Required Problems

1. Consider the following matrices:

$$A = \begin{bmatrix} 2 & 8 \\ 3 & 0 \\ 5 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 \\ 3 & 8 \end{bmatrix} \qquad C = \begin{bmatrix} 7 & 2 \\ 6 & 3 \end{bmatrix}$$

   (a) Calculate AB.
   (b) Calculate CB.
   (c) Is it true that $CB = BC$? Justify your response.

2. Find the determinant of each of the following matrices.

   (a) $D = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$

   (b) $E = \begin{bmatrix} 8 & 1 & 3 \\ 4 & 0 & 1 \\ 6 & 0 & 3 \end{bmatrix}$

3. Find the inverse of the following matrix:

$$F = \begin{bmatrix} 5 & 2 \\ 0 & 1 \end{bmatrix}$$

4. Find eigenvalues and eigenvectors associated with the matrix:

$$G = \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix}$$

Practice Problems

5. Consider the following column vectors:

$$\mathbf{v}_1 = \begin{bmatrix} 4 \\ -2 \\ 3 \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 6 \\ 0 \\ 2 \end{bmatrix} \qquad \mathbf{v}_4 = \begin{bmatrix} 1 \\ -5 \\ 3 \\ 2 \end{bmatrix}$$

   Are these vectors linearly independent? Justify your response.

6. Find the determinant of the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 0 & 9 \\ 2 & 3 & 4 & 6 \\ 1 & 6 & 0 & -1 \\ 0 & -5 & 0 & 8 \end{bmatrix}$$

7. Find the inverse of each of the following the matrices.

   (a) $B = \begin{bmatrix} -1 & 0 \\ 9 & 2 \end{bmatrix}$

   (b) $C = \begin{bmatrix} 4 & 1 & -1 \\ 0 & 3 & 2 \\ 3 & 0 & 7 \end{bmatrix}$

8. Consider the linear system given by $\mathbf{Y} = X\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, where $\mathbf{Y}$ and $\boldsymbol{\varepsilon}$ are $n \times 1$ vectors, X is a $n \times k$ matrix of full rank, and $\boldsymbol{\beta}$ is a $k \times 1$ vector.

   (a) Suppose that $\boldsymbol{\varepsilon}$ is orthogonal to each of the columns in $\mathbf{X}$ (i.e., $\mathbf{X}'\boldsymbol{\varepsilon} = \mathbf{0}$). Using matrix algebra, solve for $\boldsymbol{\beta}$.

   (b) Further, suppose that $\boldsymbol{\beta}$ is a 2×1 vector and the matrix X is $n \times 2$, such that:

   $$\boldsymbol{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \qquad \mathbf{X} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}$$

   In addition, assume that $\sum_{i=1}^{n} x_i = 0$. Find expressions for β_1 and β_2.

9. Use Cramer's rule, row operations, or matrix inversion to solve the following linear systems:

   (a) $8x_1 - x_2 = 16$
   $2x_2 + 5x_3 = 5$
   $2x_1 + 3x_3 = 7$

   (b) $-x_1 + 3x_2 + 2x_3 = 24$
   $x_1 + x_3 = 6$
   $5x_2 - x_3 = 8$

   (c) $4x + 3y - 2z = 1$
   $x + 2y = 6$
   $3x + z = 4$

   (d) $-x + y + z = a$
   $x - y + z = b$
   $x + y - z = c$

10. Suppose there is a perfectly competitive firm with a production function $y = f(x_1, x_2)$, increasing in both arguments. The firm sells output y at the market price p. The firm purchases an input x_1 and price w_1; x_2, however, represents the entrepreneur's input and is limited to $\bar{x}_2$. Thus, the firms maximization problem, framed as a Lagrangian, is given by

    $$\mathcal{L} = p \cdot f(x_1, x_2) - w_1 \cdot x_1 + \lambda(\bar{x}_2 - x_2)$$

    This yields the first order conditions:

    $$\begin{aligned} \mathcal{L}_1 &= p \cdot f_1(x_1, x_2) - w_1 = 0 \\ \mathcal{L}_2 &= p \cdot f_2(x_1, x_2) - \lambda = 0 \\ \mathcal{L}_\lambda &= \bar{x}_2 - x_2 = 0 \end{aligned}$$

(a) What is the sufficient second order condition?

Given that the profit-maximizing levels of x_1, x_2, and λ have been found as functions of p, w_1, and $\bar{x}_2$, e.g., $x_1^* = g(p, w_1, \bar{x}_2)$, the solutions may be plugged back into the first order conditions yielding:

$$\begin{aligned} pf_1(x_1^*, x_2^*) - w_1 &\equiv 0 \\ pf_2(x_1^*, x_2^*) - \lambda^* &\equiv 0 \\ \bar{x}_2 - x_2^* &\equiv 0 \end{aligned}$$

(b) Use Cramer's Rule to find a comparative statics prediction for $\frac{\partial x_1^*}{\partial w_1}$ (Hint: use the chain rule on the system of identities, differentiating w.r.t. w_1).

11. Find the eigenvalues and eigenvectors of each of the following matrices:

    (a) $A = \begin{bmatrix} -2 & 2 \\ 2 & -4 \end{bmatrix}$

    (b) $B = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

12. Given a quadratic form $\mathbf{x}'C\mathbf{x}$, where C is a symmetric 2×2 matrix:

$$C = \begin{bmatrix} a & d \\ d & b \end{bmatrix}$$

Show that the following are true (Hint: use the quadratic equation):

    (a) Both eigenvalues must be real (i.e., they cannot involve $\sqrt{-1}$).

    (b) C has repeated roots if and only if it is of the form $C = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$

13. Let A be a $m \times n$ matrix. Show that $(A^T)^T = A$.

14. Let B and C be $m \times n$ matrices. Show that $k(B + C) = kB + kC$

15. Let D be a $n \times n$ invertible matrix. Show that D^{-1} is unique.

16. Let E be a $n \times n$ matrix. Show that if E is idempotent, $I_n - E$ is idempotent as well.

17. Show that for any 2×2 matrix A

$$|A| = \frac{1}{2} \begin{vmatrix} \mathrm{tr}(A) & 1 \\ \mathrm{tr}(A^2) & \mathrm{tr}(A) \end{vmatrix}$$

18. State whether each of the following statements is true or false. If it is false, provide a counterexample.

    (a) No system of linear equations can have exactly k solutions for any $k \geq 2$.

    (b) Any system of n linear equations in n unknowns has at least one solution.

    (c) Any system of n linear equations in n unknowns has at most one solution.

    (d) If $A\underline{x} = \underline{0}$ has a solution, then $A\underline{x} = \underline{b}$ has a solution.

    (e) If an $n \times n$ matrix A is invertible, then $A\underline{x} = \underline{0}$ has the unique solution $\underline{x} = \underline{0}$.

    (f) If an $n \times n$ matrix A is full rank, then $A\underline{x} = \underline{b}$ has a solution.

    (g) If an $n \times n$ matrix A has rank less than n, then $A\underline{x} = \underline{b}$ has no solution.