Ratio Cheatsheet

Ratios can be thought of as either *pitches* or *intervals*, depending on the context. In both cases, the math is the same. There three parts to getting new ratios:

- **Ascend (multiply) or descend (divide)**
  - To find the interval between two pitches expressed as intervals, descend the ratio of the smaller from the larger (i.e divide the bigger ratio by the smaller).
- **Reduce**: reduce both the numerator and denominator by their highest common factor
- **Bring into an octave**: if needed, independently divide/multiply the numerator and/or denominator by a factors of 2 (octaves) to bring the ratio into the interval of 1 to 2.

Inversion
To invert an interval (get the opposite), all you do is take the reciprocal. Example: the inversion of $\frac{3}{2}$ is $\frac{2}{3}$. Then we bring that into an octave by multiplying the numerator by 2: $\frac{4}{3}$. So the inversion of a fifth is a fourth.

Something to remember: we only want integers in our ratios. You'll see otherwise but it's confusing.

Ascending and Descending

Ascending

When we have a pitch as a ratio and we want to ascend a given interval also expressed as a ratio, we want to *multiply* the two together.

Example

$\text{Start at } \frac{4}{3} \text{ and go up } \frac{3}{2}$

Let's assume that we're at the pitch $\frac{4}{3}$ (F, if our "base" is C) and we want to go up by $\frac{3}{2}$ (a "perfect fifth"). Since we want to ascend, we need to multiply:

$$\frac{4}{3} \times \frac{3}{2} = \frac{12}{6}$$

Then we need to reduce our new ratio to make it as simple as possible, as well as bring it into the octave.

$$\frac{12}{6} = \frac{12/6}{6/6} = \frac{2}{1}$$

The greatest common factor (GCF) of 12 and 6 was 6, so I factored that out. Then we're left with $\frac{2}{1}$ which is already good.

Descending

When we have a pitch and we want to descend by a given interval also expressed as a ratio, we want to *divide* the ratios. Recall that dividing fractions is the same thing as multiplying by the reciprocal.

Example

Start at $\frac{16}{9}$ and go down $\frac{3}{2}$.

Since we're descending, we need to divide, which is the same as multiplying by the reciprocal:

$$\frac{16}{9} \div \frac{3}{2} = \frac{16}{9} \times \frac{2}{3} = \frac{32}{27}$$

Then we check to see if we need to reduce our ratio. The GCF of 32 and 27 is 1 so there's nothing to do with reducing. Now we check to see if it's in the interval 1 to 2. Since $32/27 = 1.18518518519$ it's already in an octave so we're done. This means that if we start at the pitch $\frac{16}{9}$ and go down $\frac{3}{2}$, we get the pitch $\frac{32}{27}$.

Proof: the Greek enharmonic tetrachord is $\frac{4}{3}$ (a perfect fouth)

The intervals that make up the Greek enharmonic tetrachord are 28/27, 36/35 and 5/4. Since we need to add the ratios together in “interval-space”, we need to multiply. So, let’s start with the first two:

$$\frac{28}{27} \times \frac{36}{35} = \frac{1008}{945}$$

Now we need to reduce our result. Before I start dealing with octaves, I’m going to figure out what the common factors are, then choose the biggest one. The factors of 945 are: 1, 3, 5, 7, 9, 15, 21, 27, 35, 45, 63, 105, 135, 189, 315, 945. The factors of 1008 are: 1, 2, 3, 4, 6, 7, 8, 9, 12, 14, 16, 18, 21, 24, 28, 36, 42, 48, 56, 63, 72, 84, 112, 126, 144, 168, 252, 336, 504, 1008. So the common factors are 1, 3, 7, 9, 21, 63. So I’ll reduce both the numerator and denominator by a factor of 63, the largest common factor:

$$\frac{1008/63}{945/63} = \frac{16}{15}$$

Since $\frac{16}{15}$ is already between 1 and 2, there’s nothing more to do. Let’s keep going: go up 5/4 (the last interval in our tetrachord) from 16/15. I’m going multiply (“go up”) and reduce in one equation here to show my thinking:

$$\frac{16}{15} \times \frac{5}{4} = \frac{80}{60} = \frac{8}{6} = \frac{8/2}{6/2} = \frac{4}{3}$$

So we can see that when we ascend all of the intervals of the Greek enharmonic tetrachord, we get a $\frac{4}{3}$ (i.e. a perfect fourth).

Inverting Intervals

To get the inverse of an interval, simply take the reciprocal and bring into an octave. Let’s say we want to get the inverse of the harmonic seventh, $\frac{7}{4}$. To do that, we literally invert the ratio and bring into an octave:

$$\frac{4}{7} = \frac{4 \times 2}{7} = \frac{8}{7}$$

Since $\frac{4}{7}$ was not in the interval between 1 and 2, I had to either divide the denominator by 2 or multiply the numerator by 2. Since dividing the denominator would give me a decimal number, I decided to multiply to multiply the numerator by two (go up an octave) which gave me a reduced ratio in the interval I needed.