$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}.$$

This function is continuous at the origin; many books prove this as an application of the Squeeze Theorem. Because $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$,, it follows that $-|x| \leq \sin\left(\frac{1}{x}\right) \leq |x|$.. The Squeeze Theorem says

$$\lim_{x\to 0} -|x| \leq \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) \leq \lim_{x\to 0} |x|$$

$$0 \leq \lim_{x\to 0} \underbrace{x\sin\left(\frac{1}{x}\right)}_{f(x)} \leq 0$$

Hence $\lim_{x\to 0} f(x) = f(0) = 0$, so f is continuous at $x = 0$.

However, $f(x)$ is *not* differentiable at $x = 0$. We can verify this using the limit definition of the derivative:

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - 0}{h}$$

$$= \lim_{h \to 0} \frac{h \sin\left(\frac{1}{h}\right)}{h} = \lim_{h \to 0} \frac{\sin\left(\frac{1}{h}\right)}{1}$$

$$= \boxed{\lim_{h \to 0} \sin\left(\frac{1}{h}\right)}$$

which does not exist. You can see this in the applet on the left. Move your mouse over the picture to the left to start the animation; you may also have to click on the picture before it starts.

※ 특정 점에서 '연속'하다 할지라도, 해당 점에서
미분이 불가능한 경우가 존재한다. (위 예시)

핵심!!!

※ 특정 점에서 미분이 가능하다면, 해당 점에서 '연속'한 것이다.