Required Problems

1. Determine the definiteness of the following symmetric matrices:

   (a) $A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

   (b) $B = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 5 \\ 3 & 0 & 4 & 0 \\ 0 & 5 & 0 & 6 \end{bmatrix}$

2. Find the least squares solution to $X\mathbf{b} = \mathbf{y}$, i.e., by finding the estimate $\hat{\mathbf{b}}$ such that $X\hat{\mathbf{b}} = \hat{\mathbf{y}}$ (where $\hat{\mathbf{y}}$ is the projection of $\mathbf{y}$ onto $\text{col}(X)$) using the following information:

$$X = \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

3. In the game of dominoes, each piece is marked with two numbers. The pieces are symmetrical, so that the number pair is not ordered (e.g., (2, 6) = (6, 2)). Further, duplicated are allowed (e.g., (2, 2)). How many different pieces can be formed using the numbers $1, 2, \ldots, n$?

4. Suppose that 5% of men and 0.25% of women are color blind. A person is chosen at random and that person is color-blind. What is the probability that the person is male (assuming males and females are equal in number)?

5. If the random variable X follows a geometric distribution, its PMF is given by

$$f_X(x) = (1-p)^x p, \qquad x \in \{0, 1, 2, 3 \ldots\}$$

   where the parameter $p \in (0, 1)$. Find the CDF of X.

Practice Problems

6. Determine the definiteness of the following symmetric matrices:

   (a) $A = \begin{bmatrix} -3 & 4 \\ 4 & -5 \end{bmatrix}$

   (b) $B = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{bmatrix}$

   (c) $C = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

7. Recall the conditions for $d(x, y)$ to quality as a metric. Verify that the function d is a metric where

$$d(x, y) = \frac{|x - y|}{1 + |x - y|}$$

8. Consider the following vectors:

$$\mathbf{y} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{u} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

Find the orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}$.

9. Let S be the span of a $n \times m$ matrix X, and let its orthogonal complement be $S^{\perp}$. Define $P_x = X(X^T X)^{-1} X^T$. For $X \in S$, $E \in S^{\perp}$, show each of the following holds:
   (a) $(I_n - P_x)X = 0$
   (b) $(I_n - P_x)E = E$
   (c) $P_x P_x = P_x$
   (d) $(I_n - P_x)P_x = 0$

10. People possess the blood types A, B, AB, or O. Further, each individual has a Rhesus factor (+) or does not (−). A medical technician is recording blood types and Rhesus factors. List the sample space for this "experiment."

11. An oil prospecting firm hits oil or gas on 10% of its drillings. If each drilling is an independent event (and we assume all potential wells produce with equal probability), what is the probability that the oil company will hit oil or gas?
   (a) on both drillings?
   (b) on the first drilling but not the second?
   (c) on at least one of the drillings?

12. If $P(A) = \frac{1}{3}$ and $P(B^c) = \frac{1}{4}$, can A and B be disjoint? Explain.

13. How many different sets of initials can be formed (using the English alphabet) if every person has one surname (last name) and
   (a) Exactly two given names (first names)?
   (b) Either one or two given names?
   (c) Either one or two or three given names?

14. If two events A and B are such that $P(A) = 0.5$, $P(B) = 0.3$, and $P(A \cap B) = 0.1$, find the following:
   (a) $P(A|B)$
   (b) $P(B|A)$
   (c) $P(A|A \cup B)$
   (d) $P(A|A \cap B)$
   (e) $P(A \cap B|A \cup B)$

15. Let Y be a random variable with $p(y)$ given in the table below:

y	1	2	3	4
$p(y)$	0.4	0.3	0.2	0.1

Give the cumulative distribution function $F_Y(y)$. Be sure to specify the value of $F_Y(y)$ for all $y \in (-\infty, \infty)$.