ECON 204C - Macroeconomic Theory

Aiyagari Model

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Learning Objective

- Aiyagari Model (1994)
    - Heterogeneous agents
    - A single exogenous vehicle for borrowing and lending
    - Limits on amounts individual agents may borrow
    - Stationary Rational Expectations Equilibrium
    - Computation using Discrete State Dynamic Programming

Aiyagari Model - Household's Problem

$$
\begin{aligned}
v(a,z;\Omega) = \max_{c\geqslant 0,\ a'\in\mathcal{A}} & \left\{u(c) + \beta\mathbb{E}_z\left[v(a',z';\Omega)\right]\right\} \\
s.t. \quad & c + a' \leqslant wz + \left[1+r\right]a \\
& c \geqslant 0 \\
& a' \geqslant -\phi
\end{aligned}
$$

The exogenous process $\{z_t\}$ follows a finite state Markov chain with given stochastic matrix P. Let $a'(a,z)$ be an associated policy function for saving, then price to capital stock is given by

$$
K^s(r) = \sum_{(a,z)\in\mathcal{A}\times\mathcal{Z}} \Omega(a,z)a'(a,z)
$$

Aiyagari Model - Firm's Problem

$$\max_{K,\,N} \left\{ AK^{\alpha}N^{1-\alpha} - (r+\delta)K - wN \right\}$$

$$A\alpha \left(\frac{N^D}{K^D}\right)^{1-\alpha} - (r+\delta) = 0 \qquad \text{(FOC w.r.t. K)}$$

$$A(1-\alpha) \left(\frac{K^D}{N^D}\right)^{\alpha} - w = 0 \qquad \text{(FOC w.r.t. N)}$$

Equilibrium wage associated with a given interest rate is given by

$$w(r) = A(1-\alpha) \left\{ \left(\frac{A\alpha}{r+\delta}\right) \right\}^{\frac{\alpha}{1-\alpha}}$$

Inverse demand for capital is given by

$$r(K^D) = A\alpha \left(\frac{N^D}{K^D}\right)^{1-\alpha} - \delta$$

Aiyagari Model - Stationary Equilibrium

Definition A stationary recursive competitive equilibrium is a value function $v : \mathcal{A} \times \mathcal{Z} \to \mathbb{R}$, policy function for the household $c : \mathcal{A} \times \mathcal{Z} \to \mathbb{R}$ and $a' : \mathcal{A} \times \mathcal{Z} \to \mathbb{R}$, firm's choices N^* and K^*, prices r and w, and stationary measure Ω^* such that

1. Given (r, w), c and a' solve the household's problem and v is the associated value function.
2. Given (r, w), the firm chooses optimally its capital K and its labor N.
3. Capital and labor markets clear:

$$\sum_{(a,z)\in A\times Z} a'(a,z)\Omega^*(a,z) = K^* \qquad \sum_{(a,z)\in A\times Z} z\Omega^*(a,z) = N^*$$

4. Ω^* is consistent with $a'(a, z)$.

Discrete State Dynamic Programming

$$v(s) = \max_{d \in D(s)} \left\{ r(s,d) + \beta \sum_{s' \in S} v(s') Q(s,d,s') \right\}$$

- s is the state variable.
- d is the action.
- β is a discount factor.
- $r(s,d)$ is a current reward when the state is s and the action chosen is d.
- $Q(s,d,s')$ is a transitional probability.

Aiyagari Model - Discrete State Dynamic Programming

$$\mathcal{Z} = \{.1, 1\}$$

$$\mathcal{A} = \{1e - 10, \cdots, 20\} \quad \text{where} \quad n(\mathcal{A}) = 200$$

$$P = \begin{pmatrix} .9 & .1 \\ .1 & .9 \end{pmatrix}$$

$$u(c) = \ln(c)$$

$$\beta = .96$$

$$\alpha = .33$$

$$\delta = .05$$

$$A = 1$$

Aiyagari Model - Discrete State Dynamic Programming

$$r(s,d) = r(a,z,a') = \ln c = \begin{cases} \ln\left(wz + [1+r]a - a'\right) & \text{if} \quad c > 0 \\ \\ -\infty & \text{otherwise} \end{cases}$$

$$Q(s,d,s') = Q(a,z,a',z') = P_{zz'}$$

Trick on indexing

$$S = \left\{(a_{i_{\mathcal{A}}}, z_{i_{\mathcal{Z}}})\right\} = \left\{ \underbrace{(a_0,z_0)}_{i_S=0}, \underbrace{(a_0,z_1)}_{i_S=1}, \underbrace{(a_1,z_0)}_{i_S=2}, \underbrace{(a_1,z_1)}_{i_S=3}, \cdots, \underbrace{(a_{199},z_0)}_{i_S=398}, \underbrace{(a_{199},z_1)}_{i_S=399} \right\}$$

$$i_S = i_{\mathcal{A}} \times 2 + i_{\mathcal{Z}}$$

Aiyagari Model - Discrete State Dynamic Programming

$$K^S = \sum_{(a,z)\in\mathcal{A}\times\mathcal{Z}} a'(a,z)\Omega^*(a,z) = \sum_{a\in\mathcal{A}} a \times Pr(a) \quad \text{where} \quad Pr(a) = \sum_{z\in\mathcal{Z}} \Omega^*(a,z)$$

$$N^S = \sum_{(a,z)\in\mathcal{A}\times\mathcal{Z}} z\Omega^*(a,z) = \sum_{z\in\mathcal{Z}} z \times Pr(z) \quad \text{where} \quad Pr(z) = \sum_{a\in\mathcal{A}} \Omega^*(a,z)$$

Reference

Aiyagari, S. R. (1994). Uninsured idiosyncratic risk and aggregate saving. The Quarterly Journal of Economics, 109(3), 659-684.

Sargent, T. S. & Stachurski, J. (2020, March 30). The Aiyagari Model. Retrieved from `https://python-intro.quantecon.org/aiyagari.html`