(7) Do problem Sundstrom 4.2.17

Evaluation of proofs

See the instructions for Exercise (19) on page 100 from Section 3.1.

(**a**) For each natural number n with $n \geq 2, 2^n > 1 + n$.

Proof. We let k be a natural number and assume that $2^k > 1 + k$. Multiplying both sides of this inequality by 2, we see that $2^{k+1} > 2 + 2k$. However, $2 + 2k > 2 + k$ and, hence,

$$2^{k+1} > 1 + (k + 1)$$

By mathematical induction, we conclude that $2^n > 1 + n$. ■

(**b**) Each natural number greater than or equal to 6 can be written as the sum of natural numbers, each of which is a 2 or a 5.

Proof. We will use a proof by induction. For each natural number n, we let $P(n)$ be, "There exist non-negative integers x and y such that $n = 2x + 5y$." Since

$$6 = 3 \cdot 2 + 0 \cdot 5 \qquad 7 = 2 + 5$$
$$8 = 4 \cdot 2 + 0 \cdot 5 \qquad 9 = 2 \cdot 2 + 1 \cdot 5$$

we see that $P(6)$, $P(7)$, $P(8)$, and $P(9)$ are true.

We now suppose that for some natural number k with $k \geq 10$ that $P(6)$, $P(7), \ldots P(k)$ are true. Now

$$k + 1 = (k - 4) + 5$$

Since $k \geq 10$, we see that $k - 4 \geq 6$ and, hence, $P(k - 4)$ is true. So $k - 4 = 2x + 5y$ and, hence,

$$k - 1 = (k - 4) + 5$$

Since $k \geq 10$, we see that $k - 4 \geq 6$ and, hence, $P(k - 4)$ is true. So $k - 4 = 2x + 5y$ and, hence,

$$\begin{aligned} k + 1 &= (2x + 5y) + 5 \\ &= 2x + 5(y + 1) \end{aligned}$$

This proves that $P(k + 1)$ is true, and hence, by the Second Principle of Mathematical Induction, we have proved that for each natural number n with $n \geq 6$, there exist non-negative integers x and y such that $n = 2x + 5y$. ■

(a) The proposition is correct but the proof is invalid as it does not prove the basis step of the Principle of Mathematical Induction. We will prove the proposition.

Proposition. For each natural number n with $n \geq 2, 2^n > 1 + n$.

Proof. We will prove the proposition by using the Principle of Mathematical Induction. For each $n \in \mathbb{N}$ such that $n \geq 2$, we let $P(n)$ be

$$2^n > 1 + n.$$

We will first prove the basis step. That is, we prove the case when $n = 2$. Note that $2^2 = 4 > 1 + 2 = 3$. Since this inequality is true, we have proven $P(2)$ and consequently, the basis step.

We will next prove the inductive step. That is, we will prove that for each $k \in \mathbb{N}$ such that $k \geq 2$ if $P(k)$ is true, then $P(k+1)$ is true. We assume that $P(k)$ is

$$2^k > 1 + k \tag{1}$$

and that $P(k+1)$ is

$$\begin{aligned} 2^{k+1} &> 1 + (k+1) \\ &> 2 + k. \end{aligned} \tag{2}$$

Multiplying both sides of $P(k)$ by 2, we obtain

$$\begin{aligned} 2^k \cdot 2 &> 2(1+k) \\ 2^{k+1} &> 2 + 2k \end{aligned}$$

Since $2 + 2k > 2 + k$, it follows that

$$2^{k+1} > 2 + 2k > 2 + k$$

Since $2^{k+1} > 2 + k$ is the same as equation (2), we have shown that if $P(k)$ is true, then $P(k+1)$ is true. Hence, we have proved the inductive step and consequently, we have proved the proposition by the Principle of Mathematical Induction. ■

(b) This is a true proposition with a correct proof, but the proof could be written better. We will prove the proposition.

Proposition. Each natural number greater than or equal to 6 can be written as the sum of natural numbers, each of which is a 2 or a 5.

Proof. We will prove the proposition by the Principle of Mathematical Induction. For each $n \in \mathbb{N}$, we let $P(n)$ be, "There exist non-negative integers x and y such that $n = 2x + 5y$." We begin with the basis step. Since

$$6 = 3 \cdot 2 + 0 \cdot 5 \qquad 7 = 2 + 5$$

$$8 = 4 \cdot 2 + 0 \cdot 5 \qquad 9 = 2 \cdot 2 + 1 \cdot 5$$

we see that $P(6), P(7), P(8)$, and $P(9)$ are true.

We next prove the inductive step. We assume that for each $k \in \mathbb{N}$ such that $k \geq 10$, if $P(k)$ is true, then $P(k+1)$ is true. We assume that

$$k - 4 = 2x + 5y$$

Then,

$$\begin{aligned} k + 1 &= (k - 4) + 5 \\ &= (2x + 5y) + 5 \\ &= 2x + 5(y + 1) \end{aligned}$$

Since $y + 1$ is a non-negative integer, $P(k+1)$ has been shown to be expressed in the form of $P(n)$. That is, we have proved that if $P(k)$ is true, then $P(k+1)$ is true. Since we have proved the basis step and the inductive step, we have proved the proposition by the Principle of Mathematical Induction. ■