Darwinian Operator

Is it helpful to think of adaptive dynamics (and population genetics, microeconomics, some other things) in terms of a selection operator?

It would be defined very similar to the selection gradient, and ideally could unify pop gen, adaptive dynamics, and some other formulations. It's very tempting to call it a Darwinian, by analogy to the Lagrangian and Hamiltonian of physics, and consider how it behaves under changes of parametrization and constraints, very much like the Lagrangian. It should generalize to non-Darwinian adaptation situations, though, but maybe I'll give in to temptation anyway.

So the basic AD situation is an ecological dynamics

$$\frac{\dot{X}(\mathbf{u}|\mathbf{E})}{X(\mathbf{u}|\mathbf{E})} = f(X;\mathbf{u}|\mathbf{E})$$

that has an invasion exponent

$$\mathscr{I}(\mathbf{u}|\mathbf{E}) = \lim_{X\to 0} f(X;\mathbf{u}|\mathbf{E}).$$

The adaptive dynamics of $\mathbf{u}$ basically follows:

$$\dot{\mathbf{u}} = \gamma\hat{X}(\mathbf{u}|\mathbf{E})\partial_1\mathscr{I}(\mathbf{u}|\mathbf{E}).$$

For this we may want to write

$$\dot{\mathbf{u}} = \gamma\hat{X}(\mathbf{u}|\mathbf{E})\mathscr{D}[f,\mathbf{E}](\mathbf{u})$$

by defining

$$\mathscr{D}[f,\mathbf{E}](\mathbf{u}) = \frac{\partial}{\partial\mathbf{u}}\lim_{X\to 0} f(X;\mathbf{u}|\mathbf{E})$$

??

As we know, the invasion rate is 0 when $\mathbf{u}$ is one of the resident strains denoted in $\mathbf{E}$, so that the gradient $\mathscr{D}[f,\mathbf{E}](\mathbf{u})$ is the direction of positive invasion exponent.

That's the selection gradient $\mathbf{S}$ that I've written about aplenty. Do we gain anything by considering the $\mathscr{D}$ operator that generates it?

What do we have when there's a change of variables?

Say we have a parametrization $\mathbf{p}(\mathbf{u})$: this may be a nonsingular change of parameters, or it could be a constraint in the sense of a projection of $\mathbf{u}$ into a more-dimensional space. Either way there's a Jacobian matrix $\partial\mathbf{p}/\partial\mathbf{u}$, and

$$\mathscr{D}[f,\mathbf{E}](\mathbf{u}) = \frac{\partial\mathbf{p}}{\partial\mathbf{u}}\mathscr{D}[f,\mathbf{E}](\mathbf{p}).$$

The $\mathscr{D}$ on the right side is a different operator, on a different domain; it's a different f that it's operating on, and it yields an operator on the $\mathbf{p}$ vectors rather than $\mathbf{u}$.

The Jacobian may or not be invertible and movable to the other side.