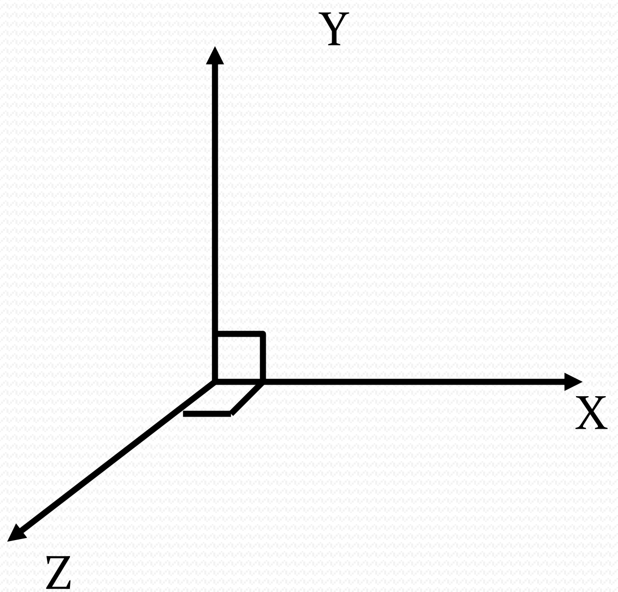
Math

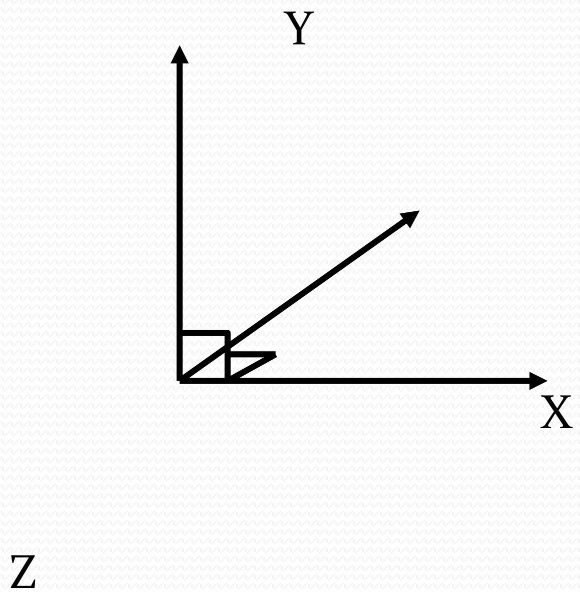
Basic Maths

- In computer graphics we need mathematics both for describing our scenes and also for performing operations on it, such as projecting and transforming it.
- Coordinate systems (right- and left-handed), serves as a reference point.
- 3 axis labelled x, y, z at right angles.

Co-ordinate Systems



Right-Handed System
(Z comes out of the screen)



Left-Handed System
(Z goes in to the screen)

Points, P (x, y, z)

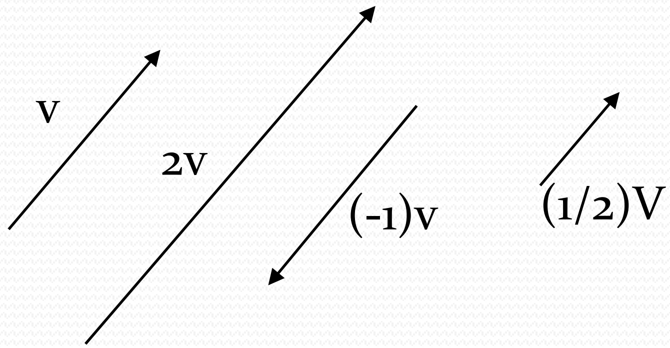
- Gives us a position in relation to the origin of our coordinate system

Vectors, V (x, y, z)

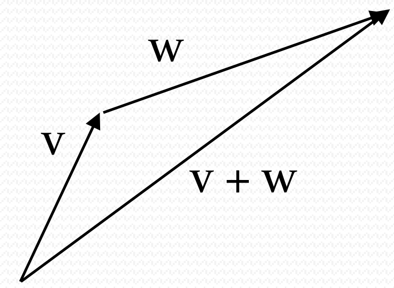
- Is a *direction* in 3D space

- Points != Vectors
  - *Point – Point = Vector*
  - *Vector+Vector = Vector*
  - *Point + Vector = Point*
  - *Point + Point = ?*

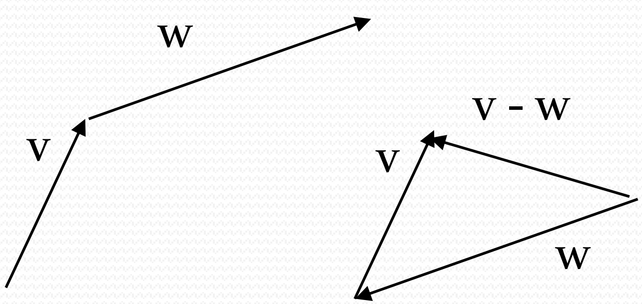
Vectors, V (x, y, z)



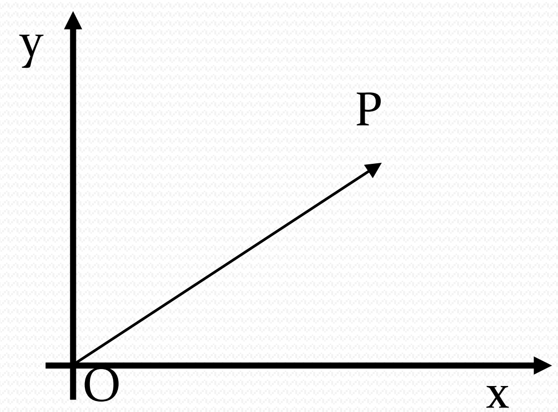
Scalar multiplication of vectors (they remain parallel)



Vector addition sum v + w



Vector difference
v - w = v + (-w)



Vector $\overrightarrow{OP}$

Vectors $\underline{V}$

- Length (modulus) of a vector $\underline{V}$ (x, y, z)
  - $|\underline{V}| = \sqrt{x^2 + y^2 + z^2}$
- A unit vector
  - $$\hat{V} = \frac{\text{vector V}}{\text{modulus of V}} = \frac{\underline{V}}{|\underline{V}|}$$

Dot Product

- $a \cdot b = |a|\ |b| \cos\theta$

  $\therefore \cos\theta = a \cdot b / |a|\ |b|$

- $a \cdot b = x_a \cdot x_b + y_a \cdot y_b + z_a \cdot z_b$

- what happens when the vectors are unit
- if dot product == 0 or == 1?
- This is purely a scalar number not a vector

Cross Product

- The result is not a scalar but a vector which is normal to the plane of the other 2
- direction is found using the determinant

$$\begin{vmatrix} i & j & k \\ x_v & y_v & z_v \\ x_u & y_u & z_u \end{vmatrix} \qquad \begin{vmatrix} i & j & k \\ x_v & y_v & z_v \\ x_u & y_u & z_u \end{vmatrix} \qquad \begin{vmatrix} i & j & k \\ x_v & y_v & z_v \\ x_u & y_u & z_u \end{vmatrix}$$

  - $i(y_v z_u - z_v y_u)$, $-j(x_v z_u - z_v x_u)$, $k(x_v y_u - y_v x_u)$
- size is a x b = $|a||b|\sin\theta$
- cross product of vector with it self is null

Parametric equation of a line (ray)

Given two points $P_0 = (x_0, y_0, z_0)$ and $P_1 = (x_1, y_1, z_1)$ the line passing through them can be expressed as:

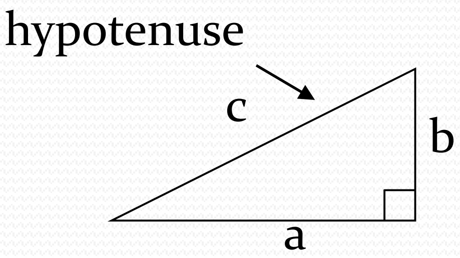
$$P(t) = P_0 + t(P_1 - P_0) = \begin{cases} x(t) = x_0 + t(x_1 - x_0) \\ y(t) = y_0 + t(y_1 - y_0) \\ z(t) = z_0 + t(z_1 - z_0) \end{cases}$$

With $-\infty < t < \infty$

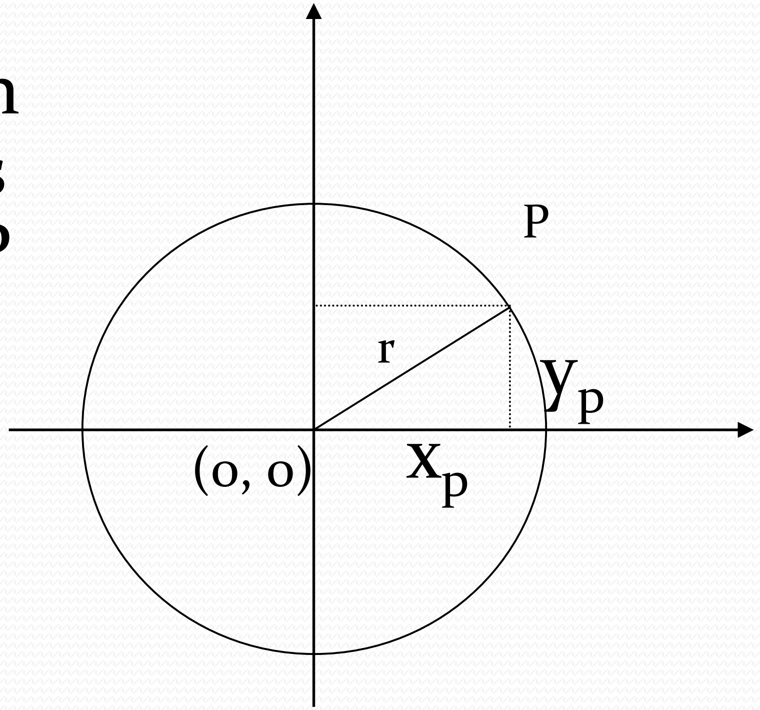
Equation of a sphere

- Pythagoras Theorem:

$a^2 + b^2 = c^2$



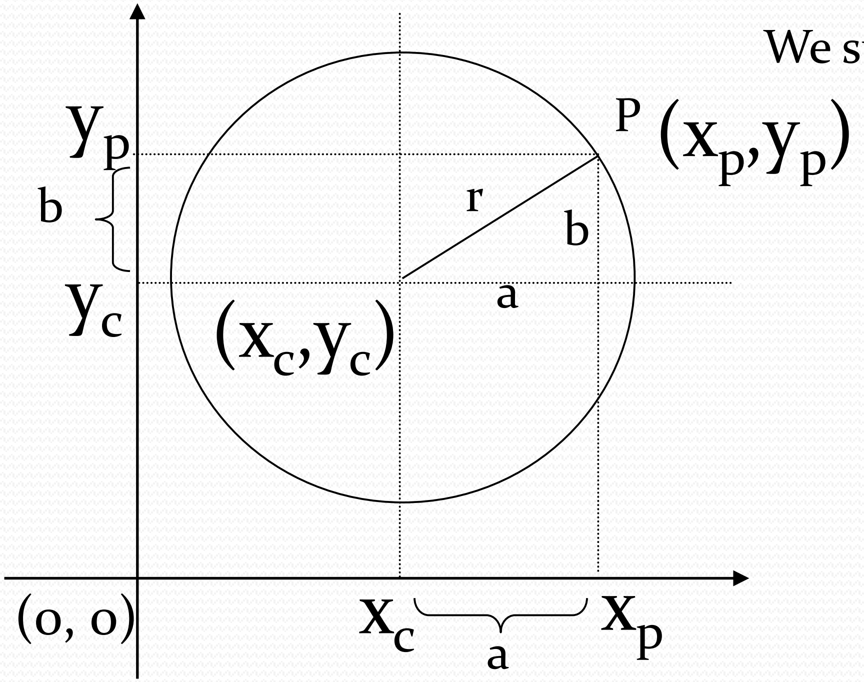
- Given a circle through the origin with radius r, then for any point P on it we have:



$x^2 + y^2 = r^2$

Equation of a sphere

* If the circle is not centered on the origin:



We still have

$$a^2 + b^2 = r^2$$

but

$$a = x_p - x_c$$

$$b = y_p - y_c$$

So for the general case

$$(x - x_c)^2 + (y - y_c)^2 = r^2$$

Equation of a sphere

- Pythagoras theorem generalises to 3D giving

  $a^2 + b^2 + c^2 = d^2$ Based on that we can easily prove that the general equation of a sphere is:

$$\boxed{(x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2 = r^2}$$

and at origin: $\boxed{x^2 + y^2 + z^2 = r^2}$

Vectors and Matrices

- Matrix is an array of numbers with dimensions M (rows) by N (columns)
  - 3 by 6 matrix
  - element 2,3 is (3)

$$\begin{pmatrix} 3 & 0 & 0 & -2 & 1 & -2 \\ 1 & 1 & 3 & 4 & 1 & -1 \\ -5 & 2 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- Vector can be considered a 1 x M matrix
  - $v = \begin{pmatrix} x & y & z \end{pmatrix}$

Types of Matrix

- Identity matrices - I

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Diagonal

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix}$$

- Symmetric

$$\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

- Diagonal matrices are (of course) symmetric
- Identity matrices are (of course) diagonal

Operation on Matrices

- Addition
  - Done elementwise

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} a+p & b+q \\ c+r & d+s \end{pmatrix}$$

- Transpose
  - "Flip" (M by N becomes N by M)

$$\begin{pmatrix} 1 & 4 & 9 \\ 5 & 2 & 8 \\ 6 & 7 & 3 \end{pmatrix}^T = \begin{pmatrix} 1 & 5 & 6 \\ 4 & 2 & 7 \\ 9 & 8 & 3 \end{pmatrix}$$

Operations on Matrices

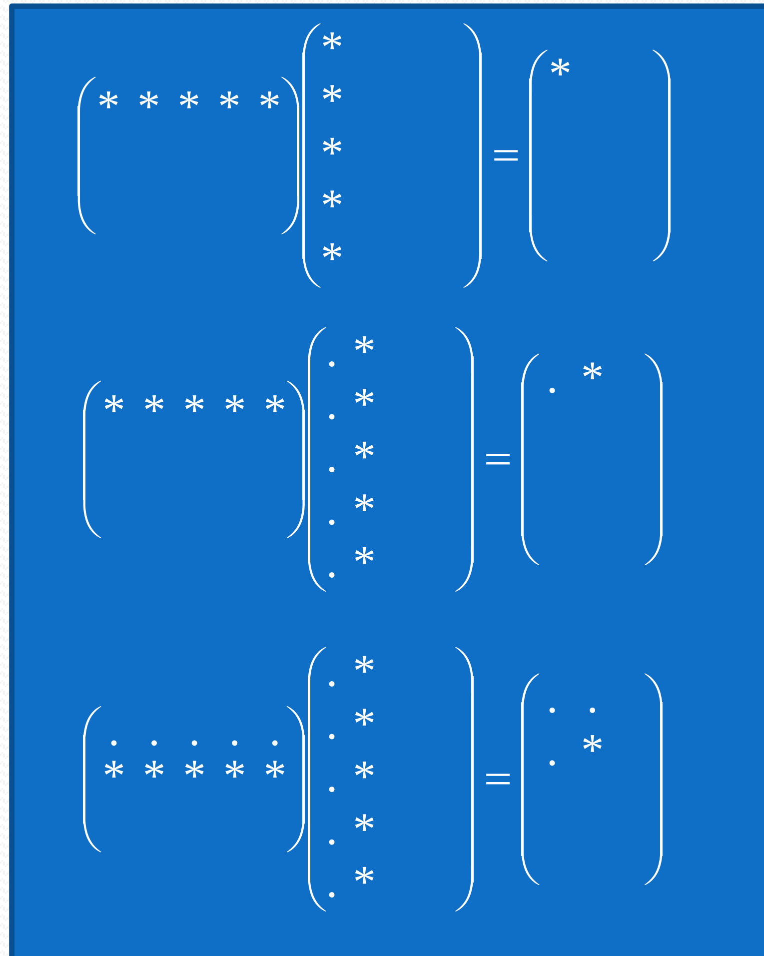
- Multiplication
  - Only possible to multiply of dimensions
    - x_1 by y_1 and x_2 by y_2 iff $y_1 = x_2$
      - resulting matrix is x_1 by y_2
    - e.g. Matrix A is 2 by 3 and Matrix by 3 by 4
      - resulting matrix is 2 by 4
    - Just because A x B is possible doesn't mean B x A is possible!

Matrix Multiplication Order

- A is n by k , B is k by m
- C = A x B defined by

$$c_{ij} = \sum_{l=1}^{k} a_{il} b_{lj}$$

- BxA not necessarily equal to AxB



Example Multiplications

$$\begin{pmatrix} 2 & 3 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \_\_ & \_\_ \\ \_\_ & \_\_ \end{pmatrix}$$

$$\begin{pmatrix} 2 & -2 & 3 \\ -3 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{pmatrix}$$

Inverse

- If $A \times B = I$ and $B \times A = I$ then
  $A = B^{-1}$ and $B = A^{-1}$