Problem Set 3
Math Camp, Spring 2020, UCSB
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Due: Wednesday, Sep 16, 2020

This problem set will help you review the key concepts from the course so far. Feel free to use any resources from the course or internet and submit them by email to the instructor (before midnight on the due date).

1. **Chebychev's Inequality**
   Let $\mathbf{X}$ be a random vector and let $r > 0$ be a positive real number. Assume that $g : \mathbb{R}^n \to \mathbb{R}_+$ takes nonnegative values.

   (a) Check that the following inequality holds for any realization of random vector $\mathbf{X}$.

   $$\frac{g(\mathbf{X})}{r} \geq \mathbb{I}\{g(\mathbf{X}) \geq r\}$$

   - If $g(\mathbf{X}) \geq r$,
   $$\frac{g(\mathbf{X})}{r} \geq 1$$
   - Otherwise,
   $$\frac{g(\mathbf{X})}{r} \geq 0$$

   (b) Check that

   $$\mathbb{E}\Big[\mathbb{I}\{g(\mathbf{X}) \geq r\}\Big] = P\left(g(\mathbf{X}) \geq r\right)$$

   $$\begin{aligned} \mathbb{E}\Big[\mathbb{I}\{g(\mathbf{X}) \geq r\}\Big] &= 1 \times P\left(g(\mathbf{X}) \geq r\right) \; + \; 0 \times P\left(g(\mathbf{X}) < r\right) \\ &= P\left(g(\mathbf{X}) \geq r\right) \end{aligned}$$

   (c) Check that

   $$\mathbb{E}\left[\frac{g(\mathbf{X})}{r}\right] \geq P\left(g(\mathbf{X}) \geq r\right)$$

   $$\begin{aligned} \frac{g(\mathbf{X})}{r} - \mathbb{I}\{g(\mathbf{X}) \geq r\} &\geq 0 && \text{(a)} \\ \mathbb{E}\left[\frac{g(\mathbf{X})}{r} - \mathbb{I}\{g(\mathbf{X}) \geq r\}\right] &\geq 0 && \text{(Weakly positive integrand)} \\ \mathbb{E}\left[\frac{g(\mathbf{X})}{r}\right] &\geq P\left(g(\mathbf{X}) \geq r\right) && \text{(Linear Operator + b)} \end{aligned}$$

(d) Show that

$$\lim_{n\to\infty} \mathbb{E}[\hat{\theta}_n - \theta] = 0 \quad \Rightarrow \quad P\left\{ \left|\hat{\theta}_n - \theta\right| < \epsilon \right\} \overset{n\to\infty}{\longrightarrow} 1 \quad \text{for any} \quad \epsilon > 0$$

Hint: Let $g(\mathbf{X}) := \left(\hat{\theta}_n - \theta\right)^2$ and $r := \epsilon^2$. In addition,

$$\left(\hat{\theta}_n - \theta\right)^2 < \epsilon^2 \quad \Leftrightarrow \quad \left|\hat{\theta}_n - \theta\right| < \epsilon$$

Use Sandwich theorem.

$$P\left\{ \left|\hat{\theta}_n - \theta\right| < \epsilon \right\} \leq \frac{\mathbb{E}\left[\left(\hat{\theta}_n - \theta\right)^2 < \epsilon^2\right]}{\epsilon^2}$$

$$P\left\{ \left|\hat{\theta}_n - \theta\right| < \epsilon \right\} \leq \frac{MSE(\hat{\theta}_n)}{\epsilon^2}$$

$$\lim_{n\to\infty} \mathbb{E}[\hat{\theta}_n - \theta] = 0 \quad \Rightarrow \quad P\left\{ \left|\hat{\theta}_n - \theta\right| < \epsilon \right\} \overset{n\to\infty}{\longrightarrow} 0 \qquad \text{(Sandwich Theorem)}$$

2. Let $X_1, \cdots, X_n$ be a random sample from a distribution with PMF

$$f(x_i|\theta) = \begin{cases} \theta(1-\theta)^{x_i - 1} & \text{if } x_i = 1, 2, \cdots \\ 0 & \text{otherwise} \end{cases}$$

where $\theta \in (0, 1)$.

- Find the method of moments estimator for θ.
  Hint: Think about $\mathbb{E}[X] - (1-\theta)\mathbb{E}[X]$.

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{x=1}^{\infty} x\theta(1-\theta)^{x-1} \\
(1-\theta)\mathbb{E}[X] &= \theta\Big[(1-\theta) + 2(1-\theta)^2 + \cdots\Big] \\
\mathbb{E}[X] - (1-\theta)\mathbb{E}[X] &= \theta\Big[1 + (1-\theta) + (1-\theta)^2 + \cdots\Big] \\
\mathbb{E}[X] - (1-\theta)\mathbb{E}[X] &= \theta \times \frac{1}{1-(1-\theta)} && \text{(Geometric Series)} \\
\mathbb{E}[X] &= \frac{1}{\theta}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} X_i &= \frac{1}{\hat{\theta}_n} \\
\hat{\theta}_n^{mm} &= \frac{1}{\bar{X}_n}
\end{aligned}$$

- Find the maximum likelihood estimator for θ.

$$\begin{aligned}
\mathfrak{L}(\theta|\mathbf{x}) &= \prod_{i=1}^{n} \theta(1-\theta)^{x_i - 1} = \theta^n (1-\theta)^{\sum_{i=1}^{n} x_i - n} \\
l(\theta|\mathbf{x}) &= n\ln(\theta) + \left(\sum_{i=1}^{n} x_i - n\right)\ln(1-\theta) \\
\frac{dl}{d\theta} &= \frac{n}{\theta} - \frac{\sum_{i=1}^{n} x_i - n}{1-\theta} \\
\frac{n}{\hat{\theta}_n^{ml}} - \frac{\sum_{i=1}^{n} x_i - n}{1-\hat{\theta}_n^{ml}} &= 0 \\
\hat{\theta}_n^{ml} &= \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{X}_n}
\end{aligned}$$

3. Suppose a coin flip lands on heads with probability θ.

$$f(x_i|\theta) = \begin{cases} \theta & \text{if } x_i = 1 \quad \text{(Head)} \\ 1-\theta & \text{if } x_i = 0 \quad \text{(Tail)} \\ 0 & \text{otherwise} \end{cases}$$

Let $X_1, \cdots, X_n$ be a random sample and let X be a random vector with the same probability distribution as X_i's.

- Apply CLT and derive the probability distribution of $\sqrt{n}(\bar{X}_n - \theta)$.

$$\sqrt{n}(\bar{X}_n - \theta) \xrightarrow{d} N\Big(0, \theta(1-\theta)\Big)$$

- (Frequentist Approach) Two mutually exclusive hypotheses on θ are given by

$$H_0 : \theta_0 \in \{.5\} \quad \text{and} \quad H_1 : \theta_A \in \{.8\}$$

Consider the following test function with level $\alpha = .05$:

$$T_n(\alpha) = \mathbb{I}\left\{ \left| \frac{\sqrt{n}\,\left(\bar{X}_n - \theta\right)}{\sqrt{\theta(1-\theta)}} \right| > q_{1-\frac{\alpha}{2}} \right\} \quad \text{where} \quad \Phi\left(q_{1-\frac{\alpha}{2}}\right) = 1 - \frac{\alpha}{2}$$

Assume that we observed 21 heads out of 32 tosses. What is your p-value and your decision?
p-value = .039 and we reject the null hypothesis.

- (Bayesian Approach) Assume that your prior is given by

$$\pi(H_0) = \pi(H_1) = .5$$

Assume that we observed 21 heads out of 32 tosses. What is your $\hat{\theta}_n$? What is the posterior probability on H_0? Do you think the evidence is in favor of the null? Hint: Derive the likelihood $P(\hat{\theta}_n|H_0)$ and $P(\hat{\theta}_n|H_1)$. Use Bayes' rule to derive the posterior.

$$\hat{\theta}_n = .65625$$

$$P(\hat{\theta}_n = .65625|H_0) = \binom{32}{21} \times .5^2 \times 0.5^1 1 \approx .030$$
$$P(\hat{\theta}_n = .65625|H_1) = \binom{32}{21} \times .8^2 \times 0.2^1 1 \approx .024$$
$$P(H_0|\hat{\theta}_n = .65625) = \frac{P(\hat{\theta}_n = .65625|H_0)\pi(H_0)}{P(\hat{\theta}_n = .65625|H_0)\pi(H_0) + P(\hat{\theta}_n = .65625|H_1)\pi(H_1)} \approx .55$$

Evidence is midly in favor of the null, $P(H_0|\hat{\theta}_n = .65625) > \pi(H_0)$.