3. Do CLRS 4-1 a, b, c. Show the solution using ***both*** the Master Theorem *and* proof by substitution/induction.

4-1 Recurrence examples

Give asymptotic upper and lower bounds for $T(n)$ in each of the following recurrences. Assume that $T(n)$ is constant for $n \leq 2$. Make your bounds as tight as possible, and justify your answers.

a. $T(n) = 2T(n/2) + n^4$.

We make our guess by substitution over k iterations,

$$\begin{aligned}
T(n) &= 2T(n/2) + n^4 \\
&= 2[2T(n/2^2) + (n/2)^4] + n^4 \\
&= 2^2T(n/2^2) + \frac{2n^4}{2^4} + n^4 \\
&= 2^2[2T(n/2^3) + (n/2^2)^4] + \frac{2n^4}{2^4} + n^4 \\
&= 2^3T(n/2^3) + \frac{2^2n^4}{2^{2\cdot4}} + \frac{2n^4}{2^4} + n^4 \\
&\vdots \\
&= 2^kT(n/2^k) + n^4\sum_{i=0}^{k-1}\frac{2^i}{2^{4i}} \\
&= 2^kT(n/2^k) + n^4\sum_{i=0}^{k-1}(\frac{2}{2^4})^i \\
&= 2^kT(n/2^k) + n^4\sum_{i=0}^{k-1}(\frac{1}{8})^i \\
&< 2^kT(n/2^k) + n^4\sum_{i=0}^{\infty}(\frac{1}{8})^i
\end{aligned}$$

Assuming $n = 2^k$ and $T(1) = c$,

$$\begin{aligned}
T(n) &< 2^{\lg n}T(1) + n^4(\frac{1}{1-\frac{1}{8}}) \\
&< 2^{\lg n}T(1) + n^4(\frac{1}{7/8}) \\
&< 2^{\lg n}T(1) + n^4(\frac{8}{7}) \\
&< cn + \frac{8}{7}n^4 \\
&< dn^4 \\
&= O(n^4)
\end{aligned}$$

Hence, we choose $T(n) = \Theta(n^4)$ as our guess.

We will first prove by the Master Theorem.

Conjecture. $T(n) = 2T(n/2) + n^4 = \Theta(n^4)$.

Proof. We assume that $a = 2, b = 2, f(n) = n^4$, and $\log_b a = \lg 2 = 1$. We note that $f(n) = \Omega(n^4)$ and that if we let $\epsilon = 14$, then $\Omega(n^{\log_b a+\epsilon}) = \Omega(n^{\lg 16}) = \Omega(n^4)$. Since $f(n) = \Omega(n^{\log_b a+\epsilon})$ for $\epsilon = 14 > 0$ and since $af(n/b) = 2(n/2)^4 = 2n^4/16 = n^4/8 \leq cn^4$ for some constants $c < 1$ and all sufficiently large n, then $T(n) = \Theta(n^4)$. Hence, we have proven the conjecture by the Master Theorem. ■

We will next prove by substitution and the Principle of Mathematical Induction.

Conjecture. $T(n) = 2T(n/2) + n^4 = \Theta(n^4)$.

Proof. We will first prove that $T(n) = O(n^4)$. That is, we will prove that $T(n) \leq dn^4$, where $d = [(c/8) + 1]$, by the Principle of Mathematical Induction.

We begin with the basis case, $n = 1$. We let $T(1) = c'$ such that $c' \leq d$. Hence, $T(1) \leq d$ and consequently, the basis case has been proven to be true.

We next prove the inductive step by strong induction. That is, we assume that $T(j) \leq dj^4$ for $j < k$ and prove that $T(k) \leq dk^4$. Then,

$$\begin{aligned} T(k) = 2T(k/2) + k^4 &\leq 2c(k/2)^4 + k^4 \\ &\leq (c/8)k^4 + k^4 \\ &\leq [(c/8) + 1]k^4 \\ &\leq dk^4 \end{aligned}$$

Then, $T(k) \leq dk^4$, where $d = [(c/8) + 1]$, and consequently, have have proven that $T(n) = O(n^4)$.

We will next prove that $T(n) = \Omega(n^4)$. That is, we will prove that $T(n) \geq dn^4$, where $d = [(c/8) + 1]$, by the Principle of Mathematical Induction.

We begin with the basis case, $n = 1$. We let $T(1) = c'$ such that $c' \geq d$. Hence, $T(1) \geq d$ and consequently, the basis case has been proven to be true.

We next prove the inductive step by strong induction. That is, we assume that $T(j) \geq dj^4$ for $j < k$

and prove that $T(k) \geq dk^4$. Then,

$$\begin{aligned} T(k) = 2T(k/2) + k^4 &\geq 2c(k/2)^4 + k^4 \\ &\geq (c/8)k^4 + k^4 \\ &\geq [(c/8) + 1]k^4 \\ &\geq dk^4 \end{aligned}$$

Then, $T(k) \geq dk^4$, where $d = [(c/8) + 1]$, and consequently, have have proven that $T(n) = \Omega(n^4)$.

Since $T(n) = O(n^4)$ and $T(n) = \Omega(n^4)$, then $T(n) = \Theta(n^4)$ and consequently, we have proven the conjecture to be true. ■

b. $T(n) = T(7n/10) + n$.

We make our guess by substitution over k iterations,

$$\begin{aligned} T(n) &= T(7n/10) + n \\ &= [T(7^2n/10^2) + 7n/10] + n \\ &= [T(7^3n/10^3) + 7^2n/10^2] + 7n/10 + n \\ &\vdots \\ &= T(7^kn/10^k) + n\sum_{i=0}^{k-1} \frac{7^i}{10^i} \\ &= T(7^kn/10^k) + n\sum_{i=0}^{k-1} (\frac{7}{10})^i \\ &< T(7^kn/10^k) + n\sum_{i=0}^{\infty} (\frac{7}{10})^i \end{aligned}$$

Assuming that $n = (\frac{10}{7})^k$ and $T(1) = c$,

$$\begin{aligned} T(n) &< T(1) + n(\frac{1}{1 - \frac{7}{10}}) \\ &< c + \frac{10}{3}n \\ &< dn \\ &= O(n) \end{aligned}$$

Hence, we choose $T(n) = \Theta(n)$ as our guess.

We will first prove by the Master Theorem.

Conjecture. $T(n) = T(7n/10) + n = \Theta(n)$.

Proof. We assume that $a = 1, b = 10/7, f(n) = n$, and $\log_b a = \log_{10/7} 1 = 0$. Note that $f(n) = \Omega(n)$

and for $\epsilon = 3/7$, $\Omega(n^{\log_b a + \epsilon}) = \Omega(n^{\log_{10/7} 1 + 3/7}) = \Omega(n^{\log_{10/7} 10/7}) = \Omega(n)$. Since $f(n) = \Omega(n^{\log_b a + \epsilon})$ and $7n/10 \leq cn$ for some constants $c < 1$ (for example, $c = 8/10$), then $T(n) = \Theta(n)$. Hence, we have proven the conjecture by the Master Theorem. ∎

We will next prove by substitution and the Principle of Mathematical Induction.

Conjecture. $T(n) = T(7n/10) + n = \Theta(n)$.

Proof. We will first prove that $T(n) = O(n)$. That is, we will prove that $T(n) \leq dn$, where $d = [(7c/10) + 1]$, by the Principle of Mathematical Induction.

We begin with the basis case, $n = 1$. We let $T(1) = c'$ such that $c' \leq d$. Hence, $T(1) \leq d$ and consequently, the basis case has been proven to be true.

We next prove the inductive step by strong induction. That is, we assume that $T(j) \leq dj$ for $j < k$ and prove that $T(k) \leq dk$. Then,

$$\begin{aligned} T(k) = T(7k/10) + k &\leq c(7k/10) + k \\ &\leq (7c/10)k + k \\ &\leq [(7c/10) + 1]k \\ &\leq dk \end{aligned}$$

Then, $T(k) \leq dk$, where $d = [(7c/10) + 1]$, and consequently, have have proven that $T(n) = O(n)$.

We will next prove that $T(n) = \Omega(n)$. That is, we will prove that $T(n) \geq dn$, where $d = [(7c/10) + 1]$, by the Principle of Mathematical Induction.

We begin with the basis case, $n = 1$. We let $T(1) = c'$ such that $c' \geq d$. Hence, $T(1) \geq d$ and consequently, the basis case has been proven to be true.

We next prove the inductive step by strong induction. That is, we assume that $T(j) \geq cj$ for $j < k$ and prove that $T(k) \geq dk$. Then,

$$\begin{aligned} T(k) = T(7k/10) + k &\geq c(7k/10) + k \\ &\geq (7c/10)k + k \\ &\geq [(7c/10) + 1]k \\ &\geq dk \end{aligned}$$

Then, $T(k) \geq dk$, where $d = [(7c/10) + 1]$, and consequently, have have proven that $T(n) = \Omega(n)$.

Since $T(n) = O(n)$ and $T(n) = \Omega(n)$, then $T(n) = \Theta(n)$ and consequently, we have proven the conjecture to be true. $\blacksquare$

c. $T(n) = 16T(n/4) + n^2$.

We make our guess by substitution over k iterations,

$$\begin{aligned}
T(n) &= 16T(n/4) + n^2 \\
&= 16[16T(n/4^2) + (n/4)^2] + n^2 \\
&= 16^2T(n/4^2) + 16n^2/16 + n^2 \\
&= 16^2T(n/4^2) + 2n^2 \\
&= 16^2[16T(n/4^3) + (n/4^2)^2] + 2n^2 \\
&= 16^3T(n/4^3) + 16^2n^2/16^2 + 2n^2 \\
&= 16^3T(n/4^3) + 3n^2 \\
&\vdots \\
&= 16^kT(n/4^k) + kn^2
\end{aligned}$$

Assuming that $n = 4^k$ and $T(1) = c$,

$$\begin{aligned}
T(n) &= 16^{\log_4 n}T(1) + (\log_4 n)n^2 \\
&= cn^{\log_4 16} + n^2\log_4 \\
&< dn^2\log_4 n \\
&= O(n^2\log n)
\end{aligned}$$

Hence, we choose $T(n) = \Theta(n^2\log n)$ as our guess.

We will first prove by the Master Theorem.

Conjecture. $T(n) = 16T(n/4) + n^2 = \Theta(n^2\log n)$.

Proof. We assume that $a = 16, b = 4, f(n) = n^2$, and $\log_b a = \log_4 16 = 2$. Note that $f(n) = \Theta(n^2)$ and that $\Theta(n^{\log_b a}) = \Theta(n^2)$. Since $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^2\log n)$. Hence, we have proven the conjecture to be true by the Master Theorem. $\blacksquare$

We will next prove by substitution and the Principle of Mathematical Induction.

Conjecture. $T(n) = 16T(n/4) + n^2 = \Theta(n^2\log n)$.

Proof. We will first prove that $T(n) = O(n^2\log n)$. That is, we will prove that $T(n) \leq cn^2\log_4 n$ by the Principle of Mathematical Induction.

We begin with the basis case, $n = 2$. We let $T(2) = c'$ such that $c' \leq 2c$ (that is, $c = 4c(1/2)$). Hence, $T(2) \leq c$ and consequently, the basis case has been proven to be true.

We next prove the inductive step by strong induction. That is, we assume that $T(j) \leq cj^2 \log_4 j$ for $j < k$ and prove that $T(k) \leq ck^2 \log_4 k$. Then,

$$\begin{aligned} T(k) = 16T(k/4) + k^2 &\leq 16c(k/4)^2 \log_4 (k/4) + k^2 \\ &\leq 16c(k^2/16) \log_4 (k/4) + k^2 \\ &\leq ck^2 \log_4 (k/4) + k^2 \\ &\leq ck^2 \log_4 k - ck^2 + k^2 \\ &\leq ck^2 \log_4 k \end{aligned}$$

for $c \geq 1$. Then, $T(k) \leq ck^2 \log_4 k$ and $T(n) = O(n^2 \log n)$ by the Principle of Mathematical Induction.

We will next prove that $T(n) = \Omega(n^2 \log n)$. That is, we will prove that $T(n) \geq cn^2 \log_4 n$ by the Principle of Mathematical Induction.

We begin with the basis case, $n = 2$. We let $T(2) = c'$ such that $c' \geq 2c$ (that is, $c = 4c(1/2)$). Hence, $T(2) \geq c$ and consequently, the basis case has been proven to be true.

We next prove the inductive step by strong induction. That is, we assume that $T(j) \geq cj^2 \log_4 j$ for $j < k$ and prove that $T(k) \geq ck^2 \log_4 k$. Then,

$$\begin{aligned} T(k) = 16T(k/4) + k^2 &\geq 16c(k/4)^2 \log_4 (k/4) + k^2 \\ &\geq 16c(k^2/16) \log_4 (k/4) + k^2 \\ &\geq ck^2 \log_4 (k/4) + k^2 \\ &\geq ck^2 \log_4 k - ck^2 + k^2 \\ &\geq ck^2 \log_4 k \end{aligned}$$

for $c \leq 1$. Then, $T(k) \geq ck^2 \log_4 k$ and $T(n) = \Omega(n^2 \log n)$ by the Principle of Mathematical Induction.

Since $T(n) = O(n^2 \log n)$ and $T(n) = \Omega(n^2 \log n)$, then $T(n) = \Theta(n^2 \log n)$ and consequently, we have proven the conjecture to be true. ■