

Consider the following relation on the natural numbers. For  $i, j \in \mathbb{N}$  say that  $iR_j$  if  $i$  and  $j$  share at least two unique common positive integer divisors (i.e., not relatively prime).

For example,  $2R_{10}$  since 1 and 2 are both divisors of 2 and 10. But the following is untrue:  $21R_{10}$  since the only factors of 21 are 1, 3, and 7 while 10's only factors are 1, 2, and 5; only 1 factor is common between these two sets.

Is this relation Reflexive? Symmetric? Transitive? An equivalence relation? Justify your answer.

### **Proof**

Let a set of positive divisors for an integer  $k \in \mathbb{N}$  be denoted  $D(k)$ . Then, a different statement of  $R$  could be the following: For  $i, j \in \mathbb{N}$  say that  $iR_j$  if and only if  $|D(i) \cap D(j)| \geq 2$ . Hence, it is easy to see that the relation is symmetric, as for any two sets  $\mathbb{A}$  and  $\mathbb{B}$ ,  $\mathbb{A} \cap \mathbb{B} = \mathbb{B} \cap \mathbb{A}$  by definition of set intersection. It is also easy to see why the relation is reflexive, as the set  $D(i) \cap D(i) = D(i)$ , meaning  $iR_i$  is true. However, the relation  $R$  is not transitive, as shown by the counterexample  $2R_{10}$  (by 1 and 2) and  $10R_{25}$  (by 1 and 5), but  $\neg 2R_{25}$ .