(l, m, n) $\quad x_p = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ $\quad X = X_n + X_p$

$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ $\quad AX = A(X_n + X_p)$

$= \underbrace{AX_n}_{0} + AX_p = b$

Finding the solution $AX = b$

1) $A : b \xrightarrow{G.E} [R \mid d]$ $\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_5 \end{bmatrix}$

2) Separate pivot variables / free variables

3) find the special solutions for null space from R

Ex) $\begin{bmatrix} ① & 0 & * & 0 & 3 & * & 0 \\ 0 & ① & * & 0 & 3 & * & 0 \\ 0 & 0 & 0 & ① & 1 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & ① \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ pivot: x_1, x_2, x_4, x_7
free: x_3, x_5, x_6

$x_3 \begin{bmatrix} \\ \\ \\ \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ -3 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} \\ \\ \\ \end{bmatrix}$ $\quad \begin{bmatrix} d_1 \\ d_2 \\ 0 \\ d_3 \\ 0 \\ 0 \\ d_4 \end{bmatrix}$

2.3 Linear Independence

Basis (vectors), Dimensions

→ Linear Independent

$c_1 V_1 + c_2 V_2 + \cdots + c_n V_n = 0$

→ only $c_1 = c_2 = c_n = 0$

ex) $c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}$ → Column Vectors

◎ If G.E of A generates m non-zero rows → m independent column vectors in A

· Rank of A = # of independent column vectors = # of independent row vectors
= # of pivots in G.E = Dim of $C(A)$

◎ spanning

all linear combinations of vectors $\{V_1, V_2, \cdots, V_n\}$ constitute a vector space

$= \{V_1, V_2, \cdots, V_n\}$ span vector space

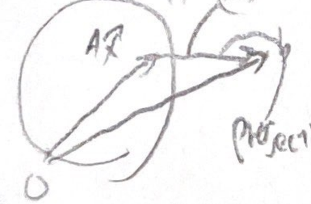
◎ Basis (vectors)

=) # of minimum linearly independent vectors to span the vector space

=) linear combination is unique from basis

◎ Basis is not unique for a vector space

$A^T(A\hat{x} - b) = 0$

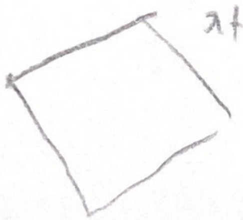


$\hat{x} = (A^TA)^{-1}A^Tb$

projection $P = Pb = \underbrace{\boxed{A(A^TA)^{-1}A^T}}_{P} b$

Projection

$x + y + z = 0$



$$A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

orthonormal basis (vectors)

$\begin{cases} V_1, V_2, \cdots, V_n \\ V_i^T V_j = 0 \end{cases}$ $\quad \|V_i\| = 1$

if V_i is orthonormal

$V_i^T \times \quad X = \sum_{i=1}^{n} c_i V_i \times V_i^T$ unique for a basis

$$c_i = V_i^T X = \frac{V_i^T X}{V_i^T V_i}$$

$$\begin{bmatrix} V_1 V_2 \cdots V_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} X \end{bmatrix}$$

◎ If given independent vectors $a_1, a_2, a_3, \cdots$ → find the orthonormal basis vector

=) Gram-Schmidt orthogonalization

① Gram-Schmidt orthogonalization

1) $a \rightarrow \frac{a}{\|a\|} = q_1$

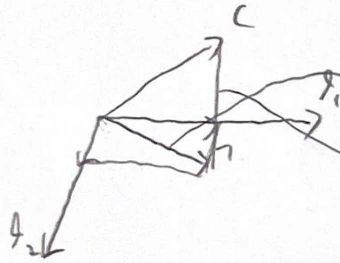
$$\frac{q_1^T b}{\underbrace{q_1^T q_1}_{1}} q_1$$

2) project b onto q_1

$$b - (q_1^T b) q_1 \perp q_1$$

$$b = (q_1^T b) q_1 + (q_2^T b) q_2$$

$$\frac{b - (q_1^T b) q_1}{\|b - (q_1^T b) q_1\|} = q_2$$



$$\left((q_1^T c) q_1 + (q_2^T c) q_2\right) \perp$$

$$c - \left((q_1^T c) q_1 + (q_2^T c) q_2\right) \perp \begin{matrix} q_1 \\ q_2 \end{matrix}$$

↓

normalization

↓

q_3

$$c = (q_1^T c) q_1 + (q_2^T c) q_2 + (q_3^T c) q_3$$

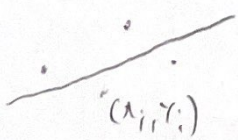
$a_1, a_2, \cdots, a_n$

1) $q_1 = \frac{a}{\|a\|}$

2) $a_j - \sum_{i=1}^{j-1} (q_i^T a_j) q_i = A_j$

3) $\frac{A_j}{\|A_j\|} = q_j$

Projection → Least square



$y_1 = a x_1 + b$

$y_2 = a x_2 + b$

$\vdots$

$y_n = a x_n + b$

$Ax = b$

$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$A^T A = \begin{bmatrix} \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & n \end{bmatrix}$$

$$\boxed{\hat{x} = (A^T A)^{-1} A b}$$

Generalized Least square

$Ax = b \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$\Rightarrow x_i \Rightarrow w_i$ $\qquad A^T A x = A^T b$

$\Rightarrow$ weight $\rightarrow$ probability $\qquad A^T W^T W A x = A^T W^T b$

$WAx = Wb$

3.4 Orthogonal Basis

- Orthogonal vectors $\rightarrow$ independent $\rightarrow$ basis vector
- Let $q_1, q_2, \cdots, q_n$ be orthonormal

$q_i^T q_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

$Q = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}$

$Q^T Q \Rightarrow Q^T = Q^{-1}$ (Left inverse) $\qquad$ Ex) Rotation, Permutation

$\begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix} = I \quad \Rightarrow \quad \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- for $q_1, q_2, \cdots, q_n \in R^n$ (square sys)

$\Rightarrow x = \sum_{i=1}^{n} c_i q_i$

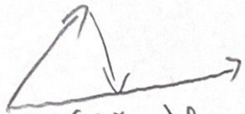
$x = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} Q \end{bmatrix}^{-1} \begin{bmatrix} x \end{bmatrix} = Q^T x$

6 Gram-Schmidt Orthogonalization

Given linearly independent vectors $\{a_1, a_2, \dots, a_n\}$ to find the orthonormal basis vectors

1) $a_1 = \dfrac{a_1}{\|a_1\|} \to q_1$



$(q_1^T a_2) q_1$

2) Project a_2 onto q_1

$$a_2 - (q_1^T a_2) q_1 \perp q_1$$

$$\frac{a_2 - (q_1^T a_2) q_1}{\| a_2 - (q_1^T a_2) q_1 \|} \to q_2$$

3) Project a_3 onto q_1 q_2

$$a_3 - \left((q_1^T a_3) q_1 + (q_2^T a_3) q_2 \right) \perp \begin{matrix} q_1 \\ q_2 \end{matrix} \to q_3$$

$$\Rightarrow a_j - \sum_{i=1}^{j-1} (q_i^T a_j) q_i \to \text{normalize} = q_j$$

$$a_j = \sum_{i=1}^{j} (q_i^T a_i) q_i$$

A = QR factorization

$$\underbrace{\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}}_{A} = \begin{bmatrix} (q_1^T a_1) q_1 & (q_1^T a_2) q_1 + (q_2^T a_2) q_2 & \cdots & (q_1^T a_n) q_1 + (q_2^T a_n) q_2 + \cdots + (q_n^T a_n) q_n \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} (q_1^T a_1) & (q_1^T a_2) & \cdots & (q_1^T a_n) \\ 0 & (q_2^T a_2) & \ddots & (q_2^T a_n) \\ \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & (q_n^T a_n) \end{bmatrix}}_{R}$$

Eigenvalue and Eigenvectors

$Ax = \lambda \underset{\text{eigen vector}}{X}$

λ → scalar multiplication → eigen value

$(A - \lambda I)x = 0$ ⇒ for non-zero x.

$\det(A - \lambda I) = 0$ → singular

$$|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & \\ \vdots & & & \end{vmatrix}$$

$$= \lambda^n - (\ \ )\lambda^{n-1} \cdots$$

Ex) $A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$

$$\begin{vmatrix} 4-\lambda & -5 \\ 2 & -3-\lambda \end{vmatrix} = (4-\lambda)(-3-\lambda) + 10 = 0$$

$\lambda = 2, -1$

$(A - \lambda I)x = 0$

x : eigenvector → $N(A - \lambda I)$

$\lambda = 2$, $\begin{bmatrix} 2 & -5 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ → null space : eigenvectors

for triangular (or diagonal) matrix,

→ eigenvalues = diagonal elements of A

$\det A = \prod_{i=1}^{n} \text{pivots} = \prod_{i=1}^{n} d_i = \prod_{i=1}^{n} \lambda_i$

Trace of $A = \sum_{i=1}^{n} a_{ii} = (a_{11} + a_{22} + \cdots + a_{nn}) = \sum_{i=1}^{n} \lambda_i$

Diagonalization of Matrix

1) $A = LU = LDU$

2) $A = QR = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix} \begin{bmatrix} (q_1^T a_1) & (q_1^T a_2) & \cdots \\ 0 & (q_2^T a_2) & \\ & 0 & \end{bmatrix}$

3) $A = S \Lambda S^{-1} \Rightarrow S = \begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix}$ $\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$

$A e_i = \lambda_i e_i$

$\begin{bmatrix} Ae_1 & Ae_2 & \cdots & Ae_n \end{bmatrix} = \begin{bmatrix} \lambda e_1 & \lambda e_2 & \cdots & \lambda e_n \end{bmatrix}$

$A \begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & & \\ 0 & \lambda_2 & \cdots & \\ 0 & & \ddots & \\ \vdots & \vdots & & \lambda_n \end{bmatrix}$

$AS = S\Lambda$

$A = S \Lambda S^{-1}$

If $\lambda_1, \lambda_2, \cdots \lambda_n$ are different

then $e_1, e_2, \cdots, e_n$ are linearly independent!