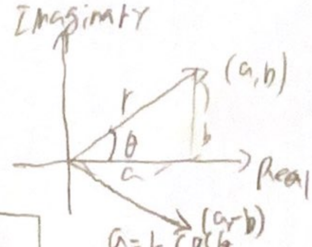
5.5 Complex Matrix

complex number $= a + jb$ $\quad j^2 = -1$

real part $\quad$ imaginary part



$r = \sqrt{a^2+b^2}$

$r = |z| = $ magnitude

$\theta = \arg\{z\} = $ argument

$\rightarrow \tan^{-1}\left(\frac{b}{a}\right)$

$a = r\cos\theta$

$b = r\sin\theta$

$z = a + jb = r(a\cos\theta + j\sin\theta)$

$= re^{j\theta}$ Euler Famula

$\rightarrow$ polar form

$\|z\|^2 = a^2+b^2 = (a+jb)(a-jb) = zz^* = z\bar{z}$

$z^* = a - jb$: symmetric on real-axis

$= re^{-j\theta}$

$|z|^2 = z \cdot z^* = re^{j\theta} \cdot re^{-j\theta} = r^2$

◦ complex vector

$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $\quad x_k = a_k + jb_k$

$\|x\|^2 = |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2 = x_1^* x_1 + x_2^* x_2 + \cdots + x_n^* x_n$

$= (x^T)^* x \rightarrow$ conjugate transpose (Hermite) $\Rightarrow H$

◎ inner Product of complex vector

$\rightarrow$ real vector : $x^T y = y^T x$

$\rightarrow$ complex vector $(x^T)^* y = x^H y \neq y^H x$

$(1+j)^*(1+2j) \neq (1+2j)^*(1+j)$

ex) $\begin{bmatrix} 2+j & 3j \\ 4+j & 5 \\ -1 & 0 \end{bmatrix} \xrightarrow{H} \begin{bmatrix} 2-j & 4-j & -1 \\ -3j & 5 & 0 \end{bmatrix}$

1) orthogonal $x^H y = 0 \ (= y^H x)$

if $x^H y = y^H x \rightarrow$ real

$(x^H y)^H$

2) $\|x\|^2 = x^H x$

3) $(AB)^H = B^H A^H$

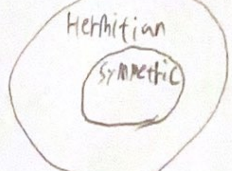
◎ Hermitian Matrix

◦ for real Matrix A

$A^T = A \Rightarrow$ Symmetric

◦ for complex matrix A

$A^H = A \rightarrow$ Hermitian



$a_{ij} = a_{ji}^* \ (i \neq j)$

$a_{ii} \Rightarrow$ real number

◦ Properties of Hermitian or Symmetric Matrix

(1) $x^H A x$ (quadratic form) is real

$(x^H A x)^H = x^H A^H x = x^H A x \rightarrow$ real number

$$[x\ y]\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{A}\begin{bmatrix} x \\ y \end{bmatrix} = x^2+y^2 = 1 \Rightarrow 원$$

$$[x\ y]\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = ax^2+by^2 = 1 \Rightarrow 타원$$

$$[x\ y]\begin{bmatrix} a\cos\theta & -b\sin\theta \\ b\sin\theta & a\cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} =$$

$R = A^H A$

$x^H R x = x^H A^H A x = (Ax)^H (Ax) = \|Ax\|^2 > 0$

(2) every eigenvalue is real

$$|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & \cdots \\ & \ddots \\ & & a_{nn}-\lambda \end{vmatrix} = (\lambda_1-\lambda)(\lambda_2-\lambda)\cdots(\lambda_n-\lambda) = \lambda^n + \alpha_1\lambda^{n-1} + \cdots + \alpha_n$$

$Ax = \lambda x$ $\quad x^H A x = \lambda x^H x = \lambda\|x\|^2$ $\quad \lambda = \dfrac{x^H A x \rightarrow \text{real}}{\|x\|^2 \rightarrow \text{real}} \Rightarrow$ real > 0

3) eigenvectors are orthogonal

$Ax_1 = \lambda_1 x_1 \quad Ax_2 = \lambda_2 x_2 \quad \lambda_1 \neq \lambda_2$

$(Ax_1)^H x_2 = x_1^H A^H x_2 = x_1^H \underbrace{A x_2}_{} = \lambda_2 x_1^H x_2$

$\|$

$(\lambda_1 x_1)^H x_2 = \lambda_1^* x_1^H x_2 = \lambda_1 x_1^H x_2 = \lambda_2 x_1^H x_2 = \underbrace{(\lambda_1 - \lambda_2)}_{\neq 0} \underbrace{x_1^H x_2}_{=0} = 0 \to$ orthogonal

Q orthonormal matrix

$Q = [q_1 \, q_2 \cdots q_n] \quad q_i^T q_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

$Q^T Q = I \quad Q^{-1} = Q^T$

$\Rightarrow$ A : symmetric, $\lambda_1, \lambda_2, \cdots, \lambda_n$

$x_1, x_2, \cdots, x_n \to$ orthonormal

$\|x_i\|^2 = 1$

$A = S \Lambda S^{-1} = Q \Lambda Q^T = Q \Lambda Q^T = [x_1 \, x_2 \cdots x_n] \begin{bmatrix} \lambda_1 & 0 & 0 \\ \vdots & \lambda_2 & \vdots \\ \vdots & & \ddots \, \lambda_n \end{bmatrix} \begin{bmatrix} x_1^T \\ x_2^T \\ \vdots \\ x_n^T \end{bmatrix}$ → spectral theorem

$= [\lambda_1 x_1 \; \lambda_2 x_2 \cdots \lambda_n x_n] \begin{bmatrix} x_1^T \\ x_2^T \\ \vdots \\ x_n^T \end{bmatrix} = \underbrace{\lambda_1 x_1 x_1^T}_{} + \lambda_2 x_2 x_2^T + \cdots + \lambda_n x_n x_n^T$

x_1 방향으로의 A Projection

$P = \frac{aa^T}{a^T a}$

U : unitary matrix

↳ complex matrix

$U^H U = U U^H = I$

$U^H = U^{-1}$

◎ angle, length, preserved

$(Qx)^T (Qy) = x^T Q^T Q y = x^T y$

$\|Qx\|^2 = (Qx)^T (Qx) = x^T Q^T Q x = \|x\|^2$

$(Ux)^H (Uy) = x^H y$

$\|Ux\|^2 = \|x\|^2$

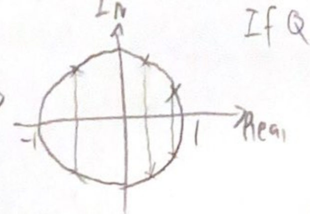
◦ eigen values $|\lambda_i| = 1$

$Ux = \lambda x$

$\|Ux\| = \|\lambda x\| = |\lambda| \|x\|$

$\|x\| = |\lambda| \|x\|$

$|\lambda| = 1$ ⟶



If Q

◦ eigen vectors are orthonormal

$Ux_1 = \lambda_1 x_1 \quad Ux_2 = \lambda_2 x_2$

$x_1^H x_2 = (Ux_1)^H (Ux_2) = (\lambda_1 x_1)(\lambda_2 x_2)$

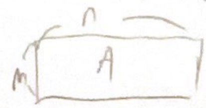
$= \lambda_1^* \lambda_2 x_1^H x_2$

$\underbrace{(1 - \lambda_1^* \lambda_2)}_{\neq 1} x_1^H x_2 = 0$

∘ Singular Value Decomposition (SVD)

$\underset{n\times n}{A} = S\Lambda S^{-1}$ for $n\times n$ square matrix

$\underset{m\times n}{A} = \underset{m\times m}{U}\,\underset{m\times n}{\Sigma}\,\underset{n\times n}{V^T}$



$C(A) \in \mathbb{R}^m \perp N(A^T) \in \mathbb{R}^m$

$C(A^T) \in \mathbb{R}^n \perp N(A) \in \mathbb{R}^n$

1) first find the eigenvalues and eigenvectors of A^TA

$\to A^TA \to$ symmetric

$x^TA^TAx = \lambda x^Tx$

$\lambda = \dfrac{\|Ax\|^2}{\|x\|^2} \geq 0$

assume that there are r non-zero eigenvalues

$\underbrace{\lambda_1, \lambda_2, \cdots, \lambda_r}, \underbrace{\lambda_{r+1}, \cdots \lambda_n}_{=0}$

$v_1 \quad v_2 \quad \cdots \quad v_r \to$ orthonormal $\in \mathbb{R}^n$

$\sqrt{\lambda_i} = \sigma_i \quad (\lambda_1, \cdots \lambda_r)$

$\hookrightarrow$ singular value

$\underset{r\times r}{\Sigma_1} = \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_r \end{bmatrix} \Rightarrow \underset{m\times n}{\Sigma} = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}$

$V_1 = [v_1\ v_2\ \cdots\ v_r] \in \mathbb{R}^n$

$\hookrightarrow C(A^T)$ row space basis

for $\lambda_{r+1} = \lambda_{r+2} = \cdots = \lambda_n = 0$

$A^TAx_j = \underset{0}{\lambda_j}x_j = 0 \quad (j = r+1, \cdots, n)$

$A^TAx_j = 0, \quad x_j \Rightarrow N(A^TA) = N(A)$

$\hookrightarrow v_{r+1}, v_{r+2}, \cdots v_n \to N(A)$ basis

$V_2 = [v_{r+1}, v_{r+2} \cdots v_n]$

$\hookrightarrow N(A)$ basis

$V = [V_1\ V_2] \quad V^TV = I$

$AV = U\Sigma V^TV$

$AV = U\Sigma$

$A[v_1\ v_2 \cdots v_r\ v_{r+1} \cdots v_n] = [Av_1\ Av_2 \cdots Av_r\ 0\ 0\ 0 \cdots]$

$\underset{m\times m}{[u_1\ u_2 \cdots u_r\ u_{r+1} \cdots u_m]} \in \mathbb{R}^m \underset{m\times n}{\begin{bmatrix} \sigma_1 & 0 & & \\ 0 & \sigma_2 & & \\ \vdots & & \ddots & \\ & & & \sigma_r \end{bmatrix}}$

$=$

$\underset{m\times n}{[\sigma_1u_1\ \sigma_2u_2 \cdots \sigma_ru_r\ 0\ 0\ 0 \cdots]}$

$Av_i = \sigma_iu_i$

$u_i = \dfrac{1}{\sigma_i}Av_i \in \mathbb{R}^m$

$u_i^Tu_j = \dfrac{1}{\sigma_i}(Av_i)^T\dfrac{1}{\sigma_j}(Av_j)$

$= \dfrac{1}{\sigma_i\sigma_j}v_i^TA^TAv_j = \dfrac{1}{\sigma_i\sigma_j}v_i^T\lambda_jv_j$

$\Rightarrow \begin{cases} i=j \Rightarrow 1 \\ i\neq j \Rightarrow 0 \end{cases}$

$\to u_i \to$ orthonormal $\to C(A)$ basis

$U_1 = [u_1\ u_2 \cdots u_r] \to C(A)$ basis

∘ $\underset{m-r}{u_{r+1}, u_{r+2} \cdots u_m} \in \mathbb{R}^m \Rightarrow$ left null space

$N(A^T) \to$ orthonormal basis

$U_2 = [u_{r+1}, u_{r+2}, \cdots, u_m]$

$\underset{m\times m}{U} = [\underset{1\text{st}}{U_1} \; \underset{r+1\sim m}{U_2}]$

$A = U\Sigma V^T = [U_1 \; U_2]\begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$

Ex)

$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \quad A^TA = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\det\begin{bmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{bmatrix} = 0$

$\lambda = 4, 0$

$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 - x_2 = 0 \Rightarrow \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{V_1}$

$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow x_1 + x_2 = 0 \rightarrow \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{V_2}$

$V_2 \rightarrow A^TA$의 Null Space

$u_1 = \frac{1}{\sigma_1}AV_1 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Left Nullspace of $A \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 + x_2 + 0 = 0 \Rightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \Rightarrow u_2$

$u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow$ orthonormal 이용해서

$$A = \overset{U}{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{\Sigma}{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}} \overset{V^T}{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}}$$

$U^TAV = \Sigma$