ECON 204C - Macroeconomic Theory

Aiyagari Model

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Learning Objective

- Aiyagari Model (1994)
  - Heterogeneous agents
  - A single exogenous vehicle for borrowing and lending
  - Limits on amounts individual agents may borrow
  - Stationary Rational Expectations Equilibrium
  - Computation using Discrete State Dynamic Programming

Aiyagari Model - Household's Problem

$$v(a, z; \Omega) = \max_{c \geq 0,\ a' \in \mathcal{A}} \Big\{ u(c) + \beta \mathbb{E}_z \big[v(a', z'; \Omega) \big] \Big\}$$

$$s.t. \qquad c + a' \leq wz + \big[1 + r\big] a$$

$$c \geq 0$$

$$a' \geq -\phi$$

The exogenous process $\{z_t\}$ follows a finite state Markov chain with given stochastic matrix P. Let $a'(a, z)$ be an associated policy function for saving, then price to capital stock is given by

$$K^s(r) = \sum_{(a,z) \in \mathcal{A} \times \mathcal{Z}} \Omega(a, z) a'(a, z)$$

Aiyagari Model - Firm's Problem

$$\max_{K,\ N} \left\{ AK^{\alpha}N^{1-\alpha} - (r+\delta)K - wN \right\}$$

$$A\alpha\left(\frac{N^D}{K^D}\right)^{1-\alpha} - (r+\delta) = 0 \qquad \text{(FOC w.r.t. K)}$$

$$A(1-\alpha)\left(\frac{K^D}{N^D}\right)^{\alpha} - w = 0 \qquad \text{(FOC w.r.t. N)}$$

Equilibrium wage associated with a given interest rate is given by

$$w(r) = A(1-\alpha)\left\{\left(\frac{A\alpha}{r+\delta}\right)\right\}^{\frac{\alpha}{1-\alpha}}$$

Inverse demand for capital is given by

$$r(K^D) = A\alpha\left(\frac{N^D}{K^D}\right)^{1-\alpha} - \delta$$

Aiyagari Model - Stationary Equilibrium

Definition A stationary recursive competitive equilibrium is a value function $v : \mathcal{A} \times \mathcal{Z} \to \mathbb{R}$, policy function for the household $c : \mathcal{A} \times \mathcal{Z} \to \mathbb{R}$ and $a' : \mathcal{A} \times \mathcal{Z} \to \mathbb{R}$, firm's choices N^* and K^*, prices r and w, and stationary measure Ω^* such that

1. Given (r, w), c and a' solve the household's problem and v is the associated value function.
2. Given (r, w), the firm chooses optimally its capital K and its labor N.
3. Capital and labor markets clear:

$$\sum_{(a,z)\in A\times Z} a'(a,z)\Omega^*(a,z) = K^* \qquad \sum_{(a,z)\in A\times Z} z\Omega^*(a,z) = N^*$$

4. Ω^* is consistent with $a'(a, z)$.

Discrete State Dynamic Programming

$$v(s) = \max_{d \in D(s)} \left\{ r(s,d) + \beta \sum_{s' \in S} v(s') Q(s,d,s') \right\}$$

- s is the state variable.
- d is the action.
- β is a discount factor.
- $r(s,d)$ is a current reward when the state is s and the action chosen is d.
- $Q(s,d,s')$ is a transitional probability.

Aiyagari Model - Discrete State Dynamic Programming

$$
\begin{aligned}
\mathcal{Z} &= \{.1, 1\} \\
\mathcal{A} &= \{1e-10, \cdots, 20\} \qquad \text{where} \qquad n(\mathcal{A}) = 200 \\
P &= \begin{pmatrix} .9 & .1 \\ .1 & .9 \end{pmatrix} \\
u(c) &= \ln(c) \\
\beta &= .96 \\
\alpha &= .33 \\
\delta &= .05 \\
A &= 1
\end{aligned}
$$

Aiyagari Model - Discrete State Dynamic Programming

$$r(s,d) = r(a,z,a') = \ln c = \begin{cases} \ln\left(wz + [1+r]a - a'\right) & \text{if} \quad c > 0 \\ \\ -\infty & \text{otherwise} \end{cases}$$

$$Q(s,d,s') = Q(a,z,a',z') = P_{zz'}$$

Trick on indexing

$$S = \left\{(a_{i_{\mathcal{A}}}, z_{i_{\mathcal{Z}}})\right\} = \left\{ \underbrace{(a_0,z_0)}_{i_S=0},\ \underbrace{(a_0,z_1)}_{i_S=1},\ \underbrace{(a_1,z_0)}_{i_S=2},\ \underbrace{(a_1,z_1)}_{i_S=3},\ \cdots,\ \underbrace{(a_{199},z_0)}_{i_S=398},\ \underbrace{(a_{199},z_1)}_{i_S=399} \right\}$$

$$i_S = i_{\mathcal{A}} \times 2 + i_{\mathcal{Z}}$$

Aiyagari Model - Discrete State Dynamic Programming

$$K^S = \sum_{(a,z)\in\mathcal{A}\times\mathcal{Z}} a'(a,z)\Omega^*(a,z) = \sum_{a\in\mathcal{A}} a \times Pr(a) \quad \text{where} \quad Pr(a) = \sum_{z\in\mathcal{Z}} \Omega^*(a,z)$$

$$N^S = \sum_{(a,z)\in\mathcal{A}\times\mathcal{Z}} z\Omega^*(a,z) = \sum_{z\in\mathcal{Z}} z \times Pr(z) \quad \text{where} \quad Pr(z) = \sum_{a\in\mathcal{A}} \Omega^*(a,z)$$

Reference

Aiyagari, S. R. (1994). Uninsured idiosyncratic risk and aggregate saving. The Quarterly Journal of Economics, 109(3), 659-684.

Sargent, T. S. & Stachurski, J. (2020, March 30). The Aiyagari Model. Retrieved from `https://python-intro.quantecon.org/aiyagari.html`