Theory of Neuronal Dynamics and Application to Machine Learning Based on Reservoir Computing
Summer 2025

Project: Forecastability of chaotic neural dynamics using reservoir computing

(25% of final your evaluation)

PROBLEM A

Biological neurons exhibit two distinct modes of activity: "spiking" and "bursting." In the spiking mode, the voltage across the neuron's membrane exhibits fast action potentials at fixed time intervals, and in bursting mode, the voltage across the membrane exhibits sequences of action potentials (bursts) separated by a relatively long time interval (slow dynamics). These modes serve distinct physiological functions, each of which has implications in computational processes, and the transitions between them are not rare. Additionally, neurons can exhibit chaotic behavior, with varying numbers of spikes per burst and irregular intervals between bursts. Despite its simplicity, the Hindmarsh-Rose (HR) mathematical neuron model replicates the diverse behaviors observed in real biological neurons, including spiking, bursting, and chaos. The dynamical system for this neuron model consists of three coupled first-order nonlinear differential equations with dimensionless variables representing the membrane potential (x), the fast sodium and potassium currents (y) responsible for spiking, and the overall transport of slow ions (z) responsible for bursting:

$$\begin{cases} \dfrac{dx}{dt} &= y + 3x^2 - x^3 - z + I, \\ \dfrac{dy}{dt} &= 1 - 5x^2 - y, \\ \dfrac{dz}{dt} &= r\Big[4\Big(x + \frac{8}{5}\Big) - z\Big]. \end{cases} \tag{S1}$$

The main control parameters of the model are the current that enters the neuron, I, and the efficiency of the slow channels to exchange ions, r. The parameters I and r can set the HR neuron in different dynamical states, including spiking, periodic bursting, and chaotic bursting. These dynamical states can be well illustrated in the bifurcation diagram.

The so-called *inter-spike intervals* (ISIs) refer to the time intervals between consecutive action potentials (spikes) in a time series of a neuron. For example, for a given value of a control parameter, the first ISI in a time series will be the time duration between the *peak* value (i.e., max value) of the first spike and the *peak* value of the second spike; the second ISI will be the time duration between the peaks of the second and third spike, and so on. Note that one can also compute the ISIs by calculating the differences between the consecutive times at which the membrane potential variable crosses a chosen threshold value. Feel free to use the method that you prefer.
ISIs are an important measure in neuroscience as they not only provide insights into neurons' firing patterns and activity, but their structure is where neurons do computations and encode the information. The duration of the ISI can vary depending on various factors, including the properties of the neuron, its intrinsic dynamics, and the input it receives from other neurons or external stimuli. By analyzing the ISI, researchers can gain information about the firing rate, regularity, and synchronization of neuronal activity, which are relevant for understanding neural coding and computing, information processing, and network dynamics in the brain.

1. By numerically integrating Eq. (S1) (using the fourth-order Runge–Kutta algorithm with initial conditions at $\big[x(t=0) = -1.0,\ y(t=0) = 2.0,\ z(t=0) = 0.5\big]$) over a duration of $T = 1500$ with a time step of $\Delta t = 0.005$ and a washed-out transient time of $T_t = 200$:

   (a) Compute a bifurcation diagram of the HR neuron model in Eq. (S1) for the control parameter I, where the vertical axis of the plot represents the logarithm of the ISI ($\log(ISI)$) of the x variable, and the horizontal axis represents the control parameter $I \in [2.5, 3.5]$, with an increment of $\Delta I = 0.005$. Fix the parameter r at $r = 0.003$. Note that you are plotting all the ISIs of the time series generated at each value of I in [2.5,3.5].

   (b) Compute a bifurcation diagram of the HR neuron model in Eq. (S1) for the control parameter r, where the vertical axis of the plot represents the logarithm of the ISIs of the x variable and the horizontal axis represents $\log(r)$, i.e., the logarithm of the control parameter $r \in [10^{-4}, 0.05]$, with increments of $\Delta r = 6.0 \times 10^{-5}$ when $r \in [10^{-4}, 10^{-3}]$, $\Delta r = 4.5 \times 10^{-5}$ when $r \in [10^{-3}, 10^{-2}]$, and $\Delta r = 2.667 \times 10^{-4}$ when $r \in [10^{-2}, 0.05]$. Fix the parameter I at $I = 3.25$.

   You can assume that a spike occurs whenever the $x(t)$ variable crosses, *from below*, the threshold value of $x_{th} = 1.0$. Note that bifurcation diagrams are plotted with dots (no lines should be connecting these dots).

2. By numerically integrating Eq. (S1) (using the fourth-order Runge–Kutta algorithm with initial conditions at $[x(t=0)=-1.0,\ y(t=0)=2.0,\ z(t=0)=0.5]$) over a duration of $T=1500$ with a time step of $\Delta t=0.005$, check whether (or not) the time series (which should be shown for $t \in [0,1500]$) of the membrane potential variable $x(t)$ will exhibit the following dynamical behaviors at the indicated value of the control parameters.

    (a) Periodic firings of spikes when $I=3.5$ and $r=0.003$.

    (b) Chaotic firings of spikes when $I=3.34$ and $r=0.003$.

    (c) Periodic firings of well-defined bursts of spikes (3 spikes per burst) when $I=1.67$ and $r=0.003$.

    (d) Periodic firings of well-defined bursts of spikes (9 spikes per burst) when $I=3.2$ and $r=0.003$.

    (e) Chaotic firings of bursts of spikes when $I=3.29$ and $r=0.003$.

PROBLEM B

For an Echo State Network (ESN) implementation, start with a reservoir network of $N_{\mathbf{res}}=300$ neurons, where the elements of the input-reservoir coupling matrix $W_{\mathbf{in}}$ are obtained from a random uniform distribution with values between -0.5 and 0.5, a leaky coefficient of $\alpha= 0.5$, biases of $b=0$, and regularization coefficient $\lambda= 10^{-6}$. The adjacency matrix $W_{\mathbf{res}}$ that determines the topology of the reservoir network consists of *Erdős-Rényi* random graph (network) that allows for self-loops and has link probability $p=0.75$ (remember the package **networkx** in Python). The elements of the weighted adjacency matrix $W_{\mathbf{res}}$ are adjusted such that it has a spectral radius of $\rho=0.85$.

The hyperparameters of this ESN include: $N_{\mathbf{res}} \in [300,1000]$, $0<p\leq 1$, $\alpha \in (0,1]$, $\rho \in (0,1.5]$, and $N_{\mathbf{warmup}} \geq 40001$.

Train this ESN to predict the future behavior of the HR neuron in Eq. (S1) in the dynamical states displayed in Problem A 2(a), 2(b), 2(c), and 2(e), after a washed-out transient time of $T_t=200$ (i.e., the first 40000 data points). Before training, it might be necessary (it is up to you to check whether it's necessary or not) that the input data $u(t)=(x(t))$ from the time series in Problem A 2(a), 2(b), 2(c), and 2(e) *be normalized* to have a mean 0 and standard deviation 1. This normalization process could ensure that the input and output signals have a consistent scale and help in the training process of the readout weight $W_{\mathbf{out}}$ using the Ridge Regression with Tikhonov regularization.

In each of the time series in Problem A 2(a), 2(b), 2(c), and 2(e) without the transient time of T_t, use 50% of the rest of the data points as the input signal $u(t)(=x(t))$ for the training phase of the reservoir.

The results you should show include the following:

1. On the vertical axis, plots the normalized root mean square error in the training phase (denoted by NRMSE_T) against each hyperparameter $N_{\mathbf{res}} \in [300,1000]$, $p \in (0,1]$ $\alpha \in (0,1]$, and $\rho \in (0,1.5]$ on the horizontal axis, for the time series in

    (a) Problem A 2(a).

    (b) Problem A 2(b).

    (c) Problem A 2(c).

    (d) Problem A 2(e).

    For the plots in Problem B 1(a), 1(b), 1(c), and 1(d), take ten equidistant values in each hyperparameter interval, including the boundary values, if possible.

2. With an optimized value of each of the hyperparameters found in Problem B 1(a), 1(b), 1(c), and 1(d), plot the normalized root mean square error in the prediction phase (denoted by NRMSE_P) against the warm-up time (denoted by t_{warmup}) where $t_{\text{warmup}}=\Delta t \cdot N_{\mathbf{warmup}}$ and $40001 \leq N_{\mathbf{warmup}} \leq 300000$, for the time series in

    (a) Problem A 2(a).

    (b) Problem A 2(b).

    (c) Problem A 2(c).

    (d) Problem A 2(e).

3. With the same hyperparameter values used in Problem B 2, plot on the <u>same</u> graph the time series of the dynamical-system generated (i.e., from Eq. (S1)) membrane potential variable $x(t)$ (in blue and for $t \in [0, 1500]$) and the ESN-generated membrane potential variable $x_r(t)$ (in red and for $t \in (200, 1500]$) for two different warm-up times T_1 and T_2 selected from Problem B 2: the first warm-up time T_1 should correspond to when the NRMSE_P is the highest, and the second warm-up time T_2 should correspond to any other time after which the high value of NRMSE_P of the first warm-up time shows (if at all) a significant drop in value (for example, you may try a second warm-up time of $T_2 = 200000$. *This is just a suggestion — you can try any other T_2 and see how well your model performs with that.*).

   On the first time series of $x(t)$ and $x_r(t)$ indicate the first warm-up time with a vertical solid line, and on the second time series of $x(t)$ and $x_r(t)$ indicate second warm-up time with a vertical dashed line [*So you have two time series to plots: one in which you show $x(t)$ and $x_r(t)$ with first warm-up time and another one in which you show $x(t)$ and $x_r(t)$ with second warm-up time*]. Do this for the time series of the dynamical-system generated membrane potential variable $x(t)$ in

   (a) Problem A 2(a).
   (b) Problem A 2(b).
   (c) Problem A 2(c).
   (d) Problem A 2(e).

4. With the values of the optimized hyperparameters used in Problem B 2, and a good second warm-up time obtained in Problem B 3, implement an ESN to predict the time series of the membrane potential variable $x(t)$ for $t \in (200, 1500]$ for values of the control parameter I in the range [2.5,3.5] with step size $\Delta I = 0.005$ when $r = 0.003$. For each of these values of I, compute the ISIs of the predicted time series and plot them ($\log(ISI)$) against the control parameter $I \in [2.5, 3.5]$. Compare the bifurcation diagram obtained in Problem A 1(a) to the one obtained in Problem B 4.

END !

NOTE:

1. There are slightly different ESN implementation methods. The tasks in this project are consistent with the ESN implementation description in Lectures 15 and 16. Carefully read and understand Lectures 15 and 16. Understanding how to correctly use the N_{warmup} in the prediction phase is crucial. Please do not hesitate to reach out to clarify any doubts/questions as you work through the project. It is acceptable to use other implementations of ENS in the project.

2. Your submission should consist of:

   (a) A PDF in which: (1) the first page presents the FULL names, StudonID, matricule number of **only** the students that actively participated in the project (being alone in a group, if you prefer, is acceptable, but I strongly recommend collaboration with some of your classmates, it is a scientific skill that also needs to be developed). However, a maximum of 7 students per group is allowed. (2) The rest of the pages show only the output (plots) of all the required graphs and short paragraphs of text explaining every plots. Please remember to label the figures. Which figure corresponds to which questions of the project? This should be clear.
   (b) The codes that you use to generate each of the outputs (figures) of the projects. Please also label these codes accordingly. For example, a Python code used to generate the bifurcation diagram of Problem A 1(a) should be labeled "Prob-A-1a.py", and so on.
   (c) Submission (via StudOn) Deadline: Sunday, 31st August 2025, at 6:00 PM

Best Wishes!
Marius Yamakou