Systems, Networks, and Strategies **Name:** ______________________

Extra Homework due May 7, 2015

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1. Climate threshold.

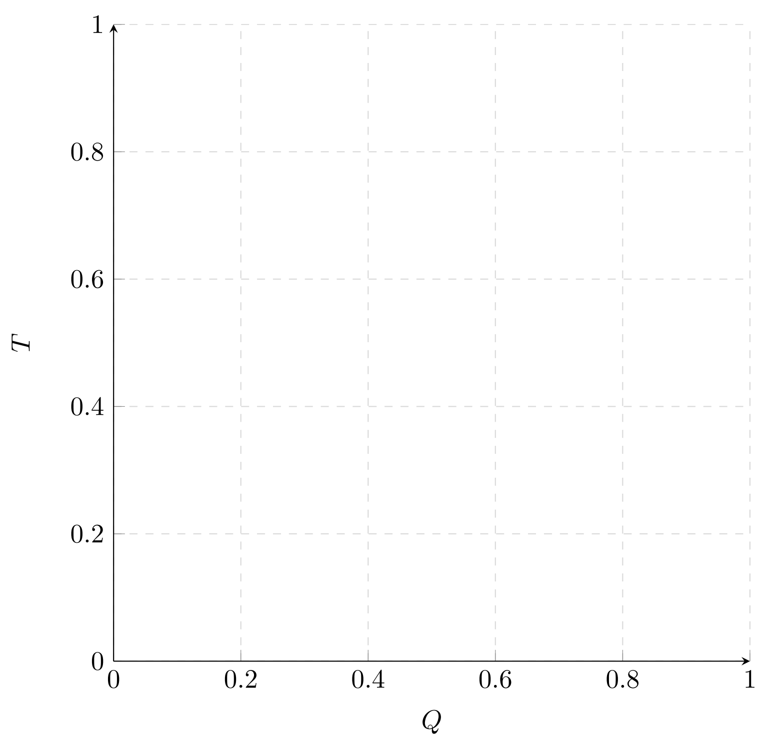
The overall temperature on Earth is affected by several positive and negative feedback loops that can have complex effects. For example, rising temperature causes more water to evaporate, which adds to the greenhouse effect that captures heat from the sun and raises the temperature more. Melting ice exposes dark-colored rock, which also captures more heat and raises the temperature. But raising temperature also causes more heat to radiate away from the earth, which keeps it from rising forever.

So when the planet receives a changing amount of solar heat (which it does all the time), the response of temperature may be complex. One simplified model of the global climate gives the relationship between total energy received from the sun, Q, and temperature, T, as

$$Q = \frac{A + BT}{c_i + \frac{1}{2}(c_f - c_i)(1 + \tanh(\gamma T))}$$

where $A = 218$ is the amount of solar energy corresponding to $T = 0°\text{C}$; $B = 1.9$ is the amount of solar energy it takes to raise the temperature by 1 degree C; $c_i = 0.35$ expresses how fast ice soaks up solar energy and $c_f = 0.7$ expresses how fast bare earth soaks up solar energy (faster because it's darker); and γ is a number that summarizes how fast the amount of ice shrinks when temperature rises. For more info about this model see https://johncarlosbaez.wordpress.com/2012/11/06/mathematics-and-the-environment-part-5/.

(a) It's hard to find T given Q, which is the natural thing to want to do, but we can plot Q given T and turn it sideways. Use an online tool or graphing calculator or whatever to get a plot of this curve, and sketch it here or attach a screenshot or printout. Use $\gamma = 0.05$.



TODO: fix ticks

(b) That plot shows that for some amounts of incoming solar energy (Q), there is only one fixed-point value of T, which predicts what the temperature will be, but at other values of Q, there are three fixed-point values of T. Which value the temperature ends up taking depends on what temperature it had before. It will be attracted to either the lower or the upper fixed point, depending on whether it is above or below the middle fixed point. If T is near the lower fixed point, it will move to that fixed point. Suppose Q is about 400, and T is at about $-40°$C. If Q increases gradually to about 450, how will T change? Sketch it on the picture and write a brief description here.

(c) If Q continues to rise to about 500, what will happen to T? Sketch it on the picture and describe it here.

(d) What will happen to T if Q declines from 500 gradually back to 400 and below?

(e) This kind of system behavior is called *hysteresis*, from the Greek for 'memory'. Why might that be?

(f) What might be happening in this system, in terms of ice and bare earth?

2. The doubling map.

I showed the beginning of this video about the Lorenz attractor and chaos in class: https://youtu.be/aAJkLh76QnM. Now watch the rest.

Let's do some work with this doubling model:

$$x \mapsto \begin{cases} 2x & \text{if } 2x < 1 \\ 2x - 1 & \text{if } 1 \leq 2x \end{cases}$$

That is, at every step, double x, and if the result is 1 or greater, subtract 1. The video explains how this is a simpler model that captures the key qualities of the Lorenz model.

If our initial condition is $x = \frac{1}{3}$, the next x is found by doubling, giving us $x = \frac{2}{3}$, and the next x after that is $\frac{4}{3} - 1 = \frac{1}{3}$. After that x will repeat those two values forever.

If the initial condition is $x = 0.2$, as in the video, the next values are 0.4, 0.8, 0.6, and then it will repeat those four values forever.

(a) What happens if the initial value is $\frac{1}{7}$?

(b) What happens if the initial value is $\frac{1}{2}$? Notice that if doubling gives 1 exactly, we subtract 1 to get a next value of 0, not 1.

(c) If we write 'L' when x is smaller than $\frac{1}{2}$ and 'R' when it is $\frac{1}{2}$ or greater, for any initial x we get a sequence of L and R describing the system's behavior. For example, starting with $x = \frac{1}{3}$ we get LRLRLR. . ., continuing forever, and starting with $x = 0.2$ it's LLRRLLRRLLRR. . . forever.

(d) What sequence do we get starting with $x = \frac{1}{2}$? Notice that the letter corresponding to $\frac{1}{2}$ itself is R, not L.

(e) What sequence do we get starting with $x = \frac{1}{4}$?

(f) What sequence do we get starting with $x = \frac{1}{8}$?

(g) What sequence do we get starting with $x = \frac{1}{2} + \frac{1}{8}$?

(h) Can you find an initial x that generates the sequence LLLR followed by an infinite number of Ls?

(i) Can you find an initial x that generates the sequence RLLR followed by an infinite number of Ls?

(j) Can you find an initial x that generates any given sequence I might name followed by an infinite number of Ls? How?

(k) Can you find an initial x that generates any given sequence I can describe? How? (This question might require a more advanced math background than the others.)

(l) Does what you've learned about the relationship between numbers and the sequences they generate help explain why the model has sensitive dependence on initial conditions? If so, please explain.