0.25 + 0.6 + 0.6 + 0.6 = 2.05.
\end{aligned}
$$

Expected Value

- **Example 2 (HW)**: Let X have PMF given by:

$$p(x) = \begin{cases} \binom{2}{x}(0.4)^x(0.6)^{2-x} & \text{for } x \in \{0, 1, 2\}, \\ 0 & \text{otherwise.} \end{cases}$$

- Note: The binomial coefficient, $\binom{n}{k}$, is defined by the expression:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \qquad k! = k \cdot (k-1) \cdot (k-2) \cdots 2 \cdot 1.$$

- For example, $0! = 1$, $1! = 1$, $2! = 2 \cdot 1 = 2$, and $3! = 3 \cdot 2 \cdot 1 = 6$.

- **Question**: Find $E(X)$.

Variance

- If X is a random variable with expected value $E(X)$, the variance of X is

$$\mathrm{Var}(X) = E\left([X - E(X)]^2\right).$$

- The standard deviation, $\mathsf{sd}(X)$, is the square root of the variance.

$$\mathsf{sd}(X) = \sqrt{\mathrm{Var}(X)}.$$

- The variance / standard deviation of a random variable, is an indication of how dispersed the probability distribution is about its center, indication of how spread out on the average.

- **Interpretation**: Start with $X - E(X)$: the difference between your prediction $E(X)$ and the actual X: Now square the prediction error to make it positive.

Variance

- The variance of X can be computed by two ways

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 \quad \text{(easy to compute)},$$
$$\mathrm{Var}(X) = E\left([X - E(X)]^2\right) \quad \text{(easy to understand)}.$$

- The variance $\mathrm{Var}(X)$ tells you the typical squared prediction error for X. The standard deviation sd(X) tells you the typical prediction error for X.

Variance

- We already know how to compute $E(X)$ so the only tricky part is $E(X^2)$. Technically this requires finding the PMF for X^2 but this is often hard. Luckily there is a shortcut:

$$E(X^2) = \sum_x x^2 p(x)$$

- Note that $p(x)$ is the PMF for X not X^2.

- This trick extends for computing $E(f(X))$ for any function $f(x)$ mapping real numbers to real numbers.

$$E(f(X)) = \sum_x f(x)p(x),$$

where $p(x)$ is PMF for X, not $Y = f(X)$.

Variance

- Computing $E(f(X)) = \sum_x f(x)p(x)$ from a PMF in a table form is particularly easy.

- **Example**: Suppose RV X has a PMF table:

x	$p(x)$
1	0.4
2	0.6

- **Question**: Find $E(X)$, sd(X), and $\mathrm{Var}(X)$.

- **Interpretation**: In words: we predict X to be $E(X)$. The typical magnitude of prediction error will be $sd(X)$. We may sometimes write this interpretation with shorthand

$$X \sim E(X) \pm \text{sd}(X) = \text{best prediction} \ \pm \ \text{typical prediction error.}$$

Variance

- **Answer**:

x^2	x	$p(x)$
1	1	0.4
4	2	0.6

$$E(X) = \sum_x xp(x) = 1 \cdot (0.4) + 2 \cdot (0.6) = 1.6,$$
$$E(X^2) = \sum_x x^2 p(x) = 1 \cdot (0.4) + 4 \cdot (0.6) = 2.8.$$

Then,

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = 2.8 - (1.6)^2 = 0.24,$$
$$\mathsf{sd}(X) = \sqrt{\mathrm{Var}(X)} = \sqrt{0.24} \approx 0.49.$$

- **Interpretation**: In words: we predict X to be $E(X)$. The typical magnitude of prediction error will be $\mathsf{sd}(X)$. We may sometimes write this interpretation with shorthand

$$X \sim E(X) \pm \mathsf{sd}(X) = 1.6 \pm 0.49.$$

Variance

- **Example (HW)**: Suppose RV W has a PMF table:

w	$p(w)$
0	35/47
2	2/47
4	10/47

- **Question**: Find $E(W^2)$, $\mathrm{Var}(W)$, $E(5-W)$, and $E(\exp(W))$?

Properties of Mean and Variance

- If Y is a linear function of X, i.e., $Y = aX + b$, where a, b are fixed constants. Then,

$$\begin{aligned} E(Y) &= E(aX + b) = E(aX) + b = aE(X) + b \\ \mathrm{Var}(Y) &= \mathrm{Var}(aX + b) = \mathrm{Var}(aX) = a^2 \mathrm{Var}(X) \\ \mathrm{sd}(Y) &= |a| \mathrm{sd}(X). \end{aligned}$$

Properties of Mean and Variance

- **Example (HW)**: Let X represent the number of units of a product sold per day at a local street market. The probability mass function (PMF) is given below:

x (Units Sold)	0	1	2	3
$p(x)$	0.1	0.3	0.4	0.2

- **Question**: Let $Y = 2X - 3$. What is the variance and standard deviation of Y?