(4) Do problem Sundstrom 3.3.20

20. Evaluation of proofs

See the instructions for Exercise (19) on page 100 from Section 3.1.

(a) **Proposition.** For each real number x, if x is irrational and m is an integer, then mx is irrational.

Proof. We assume that x is a real number and is irrational. This means that for all integers a and b with $b \neq 0$, $x \neq \frac{a}{b}$. Hence, we conclude that $mx \neq \frac{ma}{b}$ and therefore, mx is irrational. ■

(b) **Proposition.** For all real numbers x and y, if x is irrational and y is rational, then $x + y$ is irrational.

Proof. We will use a proof by contradiction. So we assume that the proposition is false, which means that there exist real numbers x and y where $x \notin \mathbb{Q}$, $y \in \mathbb{Q}$, and $x + y \in \mathbb{Q}$. Since the rational numbers are closed under subtraction and $x + y$ and y are rational, we see that

$$(x + y) - y \in \mathbb{Q}$$

However, $(x + y) - y = x$, and hence we can conclude that $x \in \mathbb{Q}$. This is a contradiction to the assumption that $x \notin \mathbb{Q}$. Therefore, the proposition is not false, and we have proven that for all real numbers x and y, if x is irrational and y is rational, then $x + y$ is irrational. ■

(c) **Proposition.** For each real number x, $x(1 - x) \leq \frac{1}{4}$.

Proof. A proof by contradiction will be used. So we assume that the proposition is false. That means there exists a real number x such that $x(1 - x) > \frac{1}{4}$. If we multiply both sides of the inequality by 4, we obtain $4x(1 - x) > 1$. However, if we let $x = 3$, we see that

$$\begin{aligned} 4x(1 - x) &> 1 \\ 4 \cdot 3(1 - 3) &> 1 \\ -12 &> 1 \end{aligned}$$

The last inequality is clearly a contradiction and so we have proved the proposition. ■

(a) This is a false proposition and consequently an incorrect proof. To show this, consider the case when $m = 0$. That is, when $mx = 0 \cdot x = 0$. Since 0 can be written in the form of $0 = \frac{a}{b}$ such that $b \neq 0$, it is considered rational. For all cases, so long as $a = 0$ and $b \neq 0$, the resultant number is 0. Therefore, the proposition is disproved.

The proof is at fault by not considering the case where $m = 0$. While the algebra in the proof is true for all $m \in \mathbb{Z}$ such that $m \neq 0$, the proposition includes the case $m = 0$ and consequently, the algebra does not prove the proposition.

(b) This is a well written and correct proof.

(c) This is a well written and correct proof.