(5) Do problem Sunstrom 4.2.12 (Use mathematical induction...!)

***12.** Prove that for each natural number n, any set with n elements has $\frac{n(n-1)}{2}$ two-element subsets.

Proposition. For each natural number n, any set with n elements has $\frac{n(n-1)}{2}$ two-element subsets.
Proof. We will prove the proposition by mathematical induction. For each $n \in \mathbb{N}$, we let $P(n)$ be

Any set with n elements has $\frac{n(n-1)}{2}$ two-element subsets.

The first step of the proof is to prove that $P(1)$ is true. Notice that $\frac{1(1-1)}{2}$ is zero subsets and in a set of only one element, there will be zero two-element subsets. This shows that $P(1)$ is true.

The inductive step of the proof will prove that if $P(k)$ is true for each $k \in \mathbb{N}$, then $P(k+1)$ is true. So we assume that $P(k)$ is true, that is

$$P(k) = \frac{k(k-1)}{2} \tag{1}$$

We now prove that $P(k+1)$ is true. That is, we prove

$$\begin{aligned} P(k+1) &= \frac{(k+1)(k+1-1)}{2} \\ &= \frac{k(k+1)}{2} \end{aligned} \tag{2}$$

The sequence of two-element subsets for n is

$$0, 1, 3, 6, 10, 15, ..., \tfrac{n(n-1)}{2}, \tfrac{(n+1)(n)}{2}$$

You may notice that for term n, $P(n) = P(n-1) + (n-1)$. Therefore, $P(k+1) = P(k) + k$. To show this, we add k to equation (1). That is,

$$\begin{aligned} P(k) + k &= \frac{k(k-1)}{2} + k \\ &= \frac{k^2 - k + 2k}{2} \\ &= \frac{k^2 + k}{2} \\ &= \frac{k(k+1)}{2} \end{aligned}$$

Using algebra, we have shown that this result is equal to equation (2). That is, we have proved that if $P(k)$ is true, then $P(k+1)$ is true. Therefore, we have proved the inductive step and consequently, we have proved by the Principle of Mathematical Induction that for each natural number n, any set with n elements has $\frac{n(n-1)}{2}$ two-element subsets. ■