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CUBELIKE PUZZLES—WHAT ARE THEY AND HOW DO YOU SOLVE THEM?

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1. Rubik's cube, Meffert's pyramid, Halpern's dodecahedron, Alexander's great dodecahedron, and the list goes on. Such objects are partitioned into smaller pieces that get mixed up when you turn their faces or layers, yet they don't fall apart. All are groups in disguise; solving them amounts to finding algorithms for factoring arbitrary elements into products of certain generators. Just what constitutes a "cubelike puzzle"? Is there a *universal algorithm* for solving them? These are questions suggested by Douglas Hofstadter's Scientific American article [12]. Concerning a puzzle he calls the IncrediBall, Hofstadter writes:

"I found that when I loosened my conceptual grip on the exact qualities of my hard-won operators for the Cube and took them more metaphorically, I could transfer some of my expertise from Cube to I-ball. Not everything transferred, needless to say. What pleased me most was when I discovered that my 'quarkscrew' and 'antiquarkscrew' were directly exportable. Of course, it took a while to determine what such an export would consist of. What is the essence of a move? What aspects of it are provincial and shedable? How can one learn to tell easily? These are difficult questions for which I do not have the answers.

I gradually learned my way around the IncrediBall by realizing that a powerful class of moves consists of turning only two overlapping 'circles' in a commutator pattern ($xyx'y'$). I therefore studied such two-circle commutators on paper until I found ones that filled all my objectives. They included quarkscrews, double swaps and 3-cycles, which form the basis of a complete solution. In doing so I came up with just barely enough notation to cover my needs, but I did not develop a complete notation for the IncrediBall. This, it seems to me, would be most useful: a standard universal notation, psychologically as well as mathematically satisfying, for all cubelike puzzles. It is, however, a very ambitious project, given that you would have to anticipate all conceivable variations on this fertile theme, which is hardly a trivial undertaking."

Concerning algorithms for solving the cube, David Singmaster [19, p. 12] writes: "...we need to proceed in two directions. First, by examining the cube and its group, we discover which patterns are possible and, second, we show that we can achieve all possible patterns." The unscrambling problem, then, is directly related to the problem of determining the structure of the underlying group. Later [19, pp. 58–9], Singmaster describes this group as a subgroup of index 12 of a direct product of wreath products.

Singmaster also writes: "...it is a remarkable phenomenon that everyone seems to find a different combination of processes and strategy." A survey of the many "how-to" books on the subject will confirm Singmaster's assertion. But common threads will be found running through all of these algorithms. These include the commutators and 3-cycles mentioned by Hofstadter.

This article is an attempt to put some algebraic order into the business of solving cubelike puzzles. Ideally, an algebraic theory would unfold that would, in an elementary way, yield highly efficient algorithms for all such puzzles. The work here is a step in that direction. Our strategy, like Singmaster's, will lead to a determination of the underlying group structures. A description of these structures is given in §6.

Our starting point will be to put the above-mentioned common threads together to form a common strategy. The strategy will then be applied to various puzzles including the general

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$n \times n \times n$ cube. The efficiency of these algorithms will not be a major consideration of this paper, although some attention is given to this topic in §7. It would be very interesting to obtain information about the length of the shortest possible algorithm. Group-theoretically, this amounts to calculating the least upper bound for the lengths of words required to express all elements of the underlying group in terms of a certain set of generators. This is a difficult problem about which little is known. (See [11, p. 35], [19, pp. 52–3]; also [20] for related results.)

In §2, common threads of cube solutions are summarized. In §3, the concept of *wreath product* is developed and examples are given that illustrate the relationship that exists between wreath products and cubelike puzzles. In §4, a general strategy is based on Propositions 1–7. These results serve as keys for solving most cubelike puzzles. Exceptions are the "two faces puzzle" of [2, p. 768] (see Example 4 below) and the "skewb" of [12, p. 20]. Applications in §5 include the cube puzzle, three different partitions of the tetrahedron, two of the octahedron, two of the dodecahedron, one of the icosahedron, and the general $n \times n \times n$ cube. The latter illustrates the essence of Proposition 1.

The only prerequisite for reading this article is an elementary knowledge of permutation groups. In particular, the reader does *not* have to know how to "do the cube". For a discussion of permutations, see almost any introductory algebra text (e.g., [3], [10]). For a fairly complete treatment of the subject of permutations and a glimpse at its evolution, see in order, [5], [6], [21], and [22]. For the general theory of groups, see, e.g., [9], [15], or [16]. A few "cube theory" references are included at the end for the interested reader.

2. Common threads. All intelligible solutions of the cube puzzle seem to have these common features:

(i) Two distinct subproblems are recognized: the *positioning problem* and the *orientation problem*. Mathematically, these relate to permutation groups and wreath products, respectively.

(ii) Two distinct orbits are recognized: the *corner cube orbit* and the *edge cube orbit*.

(iii) A special *parity-adjusting process* is needed.

(iv) Cubelets are restored *one-by-one*.

(v) Processes for restoring individual cubelets (which involve anywhere from zero to twenty quarter-turns) often involve *conjugations* and *commutators*.

A typical solution begins with operations that affect both orbits, but later restricts to those which affect only one. Likewise, cubelets are positioned and oriented simultaneously at the beginning, then separated later on. There are obviously many ways to do this and therein, no doubt, lies the explanation to Singmaster's observed "phenomenon".

3. Notation, wreath products. Throughout, $\mathscr{X}$ denotes a finite set, G a permutation group acting on $\mathscr{X}$, and Γ a subgroup of a wreath product. In §5, $\mathscr{X}$ will be interpreted as a set of unoriented puzzle pieces, G a group of permutations of such pieces, and Γ a group of permutations of oriented puzzle pieces. Permutations will act on the right, other functions on the left. By an *orbit* of $\mathscr{X}$ is meant a set of the form

$$\mathcal{O} = \mathcal{O}(x) = \{ xg : g \in G \}.$$

If $\mathcal{O}$ is an orbit, then $G|\mathcal{O}$ denotes the restriction of G to $\mathcal{O}$ and, for $g \in G$, $g|\mathcal{O}$ denotes the restriction of g to $\mathcal{O}$. We will write $\text{Act}(g)$ for the action set $\{ x \in \mathscr{X} : xg \neq x \}$, $\text{Sym}\,\mathscr{X}$ and $\text{Alt}\,\mathscr{X}$, respectively, for the symmetric and alternating groups on $\mathscr{X}$, $S_n = \text{Sym}\{1, \ldots, n\}$, $A_n = \text{Alt}\{1, \ldots, n\}$ and $\mathbb{Z}_r$ for the group of integers mod r. If S is a finite set, $|S|$ denotes its cardinality and $S^{\mathscr{X}}$ the set of functions from $\mathscr{X}$ to S. If g and h are elements of a group, then $[g, h]$ denotes the *commutator* $ghg^{-1}h^{-1}$.

Wreath products are usually studied along with *semidirect products* and/or *group extensions* as, e.g., in [13], [15], and [18]. The definition below (cf. [14, p. 32]) assumes no previous knowledge of these companion ideas. We mention, though, the important theorem of Kaloujnine and Krasner

(see [**18**, p. 100] or [**13**, p. 49]): *Every extension of a group A by a group B can be embedded in a wreath product.*

DEFINITION. Let G and H be permutation groups that act on $\mathscr{X} = \{1, \ldots, n\}$ and $\mathscr{Y} = \{1, \ldots, r\}$, respectively. Then the *wreath product of H by G*, written $H \wr G$, is the subgroup of $\text{Sym}(\mathscr{X} \times \mathscr{Y})$ generated by permutations of the following two types:

$$\pi(g) : (i, j) \rightarrow (ig, j)$$

for $g \in G$ and

$$\sigma(h_1, \ldots, h_n) : (i, j) \rightarrow (i, jh_i)$$

for $h_1, \ldots, h_n \in H$.

We may visualize the situation as follows: Suppose n decks of cards occupy positions $1, \ldots, n$ and suppose that each deck contains r cards which occupy levels $1, \ldots, r$. Then $\pi(g)$ permutes the decks according to g, maintaining card levels, and $\sigma(h_1, \ldots, h_n)$ shuffles the decks in positions $1, \ldots, n$ according to $h_1, \ldots, h_n$, respectively, maintaining deck positions.

In the applications to be considered, H will be cyclic. Accordingly, we will take $H = \mathbb{Z}_r$ for some positive integer r and use additive notation for this group. Notice that for such groups, shufflings amount to what card players call "cuts".

For $g, p \in G$ and $h_1, \ldots, h_n, k_1, \ldots, k_n \in H$, we have

$$\pi(g)\pi(p) = \pi(gp),$$

$$\sigma(h_1, \ldots, h_n)\sigma(k_1, \ldots, k_n) = \sigma(h_1 + k_1, \ldots, h_n + k_n),$$

and

$$\sigma(h_1, \ldots, h_n)\pi(g) = \pi(g)\sigma(h_{1g^{-1}}, \ldots, h_{ng^{-1}}).$$

It follows that any element α of $H \wr G$ has a unique representation of the form $\pi(\alpha')\sigma(\alpha'')$, where $\alpha' \in G$ and $\alpha'' \in H^{\mathscr{X}}$. In particular, we have

$$|H \wr G| = |G \times H^{\mathscr{X}}| = |G||H|^{|\mathscr{X}|}.$$

Henceforth, we will identify elements α of $H \wr G$ and corresponding pairs (α', α'') of $G \times H^{\mathscr{X}}$. This will simplify notation. For example, the identity 1 in $H \wr G$ will be written $(1, 0)$ or $(1, (0, \ldots, 0))$ instead of $\pi(1)\sigma(0, \ldots, 0)$. The following formulas may be routinely established:

(1) $$(\alpha\beta)' = \alpha'\beta'$$

(2) $$(\alpha^{-1})' = \alpha'^{-1}$$

(3) $$(\alpha\beta\alpha^{-1})' = \alpha'\beta'\alpha'^{-1}$$

(4) $$[\alpha, \beta]' = [\alpha', \beta']$$

(5) $$(\alpha\beta)''(i) = \alpha''(i\beta'^{-1}) + \beta''(i)$$

(6) $$(\alpha^{-1})''(i) = -\alpha''(i\alpha')$$

(7) $$(\alpha\beta\alpha^{-1})''(i) = \alpha''(i\alpha'\beta'^{-1}) + \beta''(i\alpha') - \alpha''(i\alpha')$$

(8) $$[\alpha, \beta]''(i) = \alpha''(i\beta'\alpha'\beta'^{-1}) + \beta''(i\beta'\alpha') - \alpha''(i\beta'\alpha') - \beta''(i\beta').$$

The following three examples show how wreath products relate to the puzzles under consideration. They also illustrate the abovementioned theorem of Kaloujnine and Krasner. The first example is a variation of "the three coins game" of [**4**, p. 24].

EXAMPLE 1. Three coins are placed in a row in positions $1, 2, 3$ and the following operations on

them are permitted.

α: interchange the coins in positions 1 and 2.

β: turn over the coin in position 3, then interchange coins in positions 2 and 3.

We assume that the coins have distinguishable orientations "heads" and "tails" and (unlike in [4]) that the coins themselves are distinguishable (say, penny, nickel, and dime). If we associate coin positions with deck positions and "heads" and "tails", respectively, with levels 1 and 2, it is clear that α and β may be regarded as permuting-shuffling operations on three 2-card decks; i.e., as elements of $\mathbb{Z}_2 \wr S_3$. Explicitly we have:

$$\alpha = ((12), (0, 0, 0))$$

and

$$\beta = ((23), (0, 1, 0)).$$

One may verify by direct calculation that α and β generate all 48 elements of $\mathbb{Z}_2 \wr S_3$ (however, compare this with Examples 2 and 3 below).

EXAMPLE 2. Six equilateral triangles are placed in a row in positions $1, \ldots, 6$ and the following operations on them are allowed.

α: cycle the triangles in positions 3, 4, 5, 6 in that order (i.e., $3 \to 4 \to 5 \to 6 \to 3$).

β: rotate the triangle in position 1 clockwise $120°$, then cycle the triangles in positions $1, 2, 3, 4$ in that order (i.e., $1 \to 2 \to 3 \to 4 \to 1$).

(It is to be understood here that when triangles are cycled, they are to maintain their orientations. Thus, vertices pointing upward before cycling will remain pointed upward after cycling.) If we identify the idea of having i clockwise rotations and that of occupying level i (mod 3), then α and β may be interpreted as permuting-shuffling operations on six 3-card decks. Hence, the group generated by α and β may be regarded as a subgroup Γ of $\mathbb{Z}_3 \wr S_6$. Here,

$$\alpha = ((3456), (0, 0, 0, 0, 0, 0))$$

and

$$\beta = ((1234), (0, 1, 0, 0, 0, 0)),$$

and it may be verified that Γ has index 6 in $\mathbb{Z}_3 \wr S_6$. Indeed, an argument similar to [19, pp. 55–7] will show that Γ is isomorphic to $\mathbb{Z}_3 \wr S_5$. This example is related to the "Tricky Six Puzzle" of [2, p. 759] (see also [19, p. 57], [23], and Example 3 below).

EXAMPLE 3 (*the 2-faces group*). Six blocks are arranged in an L-shaped stack as shown in Fig. 1(a), and the following operations are allowed (see Fig. 1(a)).

ϕ: rotate the front layer $90°$ clockwise.

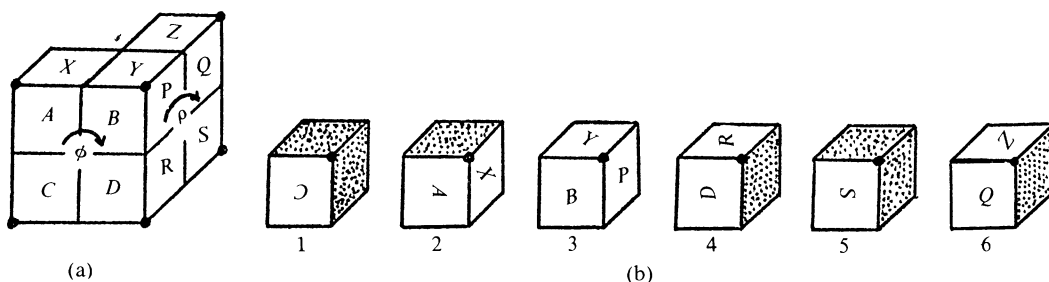ρ: rotate the right-hand layer $90°$ clockwise.

FIG. 1

This example is closely related to Example 2. To see this, note that ϕ and ρ take darkened corners to darkened corners (see Fig. 1(a)), and that blocks can rotate about these corners (e.g., $\phi\rho: B \to Y \to P \to B$). Now imagine the six blocks as spread out before us in some arrangement, say, as in Fig. 1(b). We may then view ϕ and ρ as permuting-rotating operations as in Example 2. Clearly, the representations of ϕ and ρ will depend on the arrangement, but the point is that we may regard the group Γ generated by ϕ and ρ as a subgroup of $\mathbb{Z}_3 \wr S_6$. In terms of the arrangement shown,

$$\phi = ((1234),(0,0,0,0,0,0))$$

and

$$\rho = ((3654),(0,0,1,1,0,1)).$$

It may be verified (see [**19**, pp. 55–7]) that Γ has index 18 in $\mathbb{Z}_3 \wr S_6$.

4. A common strategy. In the puzzles we are considering there are various pieces which can occupy certain positions with certain orientations. The basic moves of the puzzle may be viewed as acting on both the positions and the oriented positions. When we view these moves as acting on positions, we will obtain orbits $\mathcal{O}_1,\ldots,\mathcal{O}_N$. When we view the moves as acting on the oriented positions restricted to an orbit $\mathcal{O}_i$, the situation will be analogous to Example 3. Thus, these puzzles may be regarded as subdirect products of wreath products of the form $\mathbb{Z}_r \wr S_n$.

Our strategy will be to proceed through the orbits $\mathcal{O}_1,\ldots,\mathcal{O}_N$ restoring those pieces in a given orbit without disturbing the other pieces. To accomplish this, we will first need to make a parity adjustment. Let α be the product of moves that scrambled up our puzzle. We will produce a product π of basic moves such that $\alpha\pi$ permutes the pieces *evenly* on every orbit. We will then produce products $\alpha_1,\ldots,\alpha_N$ of basic moves such that α_i acts on $\mathcal{O}_j$ as the identity for $j \neq i$, and α_i acts on $\mathcal{O}_i$ as the inverse of $\alpha\pi$. (In fact, we need only require that α_i be the identity on $\mathcal{O}_j$ for $j < i$ and that α_i act on $\mathcal{O}_i$ as the inverse of $\alpha\pi\alpha_1 \cdots \alpha_{i-1}$. This improvement to the strategy will be discussed in §7.) Thus, the product $\alpha\pi\alpha_1 \cdots \alpha_N$ will unscramble the puzzle.

The following result shows that a parity adjustment of the type described above is always possible. We define $\nu(g,\mathcal{O})$ to be 0 or 1 according as $g|\mathcal{O}$ is even or odd.

PROPOSITION 1. *For any permutation group G there exist orbits $\mathcal{O}_1,\ldots,\mathcal{O}_m$ and elements $h_1,\ldots,h_m \in G$ such that*
   (i) $\nu(h_i,\mathcal{O}_j) = \delta_{ij}$ *($i,j = 1,\ldots,m$) and*
   (ii) *for any orbit $\mathcal{O}$ there exist $c_1,\ldots,c_m \in \{0,1\}$ such that*

$$\nu(h,\mathcal{O}) = \sum_{i=1}^{m} c_i \nu(h,\mathcal{O}_i)$$

mod 2 *for every $h \in G$.*
   Moreover, if $\mathcal{O}_1,\ldots,\mathcal{O}_m$ and $h_1,\ldots,h_m$ are as above, then
   (iii) *for any $g \in G$, $\nu(gp,\mathcal{O}) = 0$ for every orbit $\mathcal{O}$ where p is the product of the h_i's, taken in order, for which $\nu(g,\mathcal{O}_i) = 1$.*

Proof. The following argument, contributed by T. Shores, is "programmable". Let $G = \{g_1,\ldots,g_k\}$, let $\{\mathcal{X}_1,\ldots,\mathcal{X}_l\}$ be the set of orbits of G, and consider the $k \times l$ matrix $M = [\nu(g_i,\mathcal{X}_j)]$. Since the mapping $g \to [\nu(g,\mathcal{X}_j)]$ is a homomorphism from G into $Z_2^{(l)}$, the row-reduced echelon matrix E of M has the form: $E = [\nu(g_{r_i},\mathcal{X}_j)]$. If E has m nonzero rows, let $t_1,\ldots,t_m$ be the columns corresponding to the leading 1's in these rows. Then (i) and (ii) follow with $h_i = g_{r_i}$ and $\mathcal{O}_i = \mathcal{X}_{t_i}$. Also for g and p as in (iii), we have $\nu(gp,\mathcal{O}_i) = 0$ for $i = 1,\ldots,m$ and hence $\nu(gp,\mathcal{O}) = 0$ for every orbit $\mathcal{O}$ by (ii). ∎

Since the strategy outlined above deals with each orbit separately, we will concentrate for the remainder of this section on the action of a wreath product on a single orbit.

Recall that for $\alpha \in H \wr G$, α' and α'' denote the "permuting" and "shuffling" parts of α, respectively, i.e., $\alpha = (\alpha', \alpha'')$ where $\alpha' \in G$ and $\alpha'' \in H^{\mathscr{X}}$. We will also write $\text{Act}'(\alpha)$ for the projection of $\text{Act}(\alpha)$ on $\mathscr{X}$. Thus,

$$\text{Act}'(\alpha) = \{ i \in \mathscr{X} : i\alpha' \neq i \quad \text{or} \quad \alpha''(i) \neq 0 \}.$$

An element $\alpha \in H \wr G$ will be called a k-cycle if α' is a k-cycle, $\text{Act}'(\alpha) = \text{Act}(\alpha')$, and α has order k. Cubelike puzzles are generated by products of such cycles. *Examples*: (i) ϕ and ρ of Example 3 are 4-cycles. (ii) β of Example 1 [Example 2] is not a cycle because β' is a 2-cycle [4-cycle] and the order of β is 4 [12].

By formula (5),

$$\left(\alpha^k \right)''(i) = \sum_{j=0}^{k-1} \alpha''(i\alpha'^{-j}) \quad \text{for} \quad i = 1, \ldots, n.$$

Hence, it follows that

$$(9) \qquad \qquad \sum_{i=1}^{n} \alpha''(i) = 0$$

holds for any cycle α. Since property (9) is clearly preserved under multiplication in a wreath product, Proposition 2 below follows. The result, almost trivial in this context, translates into puzzle "conservation laws" like "the number of flipped cubies is always even" and "total clockwise twist equals total counterclockwise twist mod 3". Elaborate arguments have been given to explain these phenomena. See, e.g., [11, p. 28], [19, p. 17], and [2, p. 762].

PROPOSITION 2. *If Γ is a subgroup of $H \wr G$ which is generated by cycles, then* (9) *holds for all* $\alpha \in \Gamma$.

By a *complete set of 3-cycles for* $a, b, c \in \mathscr{X}$ will be meant a set $\{ \tau_0, \ldots, \tau_{r-1} \}$ of 3-cycles such that for each $j = 0, 1, \ldots, r-1$,

(i) $$\tau_j' = (abc)$$

and

(ii) $$\tau_j''(a) = \tau_0''(a) + j, \qquad \tau_j''(b) = \tau_0''(b), \qquad \tau_j''(c) = \tau_0''(c) - j.$$

The following result shows how to produce 3-cycles.

PROPOSITION 3. *If α, β and σ satisfy*
(i) $\text{Act}'(\alpha) \cap \text{Act}'(\beta) = \{b\}$, *where* $a\alpha' = b = c\beta'$, *and*
(ii) $\text{Act}'(\sigma) \cap \{a, b, c\} = \{c\}$, *where* $c\sigma' = c$ *and* $\sigma''(c) = 1$,
then a complete set of 3-cycles for a, b, c is given by

$$\tau_j = \sigma^j [\alpha, \beta] \sigma^{-j}$$

for $j = 0, 1, \ldots, r-1$.

Proof. Let $\tau_0 = [\alpha, \beta]$. By formulas (4) and (8) and condition (i),

$$\tau_0' = [\alpha', \beta'] = (abc), \quad \tau_0''(a) = \beta''(b) - \alpha''(b),$$

$$\tau_0''(b) = \alpha''(b), \quad \tau_0''(c) = -\beta''(b), \quad \text{and} \quad \tau_0''(i) = 0 \quad \text{for } i \neq a, b, c.$$

Hence, τ_0 is a 3-cycle. Let $\tau_1 = \sigma \tau_0 \sigma^{-1}$. Then by formulas (3) and (7) and conditions (i) and (ii),

$$\tau_1' = \sigma'(abc)\sigma'^{-1} = (abc), \quad \tau_1''(a) = \tau_0''(a) + 1,$$

$$\tau_1''(b) = \tau_0''(b), \quad \tau_1''(c) = \tau_0''(c) - 1, \quad \text{and} \quad \tau_1''(i) = 0 \quad \text{for } i \neq a, b, c.$$

The result follows by induction. ∎

Elements of the form $\mu(a, b) = (1, s)$, where $s(a) = 1$, $s(b) = -1$, and $s(i) = 0$ otherwise, play a special role in cubelike puzzles. For the case of the corner orbit of the cube puzzle, these elements are popularly called *mesons* because of an interesting link with particle physics (see [8]). Since $\tau_1\tau_0^{-1} = \mu(c, b)$, Proposition 3 also shows us how to produce these generalized mesons.

COROLLARY. *Under the assumptions of Proposition 3, we have*

$$\mu(c, b) = [\sigma, [\alpha, \beta]].$$

Typically, Proposition 3 will be applied with β having one of these two forms: $\beta = \gamma\delta\gamma^{-1}$ or $\beta = [\gamma, \delta]$. Propositions 4 and 5 below show how to deal with these special situations.

PROPOSITION 4. *If α, γ, and δ satisfy*

$$\text{Act}'(\alpha)\gamma' \cap \text{Act}'(\delta) = \{b\gamma'\},$$

then $\text{Act}'(\alpha) \cap \text{Act}'(\gamma\delta\gamma^{-1}) = \{b\}$.

Proof. By (3) and (7), $i \in \text{Act}'(\beta\gamma\beta^{-1})$ if and only if $i\beta' \in \text{Act}'(\gamma)$ and the result follows. ∎

PROPOSITION 5. *If α, γ, and δ satisfy*
(i) $\text{Act}'(\alpha) \cap \text{Act}'(\gamma) \cap \text{Act}'(\delta) = \{b\}$
(ii) $(\text{Act}'(\alpha) \cap \text{Act}'(\gamma))\gamma' \cap \text{Act}'(\delta) = \varnothing$, *and*
(iii) $(\text{Act}'(\alpha) \cap \text{Act}'(\delta))\delta' \cap \text{Act}'(\gamma) = \varnothing$,
then $\text{Act}'(\alpha) \cap \text{Act}'([\gamma, \delta]) = \{b\}$.

Proof. By hypothesis, $b\gamma'\delta' = b\gamma' \neq b\delta' = b\delta'\gamma'$; hence, $b \in \text{Act}'([\gamma', \delta'])$. Conversely, if $i \notin \text{Act}'(\gamma)$, then $i \in \text{Act}'([\gamma, \delta])$ if and only if $i\delta' \in \text{Act}'(\gamma)$. Therefore, if $i \in \text{Act}'(\alpha) \cap \text{Act}'([\gamma, \delta])$, then by (iii), we must have $i \in \text{Act}'(\gamma)$. Similarly, by (ii), $i \in \text{Act}'(\delta)$ and so $i = b$ by (i). ∎

For all of the puzzles to be considered the group G will be triply transitive and the following condition easy to verify.

PROPOSITION 6. *Let Γ be a subgroup of $H \setminus G$. If Γ contains a complete set of 3-cycles for three elements a, b, c and if for each $x \in \mathscr{X}$, $x \neq a, b, c$, there exists $\alpha \in \Gamma$ such that*

$$\text{Act}(\alpha') \cap \{a, b, c\} = \{x\alpha'\},$$

then Γ contains a complete set of 3-cycles for any three elements of $\mathscr{X}$.

Proof. From the proof of Proposition 3, we see that conjugation of a complete set of 3-cycles yields a complete set of 3-cycles. Hence, to prove the result we need only express 3-cycles of $\mathscr{X}$ as conjugates of (abc). For $x \in \mathscr{X}$, $x \neq a, b, c$, assume

$$\text{Act}(p) \cap \{a, b, c\} = \{a\} = \{xp\}.$$

Then

$$(xbc) = p(abc)p^{-1},$$

$$(axc) = (abc)^{-1}(xbc)(abc),$$

and

$$(abx) = (abc)(xbc)(abc)^{-1}.$$

From these we get all 3-cycles as follows:

$$(xyc) = (aby)^{-1}(xbc)(aby),$$

$$(xby) = (axc)^{-1}(aby)(axc),$$

$$(axy) = (ybc)^{-1}(axc)(ybc),$$

and

$$(xyz) = (zbc)^{-1}(xyc)(zbc).\ \blacksquare$$

PROPOSITION 7. *If* Γ *is a subgroup of* $H \wr G$ *that contains a complete set of 3-cycles for any three elements of* $\mathscr{X}$, *then* Γ *contains all elements* α *such that* α' *is even and* α'' *satisfies* (9). *For any such* α, *there exist 3-cycles* $\tau_1, \ldots, \tau_N \in \Gamma$ *such that* $\alpha \tau_1 \cdots \tau_N = 1$.

Proof. The proof is by induction on the number of elements of $\text{Act}'(\alpha)$. If $\alpha' \neq 1$, there exist distinct elements a, b, and c of $\text{Act}(\alpha')$ with $b = a\alpha'$. Let τ be a 3-cycle such that $\tau' = (acb)$ and $\tau''(a) = -\alpha''(b)$. Then $|\text{Act}'(\alpha\tau)| < |\text{Act}'(\alpha)|$. If $\alpha' = 1$, there exist distinct elements a and b with $\alpha''(a) \neq 0$ and $\alpha''(b) \neq 0$. Then $|\text{Act}'(\alpha\mu(a,b)^{\alpha''(b)})| < |\text{Act}'(\alpha)|$. $\blacksquare$

5. Applications. This section contains various applications. Discussion is limited to puzzles based on partitioned regular polyhedra. It is assumed that puzzle pieces are marked so that home locations and orientations are uniquely determined. To keep the notation manageable, in most cases only those generators and puzzle pieces that play direct roles in the illustrated moves will be given names. The reader may find it helpful at times to express generators as products of k-cycles using a more complete notational system. A complete notational system will be developed for the $n \times n \times n$ cube puzzle.

Except for slight variations, the devices considered here actually exist. The $n \times n \times n$ cube exists for $n = 2, 3, 4$, and 5, and there is even a transparent $5 \times 5 \times 5$ cube which is held together by magnets.

The icosahedron puzzle. We start with this example because it is simplest and displays many of the basic ideas. The generators are the twelve $72°$ vertex turns as indicated in Fig. 2(a). Products of these generators cause permutations of the twenty triangular faces. They also cause rotations as can be seen by studying the action of the move $L_2 R_2$ (see Fig. 2(b)) on the vertices of the face marked c. By the discussion in §3, we see that the combined permuting-rotating action can be regarded as taking place in the wreath product $\mathbb{Z}_3 \wr S_{20}$.
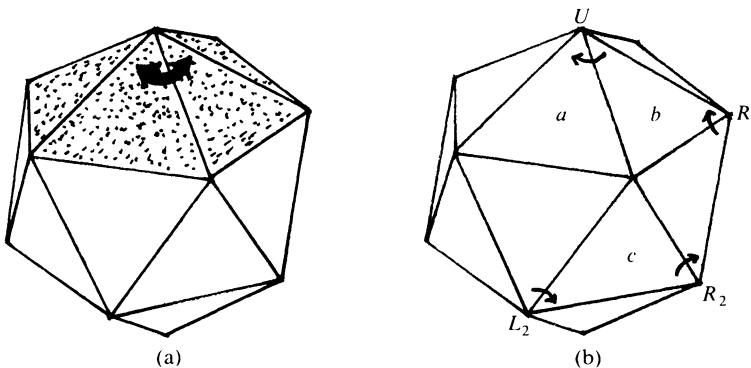

(a)   (b)

FIG. 2

Now let's see how to cycle the pieces marked a, b, c in Fig. 2(b). To apply Proposition 3 we need "disjoint" moves α and β such that α': $a \to b$ and β': $c \to b$. That $U^{-1}, R_1^{-1} R_2 R_1$ is such a pair can be seen from Fig. 2(b) and can also be verified by Proposition 4. Since $L_2 R_2$ causes a $120°$ clockwise rotation of piece c and stabilizes a and b, it follows that a complete set of 3-cycles for a, b, c is given by:

$$(abc)_j = (L_2 R_2)^j [U^{-1}, R_1^{-1} R_2 R_1] (L_2 R_2)^{-j} \quad (j = 0, \pm 1).$$

We can regard this set as a vehicle for moving piece b into location c ending up with any

orientation we please. Note that the move corresponding to $j = 0$ affects the orientation in the same way as the move $R_1^{-1}R_2^{-1}$ and the move corresponding to $j = -1$ [$j = 1$] adds a 120° clockwise [counterclockwise] rotation.

Since the generators for this puzzle are 5-cycles, all patterns correspond to even permutations and Proposition 1 plays no role. Clearly, Proposition 6 (and its proof) apply and we may proceed as in Proposition 7 to unscramble any possible configuration.
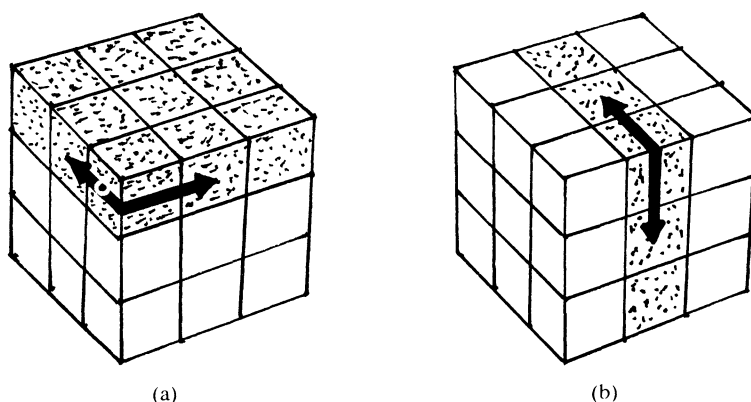

(a)   (b)

FIG. 3

The cube puzzle. This puzzle can be viewed either as being generated by the six 90° face turns indicated in Fig. 3(a) or by the nine 90° layer turns which include the three "slice turns" indicated in Fig. 3(b). Since a slice turn is clearly equivalent to a pair of parallel face turns followed by a 90° rotation of the entire cube, no new patterns arise by allowing these moves. Group-theoretically, if M_f and M_l are the groups generated by faces and layers, respectively, then M_l is an extension of M_f by the group of rigid motions of the cube.

We will work in the larger group, allowing slice turns, because it will lead to neater formulas. The price for this convenience will be the inconvenience of having to deal with a trivial orbit. The orbits are: $\mathscr{V}$ consisting of the eight corner positions, $\mathscr{E}$ consisting of the twelve edge positions, and $\mathscr{C}$, the trivial orbit, consisting of the six face centers. Corresponding to elements of $\mathscr{V}$, $\mathscr{E}$, and $\mathscr{C}$, respectively, there are 3, 2, and 4 possible orientations. We note, however, that in the most popular version of this puzzle, the six faces are solidly colored rendering center piece rotations indistinguishable.

Let h_1 be any face turn, h_2 any slice turn, and let $\mathcal{O}_1 = \mathscr{V}$, $\mathcal{O}_2 = \mathscr{C}$. Then, for $m = 2$, (i) of Proposition 1 is clearly satisfied. Moreover, if h is any generator, (ii) of Proposition 1 is satisfied by $\mathcal{O} = \mathscr{E}$ and $c_1 = c_2 = 1$. It follows that (ii) holds for all products of generators, i.e., for all moves h. The conclusion (iii) of Proposition 1 says the obvious in this case: Any element can be made even on all orbits by making at most two quarter-turns. Note that if face centers are correctly positioned, at most one quarter-turn is needed, and, if, in addition, face centers are correctly oriented, then no adjustment is needed.

As with the icosahedron puzzle, Propositions 3–4 give us the following set of 3-cycles, Proposition 6 applies to both orbits $\mathscr{V}$ and $\mathscr{E}$ and we can proceed as in Proposition 7. The notation refers to Fig. 4.

$$(abc)_j = (LD)^j[U, R^{-1}DR](LD)^{-j} \quad (j = 0, \pm 1),$$

$$(123)_j = (\rho^{-1}F^2)^j[\phi^{-1}, R^{-1}FR](\rho^{-1}F^2)^{-j} \quad (j = 0, 1).$$

Except for a possible parity adjustment at the beginning, the solution is a product of moves of the form $\gamma^i[\alpha, \beta]\gamma^{-i}$ which stabilize the oriented centers. It follows that if the center squares are
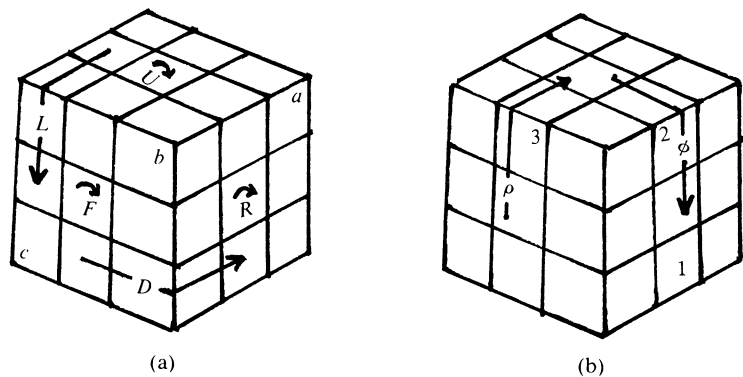
(a)   (b)

Fig. 4

Fig. 5

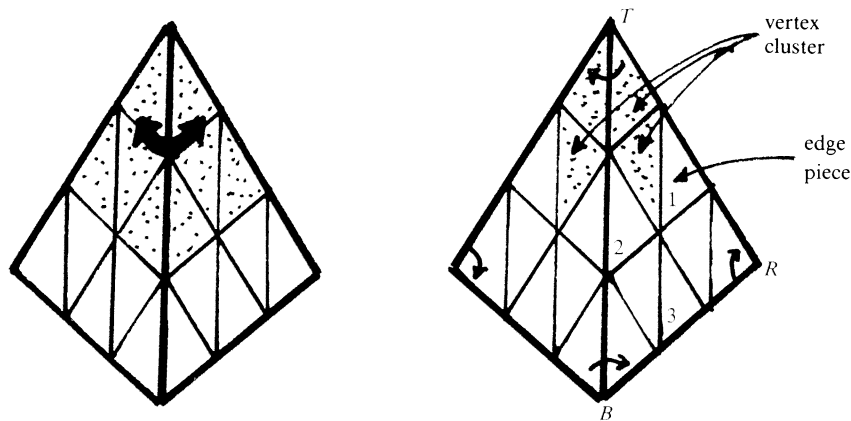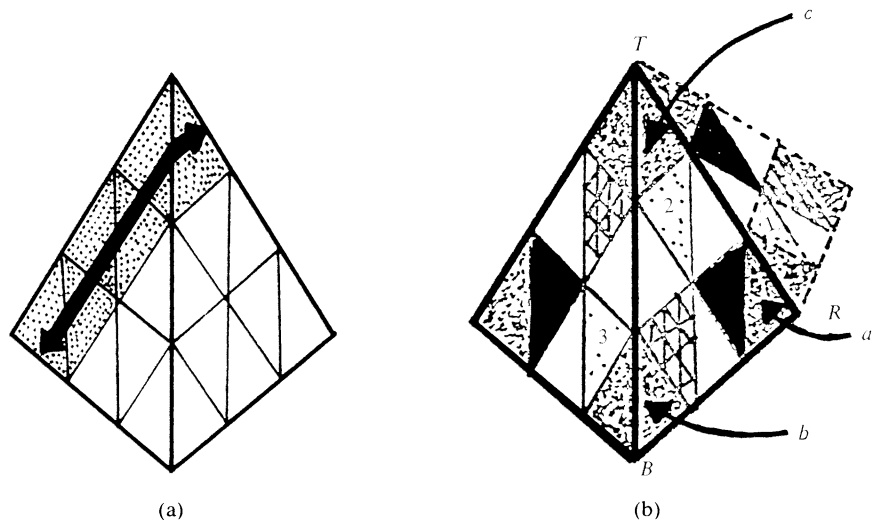(a)   (b)

Fig. 6

correctly positioned and oriented at the beginning, they will end up that way.

Tetrahedron puzzles. Two versions will be discussed. When combined, a third version (called the Master Pyraminx in [12]) results which can be solved by combining the two solutions.

The first puzzle is generated by the four 120° vertex turns as indicated in Fig. 5. Note that each vertex consists of a cluster of four pieces which rotates as a unit independent of the other vertex clusters. (For the puzzle called the Popular Pyraminx in [12] the little tetrahedron tips can also be rotated. This complication is trivial and is disregarded here.) The vertex clusters play the same pivotal role as the center squares for the cube puzzle without slices. As with the cube puzzle, it will be seen that they can be fixed up at the outset and never again worried about. Thus, there is only one orbit which consists of the six edge pieces (see Fig. 5). As before, the following complete set of 3-cycles can be obtained via Propositions 3–4, Proposition 6 applies, and we can proceed as in Proposition 7. The notation refers to Fig. 5:

$$(123)_J = (RTBT^{-1})^J [T, B^{-1}] (RTBT^{-1})^{-J} \quad (j = 0, 1).$$

The second version is generated by the 180° edge turns as indicated in Fig. 6(a). There are four orbits: $\mathscr{V}$ consisting of the four corner pieces and three face orbits $\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3$ each consisting of four face pieces. The orbits are indicated in Fig. 6(b) by different shadings.

Note that the six edge pieces remain fixed relative to one another and play the same pivotal role as the center squares in the cube puzzle and the vertex clusters in the preceding example. Clearly,

$$\nu(g, \mathscr{F}_1) + \nu(g, \mathscr{F}_2) + \nu(g, \mathscr{F}_3) = 0$$

mod 2 for any generator g and hence for all g, and Proposition 1 is satisfied with $m = 3$. Note, though, that parities are taken care of automatically by correctly orienting the edge pieces. Note also that orientations of the remaining pieces are uniquely determined.

By Propositions 3–4, we obtain:

$$(123) = [\overline{RT}, \overline{LT}\,\overline{BT}\,\overline{LT}],$$

where the notation refers to Fig. 6 and $\overline{V_1 V_2}$ denotes the turn with ends V_1, V_2. Similar 3-cycles can be found for the other two face orbits.

It is easy to see that the hypothesis of Proposition 5 is satisfied with $\alpha = \overline{BR}$, $\gamma = \overline{BT}$, and $\delta = \overline{BL}$, and, hence,

$$(abc) = [\overline{BR}, [\overline{BT}, \overline{BL}]].$$

Proposition 6 clearly applies to all orbits and the solution can be completed via Proposition 7.

Octahedron puzzles. Two versions will be discussed. The first is generated by the six 90° vertex turns as indicated in Fig. 7(a). As pointed out in [12] there is a vertex/center duality between this puzzle and the cube puzzle. The octahedron is easier because the eight corner cubelets of the cube puzzle correspond to mere points on the octahedron puzzle. Thus, a simple strategy is to first orient the vertex clusters, then use edge processes developed for the cube puzzle (compare Figs. 4 and 7(b)):

$$(123)_J = (\rho^{-1} F^2)^J [\phi^{-1}, R^{-1} FR] (\rho^{-1} F^2)^{-J} \quad (j = 0, 1).$$

The second puzzle is the face-turning puzzle described in [12]. Like the cube puzzle, we will allow slice turns (see Fig. 9). The solution given here provides for the addition of orientable pieces at the face centers, the idea being that such a center piece would rotate only when its corresponding face rotated. See Fig. 8. The puzzle without orientable center pieces is somewhat easier.

There are four orbits: $\mathscr{V}$ consisting of the six corners, $\mathscr{E}$ consisting of the twelve edges, and two face orbits $\mathscr{F}_1$ and $\mathscr{F}_2$ each consisting of twelve pieces. The orbits are indicated in Fig. 9 and

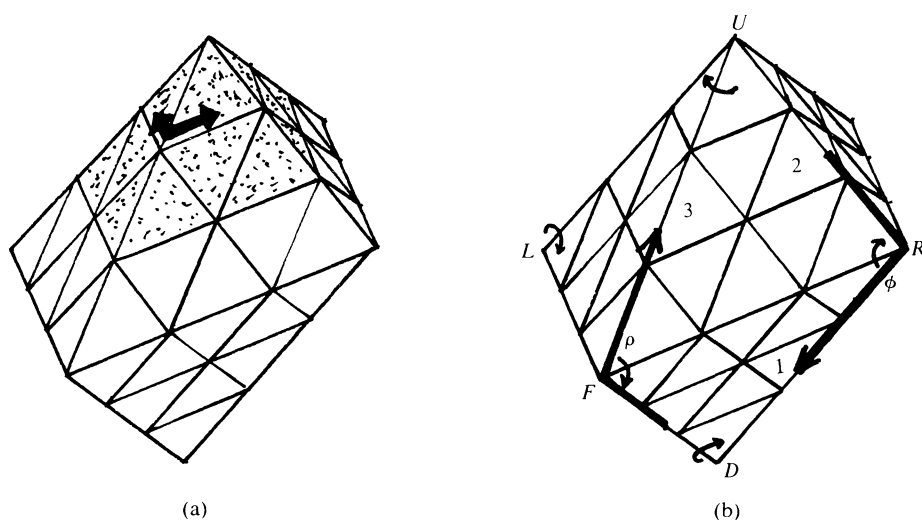(a)   (b)
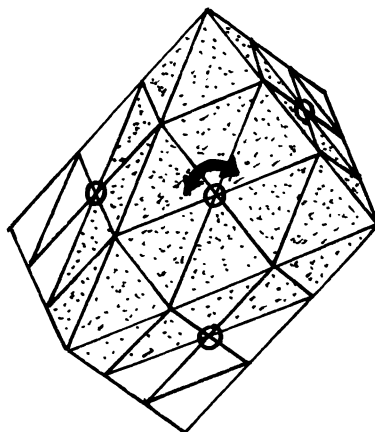
FIG. 7

FIG. 8

again in Fig. 10, which is a flattened version of the puzzle. In the flattened version, one can readily view the entire action of the generators. Of course, these can all be expressed as products of 3-cycles (five for face turns, four for slice turns) and the reader, in checking the formulas below, may well find it advantageous to do so.

Observe that edge and face pieces have only one possible orientation and that corners have two. By Propositions 3–4,

$$(e_1 e_2 e_3) = \left[F_2, \rho_1^{-1} \phi_2 \rho_1 \right],$$

and by Proposition 5,

$$(f_1 f_2 f_3) = \left[\phi_1, \left[B_1^{-1}, R_2 \right] \right],$$

and

$$(v_1 v_2 v_3)_j = (L_2 F_2)^j \left[F_1, \left[R_1^{-1} F_2^{-1} R_1, B_1 \right] \right] (L_2 F_2)^{-j} \quad (j = 0, 1).$$

A similar 3-cycle can be given for the other face orbit, and it is clear that Proposition 6 applies to all orbits. The solution can then be completed via Proposition 7.
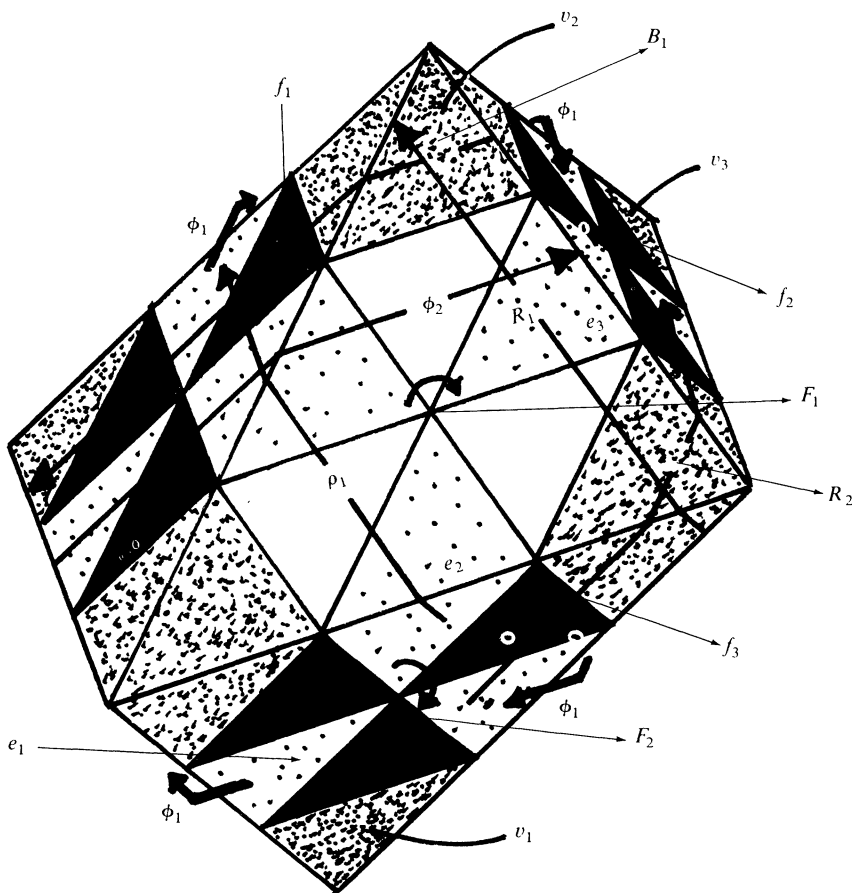
FIG. 9

Dodecahedron puzzles. Two versions will be discussed. The generators for the first one are the twelve face turns as indicated in Fig. 11(a). There are two orbits: $\mathcal{V}$ consisting of 20 corner pieces and $\mathcal{E}$ consisting of 30 edge pieces. Clearly, each generator is the product of two 5-cycles and we can proceed in the usual way, obtaining the following moves which can be used to solve the puzzle (see Fig. 11(b)):

$$(abc)_j = (R_2 R_1)^j [L_1^{-1}, U^{-1} R_1 U](R_2 R_1)^{-j} \quad (j = 0, \pm 1)$$

$$(123)_j = (L_1 L_2 F)^j [R_1^{-1}, FB^{-1}U^{-1}BF^{-1}](L_1 L_2 F)^{-j} \quad (j = 0, 1).$$

The second puzzle, shown in Fig. 12(a), is a deeper cut version of the first one, the extreme case being the Magic Crystal of [12] shown in Fig. 12(b). Noting that the face pieces a and a' act as a unit (with two possible orientations), we can see that there are three orbits $\mathcal{V}$, $\mathcal{E}$, and $\mathcal{F}$ of sizes 20, 30, and 30, respectively.

The following basic 3-cycles can be obtained via Proposition 3. Oriented variations and other details are left to the reader.

$$(xyz) = [L_2^{-1}, R_1],$$

$$(abc) = [L_1^{-1}, L_2^{-1}\rho_2^2 R_1 \phi_2^{-2} F_1 \phi_2^2 R_1^{-1} \rho_2^{-2} L_2],$$

$$(123) = [L_1^{-1}, \phi_2 R_2 L_2^{-1} R_1^{-1} R_2^{-1} R_1 \lambda_1^{-2} L_2 R_2^{-1} \phi_2^{-1}].$$
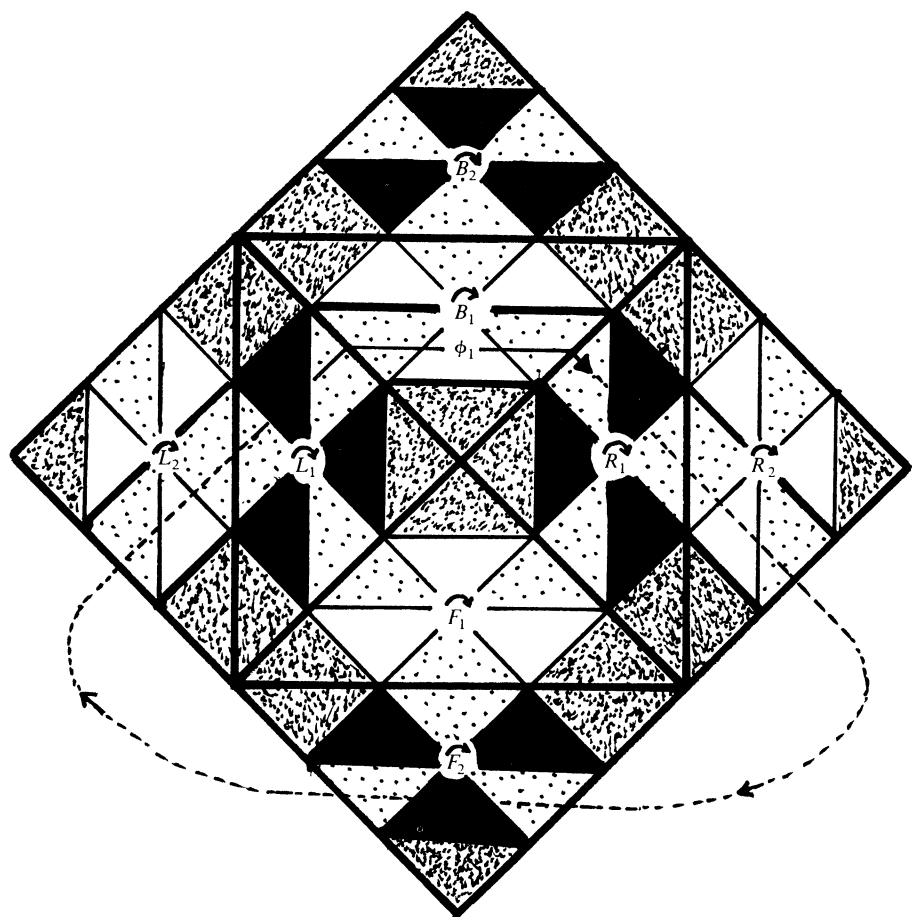
FIG. 10. Flattened octahedron puzzle with one slice turn shown.

(a)   (b)

FIG. 11

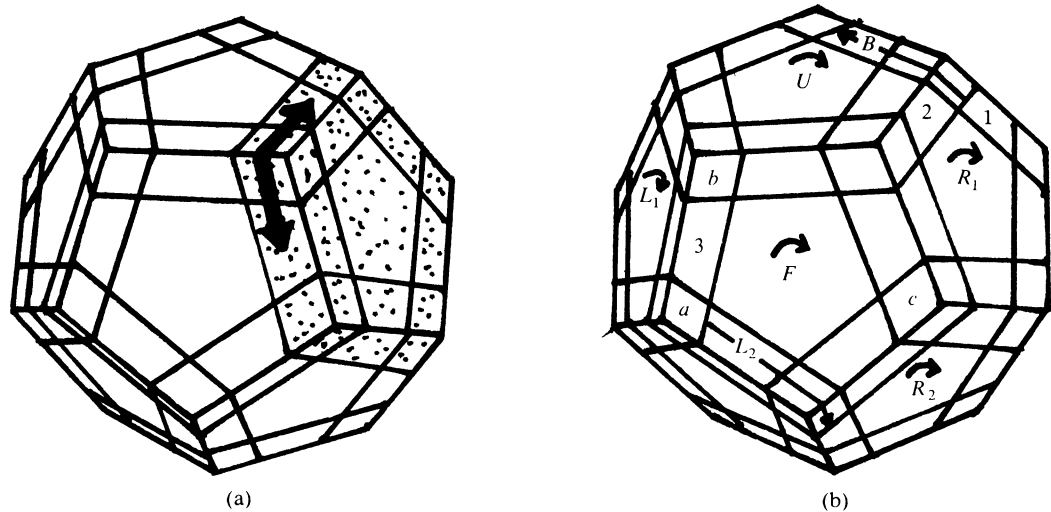(a)   (b)
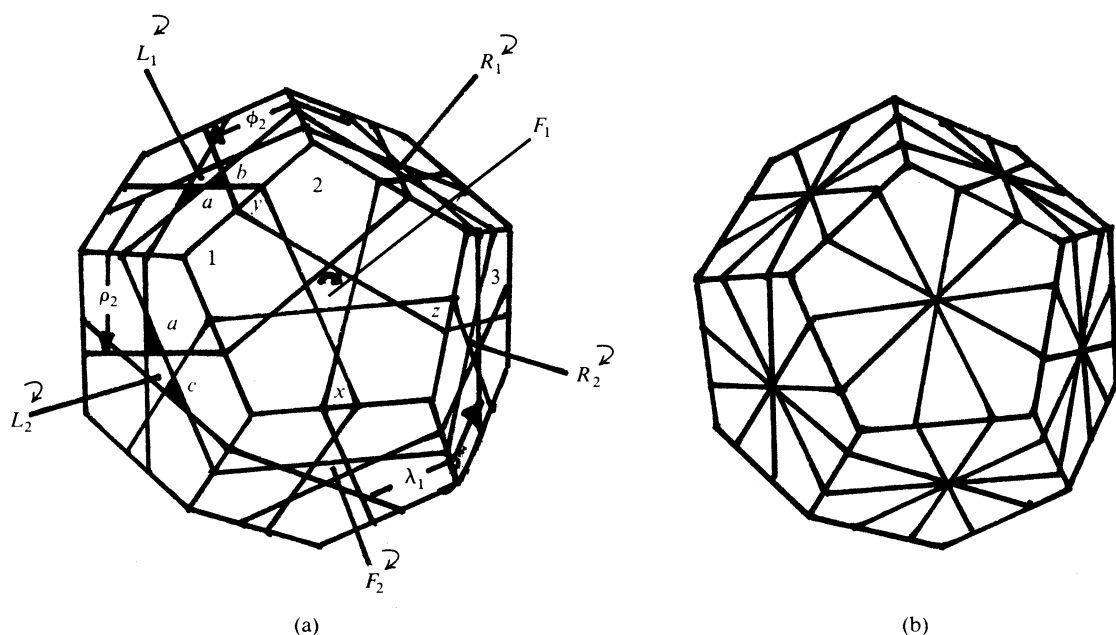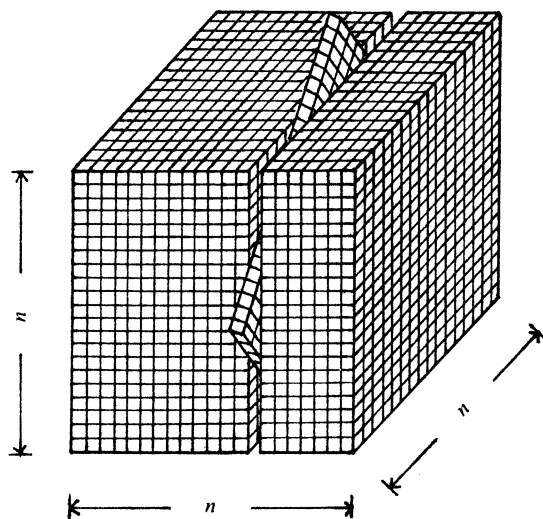
FIG. 12

FIG. 13

The $n \times n \times n$ cube puzzle. As the ultimate cubelike puzzle, consider a cube partitioned into n^3 congruent subcubes and such that each of the $3n$ layers can be rotated as indicated in Fig. 13. The algorithm to be described is for unscrambling any generated configuration of surface cubelets. It will be seen that the algorithm leaves the $(n-2)^3$ internal cubelets invariant, given that they are fixed up in advance. Thus, if one wanted, in addition, to unscramble the internal cubelets, this could be done by applying the algorithm successively to a nested sequence of cubes starting with either a $1 \times 1 \times 1$ center or a $2 \times 2 \times 2$ center (depending on the parity of n) and working outward. Each application of the algorithm would fix up a new "shell" (the surrounding cubelets, of course, would just "come along for the ride").

We assume that n is odd: $n = 2m + 1$, $m \geqslant 2$. The even case is a corollary and is left to the reader. We set up notation in an xyz-coordinate system with the center of the cube at the origin and the edges parallel to the coordinate axes. A typical cubelet is described by the set

$$\left[i - \frac{1}{2}, i + \frac{1}{2}\right] \times \left[j - \frac{1}{2}, j + \frac{1}{2}\right] \times \left[k - \frac{1}{2}, k + \frac{1}{2}\right]$$

or more briefly by the coordinates (i, j, k) of its center. The description may seem inadequate because it ignores orientations. It is easy to see, though, that except for those cubelets corresponding to the $3 \times 3 \times 3$ case, only one orientation is possible.

Each generator is a product of $m(m + 1)$ 4-cycles. Clockwise turns X_i, Y_j, Z_k $(i, j, k = 0, \pm 1, \ldots, \pm m)$ are given by:

$$X_i = \prod_{j=0}^{m} \prod_{k=1}^{m} ((i, j, k)(i, k, -j)(i, -j, -k)(i, -k, j)),$$

$$Y_j = \prod_{i=0}^{m} \prod_{k=1}^{m} ((i, j, k)(-k, j, i)(-i, j, -k)(k, j, -i)),$$

$$Z_k = \prod_{i=0}^{m} \prod_{j=1}^{m} ((i, j, k)(j, -i, k)(-i, -j, k)(-j, i, k)).$$

Here, "clockwise" is from the point of view of looking along a positive axis toward the origin.
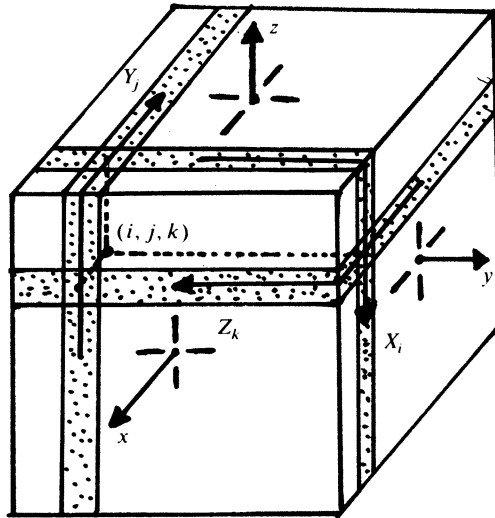

FIG. 14

See Fig. 14. Notice that the central slices X_0, Y_0, Z_0 can be omitted from the list of generators as they can be replaced by the complementary moves.
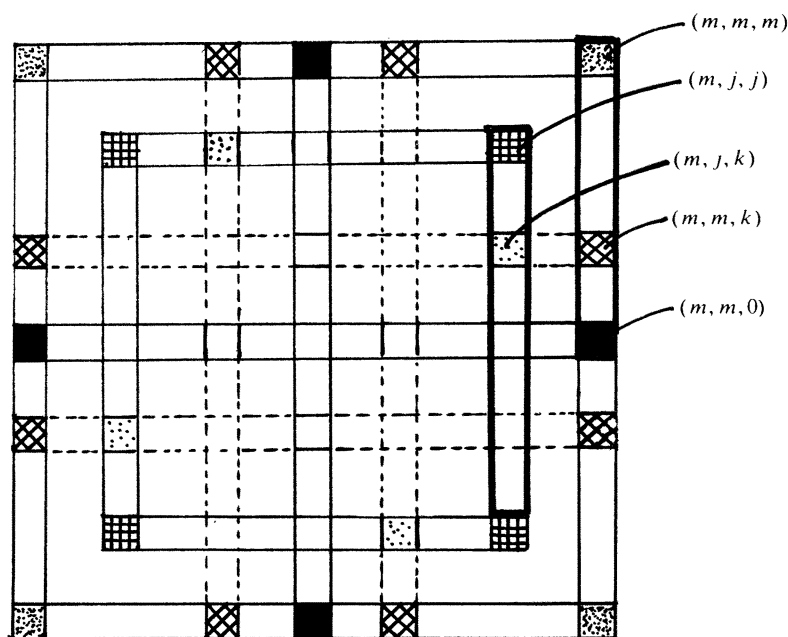
There are $m^2 + 1$ orbits (see blocked portion in Fig. 15):

$$\mathscr{V} = \mathcal{O}((m, m, m)), \quad \mathscr{E} = \mathcal{O}((m, m, 0)),$$

$$\mathscr{E}_k = \mathcal{O}((m, m, k)) \quad (k = 1, \ldots, m - 1),$$

and

$$\mathscr{F}_{jk} = \mathcal{O}((m, j, k)) \quad (j = 1, \ldots, m - 1; \ k = 0, \pm 1, \ldots, \pm(j - 1), j).$$

Orbits $\mathscr{V}$ and $\mathscr{E}$ correspond to the $3 \times 3 \times 3$ case and consist of 8 and 12 cubelets, respectively. Each $\mathscr{E}_k$ consists of 24 edge cubelets and each $\mathscr{F}_{jk}$ consists of 24 face cubelets. The shadings in Fig. 15 illustrate the four types of orbits.

FIG. 15. Orbits in the face $x = m$.

Now consider an arbitrary generated configuration g. Let $L_1, \ldots, L_N$ be any sequence of layer turns $\neq X_0, Y_0, Z_0$ such that $g = L_1 \cdots L_N$. Let $f(\mathcal{O})$ and $s(\mathcal{O})$ denote the number of L_i's which are, respectively, face and slice turns such that $\nu(L_i, \mathcal{O}) = 1$. Then, clearly,

$$\nu(g, \mathcal{O}) = f(\mathcal{O}) + s(\mathcal{O})$$

mod 2 from which we obtain the following:

$$\nu(g, \mathcal{F}_{jj}) = f(\mathcal{F}_{jj}) = f(\mathcal{E}) = \nu(g, \mathcal{E}),$$

$$\nu(g, \mathcal{F}_{jk}) = f(\mathcal{E}) + s(\mathcal{F}_{j0}) + s(\mathcal{F}_{0k})$$

$$= \nu(g, \mathcal{E}) + \nu(g, \mathcal{F}_{j0}) + \nu(g, \mathcal{F}_{k0}) \quad \text{for } k \neq j,$$

$$\nu(g, \mathcal{E}_k) = s(\mathcal{E}_k) = \nu(g, \mathcal{E}) + \nu(g, \mathcal{F}_{k0}),$$

and

$$\nu(g, \mathcal{V}) = f(\mathcal{V}) = \nu(g, \mathcal{E}).$$

Thus condition (ii) of Proposition 1 is met with $\mathcal{O}_j = \mathcal{F}_{j0}$ for $j = 1, \ldots, m-1$ and $\mathcal{O}_m = \mathcal{E}$. Furthermore, it is clear that $\nu(Y_i, \mathcal{O}_j) = \delta_{ij}$ and, therefore, condition (1) of Proposition 1 is satisfied with $h_i = Y_i$. The surprising conclusion is that all parities can be made even by making at most m quarter-turns.

Notice that if the face centers are correctly oriented at the beginning, then $\nu(g, \mathcal{E}) = 0$ and the above adjustment will not affect the face centers.

Via Propositions 3–4, we obtain the following formula which gives 3-cycles for *all* orbits simultaneously:

$$((-j, m, k)(m, j, k)(-k, m, -j)) = \left[Z_k, X_m Z_{-j} X_m^{-1} \right]$$

for $0 \leqslant |k| \leqslant j \leqslant m$, $k \neq -j$. The solution then proceeds as expected.

To see that the internal cubelets remain invariant under the algorithm if they are fixed up in
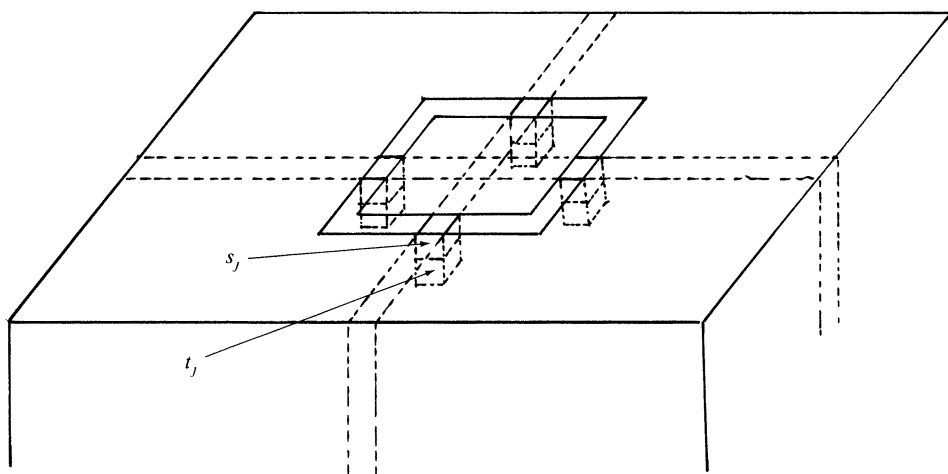
FIG. 16

advance, assume that $\nu(g, \mathcal{O}(t)) = 0$ for all internal cubelets t. Then, from Fig. 16, it is clear that

$$\nu\big(g, \mathcal{O}_j\big) = \nu\big(g, \mathcal{O}(s_j)\big) = \nu\big(g, \mathcal{O}(t_j)\big) = 0 \quad \text{for} \quad j = 1, \ldots, m-1.$$

Hence, in this situation, parity adjustments which would affect the internal cubelets are not needed. Therefore, since all other moves are commutators, the internal cubelets are unaltered by the algorithm.

6. The underlying groups. If, in the above examples, we ignore the matter of orienting pivotal centers, the underlying group P for such a puzzle can be described as a subdirect product of groups of the form $H_i \wr G_i$, where $H_i = \mathbb{Z}_{r_i}$ and G_i is either $\mathrm{Alt}\,\mathcal{X}_i$ or $\mathrm{Sym}\,\mathcal{X}_i$. By Propositions 2 and 7, $P \mid H_i \wr G_i$ has index r_i in $H_i \wr G_i$. Also, if $\{\mathcal{X}_1, \ldots, \mathcal{X}_m\}$ is a parity basis in the sense of Proposition 1, then in forming a typical element $g = g_1 \cdots g_m \cdots g_N$, one has $|\mathcal{X}_i|!$ choices for each g_i, $i = 1, \ldots, m$, and $\frac{1}{2}|\mathcal{X}_i|!$ choices for each of the remaining g_i's. It follows that the order of P is equal to

$$\frac{1}{2^{N-m}} \prod_{i=1}^{N} |\mathcal{X}_i|! r_i^{|\mathcal{X}_i|-1}.$$

In particular, we see that the $(2m+1) \times (2m+1) \times (2m+1)$ cube contains

$$\frac{8! \cdot 12! \cdot 24!^{m^2-1} \cdot 2^{11} \cdot 3^7}{2^{m^2+1-m}}$$

distinct patterns. For the $(2m) \times (2m) \times (2m)$ cube, the reader will find that there are $m^2 + 1 - m$ orbits, one of size 8 and all others of size 24, and that a parity basis has size m. Hence, taking into account the fact that the 24 rigid motions of the cube yield duplications, the $(2m) \times (2m) \times (2m)$ cube has

$$\frac{8! \cdot 24!^{m(m-1)} \cdot 3^7}{2^{(m-1)^2} \cdot 24}$$

distinct patterns. In case the faces are solidly colored, the number of distinguishable patterns decreases. The following table summarizes our findings for the cases $n = 2, 3, 4$ and 5. Conflicting values for the case $n = 4$ have appeared in print.

Colored cube size	Number of distinguishable patterns
$2 \times 2 \times 2$	$\dfrac{8! \cdot 3^7}{24}$
$3 \times 3 \times 3$	$\dfrac{8! \cdot 12! \cdot 2^{11} \cdot 3^7}{2}$
$4 \times 4 \times 4$	$\dfrac{8! \cdot 24!^2 \cdot 3^7}{2 \cdot 24 \cdot \left(4!^6/2\right)}$
$5 \times 5 \times 5$	$\dfrac{8! \cdot 12! \cdot 24!^3 \cdot 2^{11} \cdot 3^7}{2^3 \cdot \left(4!^6/2\right)^2}$

7. Efficiency. Many of the algorithms outlined in §5 can be shortened considerably by replacing 3-cycles by *relative* 3-cycles. If $\mathcal{O}_1, \ldots, \mathcal{O}_N$ is any ordering of the orbits, then the "pure" 3-cycles used to unscramble the orbit $\mathcal{O}_i$ can be replaced by 3-cycles relative to $\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_i$. These, in general, will have shorter factorizations, and can still be obtained via Proposition 3. A measurement of the efficiency of any particular ordering is given by

$$(*) \qquad m(\mathcal{O}_1, \ldots, \mathcal{O}_N) = \sum_{i=1}^{N} l_i |\mathcal{O}_i|,$$

where l_i denotes the shortest possible 3-cycle in $\mathcal{O}_i$ relative to $\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_i$.

The following table is a summary of our findings of most efficient orderings with the relevant additional 3-cycles given in the right-hand column (cf. Figs. 4, 6, 9, 11, 12, 14). The findings are based on minimizing $(*)$ over the $N!$ orderings with the l_i's taken as lengths of commutator 3-cycles.

Puzzle	Most efficient ordering	Relative 3-cycles
$3 \times 3 \times 3$ cube	$(\mathscr{E}, \mathscr{V})$	$[R, F]$
tetrahedron #2	$(\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3, \mathscr{V})$	$[\overline{RT}, \overline{BT}], [\overline{BT}, \overline{LT}], [\overline{LT}, \overline{RT}]$
octahedron #2	$(\mathscr{E}, \mathscr{V}, \mathscr{F}_1, \mathscr{F}_2)$	$[F_1, F_2], [F_1, B_1]$
dodecahedron #1	$(\mathscr{E}, \mathscr{V})$	$[R_1, R_2]$
dodecahedron #2	$(\mathscr{F}, \mathscr{V}, \mathscr{E})$	$[F_1, F_2], [L_1, R_1^{-1} R_2^{-1} R_1]$
$n \times n \times n$ cube	$(\mathcal{O}(m, 1, 0), \ldots, \mathcal{O}(m, m, 0),$ then any order)	$[X_m, Y_1], \ldots, [X_m, Y_m]$

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AN ALTERNATIVE APPROACH TO CANONICAL FORMS OF MATRICES

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0. Introduction. The purpose of this paper is to describe a new approach to the derivation of the main structure theorems for linear transformations and matrices. The approach is unconventional in that it reverses the traditional order between the study of various canonical forms and the structure of exponentials of matrices (as treated, for example, in the beautiful book [2]). It is efficient in that it leads in a unified and direct way to the main facts needed for the derivation of canonical forms.

This approach is accessible to students who have mastered the topics in linear algebra which usually precede the structure theorems (as given, for example, in Chapters I to IX of [3]). It is particularly appropriate within the context of a differential equations course, where it is often necessary to review or supplement a previous treatment of linear algebra. The exposition is given for real and complex matrices, but can easily be adapted to a more abstract treatment.

1. The structure of exp[At]. Let $A = [A_{ij}]$ be an $n \times n$ real or complex matrix and let the totality of such matrices be denoted by $M_n(R)$ or $M_n(C)$, respectively. It is customary in the treatment of linear systems of differential equations to define

$$\exp[At] = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}.$$

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