The Quest for *God's Algorithm*
by Rik van Grol

Introduction

Rubik's Cube is the subject of one of the largest —if not the largest— puzzle craze that swept the world. *Rubik's Cube* is a 3D mechanical puzzle invented by Ernõ Rubik in 1974 (see Figure 1). According to *Wikipedia* [1], by 2009 over 350 million *Rubik's Cubes* have been sold worldwide. This is more than one cube for everyone in the USA. Invented in 1974 in Hungary, the puzzle craze only started in 1980 when the puzzle was first sold outside of Hungary. In the 80's the puzzle intrigued many people. Initially solving the puzzle at all was the main issue, especially during the time that no solution books were available. But solving the puzzle in the shortest amount of time was also hot news. In the early 80's the best times were in the order of 24 seconds.

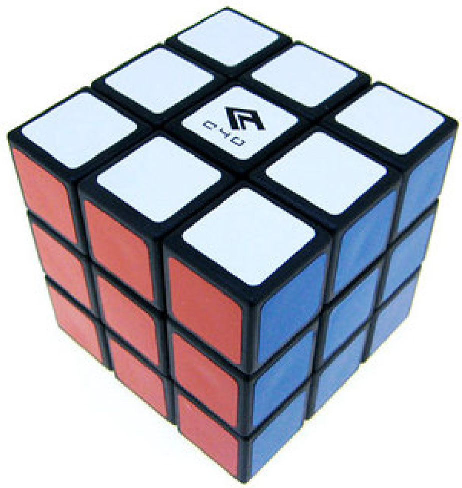

Figure 1. The 3×3×3 *Rubik's Cube*

By the end of the 80's the high point of the craze was over, but amongst many groups of people the puzzle remained very much alive. In recent years *Rubik's Cube* is making its comeback. Solving the cube the quickest —called speedcubing— is now a main issue. The fastest time is now eight seconds (eleven seconds average). Another field of interest is the development of all kinds of *Rubik Cube* variants. This has been the interest of several people making these "mods" by hand (making them very expensive). In recent years they are becoming available on a commercial and affordable basis. On the one hand the line of regular *Rubik's Cubes*: 2×2×2, 3×3×3, 4×4×4 and 5×5×5 is now being extended to 6×6×6 and 7×7×7, and only recently even to 9×9×9 and higher (commercially, but very expensive). On the other hand all kinds of variants of the *Rubik's Cube appear*, like 1×3×3, 2×2×3, 3×3×5, etc., etc. But these variants are not the subject of this paper (for more info read *The Cube* [2]).

The basis for a minimal solving time is a good algorithm to solve the cube. Searching for good algorithms has been an important subject since the early 80's. This brings me to the subject of this article: the search or quest for the optimal solution algorithm, which is also referred to as *God's Algorithm. God's Algorithm* is defined as follows:

God's Algorithm is the procedure to bring back Rubik's Cube from any random position to its solved state in the minimum number of steps.

The maximum of all minimally needed number of steps is referred to as *God's Number*. This number can be defined in several ways. The most common ones are the ones that use face turns or quarter turns. Whereas a quarter turn can either mean a positive or negative 90° turn, a face turn can also mean a 180° turn. A 180° face turn is equal to two quarter turns. In this article I will look at face turns unless stated otherwise.

An interesting aspect of *God's Algorithm* is whether it might have some pattern or regularity permitting a concise representation, or whether it simply requires a huge lookup table and is thus in some sense beyond human comprehension. In the latter case no one will ever be able to learn how to solve *Rubik's Cube* in a minimal number of moves. Still, even if *God's Algorithm* has no practical meaning, it is interesting to know what *God's Number* is.

Why is *God's Number* interesting?

If you start with a solved cube and ask someone to make a few turns and then try to restore it, you will (after some practice) be able to return it into the solved state in a minimum number of moves as long as the number of scrambling moves is not too big. With less than four scrambling moves it is easy, with four it becomes tricky and with five it is simply hard. Some people can handle six or even seven scrambling moves. If you go further, it will be practically impossible, and you will be unable to solve the cube in a minimal number of moves; most algorithms take between 50 and 100 moves. What if you would randomly turn the cube 1000 times? Will it take 1000 or more moves to get it back? No, because it still takes between 50 and 100 moves to solve *Rubik's Cube*.

***God's Number* for the 2×2×2 cube**

The 2×2×2 (see Figure 2) has been completely analysed about thirty years ago. *God's Number* for the 2×2×2 is eleven in face turns and 14 in quarter turns. To arrive at this number all permutations of the 2×2×2 have been determined and the number of turns to reach these. In Table 1 each line provides the number of unique positions (or permutations) that are reached by adding another turn to all positions of the previous line. Line 1 starts with just one position. Given that position, and holding one corner fixed it is possible to make six quarter turns (three faces can be moved because one corner is held fixed and each face can be turned a quarter in either direction. Likewise it is possible to make nine face turns (on each moveable face you can make three face turns). Given the six positions after six quarter turns 27 new positions can be reached, etc. After eleven face turns and after 14 quarter turns no new positions are reached. In total the number of positions reached is 3,674,160.

Figure 2. The 2×2×2 *Rubik's Cube*

This number of permutations of the 2×2×2 can also be calculated as follows. Eight corner cubies can be permutated in 8! ways. In principle each corner cubie can be turned in three orientations leading to 3^8 permutations. However, given the orientation of seven corners, the eighth corner is fixed, so the corners have 3^7 permutations. As the orientation of the whole cube (eight corner cubies, three orientations) is not fixed in space the total number of permutations needs to be divided by 8×3=24. Hence the total number of positions is:

$$\frac{8! \times 3^7}{8 \times 3} = 7! \times 3^6 = 3{,}674{,}160$$

From Table 1 it follows that there are only 2,644 positions for which eleven face turns are required to solve the puzzle. Assuming all positions have the same likelihood of being a starting position, on average nine face turns are required to solve the puzzle. Likewise there are only 276 positions from which 14 quarter turns are required, and on average eleven quarter turns are required to solve the puzzle.

Number of turns	Number of positions using face turns	Number of positions using quarter turns
0	1	1
1	9	6
2	54	27
3	321	120
4	1,847	534
5	9,992	2,256
6	50,136	8,969
7	227,536	33,058
8	870,072	114,149
9	1,887,748	360,508
10	623,800	930,588
11	2,644	1,350,852
12		782,536
13		90,280
14		276
Total	**3,674,160**	**3,674,160**

Table 1. The number of positions reached for the 2×2×2 after up to 14 turns

Determining *God's Number* for the 3×3×3

From the beginning in the late 70's until now the search area has been limited by two numbers: the lower bound and the upper bound. The lower bound G^{low} is determined based on proof that there are positions that need at least G^{low} turns. The upper bound G^{high} is determined by proof that no position requires more than G^{high} turns.

So, how many positions are there? The 2×2×2 cube has a manageable number of 3,674,160 positions. The 3×3×3 has a few more... With eight corner cubies and twelve edge cubies there are $8! \times 12! \times 3^8 \times 2^{12}$ different patterns, but not all patterns are possible:

- With 8 corners there are 8! corner permutations, and with 12 edges there are 12! edge permutations. However, because it is impossible to interchange two edge cubies without interchanging also two corner cubies the total number of permutations should be divided by 2.
- Corner cubies need to be turned in pairs — only seven corner cubies can be turned freely.
- Flipping of edge cubies needs to be done in pairs — only eleven edge cubies can be flipped freely.

Because of the centre pieces the orientation of the cube is fixed in space, so the number of permutations should not be divided by 24 as with the 2×2×2. Hence the number of positions of the 3×3×3 is:

$$\frac{8! \times 12!}{2} \times \frac{3^8}{3} \times \frac{2^{12}}{2} = 43{,}252{,}003{,}274{,}489{,}856{,}000 \textit{ or } 4.3 \times 10^{19}$$

This is astronomically bigger than 3,674,160. Making Table 1 for the 3×3×3 is out of the question. To determine *God's Algorithm* and *God's Number* the upper and lower bounds have been studied for three decades.

Lower bound

Using counting arguments it can be proven that there exist positions needing at least 18 moves to solve. To show this, count the number of positions achievable using at most 17 moves. It turns out that the latter number is smaller than 4.3×10^{19} [4]. This argument was made right at the start in 1979 or 1980 and was not improved upon for many years. Moreover, it is more an argument than a proof. It does not provide a concrete position that needs this many moves. At some point it was suggested that the so-called superflip would be such a position. The superflip is a state of the cube where all the cubies are in their correct position, all the corner cubies are orientated correctly, but each edge cubie is orientated the wrong way (flipped).

It took until 1992 for a solution for the superflip with 20 face turns to be found by Dik T. Winter [5]. Later, in 1995, Michael Reid proved it is minimal and thus a new lower bound for the diameter of the cube group was found [6]. Also in 1995, a solution for the superflip in 24 quarter turns was found by Michael Reid, and it was proven to be minimal by Jerry Bryan. In 1998 Michael Reid found a new position requiring more than 24 quarter turns to solve. The position, referred to by him as '*superflip composed with four spots*' needs 26 quarter turns. This leaves the lower bound at 20 face turns or 26 quarter turns.

Upper bound

Finding the upper bound requires a different kind of reasoning. The first number that I could find was the 277 moves mentioned by David Singmaster in February 1979 [4]. He basically counted the maximum number of moves in his algorithm to solve the cube. In a supplement David reports that Berlekamp, Conway & Guy came up with a different algorithm that needed at most 160 moves. In February 1979 *Conway's Cambridge Cubists* reported that the cube can be restored in at most 94 moves [4].

A breakthrough was made by Morwen B. Thistlethwaite when he came up with a novel strategy for restoring the cube. Thistlethwaite's idea was to divide the problem into subproblems. Where algorithms up to that point divided the problem by looking at the parts that remain fixed, he divided it by restricting the number of different types of moves you could execute. He divided the cube group (all possible moves with the *Rubik's Cube*) into the following chain of subgroups:

$$G_0 = \langle L,R,F,B,U,D\rangle$$
$$G_1 = \langle L,R,F,B,U2,D2\rangle$$
$$G_2 = \langle L,R,F2,B2,U2,D2\rangle$$
$$G_3 = \langle L2,R2,F2,B2,U2,D2\rangle$$
$$G_4 = \{I\}$$

Initially, Thistlethwaite showed it can always be done in at most 85 moves. In January 1980 he improved his strategy to a maximum of 80 moves. Later in 1980 Thistlethwaite proved that a maximum of 63 moves and later 52 moves are needed and believed he could reduce it to 50 or possibly 45.

After this initial rush in reducing the upper bound in the early 80's it became quiet for a long time. In 1989 Hans Kloosterman reported an algorithm of 44 moves [7], which he later in 1990 improved to 42 moves [8]. In 1992 Herbert Kociemba improved the algorithm from Thistlethwaite by reducing it to a two-phase algorithm [5].

$$G_0 = \langle L,R,F,B,U,D\rangle$$
$$G_1 = \langle L,R,F2,B2,U2,D2\rangle$$
$$G_2 = \{I\}$$

Based on Kociemba's algorithm Michael Reid significantly improved the upper bound to 29 face turns in 1995 [9].

In 1997 Richard Korf came up with a new solution approach [5]. Instead of using a fixed algorithm his strategy simultaneously searches for a solution along three lines until the solution has been found. On average the algorithm solves the cube in 18 moves. There was however no worst-case analysis.

It was quiet until 2006 when Silviu Radu reduced the upper bound to 27 [10]. Mid 2007 Gene Cooperman [11] proved 26. Finally Tomas Rokicki reduced the upper bound to 25 in March 2008 [12], to 24 in April 2008, 23 in May 2008, 22 in August 2008 and finally 20 face turns in July 2010 [13].

Tomas worked with a team and they used symmetry arguments to significantly reduce the search space and then managed to partition the space of all configurations that remained into pieces small enough to fit onto a modern computer. They made use of the enormous computing resource available through Google. After these computers became available it was a matter of weeks to complete the calculations. Remarkable!

Figure 3. The 7×7×7 *Rubik's Cube*

Epilogue

Now that the quest for *God's Number* for the 3×3×3 has been completed it might be time to look at what is ahead. As explained in the introduction, there already exist commercially available *Rubik's Cubes* up to 7×7×7 (see Figure 3) and beyond. Finding *God's Number* for such puzzles must be a problem that will last far into the future. The number of permutations of the 7×7×7 reaches up to 2.0×10^{160}.

References

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