Problem Set 1
Math Camp, Spring 2020, UCSB
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Due: Wednesday, Sep 2, 2020

This problem set will help you review the key concepts from the course so far. Feel free to use any resources from the course or internet and submit them by email to the instructor (before midnight on the due date).

1. $\mathbf{v}_1, \cdots, \mathbf{v}_k \in \mathbb{W} \subset \mathbb{R}^n$ forms a basis of $\mathbb{W}$. Then given $\mathbf{w} \in \mathbb{W}$, prove that there exists a "*unique*" $[c_1 \ \cdots \ c_k]' \in \mathbb{R}^k$ such that

$$\mathbf{w} = \sum_{i=1}^{k} c_i \mathbf{v}_i$$

Proof. Supose not. Then there exists $\mathbf{c}, \mathbf{d} \in \mathbb{R}^k$ where $\mathbf{c} \neq \mathbf{d}$ such that

$$\mathbf{w} = \sum_{i=1}^{k} c_i \mathbf{v}_i = \sum_{i=1}^{k} d_i \mathbf{v}_i$$

This implies

$$\sum_{i=1}^{k} (c_i - d_i) \mathbf{v}_i = \mathbf{0}_n$$

This contradicts to linear independence since there exists a nontrivial solution for the above vector equation. ■

2. Consider the following matrices:

$$\mathcal{A} = \begin{bmatrix} 2 & 0 \\ 3 & 8 \end{bmatrix} \qquad \mathcal{B} = \begin{bmatrix} 7 & 2 \\ 6 & 3 \end{bmatrix}$$

(a) Calculate $\mathcal{AB}$ and $\mathcal{BA}$. Is it true that $\mathcal{AB} = \mathcal{BA}$?

$$\mathcal{AB} = \begin{bmatrix} 14 & 4 \\ 69 & 30 \end{bmatrix} \qquad \mathcal{BA} = \begin{bmatrix} 20 & 16 \\ 21 & 24 \end{bmatrix}$$

Matrix multiplicatioin, in general, does not have the commutativity property.

(b) Check $(\mathcal{AB})^T = \mathcal{B}^T \mathcal{A}^T$.

(c) Check $(\mathcal{AB})^{-1} = \mathcal{B}^{-1} \mathcal{A}^{-1}$.

(d) Check $(\mathcal{A}^T)^{-1} = (\mathcal{A}^{-1})^T$.

3. Find eigenvalues and eigenvectors of the following matrix. Normalize the norm to one.

$$\mathcal{C} = \begin{bmatrix} .8 & .05 \\ .2 & .95 \end{bmatrix}$$

$$\begin{aligned} |(\mathcal{C} - \lambda I_2)| &= \begin{vmatrix} \frac{4}{5} - \lambda & \frac{1}{20} \\ \frac{1}{5} & \frac{19}{20} - \lambda \end{vmatrix} \\ &= \left(\frac{4}{5} - \lambda\right)\left(\frac{19}{20} - \lambda\right) - \frac{1}{100} \\ &= \lambda^2 - \frac{7}{4}\lambda + \frac{3}{4} \\ &= (\lambda - 1)\left(\lambda - \frac{3}{4}\right) \end{aligned}$$

$$\lambda_1 = 1 \quad \mathbf{x}_1 = \begin{bmatrix} \frac{1}{\sqrt{17}} \\ \frac{4}{\sqrt{17}} \end{bmatrix} \quad \text{and} \quad \lambda_2 = \frac{3}{4} \quad \mathbf{x}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

4. Express $-\sum_{i=1}^{n} \frac{u_i^2}{2\sigma^2}$ and $\sum_{i=1}^{n} \lambda_i u_i^2$ into quadratic forms using

$$\mathbf{u} := [u_1 \ \cdots \ u_n]', \quad \Sigma := \sigma^2 \mathcal{I}_n, \quad \text{and} \quad \Lambda := \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$-\sum_{i=1}^{n} \frac{u_i^2}{2\sigma^2} = -\frac{1}{2}\mathbf{u}^T \Sigma^{-1} \mathbf{u} \quad \text{and} \quad \sum_{i=1}^{n} \lambda_i v_i^2 = \mathbf{v}^T \Lambda \mathbf{v}$$

5. Consider the following regression equation where $n > k$:

$$
\begin{aligned}
y_i &= x_{i1}\beta_1 + x_{i2}\beta_2 + \cdots + x_{ik-1}\beta_{k-1} + \beta_k + u_i && \forall\ i = 1, \cdots, n \\
&= \mathbf{x}_i{}^T \boldsymbol{\beta} + u_i && \forall\ i = 1, \cdots, n
\end{aligned}
$$

We can rewrite this as matrix equation:

$$\mathbf{y} = \mathcal{X}\boldsymbol{\beta} + \mathbf{u}$$

where

$$
\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \qquad \mathcal{X} = \begin{bmatrix} x_{11} & \cdots & x_{1k-1} & 1 \\ \vdots & & \vdots & \vdots \\ x_{n1} & \cdots & x_{nk-1} & 1 \end{bmatrix} \qquad \boldsymbol{\beta} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_k \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}
$$

(a) Check the followings: $\mathcal{X}^T\mathcal{X} = \sum_{i=1}^n \mathbf{x}_i\mathbf{x}_i^T$ and $\mathcal{X}^T\mathbf{y} = \sum_{i=1}^n \mathbf{x}_i y_i$.

(b) Assume that $\text{rank}(\mathcal{X}) = k$. We derived OLS estimator $\widehat{\boldsymbol{\beta}} = (\mathcal{X}^T\mathcal{X})^{-1}\mathcal{X}^T\mathbf{y}$ in class. Show that

$$(\mathcal{X}^T\mathcal{X})^{-1}\mathcal{X}^T\mathbf{y} = \left(\frac{1}{n}\sum_{i=1}^n \mathbf{x}_i\mathbf{x}_i^T\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^n \mathbf{x}_i\mathbf{y}_i\right)$$

HINT: For any vector $\widehat{\boldsymbol{\beta}}$ and $n \in \mathbb{N}$, $1\widehat{\boldsymbol{\beta}} = \widehat{\boldsymbol{\beta}}$ and $1 = n^{-1}n$.

$$\widehat{\boldsymbol{\beta}} = (\mathcal{X}^T\mathcal{X})^{-1}\mathcal{X}^T\mathbf{y} = \left(\frac{1}{n}\mathcal{X}^T\mathcal{X}\right)^{-1}\left(\frac{1}{n}\mathcal{X}^T\mathbf{y}\right) = \left(\frac{1}{n}\sum_{i=1}^n \mathbf{x}_i\mathbf{x}_i^T\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^n \mathbf{x}_i\mathbf{y}_i\right)$$

(c) Show that

$$\sqrt{n}\left(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right) = \left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i^T\right)^{-1}\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{u}_i\right)$$

HINT: You may start from

$$\widehat{\boldsymbol{\beta}} = \left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i^T\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{y}_i\right) = \left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i^T\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i(\mathbf{x}_i^T\boldsymbol{\beta} + \mathbf{u}_i)\right)$$

$$\begin{aligned}
\widehat{\boldsymbol{\beta}} &= \left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i^T\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{y}_i\right) = \left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i^T\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i(\mathbf{x}_i^T\boldsymbol{\beta} + \mathbf{u}_i)\right) \\
&= \left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i^T\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i^T\right)\boldsymbol{\beta} + \left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i^T\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{u}_i\right) \\
&= \boldsymbol{\beta} + \left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i^T\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{u}_i\right)
\end{aligned}$$

(d) Let $\hat{\mathbf{u}} := \mathbf{y} - \mathcal{X}\widehat{\boldsymbol{\beta}}$. Show that $\sum_{i=1}^{n}\hat{u}_i^2 = \mathbf{u}^T\mathcal{M}_{\mathcal{X}}\mathbf{u}$ where $\mathcal{M}_{\mathcal{X}} = \mathcal{I}_n - \mathcal{X}(\mathcal{X}^T\mathcal{X})^{-1}\mathcal{X}^T$.
HINT: You may start from

$$\hat{\mathbf{u}} = \mathcal{M}_{\mathcal{X}}\mathbf{y} = \mathcal{M}_{\mathcal{X}}(\mathcal{X}\boldsymbol{\beta} + \mathbf{u})$$

$$\begin{aligned}
\hat{\mathbf{u}} &= \mathbf{y} - \mathcal{X}\widehat{\boldsymbol{\beta}} \\
&= \mathbf{y} - \mathcal{X}(\mathcal{X}^T\mathcal{X})^{-1}\mathcal{X}^T\mathbf{y} \\
&= \left(\mathcal{I}_n - \mathcal{X}(\mathcal{X}^T\mathcal{X})^{-1}\mathcal{X}^T\right)\mathbf{y} \\
&= \mathcal{M}_{\mathcal{X}}(\mathcal{X}\boldsymbol{\beta} + \mathbf{u}) \\
&= \mathcal{M}_{\mathcal{X}}\mathbf{u} && (\mathcal{M}_{\mathcal{X}}\mathcal{X} = \mathbf{0}_n) \\
\hat{\mathbf{u}}^T\hat{\mathbf{u}} &= \left(\mathcal{M}_{\mathcal{X}}\mathbf{u}\right)^T\left(\mathcal{M}_{\mathcal{X}}\mathbf{u}\right) \\
&= \mathbf{u}^T\mathcal{M}_{\mathcal{X}}^T\mathcal{M}_{\mathcal{X}}\mathbf{u} \\
&= \mathbf{u}^T\mathcal{M}_{\mathcal{X}}^2\mathbf{u} && \text{(Symmetric)} \\
&= \mathbf{u}^T\mathcal{M}_{\mathcal{X}}\mathbf{u} && \text{(Idempotent)}
\end{aligned}$$