STA 103 (SS2 2025): Problem Set 2
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TOPICS COVERED

1. Expectation formula:

$$\begin{aligned} E(aX) &= aE(X) \\ E(X+b) &= E(X)+b \\ E(X+Y) &= E(X)+E(Y) \\ E(aX+bY) &= aE(X)+bE(Y) \end{aligned}$$

2. Variance formula:

$$\begin{aligned} \mathrm{var}(aX) &= a^2\,\mathrm{var}(X) \\ \mathrm{var}(X+b) &= \mathrm{var}(X) \\ \mathrm{var}(X+Y) &= \mathrm{var}(X)+\mathrm{var}(Y)+2\,\mathrm{cov}(X,Y) \\ \mathrm{var}(X-Y) &= \mathrm{var}(X)+\mathrm{var}(Y)-2\,\mathrm{cov}(X,Y) \\ \mathrm{var}(aX+bY) &= a^2\mathrm{var}(X)+b^2\mathrm{var}(Y)+2ab\,\mathrm{cov}(X,Y) \\ \mathrm{var}(aX-bY) &= a^2\mathrm{var}(X)+b^2\mathrm{var}(Y)-2ab\,\mathrm{cov}(X,Y) \end{aligned}$$

3. Covariance properties

$$\begin{aligned} \mathrm{cov}(aX,bY) &= a\cdot b\cdot \mathrm{cov}(X,Y) \\ \mathrm{cov}(X,c) &= \mathrm{cov}(c,X)=0 \\ X \text{ \& } Y \text{ are independent} &\Longrightarrow \mathrm{cov}(X,Y)=0 \end{aligned}$$

4. Continuous probability density functions (PDF): If X has PDF $p(x)$ then

$$\begin{aligned} Pr(a\le X\le b) &= \int_a^b p(x)\,dx \\ E(X) &= \int_a^b x\,p(x)\,dx \\ E(f(X)) &= \int_a^b f(x)\,p(x)\,dx \end{aligned}$$

5. Uniform Random Variable
   - The PDF for $Y\sim \mathrm{Unif}[0,1]$ is:
   $$f(y)=\begin{cases}1, & 0\le y\le 1,\\ 0, & \text{otherwise.}\end{cases}$$
   - Generally, the PDF for $Y\sim\mathrm{Unif}[a,b]$ is:
   $$f(y)=\begin{cases}\frac{1}{(b-a)}, & a\le y\le b,\\ 0, & \text{otherwise.}\end{cases}$$
   - **Mean**:
   $$E(Y)=\frac{a+b}{2}.$$
   - **Variance**:
   $$\mathrm{Var}(Y)=\frac{(b-a)^2}{12}.$$

6. Normal Random Variables
   - The PDF for $X\sim N(\mu,\sigma^2)$ is:
   $$p(x)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$
   - **Mean**:
   $$E(X)=\mu.$$
   - **Variance**:
   $$\mathrm{Var}(X)=\sigma^2.$$
   - The PDF for standard normal variables $X\sim N(0,1)$ is when $\mu=0$ and $\sigma=1$.

7. z-scores: For any random variable X we can change units to W via

$$\begin{aligned} W &= \frac{X-a}{b} \\ X &= a+bW. \end{aligned}$$

If we use $a=E(X)$ and $b=sd(X)$ then W is called a z-score transformation of X and satisfies

$$E(W)=0 \text{ and } \mathrm{sd}(W)=1.$$

8. Special Normal/Gaussian Cases: If $X\sim\mathcal{N}(a,b)$ then

$$\begin{aligned} E(X) &= a \\ sd(X) &= \sqrt{b} \\ W &= \frac{X-a}{\sqrt{b}}\sim N(0,1) \\ Pr(c\le W\le d) &= \int_c^d \frac{e^{-x^2/2}}{\sqrt{2\pi}}dx \end{aligned}$$

9. The central limit theorem (CLT): If $X_1,\dots,X_n$ are independent random variables, all with the same PMF or PDF, then

$$\bar{X}=\frac{X_1+\cdots+X_n}{n}\approx N(a,b)$$

for large n (usually $n>30$ is good), where $a=E(\overline{X})$ and $b=\mathrm{Var}(\overline{X})$.

10. For i.i.d. RVs $X_1,\dots,X_n$, with $E(X_1)=\mu$ and $\mathrm{Var}(X_1)=\sigma^2$.

$$E(\bar{X})=E(X_1)=\mu,\quad \mathrm{Var}(\bar{X})=\frac{\mathrm{Var}(X_1)}{n}=\frac{\sigma^2}{n}.$$

PRACTICE PROBLEMS

Lecture 5

1. Let Z and W be two random variables such that $\text{Cov}(Z, W) = -4$, $\text{Var}(Z) = 10$ and $\text{Var}(W) = 5$. Find $\text{Var}(Z - W)$.

2. Suppose X and Y are two random variables with $\text{Var}(X) = 2$, $\text{Var}(Y) = 4$ and $\text{Cov}(X, Y) = -0.5$. Find the standard deviation of $nX - mY$.

3. Suppose X is a random variable. Find

$$E\left(-\frac{X}{E(X)}\right)$$

4. Suppose X and Y are independent random variables such that $\text{sd}(X) = \sigma$ and $\text{sd}(Y) = \tau$. Which of the following is the correct formula for $\text{sd}(X - Y)$?

5. If $EX = 1$ and $\text{Var}(X) = 2.5$ find EY where $Y = (2 - X)^2$.

6. Two independent investment options are offered with both of them having \$2 expected profit per share. In addition we know that the standard deviation of the first option profit is \$1 per share and the standard deviation of the second one is \$2 per share. Suppose we invest on a combination of 200 shares of the first option and 100 shares of the second one. Find the expected profit and the standard deviation of the profit.

Lecture 6

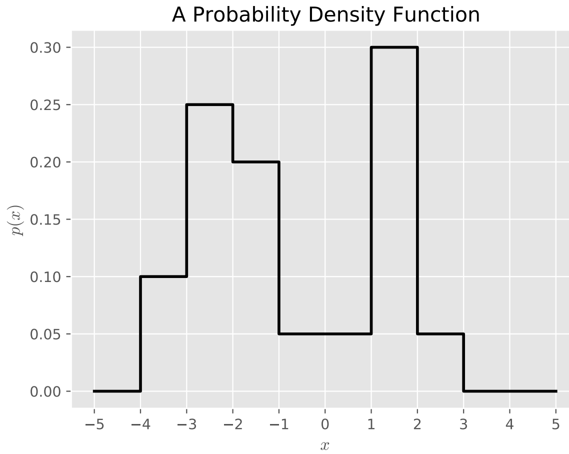

FIG. 1.

1. Suppose X is a continuous random variable with probability density function $p(x)$ shown in FIG. 1. Find $P(-2 \leq X < 2)$.

2. A computer repairman records how long it takes him to fix each computer his customers bring to him. He figures out that the time (in hours) needed to repair a computer (X) is well-modeled by the following PDF.

$$p(x) = \frac{1}{5},$$

where $x \leq 5$. A customer runs into his store 2 hours before closing and begs the repairman to fix his computer. The repairman says that he will try his best, but if he can't finish by closing time, he'll have to continue working on it tomorrow. What is the probability the customer gets his computer back before the store closes?

3. Rebecca takes the bus to school every Friday morning. The bus comes every 15 minutes. Let X denote the time Rebecca must wait for the bus after arriving at the bus stop. The PDF of X is

$$p(x) = \begin{cases} \frac{1}{15} & \text{if } 0 \leq x \leq 15; \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that Rebecca arrives at the bus stop and immediately receives a text message that her friends will drive by the bus stop in 10 minutes and can take her to school if the bus hasn't come by then. Otherwise, she will take the bus. What is the probability that Rebecca will ride with her friends to school?

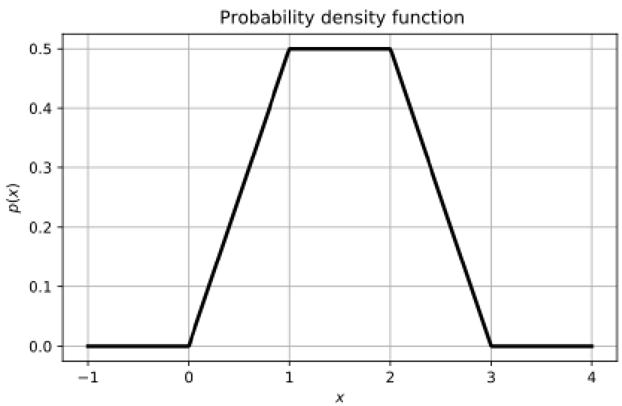

FIG. 2.

4. Suppose X is a random variable with probability density function $p(x)$ shown in Figure 2. Find $P(X < 1)$.

5. The continuous uniform distribution is characterized by the fact that all values between endpoints a and b are equally likely. Suppose X is a continuous uniform random variable. The PDF of X is given by

$$p(x) = \begin{cases} \alpha & \text{if } a \leq x \leq b; \\ 0 & \text{otherwise.} \end{cases}$$

Solve for α, and find $P(X \leq c)$, where $a \leq c \leq b$.

6. Find $Pr(0 \leq Y < 1)$ where $Y \sim N(0, 1)$.

7. Find $Pr(|W| < 0.2)$ where $W \sim N(-1, 4)$.

8. Find $Pr(-1.5 \leq Z < 0.2)$ where $Z \sim N(0, 1)$.

9. Find $Pr(Y < -5)$ where $Y \sim N(-2, 9)$.

10. Suppose Y is a Gaussian random variable. Find

$$P\left(\frac{Y-\mu}{\sigma} > -1\right)$$

where $\mu = E(Y)$ and $\sigma = sd(Y)$.

11. Suppose $X \sim \mathcal{N}(a, b)$. Find

$$P\left(-X \leq 2\sqrt{b} - a\right).$$

12. Suppose X is a Gaussian random variable. Find

$$P\Big(X \leq E(X) + \text{sd}(X)/2\Big).$$

13. Suppose X and Y are two random variables which satisfy

$$E(X) = 2, \quad \text{sd}(X) = 5 \tag{1}$$
$$E(Y) = 0, \quad \text{sd}(Y) = 2 \tag{2}$$
$$\text{Cov}(X, Y) = 1 \tag{3}$$

Which of the following random variables is the z-score for $Y - X$?

Lecture 7

1. Let $X_1, X_2, \dots, X_{25}$ denote 25 random draws (with replacement) from the following list of 5 numbers: $0, 1, 0, 0, 1$. What is $E(\overline{X})$?

2. Let $Y_1, Y_2, \dots, Y_{10}$ denote random samples, with replacement, from the following list

$$-1, -1, -1, -1, 1, 2, 2, 7.$$

What is the z-score value for the following observation

$$Y_1 + \cdots + Y_{10} = 14.8$$

3. Let $X_1, X_2, \dots, X_n$ be independent discrete random variables all with the same probability mass function. Suppose $\mathrm{Var}(X_1) = 2$ and $E(X_1) = 1$. Which of the following random variables corresponds to the Z-score of $\overline{X}$?

4. Let $X_1, X_2, \dots, X_{10}$ denote random samples, with replacement, from the following list

$$2, 2, 2, 2, 3, 4, 4, 11. \tag{4}$$

Suppose we observe that

$$X_1 + X_2 + \cdots + X_{10} = 23.93.$$

Find the z-score for this observation.

5. Suppose X_1, X_2, X_3 are three independent random variables each with expected value 0 and variance 1. Let

$$\begin{aligned} W &= X_1 + X_2 \\ V &= X_1 + X_3 \end{aligned}$$

and Z be the z-score for $W+V$. Re-write the event $\{W + V \leq 1\}$ in terms of Z.

6. Find $P(X > Y)$ where X and Y are independent random variables that satisfy $X \sim N(2, 1)$ and $Y \sim N(6, 3)$.

7. Let X be a normal random variable and that $E(X) = 2$ and $\mathrm{sd}(X) = 5.5$. What value of X corresponds to a z-score of -0.7.

8. Let X be a normal random variable and that $E(X) = -100$ and $\mathrm{sd}(X) = 5.5$. What value of X corresponds to the 90th percentile.

9. Let X be a normal random variable and that $E(X) = 1249088$ and $\mathrm{sd}(X) = 5.0124126$. What is the z-score of X that corresponds to the 40th percentile.

10. Let X be a normal random variable and that $E(X) = -5150$ and $\mathrm{sd}(X) = 10^{-50}$. Let $Z = \frac{X - E(X)}{\mathrm{sd}(X)}$ be the z-score of X. What percentile corresponds to a Z-score of 0.49.

11. Suppose $X_1, X_2, \dots, X_n$ are independent $N(\mu, \sigma^2)$ random variables. Let

$$Y = \frac{X_1 + X_2 + \cdots + X_n}{2n}.$$

Which of the following describes the random variable Y.

12. Find $\mathrm{Var}(\overline{X} - 5)$ where $X_1, \dots, X_n$ are independent random variables each with PMF given by the following table

x	$p(x)$
-1	0.2
0	0.1
1	0.7

Lecture 8

1. Suppose 13 random UCD professors walk into an elevator. The elevator has the sign: *Maximum Weight 2000 pounds*. Use the Central Limit Theorem to approximate the probability their combined weight exceeds the 2000 pound limit. You can assume the average weight of a UCD professor is 155 pounds and the standard deviation is 15 pounds.

2. Let $Y_1, Y_2, \ldots, Y_{100}$ denote random samples, with replacement, from the following list

$$-1, -1, -1, -1, 1, 2, 2, 7. \tag{5}$$

   Use the Central Limit Theorem to find

$$Pr\big(\,\overline{Y} \geq 0.675\,\big).$$

3. Consider a box of red and blue tickets. Suppose I told you that this box has 79% red tickets. Now suppose you randomly sampled, with replacement, 400 tickets from this box and got $T = 77.75$ percent red tickets.

   Use the CLT to find an approximate percentile for $T = 77.75$ under the assumption that my assertion is correct.

4. Consider a box of red and blue tickets. Suppose I told you that this box has 73% red tickets. Now suppose you randomly sampled, with replacement, 350 tickets from this box and got $T = 222$ red tickets.

   Find the z-score for the observed value of T under the assumption that my assertion is correct.