Robust Factorization of Real-world Tensor Streams with Patterns, Missing Values, and Outliers - Supplementary Document

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Abstract

In this supplementary document, we give a full proof of Theorem 2 in the main paper.

I. Proof of Theorem 2

Proof. For all $1 \leq i_N \leq I_N$ and $1 \leq j \leq R$,

$$\frac{\partial C}{\partial u^{(N)}_{i_N j}} = \sum_{(i_1,\dots,i_N)\in\Omega^{(N)}_{i_N}} 2\Big((\sum_{r=1}^{R}\prod_{l=1}^{N} u^{(l)}_{i_l r} - y^{*}_{i_1,\dots,i_N}) \prod_{l\neq N} u^{(l)}_{i_l j}\Big) + 2K_{i_N j} + 2H_{i_N j} = 0,$$

where $K_{i_N j}$ and $H_{i_N j}$ are defined as (18). It is equivalent to

$$\begin{cases}
\displaystyle\sum_{\substack{(i_1,\dots,i_N)\\ \in\Omega^{(N)}_{i_N}}} \Big(\sum_{r=1}^{R}(u^{(N)}_{i_N r}\prod_{l\neq N}u^{(l)}_{i_l r})\prod_{l\neq N}u^{(l)}_{i_l j}\Big) + (\lambda_1+\lambda_2)u^{(N)}_{i_N j} & \\
\qquad = \displaystyle\sum_{\substack{(i_1,\dots,i_N)\\ \in\Omega^{(N)}_{i_N}}} \Big(y^{*}_{i_1,\dots,i_N}\prod_{l\neq N}u^{(l)}_{i_l j}\Big) + \lambda_1 u^{(N)}_{(i_N+1)j} + \lambda_2 u^{(N)}_{(i_N+m)j} & \text{if } i_N = 1,\\
\displaystyle\sum_{\substack{(i_1,\dots,i_N)\\ \in\Omega^{(N)}_{i_N}}} \Big(\sum_{r=1}^{R}(u^{(N)}_{i_N r}\prod_{l\neq N}u^{(l)}_{i_l r})\prod_{l\neq N}u^{(l)}_{i_l j}\Big) + (2\lambda_1+\lambda_2)u^{(N)}_{i_N j} & \\
\qquad = \displaystyle\sum_{\substack{(i_1,\dots,i_N)\\ \in\Omega^{(N)}_{i_N}}} \Big(y^{*}_{i_1,\dots,i_N}\prod_{l\neq N}u^{(l)}_{i_l j}\Big) + \lambda_1 (u^{(N)}_{(i_N-1)j} + u^{(N)}_{(i_N+1)j}) + \lambda_2 u^{(N)}_{(i_N+m)j} & \text{else if } 1 < i_N \leq m,\\
\displaystyle\sum_{\substack{(i_1,\dots,i_N)\\ \in\Omega^{(N)}_{i_N}}} \Big(\sum_{r=1}^{R}(u^{(N)}_{i_N r}\prod_{l\neq N}u^{(l)}_{i_l r})\prod_{l\neq N}u^{(l)}_{i_l j}\Big) + 2(\lambda_1+\lambda_2)u^{(N)}_{i_N j} & \\
\qquad = \displaystyle\sum_{\substack{(i_1,\dots,i_N)\\ \in\Omega^{(N)}_{i_N}}} \Big(y^{*}_{i_1,\dots,i_N}\prod_{l\neq N}u^{(l)}_{i_l j}\Big) + \lambda_1 (u^{(N)}_{(i_N-1)j} + u^{(N)}_{(i_N+1)j}) + \lambda_2 (u^{(N)}_{(i_N-m)j} + u^{(N)}_{(i_N+m)j}) & \text{else if } m < i_N \leq I_N - m,\\
\displaystyle\sum_{\substack{(i_1,\dots,i_N)\\ \in\Omega^{(N)}_{i_N}}} \Big(\sum_{r=1}^{R}(u^{(N)}_{i_N r}\prod_{l\neq N}u^{(l)}_{i_l r})\prod_{l\neq N}u^{(l)}_{i_l j}\Big) + (2\lambda_1+\lambda_2)u^{(N)}_{i_N j} & \\
\qquad = \displaystyle\sum_{\substack{(i_1,\dots,i_N)\\ \in\Omega^{(N)}_{i_N}}} \Big(y^{*}_{i_1,\dots,i_N}\prod_{l\neq N}u^{(l)}_{i_l j}\Big) + \lambda_1 (u^{(N)}_{(i_N-1)j} + u^{(N)}_{(i_N+1)j}) + \lambda_2 u^{(N)}_{(i_N-m)j} & \text{else if } I_N - m < i_N \leq I_N - 1,\\
\displaystyle\sum_{\substack{(i_1,\dots,i_N)\\ \in\Omega^{(N)}_{i_N}}} \Big(\sum_{r=1}^{R}(u^{(N)}_{i_N r}\prod_{l\neq N}u^{(l)}_{i_l r})\prod_{l\neq N}u^{(l)}_{i_l j}\Big) + (\lambda_1+\lambda_2)u^{(N)}_{i_N j} & \\
\qquad = \displaystyle\sum_{\substack{(i_1,\dots,i_N)\\ \in\Omega^{(N)}_{i_N}}} \Big(y^{*}_{i_1,\dots,i_N}\prod_{l\neq N}u^{(l)}_{i_l j}\Big) + \lambda_1 u^{(N)}_{(i_N-1)j} + \lambda_2 u^{(N)}_{(i_N-m)j} & \text{otherwise.}
\end{cases} \tag{29}$$

Then, we vectorize (29) as:

$$\begin{cases} \big(\mathbf{B}_{i_N}^{(N)} + (\lambda_1+\lambda_2)\mathbf{I}_R\big)\mathbf{u}_{i_N}^{(N)} = \mathbf{c}_{i_N}^{(N)} + \lambda_1\mathbf{u}_{i_N+1}^{(N)} + \lambda_2\mathbf{u}_{i_N+m}^{(N)} & \text{if } i_N = 1, \\ \big(\mathbf{B}_{i_N}^{(N)} + (2\lambda_1+\lambda_2)\mathbf{I}_R\big)\mathbf{u}_{i_N}^{(N)} = \mathbf{c}_{i_N}^{(N)} + \lambda_1(\mathbf{u}_{i_N-1}^{(N)} + \mathbf{u}_{i_N+1}^{(N)}) + \lambda_2\mathbf{u}_{i_N+m}^{(N)} & \text{else if } 1 < i_N \leq m, \\ \big(\mathbf{B}_{i_N}^{(N)} + 2(\lambda_1+\lambda_2)\mathbf{I}_R\big)\mathbf{u}_{i_N}^{(N)} = \mathbf{c}_{i_N}^{(N)} + \lambda_1(\mathbf{u}_{i_N-1}^{(N)} + \mathbf{u}_{i_N+1}^{(N)}) + \lambda_2(\mathbf{u}_{i_N-m}^{(N)} + \mathbf{u}_{i_N+m}^{(N)}) & \text{else if } m < i_N \leq I_N - m, \\ \big(\mathbf{B}_{i_N}^{(N)} + (2\lambda_1+\lambda_2)\mathbf{I}_R\big)\mathbf{u}_{i_N}^{(N)} = \mathbf{c}_{i_N}^{(N)} + \lambda_1(\mathbf{u}_{i_N-1}^{(N)} + \mathbf{u}_{i_N+1}^{(N)}) + \lambda_2\mathbf{u}_{i_N-m}^{(N)} & \text{else if } I_N - m < i_N \leq I_N - 1, \\ \big(\mathbf{B}_{i_N}^{(N)} + (\lambda_1+\lambda_2)\mathbf{I}_R\big)\mathbf{u}_{i_N}^{(N)} = \mathbf{c}_{i_N}^{(N)} + \lambda_1\mathbf{u}_{i_N-1}^{(N)} + \lambda_2\mathbf{u}_{i_N-m}^{(N)} & \text{otherwise,} \end{cases} \tag{30}$$

where $\mathbf{B}_{i_N}^{(N)}$ and $\mathbf{c}_{i_N}^{(N)}$ are defined as (14) and (15), respectively. The solution of (30) is (17). $\square$