ECON 204C - Macroeconomic Theory **Incomplete Markets Model**
TA: Woongchan Jeon Week 4 - April 24, 2020

These practice problems are pulled from Ljungqvist and Sargent (2018).

1. Random discount factor Consider a Huggett-style incomplete markets model where individual households face discount factor shocks. A household has preferences over consumption of a single good ordered by a value function defined recursively by

$$v(\beta_t, a_t, s_t) = u(c_t) + \beta_t \mathbb{E}_t\Big[v(\beta_{t+1}, a_{t+1}, s_{t+1})\Big]$$

where $\beta_t \in (0,1)$ is the time t value of a discount factor, and a_t is time t holding of a single asset. Here v is the discounted utility for a consumer with asset holding a_t , discount factor β_t, and employment state s_t . The discount factor evolves according to a three-state Markov chain with transition probabilities $P(\beta'|\beta) = Prob(\beta_{t+1} = \beta'|\beta_t = \beta)$. The discount factor and employment state at t are both known. The household faces the sequence of budget constraints

$$c_t + qa_{t+1} \le a_t + s_t$$

where s_t evolves according to an n-state Markov chain with transition probability $Q(s'|s) = Prob(s_{t+1} = s'|s_t = s)$. The household faces the borrowing constraint $a_{t+1} \ge -\phi$ for all t. Formulate Bellman equations for the household's problem and define a stationary equilibrium for this economy.

$$v(\beta, a, s) = \max_{a',c} \left\{ u(c) + \beta \sum_{\beta'} \sum_{s'} v(\beta', a', s') P(\beta'|\beta) Q(s'|s) \right\}$$
$$s.t. \quad a' + c \le (1+r)a + ws \quad \text{and} \quad a' \ge -\phi$$

Let $a' = g(\beta, a, s)$ denote the policy function implied by the maximization on the RHS of the Bellman equation. A stationary recursive equilibrium is a value function $v(\beta, a, s)$, a policy function $g(\beta, a, s)$, a stationary distribution $\Omega(\beta, a, s)$, and price q such that:

1. Taking q as given, $v(\beta, a, s)$ and $g(\beta, a, s)$ solve the dynamic programming problem for an individual of type β, a, s.

2. The asset market clears.

$$\sum_\beta \sum_a \sum_s g(\beta, a, s)\Omega(\beta, a, s) = 0$$

3. The distribution $\Omega(\beta, a, s)$ is stationary.

$$\Omega(\beta', a', s') = \sum_\beta \sum_a \sum_s \mathcal{Q}\Big((\beta', a', s'),\ (\beta, a, s)\Big)\Omega(\beta, a, s)$$
$$\text{where} \quad \mathcal{Q}\Big((\beta', a', s'),\ (\beta, a, s)\Big) = \mathbb{I}\{a' = g(\beta, a, s)\} P(\beta'|\beta) Q(s'|s)$$

2. Unemployment There is a continuum of workers with identical probabilities λ of being fired each period when they are employed. With probability $\mu \in (0,1)$, each unemployed worker receives one offer to work at wage w drawn from the cumulative distribution function F with the support $[0, B]$. If he accepts the offer, the worker receives the offered wage each period until he is fired. With probability $1-\mu$, an unemployed worker receives no offer this period. The probability μ is determined by the function $\mu = f(U)$, where U is the unemployment rate, and $f'(U) < 0, f(0) = 1, f(1) = 0$. A worker's utility is given by $\mathbb{E}_{t=1}^{\infty} = \beta^t y_t$, where $\beta \in (0,1)$ and y_t is income in period t, which equals the wage if employed and zero otherwise. There is no unemployment compensation. Each worker regards U as fixed and constant over time in making his decisions.

a. For fixed U, write the Bellman equation for the worker. Argue that his optimal policy has the reservation wage property.

$$V^u(U) = 0 + \beta\left[f(U)\int_0^B \max\left\{V^e(w'), \int_0^1 V^u(U')df(U')\right\}dF(w') + \Big[1-f(U)\Big]\int_0^1 V^u(U')df(U')\right]$$

$$V^e(w) = w + \beta\left[\lambda\int_0^1 V^u(U')df(U') + [1-\lambda]V^e(w)\right]$$

Rewrite the value of being employed

$$V^e(w) = \left[\frac{1}{1-\beta+\lambda\beta}\right]w + \left[\frac{\lambda\beta}{1-\beta+\lambda\beta}\right]\int_0^1 V^u(U')df(U')$$

Substitue this into the value function of being unemployed.

$$V^u(U) = \beta\left[f(U)\int_0^B \max\left\{Aw + \lambda\beta A\mathbb{E}[V^u],\ \mathbb{E}[V^u]\right\}dF(w') + \Big[1-f(U)\Big]\mathbb{E}[V^u]\right]$$

where $A = \dfrac{1}{1-\beta+\lambda\beta}$

$$\mathbb{E}[V^u] = \int_0^1 V^u(U')df(U')$$

Conditional on receiving an offer, the unemployed worker accepts the draw w if and only if $w \geq R$, where

$$R = \beta\left[\frac{1-\lambda\beta A}{A}\right]\mathbb{E}[V^u]$$

Let $\mathbb{I}\{w \geq R\}$ denote the policy function implied by the maximization on the RHS of the Bellman equation for the unemployed.

b. Given the typical worker's policy (i.e., his reservation wage), display a difference equation for the unemployment rate. Show that a stationary unemployment

$$\lambda(1-U) = Uf(U)[1-F(\bar{w})],$$

where $\bar{w}$ is the reservation wage.

$$\begin{aligned} U' &= (1-U)\lambda + U\Big[1 - f(U)[1-F(R)]\Big] \\ \lambda(1-U) &= Uf(U)[1-F(\bar{w})] \end{aligned} \qquad (U' = U)$$

c. Define a stationary equilibrium.

A stationary equilibrium is a stationary unemployment rate U, value functions $\{V^u(U), V^e(w)\}$, and a policy function $\mathbb{I}\{w \geq R\}$ such that

(a) Taking U as given, $V^u(U)$, $V^e(w)$, and $\mathbb{I}\{w \geq R\}$ solve the dynamic programming problem.

(b) The equilibrium unemployment rate is stationary.

$$\lambda(1-U) = Uf(U)[1-F(\bar{w})]$$