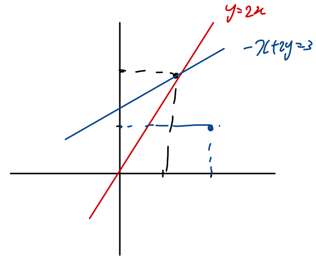
Row picture
Column picture
& Sub space

Plug into

$2y = x+3$

$\frac{x+3}{2}$



The geometry of linear equations

The fundamental problem of linear algebra is to solve n linear equations in n unknowns; for example:

n개의 선형 방정식을 푸는 것.

$$
\begin{aligned}
2x - y &= 0 \\
-x + 2y &= 3.
\end{aligned}
$$

$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$

$A \quad X = b$ → vector.
↳ vector of unknowns.
↳ matrix of coefficients

In this first lecture on linear algebra we view this problem in three ways.

The system above is two dimensional ($n = 2$). By adding a third variable z we could expand it to three dimensions.

Row Picture

→ one eq at a time 2x2,

$Ax = b$.
coefficient

one row at a time

Plot the points that satisfy each equation. The intersection of the plots (if they do intersect) represents the solution to the system of equations. Looking at Figure 1 we see that the solution to this system of equations is $x = 1, y = 2$.

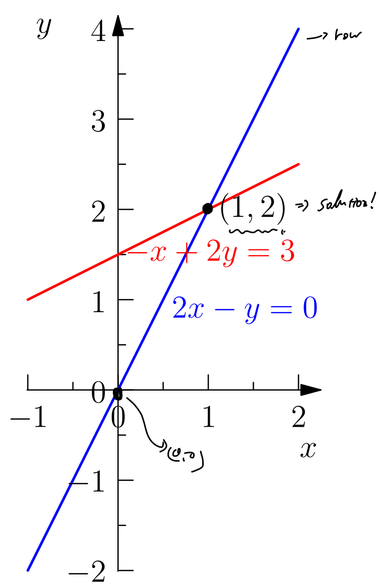


Figure 1: The lines $2x - y = 0$ and $-x + 2y = 3$ intersect at the point $(1, 2)$.

We plug this solution in to the original system of equations to check our work:

$$
\begin{aligned}
2 \cdot 1 - 2 &= 0 \\
-1 + 2 \cdot 2 &= 3.
\end{aligned}
$$

The solution to a three dimensional system of equations is the common point of intersection of three planes (if there is one).

row/column 이해

⊛ => Such an important

columns!

Column Picture

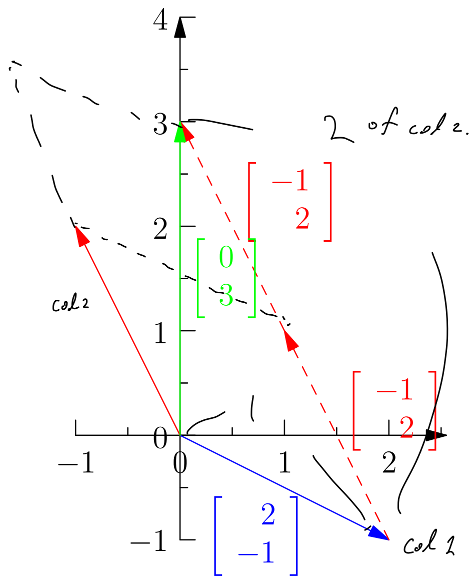
In the column picture we rewrite the system of linear equations as a single equation by turning the coefficients in the columns of the system into vectors:

$$x\begin{bmatrix} 2 \\ -1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

vectors!

Given two vectors $\mathbf{c}$ and $\mathbf{d}$ and scalars x and y, the sum $x\mathbf{c} + y\mathbf{d}$ is called a *linear combination* of $\mathbf{c}$ and $\mathbf{d}$. Linear combinations are important throughout this course.

Geometrically, we want to find numbers x and y so that x copies of vector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ added to y copies of vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ equals the vector $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$. As we see from Figure 2, $x = 1$ and $y = 2$, agreeing with the row picture in Figure 2.



합치면!

Figure 2: A linear combination of the column vectors equals the vector $\mathbf{b}$.

In three dimensions, the column picture requires us to find a linear combination of three 3-dimensional vectors that equals the vector $\mathbf{b}$.

A

Matrix Picture

We write the system of equations

$$\begin{aligned} 2x - y &= 0 \\ -x + 2y &= 3 \end{aligned}$$

matrix × vector

⊛ Can I solve $Ax=b$ for every b?

Do the linear combinations of the columns fill 3-D space?

For this A, Answer => Yes.

셋 중 하나가 다른 둘 중에 하나를 포함하면 → 풀 수 X.

세 평면이 있는데 적어도 하나가 나머지 둘 중에 하나에 포함되는 케이스면 b를 구할 수 X

singular case.

column이 동일하면

5주

4주

3주

(2주에 끝낸다

as a single equation by using matrices and vectors:

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

The matrix $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ is called the *coefficient matrix*. The vector $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ is the vector of unknowns. The values on the right hand side of the equations form the vector $\mathbf{b}$:

$$A\mathbf{x} = \mathbf{b}.$$

The three dimensional matrix picture is very like the two dimensional one, except that the vectors and matrices increase in size.

Matrix Multiplication

How do we multiply a matrix A by a vector $\mathbf{x}$?

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \ ?$$

$$1 \times \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \times \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \times 1 + 10 \\ 1 \times 1 + 2 \times 3 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}$$

One method is to think of the entries of $\mathbf{x}$ as the coefficients of a linear combination of the column vectors of the matrix:

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}.$$

This technique shows that $A\mathbf{x}$ is a linear combination of the columns of A.

You may also calculate the product $A\mathbf{x}$ by taking the dot product of each row of A with the vector $\mathbf{x}$:

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 + 5 \cdot 2 \\ 1 \cdot 1 + 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}.$$

Linear Independence

In the column and matrix pictures, the right hand side of the equation is a vector $\mathbf{b}$. Given a matrix A, can we solve:

$$A\mathbf{x} = \mathbf{b}$$

for every possible vector $\mathbf{b}$? In other words, do the linear combinations of the column vectors fill the xy-plane (or space, in the three dimensional case)?

If the answer is "no", we say that A is a *singular matrix*. In this singular case its column vectors are *linearly dependent*; all linear combinations of those vectors lie on a point or line (in two dimensions) or on a point, line or plane (in three dimensions). The combinations don't fill the whole space.

18.06SC Linear Algebra
Fall 2011

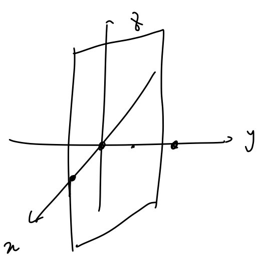
$$2x - y = 0$$
$$-x + 2y - z = -1$$
$$-3y + 4z = 4$$

$$x\begin{bmatrix}2\\-1\\0\end{bmatrix} + y\begin{bmatrix}-1\\2\\-1\end{bmatrix} + z\begin{bmatrix}0\\-3\\4\end{bmatrix} = \begin{bmatrix}0\\-1\\4\end{bmatrix}$$

↳ linear combination of three vectors.

$$\begin{bmatrix}2 & -1 & 0\\-1 & 2 & -1\\0 & -3 & 4\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\-1\\4\end{bmatrix}$$

Row pic



→ linear combinations (add, subtract, ...)

$\vec{u}, \vec{v}, \vec{w}$ → $\boxed{\text{linear combination}}$ → $x_1 u + x_2 v + x_3 w = b$

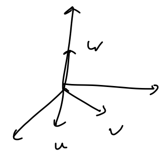
x_1 ↓ Scalar

Vector → Matrices → subspace

$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Abstraction.

↳ 여기는 3차원이지만 10, 100차원도 가능 (시각화는 못할 뿐)



An overview of key ideas

This is an overview of linear algebra given at the start of a course on the mathematics of engineering.

Linear algebra progresses from vectors to matrices to subspaces.

Vectors

What do you do with vectors? Take combinations.

We can multiply vectors by scalars, add, and subtract. Given vectors **u**, **v** and **w** we can form the *linear combination* $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{b}$.

An example in $\mathbb{R}^3$ would be:

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The collection of all multiples of **u** forms a line through the origin. The collection of all multiples of **v** forms another line. The collection of all combinations of **u** and **v** forms a plane. Taking *all combinations* of some vectors creates a *subspace*.

We could continue like this, or we can use a matrix to add in all multiples of **w**.

Matrices

Create a matrix A with vectors **u**, **v** and **w** in its columns:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

The product:

$$A\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 + x_2 \\ -x_2 + x_3 \end{bmatrix}$$

equals the sum $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{b}$. The product of a matrix and a vector is a combination of the columns of the matrix. (This particular matrix A is a *difference matrix* because the components of $A\mathbf{x}$ are differences of the components of that vector.)

When we say $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{b}$ we're thinking about multiplying numbers by vectors; when we say $A\mathbf{x} = \mathbf{b}$ we're thinking about multiplying a matrix (whose columns are **u**, **v** and **w**) by the numbers. The calculations are the same, but our perspective has changed.

number x vector → matrix x number

For any input vector **x**, the output of the operation "multiplication by A" is some vector **b**:

$$A\begin{bmatrix}1\\4\\9\end{bmatrix}=\begin{bmatrix}1\\3\\5\end{bmatrix}.$$

A deeper question is to start with a vector **b** and ask "for what vectors **x** does $A\mathbf{x}=\mathbf{b}$?" In our example, this means solving three equations in three unknowns. Solving:

$$A\mathbf{x}=\begin{bmatrix}1&0&0\\-1&1&0\\0&-1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_1\\x_2-x_1\\x_3-x_2\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

is equivalent to solving:

b_1+b_2

$b_1+b_2+b_3$

$$\begin{aligned}x_1&=b_1\\x_2-x_1&=b_2\\x_3-x_2&=b_3.\end{aligned}$$

We see that $x_1=b_1$ and so x_2 must equal b_1+b_2. In vector form, the solution is:

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}b_1\\b_1+b_2\\b_1+b_2+b_3\end{bmatrix}.$$

matrix x vector

But this just says:

inverse matrix

$$\mathbf{x}=\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix},$$

or $\mathbf{x}=A^{-1}\mathbf{b}$. If the matrix A is invertible, we can multiply on both sides by A^{-1} to find the unique solution $\mathbf{x}$ to $A\mathbf{x}=\mathbf{b}$. We might say that A represents a transform $\mathbf{x}\to\mathbf{b}$ that has an inverse transform $\mathbf{b}\to\mathbf{x}$.

In particular, if $\mathbf{b}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ then $\mathbf{x}=\begin{bmatrix}0\\0\\0\end{bmatrix}$.

one operation

The second example has the same columns **u** and **v** and replaces column vector **w**:

$$C=\begin{bmatrix}1&0&-1\\-1&1&0\\0&-1&1\end{bmatrix}.$$

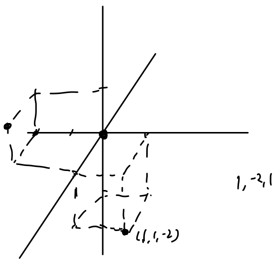
w만 $\begin{bmatrix}0\\0\\1\end{bmatrix}$에서 $\begin{bmatrix}-1\\0\\1\end{bmatrix}$로 변경

Then:

$$C\mathbf{x}=\begin{bmatrix}1&0&-1\\-1&1&0\\0&-1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_1-x_3\\x_2-x_1\\x_3-x_2\end{bmatrix}$$

and our system of three equations in three unknowns becomes circular.

$Cx=0$ 이면 x는 line solution (c, c, c)



이 경우에는 C의 역행렬을 곱해서
x의 0이 아닌 답을 찾을수 X
↳ C^{-1} 곱하면 우리긴 역행렬이니까.

Where before $A\mathbf{x} = \mathbf{0}$ implied $\mathbf{x} = \mathbf{0}$, there are non-zero vectors $\mathbf{x}$ for which $C\mathbf{x} = \mathbf{0}$. For any vector $\mathbf{x}$ with $x_1 = x_2 = x_3$, $C\mathbf{x} = \mathbf{0}$. This is a significant difference; we can't multiply both sides of $C\mathbf{x} = \mathbf{0}$ by an inverse to find a non-zero solution $\mathbf{x}$.

The system of equations encoded in $C\mathbf{x} = \mathbf{b}$ is:

$$\begin{aligned} x_1 - x_3 &= b_1 \\ x_2 - x_1 &= b_2 \\ x_3 - x_2 &= b_3. \end{aligned}$$

If we add these three equations together, we get:

$$\boxed{0 = b_1 +_2 + b_3.}$$

This tells us that $C\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$ only when the components of $\mathbf{b}$ sum to 0. In a physical system, this might tell us that the system is stable as long as the forces on it are balanced.

같은 plane에 놓여있지 않는 애들은 linear comb가 X
⇒ b에 대해 Cx=b를 만족하는 x가 없다.

Subspaces

Geometrically, the columns of C lie in the same plane (they are *dependent*; the columns of A are *independent*). There are many vectors in $\mathbb{R}^3$ which do not lie in that plane. Those vectors cannot be written as a linear combination of the columns of C and so correspond to values of $\mathbf{b}$ for which $C\mathbf{x} = \mathbf{b}$ has no solution $\mathbf{x}$. The linear combinations of the columns of C form a two dimensional *subspace* of $\mathbb{R}^3$. ⇒ C의 컬럼의 선형결합은 3차원 공간에서 2차원 평면으로 나타낸다.

all combs of u, v, w
‖ "all vectors Cx"
당연! ↓ 선형!

This plane of combinations of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ can be described as "all vectors $C\mathbf{x}$". But we know that the vectors $\mathbf{b}$ for which $C\mathbf{x} = \mathbf{b}$ satisfy the condition $b_1 + b_2 + b_3 = 0$. So the plane of all combinations of $\mathbf{u}$ and $\mathbf{v}$ consists of all vectors whose components sum to 0.

If we take all combinations of:

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \text{ and } \mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

이 세개의 상수배로 R^3 전체를 나타낼수 있다고.

$$\underbrace{\begin{bmatrix} u & v & w \end{bmatrix}}_{\text{벡터}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{C}_{\text{상수 매트릭스}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

↓ 벡터의 선형결합

we get the entire space $\mathbb{R}^3$; the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^3$. We say that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ form a *basis* for $\mathbb{R}^3$.

A *basis* for $\mathbb{R}^n$ is a collection of n independent vectors in $\mathbb{R}^n$. Equivalently, a basis is a collection of n vectors whose combinations cover the whole space. Or, a collection of vectors forms a basis whenever a matrix which has those vectors as its columns is invertible.

※ linear comb이 어떤 plane을 구성하는 모든 점을 가리킬수 있다
‖
기저(basis)는 바탕으로 한 벡터의 조합계를 형성할수 있다.

$k u = v$ 가 되지 않는 (= 평행하지 않은)
n개의 벡터들로 R^n 의 공간을 표현할수 있음.

A *vector space* is a collection of vectors that is closed under linear combinations. A *subspace* is a vector space inside another vector space; a plane through the origin in $\mathbb{R}^3$ is an example of a subspace. A subspace could be equal to the space it's contained in; the smallest subspace contains only the zero vector.

The subspaces of $\mathbb{R}^3$ are:

↓
R^0 R^1, R^2, R^3

- the origin, → 점
- a line through the origin, 선
- a plane through the origin, 면
- all of $\mathbb{R}^3$. 공간

Conclusion

When you look at a matrix, try to see "what is it doing?"

Matrices can be rectangular; we can have seven equations in three unknowns. Rectangular matrices are not invertible, but the symmetric, square matrix $A^T A$ that often appears when studying rectangular matrices may be invertible.

ex) $7 \times 3 = A$

$A^T A =$ $A^T = 3 \times 7$

$A^T A = (3 \times 7) \times (7 \times 3) = 3 \times 3$ → 역행렬 가질수 있음!

※ 행렬이 n×n이 아니면 역행렬 가질수 X.

n×n 이면 가질 "수도" 있다.

Elimination with matrices

Method of Elimination

Elimination is the technique most commonly used by computer software to solve systems of linear equations. It finds a solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$ whenever the matrix A is invertible. In the example used in class,

Elimination – success / failure

first pivot → easy number!

$$A = \begin{bmatrix} \boxed{1} & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 2 \\ 12 \\ 2 \end{bmatrix}.$$

3× 1 2 1
3 6 3
− 3 8 1
0 2 −2

지금 하고 있는 게 Elimination!

The number 1 in the upper left corner of A is called the *first pivot*. We recopy the first row, then multiply the numbers in it by an appropriate value (in this case 3) and subtract those values from the numbers in the second row. The first number in the second row becomes 0. We have thus *eliminated* the 3 in row 2 column 1.

첫번째 행에 3을 곱하고 (2번째 행의 첫번째 열 값과 맞추기 위해) first pivot 값을

2번째 행 − 1번째 행

The next step is to perform another elimination to get a 0 in row 3 column 1; here this is already the case.

The *second pivot* is the value 2 which now appears in row 2 column 2. We find a multiplier (in this case 2) by which we multiply the second row to eliminate the 4 in row 3 column 2. The *third pivot* is then the 5 now in row 3 column 3.

3 8 1 ← 2번째 행
− 3 6 3 ← 1번째 행
0 2 −2

We started with an invertible matrix A and ended with an *upper triangular matrix* U; the lower left portion of U is filled with zeros. Pivots 1, 2, 5 are on the diagonal of U.

1 2 1
0 2 −2
0 4 1

point: 같은 열의 pivot를 0으로 바꾸는 게 point임.

2열에서 여기 y, z에 대한 식.

Second pivot ⇒ 2 (2행 2열)

first pivot, second pivot, ×2 하고 빼, 아래서 위로 빼

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 4 & 1 \end{bmatrix} \longrightarrow U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

1−(−4)

third pivot

6 = 12 − 2×3

We repeat the multiplications and subtractions with the vector $\mathbf{b} = \begin{bmatrix} 2 \\ 12 \\ 2 \end{bmatrix}$.

For example, we multiply the 2 in the first position by 3 and subtract from 12 to get 6 in the second position. When calculating by hand we can do this efficiently by *augmenting* the matrix A, appending the vector $\mathbf{b}$ as a fourth or final column. The method of elimination transforms the equation $A\mathbf{x} = \mathbf{b}$ into a new equation $U\mathbf{x} = \mathbf{c}$. In the example above, $U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$ comes from A and $\mathbf{c} = \begin{bmatrix} 2 \\ 6 \\ -10 \end{bmatrix}$ comes from $\mathbf{b}$.

만약 first pivot이 0이면? 못풀어? ㄴㄴ → 행을 바꾸면 된다!

The equation $U\mathbf{x} = c$ is easy to solve by *back substitution*; in our example, $z = -2$, $y = 1$ and $x = 2$. This is also a solution to the original system $A\mathbf{x} = \mathbf{b}$.

→ Back-substitution

augment: A 옆에 b 붙여서, → as extra column

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{bmatrix} \xrightarrow{①:\ 2행 - 1행\times 3} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 4 & 1 & 2 \end{bmatrix} \xrightarrow{②:\ 3행 - 2행\times 2} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 5 & -10 \end{bmatrix}$$

c!

$x + 2y + z = 2$
$2y - 2z = 6$
$5z = -10$

$x + 2 - 2 = 2$, $x = 2$
$2y + 4 = 6$, $y = 1$
$z = -2$

how it's possible? → 각 선형에 곱한 거임.

$$\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}\begin{bmatrix}3\\4\\5\end{bmatrix} = 3\times col\,1 + 4\times col\,2 + 5\times col_3$$

$$\hookrightarrow\ 3\times\underbrace{\begin{bmatrix}-\\-\\-\end{bmatrix}}_{col_1} + 4\times\underbrace{\begin{bmatrix}-\\-\\-\end{bmatrix}}_{col_2} + 5\times\underbrace{\begin{bmatrix}-\\-\\-\end{bmatrix}}_{col_3}$$

$$\underbrace{\begin{bmatrix}1 & 2 & 7\end{bmatrix}}_{1\times 3}\underbrace{\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}}_{3\times 3} = 1\times row\,1 + 2\times row\,2 + 7\times row\,3$$

The *determinant* of U is the product of the pivots. We will see this again.

Pivots may not be 0. If there is a zero in the pivot position, we must exchange that row with one below to get a non-zero value in the pivot position.

If there is a zero in the pivot position and no non-zero value below it, then the matrix A is not invertible. Elimination can not be used to find a unique solution to the system of equations – it doesn't exist.

Matrices: subtract 3×row 1 from row 2.

Elimination Matrices

The product of a matrix (3x3) and a column vector (3x1) is a column vector (3x1) that is a linear combination of the columns of the matrix.

The product of a row (1x3) and a matrix (3x3) is a row (1x3) that is a linear combination of the rows of the matrix.

We can subtract 3 times row 1 of matrix A from row 2 of A by calculating the matrix product:

Step 1: subtract 3×row 1 from row 2

$$\underbrace{\begin{bmatrix}1&0&0\\-3&1&0\\0&0&1\end{bmatrix}}_{E_{21}}\begin{bmatrix}1&2&1\\3&8&1\\0&4&1\end{bmatrix} = \begin{bmatrix}1&2&1\\0&2&-2\\0&4&1\end{bmatrix}$$

단위행렬(E)에서 값을 바꿈.

똑같으니까 패스

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}.$$

이렇게 가려면?

The *elimination matrix* used to eliminate the entry in row m column n is denoted E_{mn}. The calculation above took us from A to $E_{21}A$. The three elimination steps leading to U were: $E_{32}(E_{31}(E_{21}A)) = U$, where $E_{31} = I$. Thus $\boxed{E_{32}(E_{21}A) = U.}$

Step 2: subtract 2×row 2 from row 3

$$E_{32}\begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix}\begin{bmatrix}1&2&1\\0&2&-2\\0&4&1\end{bmatrix} = \begin{bmatrix}1&2&1\\0&2&-2\\0&0&5\end{bmatrix}$$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix}$$

Matrix multiplication is *associative*, so we can also write $(E_{32}E_{21})A = U$. The product $E_{32}E_{21}$ tells us how to get from A to U. The *inverse* of the matrix $E_{32}E_{21}$ tells us how to get from U to A.

If we solve $U\mathbf{x} = EA\mathbf{x} = E\mathbf{b}$, then it is also true that $A\mathbf{x} = \mathbf{b}$. This is why the method of elimination works: all steps can be reversed.

A *permutation matrix* exchanges two rows of a matrix; for example,

Exchange rows 1 and 2

$$\underbrace{\begin{bmatrix}0&1\\1&0\end{bmatrix}}_{P}\begin{bmatrix}a&b\\c&d\end{bmatrix} = \begin{bmatrix}c&d\\a&b\end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$E_{32} = \begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix}$$

$$E_{21} = \begin{bmatrix}1&0&0\\-3&1&0\\0&0&1\end{bmatrix}$$

$$E_{32}\times E_{21} = \begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix}\begin{bmatrix}1&0&0\\-3&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}1&0&0\\-3&1&0\\6&-2&1\end{bmatrix}$$

The first and second rows of the matrix PA are the second and first rows of the matrix A. The matrix P is constructed by exchanging rows of the identity matrix.

To exchange the columns of a matrix, multiply on the right (as in AP) by a permutation matrix.

Note that matrix multiplication is not *commutative*: $PA \neq AP$.

Inverses

We have a matrix:

$U \to A$!

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix}a&b\\c&d\end{bmatrix}\underbrace{\begin{bmatrix}0&1\\1&0\end{bmatrix}}_{} = \begin{bmatrix}b&a\\d&c\end{bmatrix}$$

what matrix does that job?

which subtracts 3 times row 1 from row 2. To "undo" this operation we must add 3 times row 1 to row 2 using the inverse matrix:

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In fact, $E_{21}^{-1}E_{21} = I$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$