LU 분할.

◎ Elementary Matrix in Gauss, Elimination

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -l_{21} & & \\ 0 & 0 & 1 \end{bmatrix}$$

↳ ② − ① × l_{21}

E_{21} : ② − l_{21} × ① → ②′

⇓

② = ②′ + l_{21} × ①

$E_{21}A$ → ②′

$E_{21}^{-1}A'$ ⇒ A ②

ex) $2u + v + w = 5$

② $4u - 6v = -2$

⇓

②′ $-8v - 2w = -12$

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$E_{32}E_{31}E_{21}A = U$

$A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U$

Lower triangluar Matrice

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_{31} & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & l_{32} & 1 \end{bmatrix}U = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}U$$

↳ upper triangulular MAT

$= LU$

Triangular Factors

$Ax = b$ → $L^{-1}Ax = L^{-1}b = c$

$A = LU$ → $Ux = c$

⇒ LU factorization (decomposition)

$A = LU,\ Lc = b$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

ex) $L = \begin{bmatrix} 1 & 0 & 0 \\ 7 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$$

⇓

$$U = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

→ $\det(A) = \det(U) \times \det(L)$

ex) $A = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$

$= LI$

$$U = \begin{bmatrix} d_1 & u_{12} & u_{13} & \cdots & u_{1n} \\ & d_2 & & & \\ & & d_3 & & \\ O & & & \ddots & d_n \end{bmatrix} = \underbrace{\begin{bmatrix} d_1 & & O \\ & d_2 & \\ O & & \ddots\ d_n \end{bmatrix}}_{D}\begin{bmatrix} 1 & \frac{u_{12}}{d_1} & \frac{u_{13}}{d_1} & \frac{u_{14}}{d_1} & \cdots \\ & 1 & \frac{u_{23}}{d_2} & & \\ O & & & \ddots & 1 \end{bmatrix}$$

$A = LU \Rightarrow LDU$

$$D^n = \begin{bmatrix} d_1^n & & O \\ & d_2^n & \\ & & d_3^n & \\ O & & & \ddots\ d_n^n \end{bmatrix}$$

LU factorization is unique

Permutation Matrix has the same rows with I

=> There is a single "1" in every row and column

$P_{21} P_{32} \neq P_{32} P_{21}$

$P^{-1} = P^{T}$ $\quad$ $A = P^{T} L U$

$PA = LU$ ↗

$\#$ of unknowns $\geq$ $\#$ of eqns

↳ { infinitely many sol
  no solution

○ Vector space

Column space of A ($C(A)$)

=> set of all linear combination from column vectors of A

$Ax = b$

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = b$$

$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$

If $b \in C(A)$ then there is at least one solution

if $b_1, b_2 \in C(A)$

$Ax_1 = b_1$

$Ax_2 = b_2$

$b_1 + b_2 = b$ $\quad$ $Ax_1 + Ax_2 = A\underbrace{(x_1 + x_2)}_{x} = b$

ex) $b = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ $Ax = b$

$Ax_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

$Ax_2 = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$ $\Rightarrow x_1 + x_2 + x_3 = x$

$Ax_3 = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$

$b = 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Null Space of A (N(A))

$\Rightarrow$ Set of vectors such that $Ax=0$

$N(A) = \{x \mid Ax=0\}$

2.2 Solving $Ax=0$ and $Ax=b$

$$\boxed{A_{m\times n}}\ \boxed{x} = \boxed{b}$$

$m<n$ # of eqns < # of unknowns.

◎ Echelon form.

ex) $\begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Dimension of C(A) ≠ Dimension of N(A)

$$\Rightarrow \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot

Row Reduced Form

$\begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

special solution

$u+3v-z=0$ $\quad u=-3v+z$

$w+z=0$ $\quad w=-z$ $\Rightarrow \begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix} = \begin{bmatrix} -3v+z \\ v \\ -z \\ z \end{bmatrix} = v\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + z\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \in N(A)$

pivot variables: u, w

free variable: v, z

Dimensions of vector space.

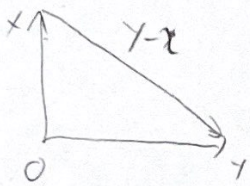
$\begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix}$ in R^4 $\quad$ Dim(N(A)) = # of independent special vectors = 2

나머지 2차원 : row sp

orthogonality

- length of vectors $\|x\|$

$\Rightarrow \|x\|^2 = x^T x = \sum_{i=1}^{n} x_i^2$



$\Rightarrow \|y-x\|^2 = \|x\|^2 + \|y\|^2$

$(y-x)^T(y-x) = x^T x + y^T y$

$\Rightarrow \cancel{y^T y} - y^T x - x^T y + \cancel{x^T x} = \cancel{x^T x} + \cancel{y^T y}$

$\Rightarrow x^T y = 0$

1) $x^T y = 0 \rightarrow$ angle $= 90^\circ$

2) $x^T y < 0 \rightarrow$ angle $> 90^\circ$

3) $x^T y > 0 \rightarrow$ angle $< 90^\circ$

◎ If non-zero vectors $v_0, v_1, \cdots v_n$ are orthogonal

$v_i^T v_j = 0 \ (i \neq j)$

$\|v_i\| \neq 0$ $\Rightarrow$ then the vectors are linearly independent!

$\Downarrow$

$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0 \Rightarrow c_1 \cdots c_n = 0$ 일때만 만족 $\Rightarrow$ linearly independent

$v_i^T (c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) = 0$

$c_i \|v_i\|^2 = 0$ for $\forall i$

$\underline{c_i = 0}$

◎ Row space $\perp$ Null space

$C(A^T)$

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

$\sum_{k=1}^{n} c_k a_k \perp x$

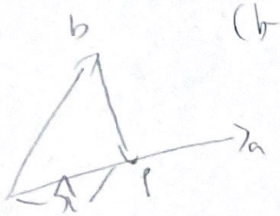
◎ Column space $\perp$ Left Null space

$A_{m \times n}$

Dim of V.S = rank of A(r)

$\mathrm{Dim}(C(A)) + \mathrm{Dim}(N(A^T)) = m$, $\mathrm{Dim}(C(A^T)) + \mathrm{Dim}(N(A)) = n$

$a^T b = \|a\|\|b\| \cos\theta$ => 내적



$(b - \hat{x}a) \perp a$

$a^T(b - \hat{x}a) = 0$

$\hat{x} = \frac{a^T b}{a^T a}$

$p = \hat{x}a = \frac{a^T b}{a^T a} a$

$= \left(\frac{aa^T}{a^T a}\right) b \rightarrow P = \frac{aa^T}{a^T a}$ => Projection Matrice

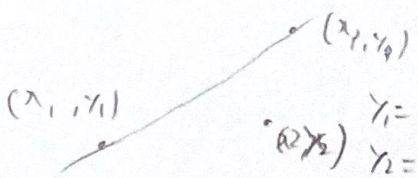
$P^T = \left(\frac{aa^T}{a^T a}\right)^T = \frac{aa^T}{a^T a} = P$

$P^2 = \frac{aa^T aa^T}{(a^T a)(a^T a)} = \frac{aa^T}{a^T a} = P$

$\boxed{A}\,\boxed{x} = b$ # of unknowns < # of eqn

=> $\min \|Ax - b\|$ -> Least Square Problem

Least Square for line fitting



$y_1 = ax_1 + b$

$y_2 = ax_2 + b$

$y_4 = ax_4 + b$

$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \Rightarrow Ax - b = \begin{bmatrix} ax_1 + b - y_1 \\ ax_2 + b - y_2 \\ \vdots \\ ax_n + b - y_n \end{bmatrix}$$

$\|Ax - b\|^2 = \sum_{i=1}^{n} (ax_i + b - y_i)^2$

$Ax = b$

$\|Ax - b\|^2$