

# **ECON 204C - Macroeconomic Theory**

**Asset Pricing**

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# Learning Objective

- Asset Pricing
  - Law of iterated expectation
  - Representative agent consumption-based asset pricing
  - CRRA and risk premium
  - Elasticity of Substitution

# Law of Iterated Expectation

$$\mathbb{E} \left[ \mathbb{E}[Y|X, Z] \mid X \right] = \mathbb{E}[Y|X]$$

$$\mathbb{E}_t[X_{t+2}] = \mathbb{E}[X_{t+2}|I_t]$$

$$\mathbb{E}_{t+1}[X_{t+2}] = \mathbb{E}[X_{t+2}|I_{t+1}]$$

$$I_t \subset I_{t+1}$$

$$\mathbb{E}_t \left[ \mathbb{E}_{t+1}[X_{t+2}] \right] = \mathbb{E}_t[X_{t+2}]$$

The smaller information set wins!

# Representative Agent Consumption-based Asset Pricing

$$\begin{aligned} \max_{\{c_t, \{a_{j,t+1}\}_j\}_{t=0}^{\infty}} \quad & \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma} \\ \text{s.t.} \quad & c_t + \sum_j p_{j,t} a_{j,t+1} = y_t + \sum_j (p_{j,t} + d_{j,t}) a_{j,t} \end{aligned}$$

$$\underbrace{p_{j,t} u'(c_t)}_{MC} = \mathbb{E}_t \left[ \underbrace{\beta u'(c_{t+1}) [p_{j,t+1} + d_{j,t+1}]}_{MB} \right] \quad (\text{Euler equation})$$

$$p_{j,t} = \mathbb{E}_t \left[ \underbrace{\beta \frac{u'(c_{t+1})}{u'(c_t)}}_{\text{Stochastic Discount Factor (SDF)}} [p_{j,t+1} + d_{j,t+1}] \right] \quad (\text{Asset Pricing})$$

$$p_{j,t} = \mathbb{E}_t \left[ \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} [p_{j,t+1} + d_{j,t+1}] \right]$$

# Representative Agent Consumption-based Asset Pricing

$$\begin{aligned}
 p_{j,t} &= \mathbb{E}_t \left[ \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} [p_{j,t+1} + d_{j,t+1}] \right] \\
 &= \mathbb{E}_t \left[ \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} p_{j,t+1} \right] + \mathbb{E}_t \left[ \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} d_{j,t+1} \right] \\
 &= \mathbb{E}_t \left[ \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} \mathbb{E}_{t+1} \left[ \beta \left( \frac{c_{t+2}}{c_{t+1}} \right)^{-\gamma} [p_{j,t+2} + d_{j,t+2}] \right] \right] + \mathbb{E}_t \left[ \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} d_{j,t+1} \right] && \text{(Recursion)} \\
 &= \mathbb{E}_t \left[ \mathbb{E}_{t+1} \left[ \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} \beta \left( \frac{c_{t+2}}{c_{t+1}} \right)^{-\gamma} [p_{j,t+2} + d_{j,t+2}] \right] \right] + \mathbb{E}_t \left[ \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} d_{j,t+1} \right] && \text{(Information)} \\
 &= \mathbb{E}_t \left[ \beta^2 \left( \frac{c_{t+2}}{c_t} \right)^{-\gamma} [p_{j,t+2} + d_{j,t+2}] \right] + \mathbb{E}_t \left[ \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} d_{j,t+1} \right] && \text{(LIE)} \\
 &= \mathbb{E}_t \left[ \beta^2 \left( \frac{c_{t+2}}{c_t} \right)^{-\gamma} p_{j,t+2} \right] + \sum_{\tau=1}^2 \mathbb{E}_t \left[ \beta^{\tau} \left( \frac{c_{t+\tau}}{c_t} \right)^{-\gamma} d_{j,t+\tau} \right] \\
 &= \dots \\
 &= \mathbb{E}_t \left[ \lim_{k \rightarrow \infty} \beta^k \left( \frac{c_{t+k}}{c_t} \right)^{-\gamma} p_{j,t+k} \right] + \sum_{\tau=1}^{\infty} \mathbb{E}_t \left[ \beta^{\tau} \left( \frac{c_{t+\tau}}{c_t} \right)^{-\gamma} d_{j,t+\tau} \right] \\
 &= \sum_{\tau=1}^{\infty} \mathbb{E}_t \left[ \beta^{\tau} \left( \frac{c_{t+\tau}}{c_t} \right)^{-\gamma} d_{j,t+\tau} \right] && \text{(TVC)}
 \end{aligned}$$

# Representative Agent Consumption-based Asset Pricing

$$p_{j,t} = \mathbb{E}_t \left[ \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} [p_{j,t+1} + d_{j,t+1}] \right]$$

$$p_{j,t} = \mathbb{E}_t \left[ \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} \left[ \frac{p_{j,t+1}}{d_{j,t+1}} + 1 \right] d_{j,t+1} \right]$$

(Factor out  $d_{j,t+1}$ )

$$\frac{p_{j,t}}{d_{j,t}} = \mathbb{E}_t \left[ \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} \left[ \frac{p_{j,t+1}}{d_{j,t+1}} + 1 \right] \frac{d_{j,t+1}}{d_{j,t}} \right]$$

$= \dots$

$$= \sum_{\tau=1}^{\infty} \mathbb{E}_t \left[ \beta^{\tau} \left( \frac{c_{t+\tau}}{c_t} \right)^{-\gamma} \frac{d_{j,t+\tau}}{d_{j,t}} \right]$$

(Recursion + TVC)

# Constant Relative Risk Aversion and Risk Premium

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma} \quad \text{and} \quad \gamma = -\frac{u''(c)}{u'(c)}c$$

Consider offering two alternatives to a consumer who starts off with risk-free consumption level  $c_0$ . She can receive

1.  $c_0 + \Delta_c$  with certainty, or

$$L_1(\Delta_c) = (c_0 + \Delta_c : 1)$$

2. a lottery paying  $c_0 - y$  with probability .5 and  $c_0 + y$  with probability .5

$$L_2 = (c_0 - y, c_0 + y : .5, .5)$$

# Constant Relative Risk Aversion and Risk Premium

For given values of  $y$  and  $c_0$ , we want to find  $\Delta_c = \Delta_c(y, c_0)$  that leaves the consumer indifferent between these two lotteries.

$$L_1(\Delta_c(y, c_0)) \sim L_2$$
$$u(c_0 - \Delta_c(y, c_0)) = .5u(c_0 - y) + .5u(c_0 + y)$$

Taylor expansion of  $u$  around the point  $a$

$$u(x) \approx u(a) + u'(a)(x - a) \quad \text{(First order approximation)}$$

$$u(x) \approx u(a) + u'(a)(x - a) + \frac{1}{2}u''(a)(x - a)^2 \quad \text{(Second order approximation)}$$



# Constant Relative Risk Aversion and Risk Premium

$$u(c_0 - \Delta_c(y, c_0)) = .5u(c_0 - y) + .5u(c_0 + y)$$

First order approximate LHS around  $c_0$ .

$$u(c_0 - \Delta_c(y, c_0)) \approx u(c_0) - u'(c_0)\Delta_c(y, c_0)$$

Let  $Y$  denote random variable taking value  $y$  with probability .5 and  $-y$  with probability .5. Second order approximate RHS around  $c_0$ .

$$u(c_0 + Y) \approx u(c_0) + u'(c_0)Y + \frac{1}{2}u''(c_0)Y^2$$
$$\mathbb{E}[u(c_0 + Y)] \approx u(c_0) + \frac{1}{2}u''(c_0) \underbrace{y^2}_{=Var(Y)}$$

# Constant Relative Risk Aversion and Risk Premium

Ignoring the higher-order terms gives

$$\begin{aligned}\Delta_c(y, c_0) &\approx \frac{1}{2}y^2 \left( -\frac{u''(c_0)}{u'(c_0)} \right) \\ \Delta_c(y, c_0) &\approx \frac{1}{2}y^2 \frac{\gamma}{c_0} \\ \frac{\Delta_c(y, c_0)}{y} &\approx \frac{1}{2}\gamma \frac{y}{c_0}\end{aligned}\tag{CRRA}$$

- LHS is the percentage premium that the consumer is willing to pay to avoid a fair bet of size  $y$
- RHS is one half  $\gamma$  times the ratio of the size of the bet  $y$  to his initial consumption level  $c_0$ .

## Constant Relative Risk Aversion and Risk Premium

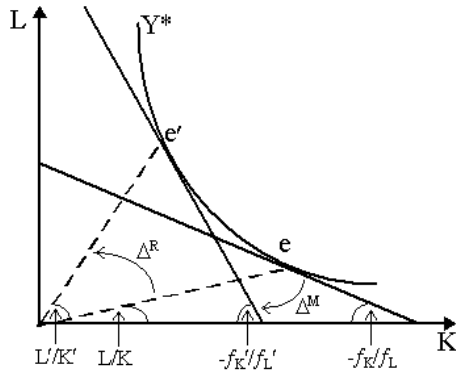
Following Cochrane (1997), think of confronting someone with initial consumption of \$ 50,000 per year with a 50 - 50 chance of winning or losing  $y$  dollars.

$\gamma \setminus y$	10	100	1000	5000
2	.02	.2	20	500
5	.05	5	50	1217
10	.1	1	100	2212

[ Risk premium  $\Delta_c(y, c_0 = 50000)$  ]

For values of  $\gamma$  even as high as 5, risk premiums are too big. This result is one important source of macroeconomists' prejudice that  $\gamma$  should not be much higher than 2 or 3.

# Elasticity of Substitution



$$\Delta M = \Delta \left( -\frac{dL}{dK} \Big|_{Y^*=F(K^*,L^*)} \right) = \Delta \left( \frac{F_K(K^*,L^*)}{F_L(K^*,L^*)} \right)$$

$$\Delta R = \Delta \left( \frac{L}{K} \right)$$

$$\text{Elasticity} = \frac{\frac{d(L/K)}{(L/K)}}{\frac{d(F_K/F_L)}{(F_K/F_L)}} = \frac{d \ln(L/K)}{d \ln(F_K/F_L)}$$

## Constant Elasticity of Substitution (CES)

$$Y = F(K, L) = \left( \alpha_L L^{\frac{\varepsilon-1}{\varepsilon}} + \alpha_K K^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

$$MRTS_{L,K} = - \frac{dL}{dK} \bigg|_{Y=F(K,L)} = \frac{F_K}{F_L} = \frac{\alpha_K}{\alpha_L} \left( \frac{L}{K} \right)^{\frac{1}{\varepsilon}}$$

$$\frac{L}{K} = \left( \frac{\alpha_L}{\alpha_K} \right)^{\varepsilon} \left( \frac{F_K}{F_L} \right)^{\varepsilon}$$

$$\ln(L/K) = \varepsilon \ln(\alpha_L/\alpha_K) + \varepsilon \ln(F_K/F_L)$$

$$\frac{d \ln(L/K)}{d \ln(F_K/F_L)} = \varepsilon$$

## CRRA and IES

$$U = U(c_1, c_2) = u(c_1) + \beta u(c_2) \quad \text{where} \quad u(x) = \frac{x^{1-\gamma}}{1-\gamma}$$

$$MRS_{2,1} = -\frac{dc_2}{dc_1} \Big|_{U=U(c_1,c_2)} = \frac{U_1}{U_2} = \frac{u'(c_1)}{\beta u'(c_2)} = \frac{c_1^{-\gamma}}{\beta c_2^{-\gamma}} = \frac{1}{\beta} \left( \frac{c_1}{c_2} \right)^{-\gamma}$$

$$\frac{c_2}{c_1} = \beta^{1/\gamma} \left( \frac{U_1}{U_2} \right)^{1/\gamma}$$

$$\ln(c_2/c_1) = \frac{1}{\gamma} \ln \beta + \frac{1}{\gamma} \ln(U_1/U_2)$$

$$\frac{d \ln(c_2/c_1)}{d \ln(U_1/U_2)} = \frac{1}{\gamma}$$

# Reference

Ljungqvist, L., & Sargent, T. J. (2018). Recursive macroeconomic theory. MIT press.