**Required Problems**

1. Let $Y$ be a continuous random variable with PDF

$$f_Y(y) = \begin{cases} (3/2)y^2 + y & \text{if } 0 \le y \le 1 \\ 0 & \text{else} \end{cases}$$

   (a) Find the mean of $Y$.
   (b) Find the variance of $Y$.

2. Let $Y$ be a random variable with probability density function given by

$$f_Y(y) = 2(1-y), \qquad y \in [0,1]$$

   (a) Find the PDF of $U = 2Y - 1$.
   (b) Find the PDF of $W = 1 - 2Y$.
   (c) Find the PDF of $Z = Y^2$.

3. Consider the multivariate distribution characterized by the PDF

$$f_{XY} = 6(1-y), \qquad 0 \le x \le y \le 1$$

   (a) Find the conditional expectation $\mathbb{E}[X|Y=y]$.
   (b) Find the covariance of $X$ and $Y$.

4. Let $X_1 \dots, X_n$ be a random sample from a distribution with PMF

$$f(x_i|\theta) = \begin{cases} \theta(1-\theta)^{x_i-1} & \text{if } x = 1, 2, 3 \dots \\ 0 & \text{else} \end{cases}$$

   where $\theta \in (0,1)$.

   (a) Find the method of moments estimator for $\theta$.
   (b) Find the maximum likelihood estimator for $\theta$.

**Practice Problems**

5. Let $X$ be a discrete random variable with PMF $f_X(x)$, given in the following table.

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $f_X(x)$ | 0.4 | 0.3 | 0.2 | 0.1 |

   Find the following:

   (a) $\mathbb{E}[X]$
   (b) $\mathbb{E}[1/X]$
   (c) $\mathbb{E}[X^2 - 1]$
   (d) $\text{Var}[X]$

6. A binomially-distributed random variable $X$ has the PMF

$$f_X(x) = \binom{n}{x} p^x (1-p)^{n-x}, \qquad x \in \{0, 1, 2, \dots, n\}$$

where $0 < p < 1$.

(a) Show that the expected value of $X$ is $np$.

(b) Show that the variance of $X$ is $np(1-p)$.

7. Let $Y$ be a random variable with mean $\mu$ and variance $\sigma^2$. If $a$ and $b$ are constants, show that:

(a) $\mathbb{E}[aY + b] = a\mu + b$

(b) $\text{Var}(aY + b) = a^2\sigma^2$

8. Let $X$ be a standard normal random variable with PDF

$$f_X(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}, \qquad x \in (-\infty, \infty)$$

(a) Show that the MGF of $X$ is $M_X(t) = \exp\{(1/2)t^2\}$ (hint: complete the square in the exponent).

(b) Use the MGF of $X$ to calculate $\mathbb{E}[X]$ and $\mathbb{E}[X^2]$.

(c) Use the MGF to show that the sum of two independent standard normal random variables is distributed normally with $\mu = 0$ and $\sigma^2 = 2$.

9. For each of the following random variables, find the PDF of $Y$.

(a) $f_X(x) = \frac{1}{2}(1 + x), \quad x \in (-1, 1)$ and $Y = X^2$

(b) $f_X(x) = 60x^3(1 - x)^2, \quad x \in (0, 1)$ and $Y = \log(X)$

(c) $f_X(x) = (1 + x)^2/9, \quad x \in (-1, 2)$ and $Y = (X - 1)^2$

10. Consider the joint distribution of $Y_1$ and $Y_2$ given by the following table:

| | | $x_1$ = 0 | $x_1$ = 1 |
|---|---|---|---|
| $x_2$ | 0 | 0.38 | 0.17 |
| | 1 | 0.14 | 0.02 |
| | 2 | 0.24 | 0.05 |

(a) Find the marginal PMFs for $X_1$ and $X_2$.

(b) Find the conditional PMF for $X_2$ given $X_1 = 0$.

11. Consider the joint PDF:

$$f_{Y_1 Y_2}(y_1, y_2) = 3y_1, \qquad 0 \le y_2 \le y_1 \le 1$$

(a) Find the marginal PDFs for $Y_1$ and $Y_2$.

(b) Find the conditional PDF of $Y_2$ given $Y_1 = y_1$.

(c) What is the probability that $Y_2 \ge 1/2$ given that $Y_1 = 3/4$?

12. For each of the following joint PDFs, determine whether or not the random variables $x$ and $y$ are independent.

    (a) $f_{XY}(x,y) = 1$, where $x \in [0,1]$ and $y \in [0,1]$

    (b) $f_{XY}(x,y) = e^{-(x+y)}$, where $x > 0$ and $y > 0$

    (c) $f_{XY}(x,y) = 6(1-y)$, where $0 \leq x \leq y \leq 1$

13. Show that $\mathbb{E}\big[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\big] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$.

14. Let $X_1, \ldots, X_n$ be a random sample from a distribution with mean $\mu$ and $\sigma^2 < \infty$. Show that:

    (a) $\mathbb{E}[\bar{X}] = \mu$

    (b) $Var(\bar{X}) = \frac{\sigma^2}{n}$

    (c) $\sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2 = \left(\sum_{i=1}^{n} X_i^2\right) - \bar{X}^2$

    (d) $\mathbb{E}[S^2] = \sigma^2$

15. Let $Y_1 \ldots, Y_n$ be a random sample from a distribution with PDF

$$f(y_i|\theta) = \begin{cases} (\theta+1)y^{-(\theta+2)} & \text{if } y > 1 \\ 0 & \text{else} \end{cases}$$

    where $\theta \in (0, \infty)$.

    (a) Find the method of moments estimator for $\theta$.

    (b) Find the maximum likelihood estimator for $\theta$.

16. For each of the following PDFs, let $X_1, \ldots, X_n$ be a random sample. Find a complete sufficient statistic for the unknown parameter(s).

    (a) $f(x|\theta) = \dfrac{\theta}{(1+x)^{1+\theta}}$, where $0 < x < \infty$ and $\theta > 0$

    (b) $f(x|\beta) = \dfrac{\ln[\beta]\beta^x}{\beta - 1}$, where $0 < x < 1$ and $\beta > 1$

    (c) $f(x|\theta) = \exp\{-(x-\theta)\}\exp\left\{-e^{-(x-\theta)}\right\}$, where $-\infty < x < \infty$ and $-\infty < \theta < \infty$

    (d) $f(x|\alpha,\beta) = \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$ where $0 < x < 1$ and $\alpha, \beta > 0$