- Difference equation:-

Difference equations model processes in which we know relationships between changes or differences of the sequences rather than rates of changes which leads to the differential equations.

Example: (Fibonacci Relation)

Suppose there is only one pair of rabbits male and female just born. Further suppose that every month each pair of rabbits that are one month old, produces a new pair of offspring of opposite sexes.

So, now, if F_n is the number of pairs of rabbits after n months then we have the relation,

$$F_{n+2} = F_{n+1} + F_n \quad , \quad n \geq 0 \qquad \text{———(i)}$$

provided $F_0 = F_1 = 1$.

This is an example of a difference equation. From (i) we can write,

$$F_{n+2} - F_{n+1} - F_n = 0$$

This is second order linear homogeneous difference equations with constant co-efficients. This is second order because the highest term F_{n+2}. This is of constant co-efficients because all the co-efficients are constants. This is linear because all the terms are linear. Also, this is homogeneous because the R.H.S is zero.

- **Kth order linear difference equations with Constant Co-efficients:-**

A Kth order linear difference equations with Constant Co-efficients is of the form:

$$a_K y_{n+K} + a_{K-1} y_{n+K-1} + \cdots + a_1 y_{n+1} + a_0 y_n = f_n \qquad \text{(ii)}$$

Where $n = 0, 1, 2, \ldots$ Here $a_0, a_1, a_2, \ldots, a_{K-1}, a_K$ are Constants.

If $f_n = 0$ for all n then this difference equation is said to be homogeneous.

If $f_n \neq 0$ for some n, then this difference equation is said to be non-homogeneous.

In the above equation (ii) if $a_K = 1$ then the difference equation is said to be in the standard form.

- **Solution of first order linear homogeneous equation:-**

Consider the following standard first order linear homogeneous difference equation with Constant Co-efficients:

$$y_{n+1} - b y_n = 0, \quad n \geq 0 \quad (b \text{ is a Constant.})$$

Let the Solution is of the form, $y_n = c.r^n$ where $c \neq 0$, $r \neq 0$.

$\therefore$ From the equation, $c r^{n+1} - b c r^n = 0$

$$\Rightarrow c r^n (r - b) = 0$$

$$\Rightarrow r - b = 0 \quad [\because c r^n \neq 0]$$

$$\Rightarrow r = b.$$

$\therefore$ So, the general sol^n is $y_n = c.b^n$. Where C is a Constant.

In addition if there is a condition $y_0 = d$ [that is the first term of the sequence y_0 is given as Constant d.]

$\therefore$ So, we get, $y_0 = c.b^0$

$$\Rightarrow d = c \Rightarrow c = d.$$

$\therefore$ In this case, the particular Solution becomes, $y_n = d b^n$.

Solution of Second order linear homogeneous equation:-

Consider the following second order linear homogeneous equation:

$$a_2 y_{n+2} + a_1 y_{n+1} + a_0 y_n = 0 \quad , \quad n \geq 0.$$

Let, $y_n = c.r^n$ be the solution. Then from the equation we get,

$$a_2 . c.r^{n+2} + a_1 .c.r^{n+1} + a_0 .c.r^n = 0$$

$$\Rightarrow cr^n (a_2 r^2 + a_1 r + a_0) = 0$$

$$\Rightarrow a_2 r^2 + a_1 r + a_0 = 0 \quad [\because c \neq 0, \ r \neq 0].$$

This is called the auxiliary equation (Quadratic equation)

Now, three different cases may occur.

Case 1: Let all the roots are real and distinct, say r_1, r_2. Then the general solution becomes:

$$y_n = c_1 . r_1^n + c_2 . r_2^n \quad , \quad n \geq 0.$$

Case 2: Let, all the roots are real and equal, say r. Then the general solution becomes:

$$y_n = c_1 . r^n + c_2 . n . r^n = (c_1 + c_2 . n) r^n .$$

c_1, c_2 are constants.

Case 3: Let the roots are complex numbers say $a \pm ib$. Then the general solution becomes:

$$y_n = r^n (c_1 \cos n\theta + c_2 \sin n\theta)$$

Where $r = \sqrt{a^2 + b^2}$, $\tan\theta = \frac{b}{a}$.

Note: In Case 2, if the auxiliary equation is a cubic equation and the equal root is r then the solution is, $y_n = c_1 . r^n + c_2 . n . r^n + c_3 . n^2 . r^n$. ($c_1, c_2, c_3$ are constants).

Example: Find the general solution of the difference equation $2a_n - 3a_{n-1} = 0$, $n \geq 1$ and $a_4 = 81$.

Sol^n: First we will make this equation in Standard form.

$$2a_n - 3a_{n-1} = 0$$

$$\Rightarrow 2a_{n+1} - 3a_n = 0 \quad [\text{Replacing } n \text{ by } n+1]$$

$$\Rightarrow a_{n+1} - \frac{3}{2} a_n = 0 \quad [\text{Dividing by 2 both sides. Because in Standard form, co-efficient of } a_{n+1} \text{ will be } 1].$$

Let, the solution of the form $a_n = c.r^n$ where $c \neq 0$, $r \neq 0$.

$\therefore$ From the equation:

$$c.r^{n+1} - \frac{3}{2}.c.r^n = 0$$

$$\Rightarrow c.r^n\left(r - \frac{3}{2}\right) = 0$$

$$\Rightarrow r - \frac{3}{2} = 0 \quad [\because cr^n \neq 0]$$

$$\Rightarrow r = \frac{3}{2}.$$

$\therefore$ The general solution is, $a_n = c.\left(\frac{3}{2}\right)^n$, $n \geq 0$.

Now,

$$a_4 = c.\left(\frac{3}{2}\right)^4$$

$$\Rightarrow 81 = c.\left(\frac{3}{2}\right)^4 \quad [\because \text{it is given } a_4 = 81]$$

$$\Rightarrow 81 = c \times \frac{81}{16}$$

$$\Rightarrow c = 16$$

$\therefore$ The solution is

Example: Solve the difference equation: $F_{n+2} = F_{n+1} + F_n$, $n \geq 0$, $F_0 = F_1 = 1$.

Sol^n: Here the equation is, $F_{n+2} = F_{n+1} + F_n$

$$\Rightarrow F_{n+2} - F_{n+1} - F_n = 0$$

This is of Standard Second order linear homogeneous difference equations with constant co-efficients.

Let $F_n = c.r^n$ be the solution.

$\therefore$ So, from the equation,

$$c.r^{n+2} - c.r^{n+1} - cr^n = 0$$

$$\Rightarrow cr^n(r^2 - r - 1) = 0$$

$$\Rightarrow r^2 - r - 1 = 0 \quad [\because cr^n \neq 0.]$$

$$r = \frac{1 \pm \sqrt{1 - 4.1.(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

$\therefore$ So, the general solution becomes:

$$F_n = c_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n, \quad n \geq 0$$

where c_1, c_2 are constants.

Now, from the condition, $F_0 = 1$, $F_1 = 1$.

$$\therefore F_0 = c_1 + c_2 \Rightarrow c_1 + c_2 = 1 \quad \text{——— (i)}$$

$$\therefore F_1 = c_1 \left(\frac{1+\sqrt{5}}{2}\right) + c_2 \left(\frac{1-\sqrt{5}}{2}\right)$$

$$\Rightarrow c_1 \left(\frac{1+\sqrt{5}}{2}\right) + c_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1 \quad \text{——— (ii)}$$

Solving (i) and (ii) we get,

$$c_1 = \frac{1+\sqrt{5}}{2\sqrt{5}}$$

$$c_2 = \frac{\sqrt{5}-1}{2\sqrt{5}}.$$

$\therefore$ The solution is,

$$F_n = \frac{1+\sqrt{5}}{2\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{\sqrt{5}-1}{2\sqrt{5}}\right) \cdot \left(\frac{1-\sqrt{5}}{2}\right)^n, \quad n \geq 0.$$

Exercise: i) Solve the difference equation:

$$y_{n+3} - y_{n+2} + y_{n+1} - y_n = 0 \quad \text{with}$$

the condition $y_0 = 1$, $y_1 = 0$, $y_2 = 2$.

(ii) $y_{n+3} - 12y_{n+2} + 48y_{n+1} - 64y_n = 0$.

Solution of non-homogeneous difference equations by method of undetermined Co-efficients:-

Consider the following Kth order linear difference equation which is non-homogeneous:

$$a_K y_{n+K} + a_{K-1} y_{n+K-1} + \cdots + a_1 y_{n+1} + a_0 y_n = f(n)$$

where $a_0, a_1, \ldots, a_{K-1}, a_K$ are Constants and $f(n) \neq 0$.

Here for the method of undetermined co-efficients, we will discuss some forms of $f(n)$.

1. If $f(n) = d^n$, where d is a Constant then we take the trial solution as $y_n = ad^n$ where the value of a to be determined.

2. If $f(n) = n + c$ (where c is a Constant) then we take trail solution as $y_n = a_1 n + b_1$ where value of a_1, b_1 to be determined.

3. If $f(n) = n^2 + c$ (where c is a Constant) then we take the trail solution as $y_n = a_1 n^2 + b_1 n + c_1$ where a_1, b_1, c_1 to be determined.

And in the case of non-homogeneous difference equations the general solution will be y_n = Complementary function + PI.

Now, we discuss examples.

Example: Solve the difference equation:

$$y_{n+2} + 4y_{n+1} + 3y_n = 3^n \text{ with } y_0 = 0,\ y_1 = 1.$$

Sol^n: First we will consider the homogeneous equation:

$$y_{n+2} + 4y_{n+1} + 3y_n = 0.$$

Let the Solution is of the form:

$$y_n = c.r^n \text{ where } c \neq 0,\ r \neq 0.$$

So, from the equation we get,

$$c.r^{n+2} + 4c.r^{n+1} + 3c.r^n = 0$$

$$\Rightarrow cr^n(r^2+4r+3) = 0$$

$$\Rightarrow r^2+4r+3 = 0 \quad [\because cr^n \neq 0]$$

$$\Rightarrow (r+1)(r+3) = 0$$

$$\Rightarrow r = -1, -3.$$

$\therefore$ As the roots are real and distinct, so,

$$CF = C_1(-1)^n + C_2.(-3)^n.$$

Now, let the trial Solution is:

$$y_n = a_1.3^n$$

$$\therefore \text{ So, } y_{n+2} = a_1.3^{n+2}.$$

$$y_{n+1} = a_1.3^{n+1}.$$

$\therefore$ So, from the main equation we get,

$$y_{n+2} + 4y_{n+1} + 3y_n = 3^n$$

$$\Rightarrow a_1.3^{n+2} + 4a_1.3^{n+1} + 3a_1.3^n = 3^n$$

$$\Rightarrow 9a_1.3^n + 12a_1.3^n + 3a_1.3^n = 3^n$$

$$\Rightarrow 24a_1.3^n = 3^n$$

$\therefore$ So, Comparing both sides, we have,

$$24a_1 = 1 \Rightarrow a_1 = \frac{1}{24}.$$

$$\therefore PI = \frac{1}{24}.3^n.$$

$\therefore$ The general solution will be:

$$y_n = CF + PI$$

$$\Rightarrow y_n = C_1.(-1)^n + C_2.(-3)^n + \frac{1}{24}3^n.$$

Now, it is given $y_0 = 0$, $y_1 = 1$.

put, $n=0$, $\quad y_0 = c_1 + c_2 + \frac{1}{24}$

$$\Rightarrow c_1 + c_2 + \frac{1}{24} = 0$$

$$\Rightarrow c_1 + c_2 = -\frac{1}{24} \quad \text{———— (i)}.$$

Now put $n=1$, $\quad y_1 = -c_1 - 3c_2 + \frac{1}{8}$

$$\Rightarrow -c_1 - 3c_2 + \frac{1}{8} = 1$$

$$\Rightarrow -c_1 - 3c_2 = \frac{7}{8} \quad \text{———— (ii)}.$$

Solving these two equations (Check!) we get, $c_1 = \frac{9}{24} = \frac{3}{8}$

$$c_2 = -\frac{5}{12}.$$

$\therefore$ The solution is : $\quad y_n = \frac{3}{8}\cdot(-1)^n - \frac{5}{12}(-3)^n + \frac{1}{24}\cdot 3^n.$

Example: Solve the following difference equation by method of undetermined co-efficients:

$$y_{n+2} - 4y_{n+1} + 4y_n = n$$

Sol^n: Let the form of the solution is $y_n = c.r^n$ where $c \neq 0, r \neq 0$.
From the homogeneous equation we have,

$$y_{n+2} - 4y_{n+1} + 4y_n = 0$$

$$\Rightarrow c.r^{n+2} - 4.cr^{n+1} + 4.cr^n = 0$$

$$\Rightarrow cr^n(r^2 - 4r + 4) = 0$$

$$\Rightarrow r^2 - 4r + 4 = 0 \quad [\because cr^n \neq 0]$$

$$\Rightarrow (r-2)^2 = 0$$

$$\Rightarrow r = 2, 2$$

$\therefore$ $CF = c_1.2^n + c_2.n.2^n = (c_1 + c_2.n)2^n$ where c_1, c_2 are constants.

Now, let the trial solution be, $y_n = a_1 n + b_1$ where a_1, b_1 to be determined.

$$\therefore y_{n+2} = a_1(n+2) + b_1$$

$$\therefore y_{n+1} = a_1(n+1) + b_1$$

$\therefore$ From the main equation we have:

$$(a_1(n+2) + b_1) - 4(a_1(n+1) + b_1) + 4(a_1 n + b_1) = n$$

$$\Rightarrow a_1 n + (2a_1 + b_1 - 4a_1 - 4b_1 + 4b_1) = n$$

$$\Rightarrow a_1 n + (b_1 - 2a_1) = n.$$

$\therefore$ Comparing both sides, $a_1 = 1$, $b_1 - 2a_1 = 0$
$\Rightarrow b_1 = 2$.

$$\therefore PI = n + 2.$$

$\therefore$ The general solution becomes: $y_n = (c_1 + c_2 n)2^n + n + 2$.

Exercise: Solve the following difference equation by method of undetermined co-efficients:

i) $y_{n+2} - 6y_{n+1} + 8y_n = 2^n + 6n$.

ii). $y_{n+2} - 4y_{n+1} + 3y_n = 5^n$, with $y_0 = 0$, $y_1 = 1$.

Solving difference equation by Z-transforms:-

Step1: First we apply Z-transform to both sides of the difference equation.

Step2: We apply the formula:

$$Z(u_{n-k}) = z^{-k} U(z)$$

or $$Z(u_{n+k}) = z^{k}\left[U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} - \cdots\cdots - u_{k-1} z^{-(k-1)}\right]$$

Where remember $U(z) = Z(u_n)$.

Step3: We will find $U(z)$ by partial fraction method.

Step4: In last step, we will apply the inverse Z-transform.

Example: Solve the following difference equation by Z-transforms.

$$u_{n+2} + 4u_{n+1} + 3u_n = 3^n \text{ with } u_0 = 0,\ u_1 = 1.$$

Solⁿ: Firstly, we will apply Z-transform to both sides.

$$Z(u_{n+2} + 4u_{n+1} + 3u_n) = Z(3^n)$$

$\Rightarrow Z(u_{n+2}) + 4Z(u_{n+1}) + 3Z(u_n) = \dfrac{z}{z-3}$ $\left[\because \text{Z-transform is linear}\right.$ $\left.\because Z(a^n) = \dfrac{z}{z-a}\right]$

$$\Rightarrow z^2\left(U(z) - u_0 - \cancel{u_1} z^{-1}\right) + 4z\left(U(z) - u_0\right) + 3U(z) = \frac{z}{z-3}$$

$\Rightarrow z^2\left(U(z) - \dfrac{1}{z}\right) + 4z \cdot U(z) + 3U(z) = \dfrac{z}{z-3}$ $\left[\because u_0 = 0,\ u_1 = 1 \text{ (given)}\right]$

$$\Rightarrow U(z)\left(z^2+4z+3\right) = z + \frac{z}{z-3}$$

$$\Rightarrow U(z)\cdot(z+1)(z+3) = z\left(1+\frac{1}{z-3}\right)$$

$$\Rightarrow \frac{U(z)}{z} = \frac{1}{(z+1)(z+3)} + \frac{1}{(z+1)(z+3)(z-3)}$$

$$\Rightarrow \frac{U(z)}{z} = \frac{z-3+1}{(z+1)(z+3)(z-3)}$$

$$\Rightarrow \frac{U(z)}{z} = \frac{z-2}{(z+1)(z+3)(z-3)}$$

Now by partial fraction,

$$\frac{z-2}{(z+1)(z+3)(z-3)} = \frac{A}{z+1} + \frac{B}{z+3} + \frac{C}{z-3} = \frac{A(z^2-9)+B(z+1)(z-3)+C(z+1)(z+3)}{(z+1)(z+3)(z-3)}$$

$$\Rightarrow \frac{z-2}{(z+1)(z+3)(z-3)} = \frac{z^2(A+B+C)+z(4C-2B)+(3C-3B-9A)}{(z+1)(z+3)(z-3)}$$

Comparing both sides, $A+B+C=0$, $4C-2B=1$, $3C-3B-9A=-2$

Solving we get $C=\frac{1}{24}$, $B=-\frac{5}{12}$, $A=\frac{3}{8}$

$$\therefore \frac{U(z)}{z} = \frac{1}{24}\cdot\frac{1}{z-3} - \frac{5}{12}\cdot\frac{1}{z+3} + \frac{3}{8}\cdot\frac{1}{z+1}$$

$$\Rightarrow U(z) = \frac{1}{24}\cdot\frac{z}{z-3} - \frac{5}{12}\cdot\frac{z}{z+3} + \frac{3}{8}\cdot\frac{z}{z+1}.$$

Now, we take inverse Z-transform on both sides:

$$Z^{-1}(U(z)) = \frac{1}{24}\cdot Z^{-1}\left(\frac{z}{z-3}\right) - \frac{5}{12}\cdot Z^{-1}\left(\frac{z}{z+3}\right) + \frac{3}{8}\cdot Z^{-1}\left(\frac{z}{z+1}\right)$$

$$\Rightarrow u_n = \frac{1}{24}\cdot(3)^n - \frac{5}{12}(-3)^n + \frac{3}{8}(-1)^n.$$

$$\Rightarrow u_n = \frac{3}{8}\cdot(-1)^n - \frac{5}{12}\cdot(-3)^n + \frac{1}{24}\cdot 3^n.$$

Example: Solve the following difference equation by Z-transforms:

$$u_{n+2} + 6u_{n+1} + 9u_n = 2^n \quad \text{with } u_0 = u_1 = 0$$

Sol^n: Firstly, we will apply Z-transforms on both sides:

$$Z\left(u_{n+2} + 6u_{n+1} + 9u_n\right) = Z(2^n)$$

$$\Rightarrow Z(u_{n+2}) + 6Z(u_{n+1}) + 9Z(u_n) = \frac{z}{z-2}$$

$$\Rightarrow z^2\left(U(z) - u_0 - u_1 z^{-1}\right) + 6z\left(U(z) - u_0\right) + 9U(z) = \frac{z}{z-2}$$

$$\Rightarrow z^2 U(z) + 6z\,U(z) + 9U(z) = \frac{z}{z-2} \quad [\because u_0 = u_1 = 0]$$

$$\Rightarrow U(z)\left(z^2 + 6z + 9\right) = \frac{z}{z-2}$$

$$\Rightarrow U(z).(z+3)^2 = \frac{z}{z-2}$$

$$\Rightarrow \frac{U(z)}{z} = \frac{1}{(z-2)(z+3)^2} = \frac{A}{z-2} + \frac{B}{z+3} + \frac{C}{(z+3)^2}$$

Solving we get, $A = \frac{1}{25}$, $B = -\frac{1}{25}$, $C = -\frac{1}{5}$ (Check!)

$$\therefore \frac{U(z)}{z} = \frac{1}{25}\cdot\frac{1}{z-2} - \frac{1}{25}\cdot\frac{1}{z+3} - \frac{1}{5}\cdot\frac{1}{(z+3)^2}$$

$$\Rightarrow U(z) = \frac{1}{25}\cdot\frac{z}{z-2} - \frac{1}{25}\cdot\frac{z}{z+3} - \frac{1}{5}\cdot\frac{z}{(z+3)^2}$$

$$\Rightarrow U(z) = \frac{1}{25}\left[\frac{z}{z-2} - \frac{z}{z+3} + \frac{5}{3}\left(\frac{-3z}{(z+3)^2}\right)\right]$$

[Because here we use the fact, $Z(a^n n) = \frac{az}{(z-a)^2}$ in the last part where $a = -3$]

Now, we take the inverse Z-transform.

$$\therefore u_n = \frac{1}{25}\left[2^n - (-3)^n + \frac{5}{3}.(-3)^n n\right]$$

$\therefore$ So, this is the solution.

Exercise: Solve the following difference equations by Z-transforms:

i) $6u_{n+2} - u_{n+1} - u_n = 0$ with $u_0 = 1$, $u_1 = 1$.

ii). $u_{n+2} + 2u_{n+1} + u_n = 0$ with $u_0 = 0$, $u_1 = 0$.

iii). $u_{n+2} - 6u_{n+1} + 9u_n = 3^n$ with $u_0 = 0$, $u_1 = 0$.