

**Required Problems**

1. Consider the following matrices:

$$A = \begin{bmatrix} 2 & 8 \\ 3 & 0 \\ 5 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 0 \\ 3 & 8 \end{bmatrix} \quad C = \begin{bmatrix} 7 & 2 \\ 6 & 3 \end{bmatrix}$$

- (a) Calculate  $AB$ .
  - (b) Calculate  $CB$ .
  - (c) Is it true that  $CB = BC$ ? Justify your response.
2. Find the determinant of each of the following matrices.

(a)  $D = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$

(b)  $E = \begin{bmatrix} 8 & 1 & 3 \\ 4 & 0 & 1 \\ 6 & 0 & 3 \end{bmatrix}$

3. Find the inverse of the following matrix:

$$F = \begin{bmatrix} 5 & 2 \\ 0 & 1 \end{bmatrix}$$

4. Find eigenvalues and eigenvectors associated with the matrix:

$$G = \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix}$$

**Practice Problems**

5. Consider the following column vectors:

$$\mathbf{v}_1 = \begin{bmatrix} 4 \\ -2 \\ 3 \\ 1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 6 \\ 0 \\ 2 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ -5 \\ 3 \\ 2 \end{bmatrix}$$

Are these vectors linearly independent? Justify your response.

6. Find the determinant of the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 0 & 9 \\ 2 & 3 & 4 & 6 \\ 1 & 6 & 0 & -1 \\ 0 & -5 & 0 & 8 \end{bmatrix}$$

7. Find the inverse of each of the following the matrices.

(a)  $B = \begin{bmatrix} -1 & 0 \\ 9 & 2 \end{bmatrix}$

(b)  $C = \begin{bmatrix} 4 & 1 & -1 \\ 0 & 3 & 2 \\ 3 & 0 & 7 \end{bmatrix}$

8. Consider the linear system given by  $\mathbf{Y} = X\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ , where  $\mathbf{Y}$  and  $\boldsymbol{\varepsilon}$  are  $n \times 1$  vectors,  $X$  is a  $n \times k$  matrix of full rank, and  $\boldsymbol{\beta}$  is a  $k \times 1$  vector.

- (a) Suppose that  $\boldsymbol{\varepsilon}$  is orthogonal to each of the columns in  $\mathbf{X}$  (i.e.,  $\mathbf{X}'\boldsymbol{\varepsilon} = \mathbf{0}$ ). Using matrix algebra, solve for  $\boldsymbol{\beta}$ .

- (b) Further, suppose that  $\boldsymbol{\beta}$  is a  $2 \times 1$  vector and the matrix  $X$  is  $n \times 2$ , such that:

$$\boldsymbol{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \quad \mathbf{X} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}$$

In addition, assume that  $\sum_{i=1}^n x_i = 0$ . Find expressions for  $\beta_1$  and  $\beta_2$ .

9. Use Cramer's rule, row operations, or matrix inversion to solve the following linear systems:

(a)  $8x_1 - x_2 = 16$   
 $2x_2 + 5x_3 = 5$   
 $2x_1 + 3x_3 = 7$

(b)  $-x_1 + 3x_2 + 2x_3 = 24$   
 $x_1 + x_3 = 6$   
 $5x_2 - x_3 = 8$

(c)  $4x + 3y - 2z = 1$   
 $x + 2y = 6$   
 $3x + z = 4$

(d)  $-x + y + z = a$   
 $x - y + z = b$   
 $x + y - z = c$

10. Suppose there is a perfectly competitive firm with a production function  $y = f(x_1, x_2)$ , increasing in both arguments. The firm sells output  $y$  at the market price  $p$ . The firm purchases an input  $x_1$  and price  $w_1$ ;  $x_2$ , however, represents the entrepreneur's input and is limited to  $\bar{x}_2$ . Thus, the firm's maximization problem, framed as a Lagrangian, is given by

$$\mathcal{L} = p \cdot f(x_1, x_2) - w_1 \cdot x_1 + \lambda(\bar{x}_2 - x_2)$$

This yields the first order conditions:

$$\mathcal{L}_1 = p \cdot f_1(x_1, x_2) - w_1 = 0$$

$$\mathcal{L}_2 = p \cdot f_2(x_1, x_2) - \lambda = 0$$

$$\mathcal{L}_\lambda = \bar{x}_2 - x_2 = 0$$

- (a) What is the sufficient second order condition?

Given that the profit-maximizing levels of  $x_1$ ,  $x_2$ , and  $\lambda$  have been found as functions of  $p$ ,  $w_1$ , and  $\bar{x}_2$ , e.g.,  $x_1^* = g(p, w_1, \bar{x}_2)$ , the solutions may be plugged back into the first order conditions yielding:

$$\begin{aligned} pf_1(x_1^*, x_2^*) - w_1 &\equiv 0 \\ pf_2(x_1^*, x_2^*) - \lambda^* &\equiv 0 \\ \bar{x}_2 - x_2^* &\equiv 0 \end{aligned}$$

- (b) Use Cramer's Rule to find a comparative statics prediction for  $\frac{\partial x_1^*}{\partial w_1}$  (Hint: use the chain rule on the system of identities, differentiating w.r.t.  $w_1$ ).
11. Find the eigenvalues and eigenvectors of each of the following matrices:

(a)  $A = \begin{bmatrix} -2 & 2 \\ 2 & -4 \end{bmatrix}$

(b)  $B = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

12. Given a quadratic form  $\mathbf{x}'C\mathbf{x}$ , where  $C$  is a symmetric  $2 \times 2$  matrix:

$$C = \begin{bmatrix} a & d \\ d & b \end{bmatrix}$$

Show that the following are true (Hint: use the quadratic equation):

- (a) Both eigenvalues must be real (i.e., they cannot involve  $\sqrt{-1}$ ).
- (b)  $C$  has repeated roots if and only if it is of the form  $C = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$
13. Let  $A$  be a  $m \times n$  matrix. Show that  $(A^T)^T = A$ .
14. Let  $B$  and  $C$  be  $m \times n$  matrices. Show that  $k(B + C) = kB + kC$
15. Let  $D$  be a  $n \times n$  invertible matrix. Show that  $D^{-1}$  is unique.
16. Let  $E$  be a  $n \times n$  matrix. Show that if  $E$  is idempotent,  $I_n - E$  is idempotent as well.
17. Show that for any  $2 \times 2$  matrix  $A$

$$|A| = \frac{1}{2} \begin{vmatrix} \text{tr}(A) & 1 \\ \text{tr}(A^2) & \text{tr}(A) \end{vmatrix}$$

18. State whether each of the following statements is true or false. If it is false, provide a counterexample.

- (a) No system of linear equations can have exactly  $k$  solutions for any  $k \geq 2$ .
- (b) Any system of  $n$  linear equations in  $n$  unknowns has at least one solution.
- (c) Any system of  $n$  linear equations in  $n$  unknowns has at most one solution.
- (d) If  $A\mathbf{x} = \mathbf{0}$  has a solution, then  $A\mathbf{x} = \mathbf{b}$  has a solution.
- (e) If an  $n \times n$  matrix  $A$  is invertible, then  $A\mathbf{x} = \mathbf{0}$  has the unique solution  $\mathbf{x} = \mathbf{0}$ .
- (f) If an  $n \times n$  matrix  $A$  is full rank, then  $A\mathbf{x} = \mathbf{b}$  has a solution.
- (g) If an  $n \times n$  matrix  $A$  has rank less than  $n$ , then  $A\mathbf{x} = \mathbf{b}$  has no solution.