(6) Do problem Sundstrom 4.2.13 (Use mathematical induction...!)

13. Prove or disprove each of the following propositions.

(**a**) For each $n \in \mathbb{N}$, $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1}$.

(**b**) For each natural number n with $n \geq 3$, $\frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + ... + \frac{1}{n(n+1)} = \frac{n-2}{3n+3}$.

(**c**) For each $n \in \mathbb{N}$, $1\cdot 2 + 2\cdot 3 + 3\cdot 4 + ... + n(n+1) = \frac{n(n+1)(n+2)}{3}$.

(a) We will prove the proposition.

Proposition. For each $n \in \mathbb{N}$, $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1}$.

Proof. We will prove the proposition by using the Principle of Mathematical Induction. For each $n \in \mathbb{N}$, we let $P(n)$ be

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

We will first prove the basis step. That is, we will prove $P(1)$. Since

$$\frac{1}{1\cdot 2} = \frac{1}{(1+1)},$$

we have proven that $P(1)$ is true.

We will next prove the inductive step. That is, we prove that for each $k \in \mathbb{N}$ if $P(k)$ is true, then $P(k+1)$ is true. We assume

$$P(k) = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + ... + \frac{1}{k(k+1)} = \frac{k}{k+1} \tag{1}$$

$$P(k+1) = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + ... + \frac{1}{k(k+1)} + \frac{1}{(k+1)((k+1)+1)} = \frac{k+1}{(k+1)+1} \tag{2}$$

Note that $P(k+1) = P(k) + \frac{1}{(k+1)((k+1)+1)}$. So to prove the inductive step, we will prove that by adding $\frac{1}{(k+1)((k+1)+1)}$ to $P(k)$, it will equate to $P(k+1)$. That is,

$$\begin{aligned} P(k) + \frac{1}{(k+1)((k+1)+1)} &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\ &= \frac{k(k+2)+1}{(k+1)(k+2)} \\ &= \frac{k^2+2k+1}{(k+1)(k+2)} \\ &= \frac{(k+1)^2}{(k+1)(k+2)} \\ &= \frac{k+1}{k+2} \\ &= \frac{k+1}{(k+1)+1} \end{aligned}$$

Since the algebra results to equation (1), we have proved the inductive step by showing that if $P(k)$ is true, then $P(k+1)$ is true. By proving the basis step and the inductive step, we have proven the proposition by the Principle of Mathematical Induction. ■

(b) We will prove the proposition.

Proposition. For each natural number n with $n \geq 3$, $\frac{1}{3\cdot4} + \frac{1}{4\cdot5} + ... + \frac{1}{n(n+1)} = \frac{n-2}{3n+3}$.

Proof. We will prove the proposition by using the Principle of Mathematical Induction. For each $n \in \mathbb{N}$ such that $n \geq 3$, we let $P(n)$ be

$$\frac{1}{3\cdot4} + \frac{1}{4\cdot5} + ... + \frac{1}{n(n+1)} = \frac{n-2}{3n+3}.$$

We begin the proof with the basis step. That is, we will prove $P(3)$. Letting $n = 3$, we obtain

$$\begin{aligned}\frac{3-2}{3(3)+3} &= \frac{1}{12} \\ &= \frac{1}{3\cdot4}\end{aligned}$$

This shows that

$$\frac{1}{3\cdot4} = \frac{3-2}{3(3)+3}$$

which proves that $P(3)$ is true.

We will next prove the inductive step. That is, we prove that for each $k \in \mathbb{N}$ such that $k \geq 3$ if $P(k)$ is true, then $P(k+1)$ is true. We assume

$$P(k) = \frac{1}{3\cdot4} + \frac{1}{4\cdot5} + ... + \frac{1}{k(k+1)} = \frac{k-2}{3k+3} \tag{3}$$

$$P(k+1) = \frac{1}{3\cdot4} + \frac{1}{4\cdot5} + ... + \frac{1}{k(k+1)} + \frac{1}{(k+1)((k+1)+1)} = \frac{(k+1)-2}{3(k+1)+3} \tag{4}$$

$$= \frac{k-1}{3k+6} = \frac{k-1}{3(k+2)}$$

Note that $P(k+1) = P(k) + \frac{1}{(k+1)((k+1)+1)}$. So to prove the inductive step, we will show that by adding

$\frac{1}{(k+1)((k+1)+1)}$ to $P(k)$, it will equate to $P(k+1)$. That is,

$$\begin{aligned}
P(k) + \frac{1}{(k+1)((k+1)+1)} &= \frac{k-2}{3k+3} + \frac{1}{(k+1)((k+1)+1)} \\
&= \frac{k-2}{3k+3} + \frac{1}{(k+1)(k+2)} \\
&= \frac{(k-2)(k+2)+3}{(3k+3)(k+2)} \\
&= \frac{k^2-4+3}{(3k+3)(k+2)} \\
&= \frac{k^2-1}{(3k+3)(k+2)} \\
&= \frac{(k+1)(k-1)}{3(k+1)(k+2)} \\
&= \frac{k-1}{3(k+2)}
\end{aligned}$$

Comparing the result to equation (4), we can see that the result has equated to $P(k+1)$. Therefore, we have proved the inductive step by showing that if $P(k)$ is true, then $P(k+1)$ is true. By proving the basis step and the inductive step, we have proven the proposition by the Principle of Mathematical Induction. ■

(c) We will prove the proposition.

Proposition. For each $n \in \mathbb{N}$, $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + ... + n(n+1) = \frac{n(n+1)(n+2)}{3}$.

Proof. We will prove the proposition by using the Principle of Mathematical Induction. For each $n \in \mathbb{N}$, we let $P(n)$ be

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + ... + n(n+1) = \frac{n(n+1)(n+2)}{3}.$$

We begin the proof with the basis step. That is, we will prove $P(1)$. Note that $1 \cdot 2$ is 2. Letting $n = 1$, we obtain

$$\begin{aligned}
\frac{1 \cdot (1+1)(1+2)}{3} &= \frac{2 \cdot 3}{3} \\
&= 2.
\end{aligned}$$

This shows that

$$1 \cdot 2 = \frac{1 \cdot (1+1)(1+2)}{3}$$

and consequently, we have proved $P(1)$.

We will next prove the inductive step. That is, we will prove that for each $k \in \mathbb{N}$ if $P(k)$ is true, then

$P(k+1)$ is true. We assume that

$$P(k) = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + ... + k(k+1) = \frac{k(k+1)(k+2)}{3} \tag{5}$$

$$P(k+1) = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + ... + k(k+1) + (k+1)((k+1)+1) = \frac{(k+1)((k+1)+1)((k+1)+2)}{3} \tag{6}$$

Note that $P(k+1) = P(k) + (k+1)(k+2)$. So to prove the inductive step, we will prove that by adding $(k+1)(k+2)$ to $P(k)$, it will equate to $P(k+1)$. That is,

$$\begin{aligned} P(k) + (k+1)(k+2) &= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \\ &= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3} \\ &= \frac{(k+1)(k+2)(k+3)}{3} \\ &= \frac{(k+1)((k+1)+1)((k+1)+2)}{3} \end{aligned}$$

After algebra, the result is the same as equation (6). Therefore, we have proved the inductive step by showing that if $P(k)$ is true, then $P(k+1)$ is true. By proving the basis step and the inductive step, we have proven the proposition by the Principle of Mathematical Induction. ■