(8) Sundstrom 4.3.21

Evaluation of proofs See the instructions for Exercise (19) on page 100 from Section 3.1.

(**a**) Let f_n be the nth Fibonacci number, and let α be the positive solution of the equation $x^2 = x + 1$. So $\alpha = \frac{1+\sqrt{5}}{2}$. For each natural number n, $f_n \leq \alpha^{n-1}$.

Proof. We will use a proof by mathematical induction. For each natural number n, we let $P(n)$ be, "$f_n \leq \alpha^{n-1}$."

We first note that $P(1)$ is true since $f_1 = 1$ and $\alpha^0 = 1$. We also notice that $P(2)$ is true since $f_2 = 1$ and, hence, $f_2 \leq \alpha^1$.

We now let k be a natural number with $k \geq 2$ and assume that $P(1), P(2), \ldots, P(k)$ are all true. We now need to prove that $P(k+1)$ is true or that $f_{k+1} \leq \alpha^k$.

Since $P(k-1)$ and $P(k)$ are true, we know that $f_{k-1} \leq \alpha^{k-2}$ and $f_k \leq \alpha^{k-1}$. Therefore,

$$
\begin{aligned}
f_{k+1} &= f_k + f_{k-1} \\
f_{k+1} &\leq \alpha^{k-1} + \alpha^{k-2} \\
f_{k+1} &\leq \alpha^{k-2}(\alpha + 1).
\end{aligned}
$$

We now use the fact that $\alpha + 1 = \alpha^2$ and the preceding inequality to obtain

$$
\begin{aligned}
f_{k+1} &\leq \alpha^{k-2}\alpha^2 \\
f_{k+1} &\leq \alpha^k.
\end{aligned}
$$

This proves that if $P(1), P(2), \ldots, P(k)$ are true, then $P(k+1)$ is true. Hence, by the Second Principle of Mathematical Induction, we conclude that for each natural number n, $f_n \leq \alpha^{n-1}$. ■

(a) This is a well written and correct proof.