(9) Prove the sum of $1+2+3+\ldots+n = n(n+1)/2$ using mathematical induction.

Proposition. For each $n \in \mathbb{N}, 1+2+3+\ldots+n = \frac{n(n+1)}{2}$.

Proof. We will use mathematical induction to prove the proposition. For each $n \in \mathbb{N}$, we let $P(n)$ be

$$1+2+3+\ldots+n = \frac{n(n+1)}{2}.$$

We begin with the basis step. That is, we prove that $P(1)$ is true. Notice that $1 = \frac{1\cdot(1+1)}{2}$. Therefore, $P(1)$ is true and we have proved the basis step.

We next prove the inductive step. That is, for each $k \in \mathbb{N}$ if $P(k)$ is true, then $P(k+1)$ is true. We assume that $P(k)$ is

$$1+2+3+\ldots+k = \frac{k(k+1)}{2}. \tag{1}$$

We assume that $P(k+1)$ is

$$\begin{aligned} 1+2+3+\ldots+k+(k+1) &= \frac{(k+1)((k+1)+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned} \tag{2}$$

Notice that $P(k+1) = P(k) + (k+1)$. So, if we add $(k+1)$ to $P(k)$, we will show the algebraic expression for $P(k+1)$. So,

$$\begin{aligned} P(k)+(k+1) &= \frac{k(k+1)}{2} + k + 1 \\ &= \frac{k^2+k+2k+2}{2} \\ &= \frac{k^2+3k+2}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

Comparing the result to equation (2), we have shown that if $P(k)$ is true, then $P(k+1)$ is true. Consequently, we have proved the inductive step. Hence, we have proved the basis step and the inductive step. Therefore, we have proved the proposition by the Principle of Mathematical Induction. ∎