$$\lim_{\substack{h\to 0\\ k\to 0\\ l\to 0}} \frac{f(x+h+k+l)+f(x+h)+f(x+k)+f(x+l)}{hkl}$$

$$-\, f(x+h+k) - f(x+h+l) - f(x+k+l) \ldots$$

$$\lim_{s\to 0} \frac{f(x+3s)+3f(x+s)-3f(x+2s)-f(\ldots}{s^3}$$

$\cancel{x^2+2x+5}$

$x^3+9x^2s+ \ldots 27s^3$

$3x^3+9x^2s+ \ldots 3s^3$

$4x^3+18x^2s+26xs^2+30s^3$

$3x^3+16x^2s+36xs^2+24s^3$

DIFFERENTIAL CALCULUS

BY

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PREFACE

In this text on differential calculus I have continued the plan adopted for my Analytic Geometry, wherein a few central methods are expounded and applied to a large variety of examples to the end that the student may learn principles and gain power. In this way the differential calculus makes only a brief text suitable for a term's work and leaves for the integral calculus, which in many respects is far more important, a greater proportion of time than is ordinarily devoted to it.

As material for review and to provide problems for which answers are not given, a supplementary list, containing about half as many exercises as occur in the text, is placed at the end of the book.

I wish to acknowledge my indebtedness to Professor H. W. Tyler and Professor E. B. Wilson for advice and criticism and to Dr. Joseph Lipka for valuable assistance in preparing the manuscript and revising the proof.

H. B. PHILLIPS.

Boston, Mass., *August*, 1916.

CONTENTS

DIFFERENTIAL CALCULUS

CHAPTER I

INTRODUCTION

1. Definition of Function. — *A quantity y is called a function of a quantity x if values of y are determined by values of x.*

Thus, if $y = 1 - x^2$, y is a function of x; for a value of x determines a value of y. Similarly, the area of a circle is a function of its radius; for, the radius being given, the area is determined.

It is not necessary that only one value of the function correspond to a value of the variable. Several values may be determined. Thus, if x and y satisfy the equation

$$x^2 - 2xy + y^2 = x,$$

then y is a function of x. To each value of x correspond two values of y found by solving the equation for y.

A quantity u is called a function of several variables if u is determined when values are assigned to those variables.

Thus, if $z = x^2 + y^2$, then z is a function of x and y; for, values being given to x and y, a value of z is determined. Similarly, the volume of a cone is a function of its altitude and radius of base; for the radius and altitude being assigned, the volume is determined.

2. Kinds of Functions. — An expression containing variables is called an *explicit* function of those variables. Thus $\sqrt{x + y}$ is an explicit function of x and y. Similarly, if

$$y = \sqrt{x + 1},$$

y is an explicit function of x.

A quantity determined by an equation not solved for that quantity is called an *implicit* function. Thus, if

$$x^2 - 2\,xy + y^2 = x,$$

y is an implicit function of x. Also x is an implicit function of y.

Explicit and implicit do not denote properties of the function but of the way it is expressed. An implicit function is rendered explicit by solving. For example, the above equation is equivalent to

$$y = x \pm \sqrt{x},$$

in which y appears as an explicit function of x.

A *rational* function is one representable by an algebraic expression containing no fractional powers of variable quantities. For example,

$$\frac{x\sqrt{5} + 3}{x^2 + 2\,x}$$

is a rational function of x.

An *irrational* function is one represented by an algebraic expression which cannot be reduced to rational form. Thus $\sqrt{x + y}$ is an irrational function of x and y.

A function is called *algebraic* if it can be represented by an algebraic expression or is the solution of an algebraic equation. All the functions previously mentioned are algebraic.

Functions that are not algebraic are called *transcendental.* For example, $\sin x$ and $\log x$ are transcendental functions of x.

3. Independent and Dependent Variables. — In most problems there occur a number of variable quantities connected by equations. Arbitrary values can be assigned to some of these quantities and the others are then determined. Those taking arbitrary values are called *independent* variables; those determined are called *dependent* variables. Which variables are taken as independent and which as dependent is usually a matter of convenience. The number of independent variables is, however, determined by the equations.

For example, in plotting the curve

$$y = x^3 + x,$$

values are assigned to x and values of y are calculated. The independent variable is x and the dependent variable y. We might assign values to y and calculate values of x but that would be much more difficult.

4. Notation. — A particular function of x is often represented by the notation $f(x)$, which should be read, function of x, or f of x, not f times x. For example,

$$f(x) = \sqrt{x^2 + 1}$$

means that $f(x)$ is a symbol for $\sqrt{x^2 + 1}$. Similarly,

$$y = f(x)$$

means that y is some definite (though perhaps unknown) function of x.

If it is necessary to consider several functions in the same discussion, they are distinguished by subscripts or accents or by the use of different letters. Thus, $f_1(x)$, $f_2(x)$, $f'(x)$, $f''(x)$, $g(x)$ (read f-one of x, f-two of x, f-prime of x, f-second of x, g of x) represent (presumably) different functions of x.

Functions of several variables are expressed by writing commas between the variables. For example,

$$v = f(r, h)$$

expresses that v is a function of r and h and

$$v = f(a, b, c)$$

expresses that v is a function of a, b, c.

The f in the symbol of a function should be considered as representing an operation to be performed on the variable or variables. Thus, if

$$f(x) = \sqrt{x^2 + 1},$$

f represents the operation of squaring the variable, adding 1, and extracting the square root of the result. If x is replaced by any other quantity, the same operation is to be performed on that quantity. For example,

$$f(2) = \sqrt{2^2 + 1} = \sqrt{5}.$$
$$f(y+1) = \sqrt{(y+1)^2 + 1} = \sqrt{y^2 + 2y + 2}.$$

Similarly, if

$$f(x, y) = x^2 + xy - y^2,$$

then
$$f(1, 2) = 1^2 + 1 \cdot 2 - 2^2 = -1.$$

If
$$f(x, y, z) = x^2 + y^2 + z^2,$$

then
$$f(2, -3, 1) = 2^2 + (-3)^2 + 1 = 14.$$

EXERCISES

1. Given $x^{\frac{3}{2}} + y^{\frac{3}{2}} = a^{\frac{3}{2}}$, express y as an explicit function of x.

2. Given $\log_{10}(x) = \sin y$, express x as an explicit function of y. Also express y as an explicit function of x.

3. If $f(x) = x^2 - 3x + 2$, show that $f(1) = f(2) = 0$.

4. If $F(x) = x^4 + 2x^2 + 3$, show that $F(-a) = F(a)$.

5. If $F(x) = x + \dfrac{1}{x}$, find $F(x+1)$. Also find $F(x) + 1$.

6. If $\phi(x) = \sqrt{x^2 - 1}$, find $\phi(2x)$. Also find $2\phi(x)$.

7. If $\psi(x) = \dfrac{x+2}{2x-3}$, find $\psi\left(\dfrac{1}{x}\right)$. Also find $\dfrac{1}{\psi(x)}$.

8. If $f_1(x) = 2^x$, $f_2(x) = x^2$, find $f_1[f_2(y)]$. Also find $f_2[f_1(y)]$.

9. If $f(x, y) = x - \dfrac{1}{y}$, show that $f(2, 1) = 2f(1, 2) = 1$.

10. Given $f(x, y) = x^2 + xy$, find $f(y, x)$.

11. On how many independent variables does the volume of a right circular cylinder depend?

12. Three numbers x, y, z satisfy two equations

$$x^2 + y^2 + z^2 = 5,$$
$$x + y + z = 1.$$

How many of these numbers can be taken as independent variables?

5. Limit. — If in any process a variable quantity approaches a constant one in such a way that the difference of the two becomes and remains as small as you please, the constant is said to be the *limit* of the variable.

The use of limits is well illustrated by the incommensurable cases of geometry and the determination of the area of a circle or the volume of a cone or sphere.

6. Limit of a Function. — As a variable approaches a limit a function of that variable may approach a limit. Thus, as x approaches 1, $x^2 + 1$ approaches 2.

We shall express that a variable x approaches a limit a by the notation

$$x \doteq a.$$

The symbol $\doteq$ thus means "approaches as a limit."

Let $f(x)$ approach the limit A as x approaches a; this is expressed by

$$\lim_{x \doteq a} f(x) = A,$$

which should be read, "the limit of $f(x)$, as x approaches a, is A."

Example 1. Find the value of

$$\lim_{x \doteq 1} \left(x + \frac{1}{x}\right).$$

As x approaches 1, the quantity $x + \frac{1}{x}$ approaches $1 + \frac{1}{1}$ or 2. Hence

$$\lim_{x \doteq 1} \left(x + \frac{1}{x}\right) = 2.$$

Ex. 2. Find the value of

$$\lim_{\theta \doteq 0} \frac{\sin \theta}{1 + \cos \theta}.$$

As θ approaches zero, the function given approaches

$$\frac{0}{1 + 1} = 0.$$

Hence

$$\lim_{\theta \doteq 0} \frac{\sin \theta}{1 + \cos \theta} = 0.$$

7. Properties of Limits. — In finding the limits of functions frequent use is made of certain simple properties that follow almost immediately from the definition.

1. *The limit of the sum of a finite number of functions is equal to the sum of their limits.*

Suppose, for example, X, Y, Z are three functions approaching the limits A, B, C respectively. Then $X+Y+Z$ is approaching $A+B+C$. Consequently,

$$\lim (X+Y+Z) = A+B+C = \lim X + \lim Y + \lim Z.$$

2. *The limit of the product of a finite number of functions is equal to the product of their limits.*

If, for example, X, Y, Z approach A, B, C respectively, then XYZ approaches ABC, that is,

$$\lim XYZ = ABC = \lim X \lim Y \lim Z.$$

3. *If the limit of the denominator is not zero, the limit of the ratio of two functions is equal to the ratio of their limits.*

Let X, Y approach the limits A, B and suppose B is not zero. Then $\frac{X}{Y}$ approaches $\frac{A}{B}$, that is,

$$\lim \frac{X}{Y} = \frac{A}{B} = \frac{\lim X}{\lim Y}.$$

If B is zero and A is not zero, $\frac{A}{B}$ will be infinite. Then $\frac{X}{Y}$ cannot approach $\frac{A}{B}$ as a limit; for, however large $\frac{X}{Y}$ may become, the difference of $\frac{X}{Y}$ and infinity will not become small.

8. The Form $\frac{0}{0}$. — When x is replaced by a particular value, a function sometimes takes the form $\frac{0}{0}$. Although this symbol does not represent a definite value, the function may have a definite limit. This is usually made evident by writing the function in a different form.

Example 1. Find the value of

$$\lim_{x \doteq 1} \frac{x^2-1}{x-1}$$

When x is replaced by 1, the function takes the form

$$\frac{1-1}{1-1}=\frac{0}{0}.$$

Since, however,

$$\frac{x^2-1}{x-1}=x+1,$$

the function approaches $1+1$ or 2. Therefore

$$\lim_{x \doteq 1}\frac{x^2-1}{x-1}=2.$$

Ex. 2. Find the value of

$$\lim_{x \doteq 0}\frac{(\sqrt{1+x}-1)}{x}.$$

When $x=0$ the given function becomes

$$\frac{1-1}{0}=\frac{0}{0}.$$

Multiplying numerator and denominator by $\sqrt{1+x}+1$,

$$\frac{\sqrt{1+x}-1}{x}=\frac{x}{x\,(\sqrt{1+x}+1)}=\frac{1}{\sqrt{1+x}+1}.$$

As x approaches 0, the last expression approaches $\frac{1}{2}$. Hence

$$\lim_{x \doteq 0}\frac{(\sqrt{1+x}-1)}{x}=\frac{1}{2}.$$

9. Infinitesimal. — A variable approaching zero as a limit is called an *infinitesimal.*

Let α and β be two infinitesimals. If

$$\lim\frac{\alpha}{\beta}$$

is finite and not zero, α and β are said to be infinitesimals of the same order. If the limit is zero, α is of higher order than β. If the ratio $\frac{\alpha}{\beta}$ approaches infinity, β is of higher order than α. Roughly speaking, the higher the order, the smaller the infinitesimal.

For example, let x approach zero. The quantities

$$x,\ x^2,\ x^3,\ x^4, \text{ etc.}$$

are infinitesimals arranged in ascending order. Thus x^4 is of higher order than x^2; for

$$\lim_{x \doteq 0} \frac{x^4}{x^2} = \lim_{x \doteq 0} x^2 = 0.$$

Similarly, x^3 is of lower order than x^4, since

$$\frac{x^3}{x^4} = \frac{1}{x}$$

approaches infinity when x approaches zero.

As x approaches $\frac{\pi}{2}$, $\cos x$ and $\cot x$ are infinitesimals of the same order; for

$$\lim_{x \doteq \frac{\pi}{2}} \frac{\cos x}{\cot x} = \lim_{x \doteq 0} \sin x = 1,$$

which is finite and not zero.

EXERCISES

Find the values of the following limits:

1. $\lim_{x \doteq 0} \dfrac{x^2 - 2x + 3}{x - 5}$.

2. $\lim_{\theta \doteq \frac{\pi}{4}} \dfrac{\sin \theta + \cos \theta}{\sin 2\theta + \cos 2\theta}$.

3. $\lim_{x \doteq 1} \dfrac{x^2 - 3x + 2}{x - 1}$.

4. $\lim_{x \doteq 0} \dfrac{\sqrt{1 - x^2} - \sqrt{1 + x^2}}{x^2}$.

5. $\lim_{\theta \doteq 0} \dfrac{\sin \theta}{\tan \theta}$.

6. $\lim_{\theta \doteq 0} \dfrac{\sin \theta}{\sin 2\theta}$.

7. By the use of a table of natural sines find the value of

$$\lim_{x \doteq 0} \frac{\sin x}{x}.$$

8. Define as a limit the area within a closed curve.

9. Define as a limit the volume within a closed surface.

10. Define $\sqrt{2}$.

11. On the segment PQ (Fig. 9a) construct a series of equilateral triangles reaching from P to Q. As the number of triangles is increased, their bases approaching zero, the polygonal line $PABC$, etc., approaches PQ. Does its length approach that of PQ?

FIG. 9a.

12. Inscribe a series of cylinders in a cone as shown in Fig. 9b. As the number of cylinders increases indefinitely, their altitudes approaching zero, does the sum of the volumes of the cylinders approach that of the cone? Does the sum of the lateral areas of the cylinders approach the lateral area of the cone?

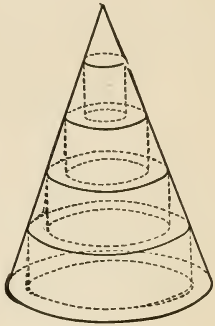


FIG. 9b.

13. Show that when x approaches zero, $\tan \frac{1}{x}$ does not approach a limit.

14. As x approaches 1, which of the infinitesimals $1 - x$ and $\sqrt{1 - x}$ is of higher order?

15. As the radius of a sphere approaches zero, show that its volume is an infinitesimal of higher order than the area of its surface and of the same order as the volume of the circumscribing cylinder.

CHAPTER II

DERIVATIVE AND DIFFERENTIAL

10. Increment. — When a variable changes value, the algebraic increase (new value minus old) is called its *increment* and is represented by the symbol Δ written before the variable.

Thus, if x changes from 2 to 4, its increment is

$$\Delta x = 4 - 2 = 2.$$

If x changes from 2 to -1,

$$\Delta x = -1 - 2 = -3.$$

When the increment is positive there is an increase in value, when negative a decrease.

Let y be a function of x. When x receives an increment Δx, an increment Δy will be determined. The increments of x and y thus correspond. To illustrate this graphically let x and y be the rectangular coördinates of a point P. An equation

$$y = f(x)$$

represents a curve. When x changes, the point P changes to some other position Q on the curve. The increments of x and y are

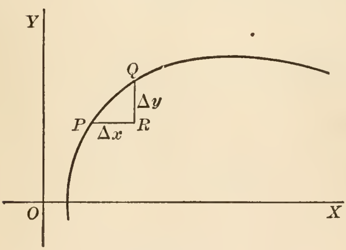


FIG. 10.

$$\Delta x = PR, \qquad \Delta y = RQ. \tag{10}$$

11. Continuous Function. — A function is called *continuous* if the increment of the function approaches zero as the increment of the variable approaches zero.

In Fig. 10, y is a continuous function of x; for, as Δx approaches zero, Q approaches P and so Δy approaches zero.

In Figs. 11a and 11b are shown two ways that a function can be *discontinuous.* In Fig. 11a the curve has a break at

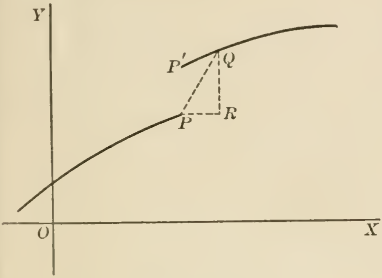


FIG. 11a.

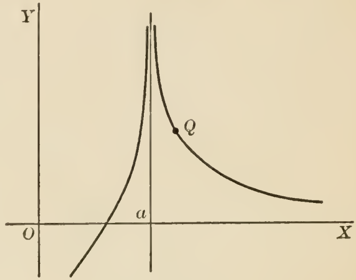


FIG. 11b.

P. As Q approaches P', $\Delta x = PR$ approaches zero, but $\Delta y = RQ$ does not. In Fig. 11b the ordinate at $x = a$ is infinite. The increment Δy occurring in the change from $x = a$ to any neighboring value is infinite.

12. Slope of a Curve. — As Q moves along a continuous curve toward P, the line PQ turns about P and usually approaches a limiting position PT. This line PT is called the *tangent* to the curve at P.

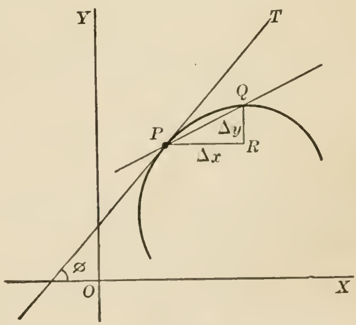


FIG. 12a.

The slope of PQ is

$$\frac{RQ}{PR} = \frac{\Delta y}{\Delta x}.$$

As Q approaches P, Δx approaches zero and the slope of PQ approaches that of PT. Therefore

$$\text{Slope of the tangent} = \tan\phi = \lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x}. \tag{12}$$

The slope of the tangent at P is called the *slope of the curve* at P.

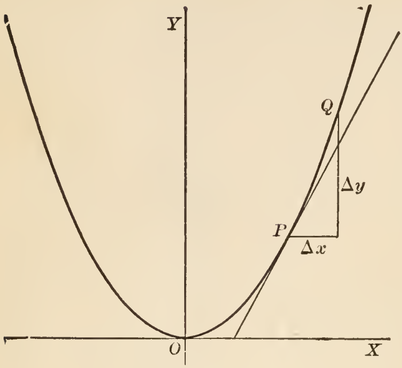


FIG. 12b.

Example. Find the slope of the parabola $y = x^2$ at the point (1, 1).

Let the coördinates of P be x, y. Those of Q are $x + \Delta x$, $y + \Delta y$. Since P and Q are both on the curve,

$$y = x^2$$

and

$$y + \Delta y = (x + \Delta x)^2 = x^2 + 2\,x\,\Delta x + (\Delta x)^2.$$

Subtracting these equations, we get

$$\Delta y = 2\,x\,\Delta x + (\Delta x)^2.$$

Dividing by Δx,

$$\frac{\Delta y}{\Delta x} = 2\,x + \Delta x.$$

As Δx approaches zero, this approaches

$$\text{Slope at } P = 2\,x.$$

This is the slope at the point with abscissa x. The slope at (1, 1) is then $2 \cdot 1 = 2$.

13. Derivative. — Let y be a function of x. If $\dfrac{\Delta y}{\Delta x}$ approaches a limit as Δx approaches zero, that limit is called the *derivative of y with respect to x*. It is represented by the notation $D_x y$, that is,

$$D_x y = \lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x}. \tag{13a}$$

If a function is represented by $f(x)$, its derivative with respect to x is often represented by $f'(x)$. Thus

$$f'(x) = \lim_{\Delta x \doteq 0} \frac{\Delta f(x)}{\Delta x} = D_x f(x). \tag{13b}$$

In Art. 12 we found that this limit represents the slope of the curve $y = f(x)$. The derivative is, in fact, a function of x whose value is the slope of the curve at the point with abscissa x.

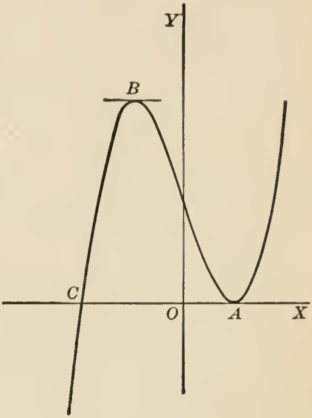


FIG. 13.

The derivative, being the limit of $\frac{\Delta y}{\Delta x}$, is approximately equal to a small change in y divided by the corresponding small change in x. It is then large or small according as the small increment of y is large or small in comparison with that of x.

If small increments of x and y have the same sign $\frac{\Delta y}{\Delta x}$ and its limit $D_x y$ are positive. If they have opposite signs $D_x y$ is negative. Therefore $D_x y$ is positive when x and y increase and decrease together and negative when one increases as the other decreases.

Example. $y = x^3 - 3x + 2$.

Let x receive an increment Δx. The new value of x is $x + \Delta x$. The new value of y is $y + \Delta y$. Since these satisfy the equation,

$$y + \Delta y = (x + \Delta x)^3 - 3(x + \Delta x) + 2.$$

Subtracting the equation

$$y = x^3 - 3x + 2,$$

we get

$$\Delta y = 3x^2 \Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - 3\Delta x.$$

Dividing by Δx,

$$\frac{\Delta y}{\Delta x} = 3x^2 + 3x\Delta x + (\Delta x)^2 - 3.$$

As Δx approaches zero this approaches the limit

$$D_x y = 3x^2 - 3.$$

The graph is shown in Fig. 13. At A (where $x = 1$) $y = 0$ and $D_xy = 3 \cdot 1 - 3 = 0$. The curve is thus tangent to the x-axis at A. The slope is also zero at B (where $x = -1$). This is the highest point on the arc AC. On the right of A and on the left of B, the slope D_xy is positive and x and y increase and decrease together. Between A and B the slope is negative and y decreases as x increases.

EXERCISES

1. Given $y = \sqrt{x}$, find the increment of y when x changes from $x = 2$ to $x = 1.9$. Show that the increments approximately satisfy the equation

$$\frac{\Delta y}{\Delta x} = \frac{1}{2\sqrt{x}}.$$

2. Given $y = \log_{10} x$, find the increments of y when x changes from 50 to 51 and from 100 to 101. Show that the second increment is approximately half the first.

3. The equation of a certain line is $y = 2x + 3$. Find its slope by calculating the limit of $\frac{\Delta y}{\Delta x}$.

4. Construct the parabola $y = x^2 - 2x$. Show that its slope at the point with abscissa x is $2(x - 1)$. Find its slope at $(4, 8)$. At what point is the slope equal to 2?

5. Construct the curve represented by the equation $y = x^4 - 2x^2$. Show that its slope at the point with abscissa x is $4x(x^2 - 1)$. At what points are the tangents parallel to the x-axis? Indicate where the slope is positive and where negative.

In each of the following exercises show that the derivative has the value given. Also find the slope of the corresponding curve at $x = -1$.

6. $y = (x + 1)(x + 2)$, $D_xy = 2x + 3$.

7. $y = x^4$, $D_xy = 4x^3$.

8. $y = x^3 - x^2$, $D_xy = 3x^2 - 2x$.

9. $y = \frac{1}{x}$, $D_xy = -\frac{1}{x^2}$.

10. If x is an acute angle, is $D_x \cos x$ positive or negative?

11. For what angles is $D_x \sin x$ positive and for what angles negative?

14. Approximate Value of the Increment of a Function.— Let y be a function of x and represent by ϵ a quantity such that

$$\frac{\Delta y}{\Delta x} = D_xy + \epsilon.$$

As Δx approaches zero, $\frac{\Delta y}{\Delta x}$ approaches $D_x y$ and so ϵ approaches zero.

The increment of y is

$$\Delta y = D_x y \, \Delta x + \epsilon \Delta x.$$

The part

$$D_x y \, \Delta x \tag{14}$$

is called the *principal part* of Δy. It differs from Δy by an amount $\epsilon \Delta x$. As Δx approaches zero, ϵ approaches zero, and so $\epsilon \Delta x$ becomes an indefinitely small fraction of Δx. It is an infinitesimal of higher order than Δx. If then the principal part is used as an approximation for Δy, the error will be only a small fraction of Δx when Δx is sufficiently small.

Example. When x changes from 2 to 2.1 find an approximate value for the change in $y = \frac{1}{x}\cdot$

In exercise 9, page 14, the derivative of $\frac{1}{x}$ was found to be $-\frac{1}{x^2}\cdot$ Hence the principal part of Δy is

$$-\frac{1}{x^2}\Delta x = -\frac{1}{4}(.1) = -0.0250.$$

The exact increment is

$$\Delta y = \frac{1}{(2.1)^2} - \frac{1}{2^2} = -0.0232.$$

The principal part represents Δy with an error less than 0.002 which is 2% of Δx.

15. Differentials. — Let x be the independent variable and let y be a function of x. The principal part of Δy is called the *differential* of y and is denoted by dy; that is,

$$dy = D_x y \, \Delta x. \tag{15a}$$

This equation defines the differential of any function y of x. In particular, if $y = x$, $D_x y = 1$, and so

$$dx = \Delta x, \tag{15b}$$

that is, *the differential of the independent variable is equal to* *its increment and the differential of any function y is equal to the product of its derivative and the increment of the independent variable.*

Combining 15a and 15b, we get

$$dy = D_x y\, dx, \tag{15c}$$

whence

$$\frac{dy}{dx} = D_x y, \tag{15d}$$

that is, *the quotient* $\frac{dy}{dx}$ *is equal to the derivative of* y *with respect to* x.

Since $D_x y$ is the slope of the curve $y = f(x)$, equations 15b and 15c express that dy and dx are the sides of the right triangle PRT (Fig. 15) with hypotenuse PT extending along the tangent at P. On this diagram, Δx and Δy are the increments

$$\Delta x = PR, \qquad \Delta y = RQ,$$

occurring in the change from P to Q. The differentials are

$$dx = PR, \qquad dy = RT.$$

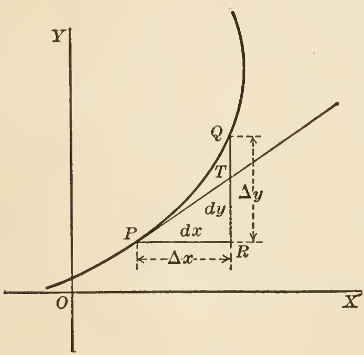


FIG. 15.

A point describing the curve is moving when it passes through P in the direction of the tangent PT. The differential dy is then the amount y *would increase* when x changes to $x + \Delta x$ if the direction of motion did not change. In general the direction of motion does change and so the actual increase $\Delta y = RQ$ is different from dy. If the increments are small the change in direction will be small and so Δy and dy will be approximately equal.

Equation 15c was obtained under the assumption that x was the independent variable. It is still valid if x and y are continuous functions of an independent variable t. For then

$$dx = D_t x\, \Delta t, \qquad dy = D_t y\, \Delta t.$$

The identity

$$\frac{\Delta y}{\Delta t} = \frac{\Delta y}{\Delta x} \cdot \frac{\Delta x}{\Delta t}$$

gives in the limit

$$D_t y = D_x y \cdot D_t x.$$

Hence

$$D_t y \, \Delta t = D_x y \cdot D_t x \, \Delta t,$$

that is,

$$dy = D_x y \, dx.$$

Example 1. Given $y = \dfrac{x+1}{x}$, find dy.

In this case

$$\Delta y = \frac{x + \Delta x + 1}{x + \Delta x} - \frac{x+1}{x} = -\frac{\Delta x}{x\,(x + \Delta x)}.$$

Consequently,

$$\frac{\Delta y}{\Delta x} = -\frac{1}{x\,(x + \Delta x)}.$$

As Δx approaches zero, this approaches

$$\frac{dy}{dx} = -\frac{1}{x^2}.$$

Therefore

$$dy = -\frac{dx}{x^2}.$$

Ex. 2. Given $x = t^2$, $y = t^3$, find $\dfrac{dy}{dx}$.

The differentials of x and y are found to be

$$dx = 2\,t\,dt, \qquad dy = 3\,t^2\,dt.$$

Division then gives,

$$\frac{dy}{dx} = \frac{3}{2}t.$$

Ex. 3. An error of 1% is made in measuring the side of a square. Find approximately the error in the calculated area.

Let x be the correct measure of the side and $x + \Delta x$ the value found by measurement. Then $dx = \Delta x = \pm 0.01\,x$.

The error in the area is approximately

$$dA = d\,(x^2) = 2\,x\,dx = \pm 0.02\,x^2 = \pm 0.02\,A,$$

which is 2% of the area.

EXERCISES

1. Let n be a positive integer and $y = x^n$. Expand

$$\Delta y = (x + \Delta x)^n - x^n$$

by using the binomial theorem. Show that

$$\frac{dy}{dx} = nx^{n-1}.$$

What is the principal part of Δy?

2. Using the results of Ex. 1, find an approximate value for the increment of x^6 when x changes from 1.1 to 1.2. Express the error as a percentage of Δx.

3. If A is the area of a circle of radius r, show that $\dfrac{dA}{dr}$ is equal to the circumference.

4. If the radius of a circle is measured and its area calculated by using the result, show that an error of 1% in the measurement of the radius will lead to an error of about 2% in the area.

5. If v is the volume of a sphere with radius r, show that $\dfrac{dv}{dr}$ is equal to the area of its surface.

6. Let v be the volume of a cylinder with radius r and altitude h. Show that if r is constant $\dfrac{dv}{dh}$ is equal to the area of the base of cylinder and if h is constant $\dfrac{dv}{dr}$ is equal to the lateral area.

7. If $y = f\,(x)$ and for all variations in x, $dx = \Delta x$, $dy = \Delta y$, show that the graph of $y = f\,(x)$ is a straight line.

8. If y is the independent variable and $x = f\,(y)$, make a diagram showing dx, dy, Δx, and Δy.

9. If the y-axis is vertical, the x-axis horizontal, a body thrown horizontally from the origin with a velocity of 50 ft. per second will in t seconds reach the point

$$x = 50\,t, \qquad y = -16\,t^2.$$

Find the slope of its path at that point.

10. A line turning about a fixed point P intersects the x-axis at A and the y-axis at B. If K_1 and K_2 are the areas of the triangles OPA and OPB, show that

$$\frac{dK_1}{dK_2} = \frac{PA^2}{PB^2}.$$

CHAPTER III

DIFFERENTIATION OF ALGEBRAIC FUNCTIONS

16. The process of finding derivatives and differentials is called *differentiation.* Instead of applying the direct method of the last chapter, differentiation is usually performed by means of certain formulas derived by that method. In this work we use the letter d for the operation of taking the differential and the symbol $\frac{d}{dx}$ for the operation of taking the derivative with respect to x. Thus

$$d(u+v) = \text{differential of } (u+v),$$

$$\frac{d}{dx}(u+v) = \text{derivative of } (u+v) \text{ with respect to } x.$$

To obtain the derivative with respect to x we proceed as in finding the differential except that d is everywhere replaced by $\frac{d}{dx}\cdot$

17. Formulas. — Let u, v, w be continuous functions of a single variable x, and c, n constants.*

I. $$dc = 0.$$

II. $$d(u+v) = du + dv.$$

III. $$d(cu) = c\,du.$$

IV. $$d(uv) = u\,dv + v\,du.$$

V. $$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}.$$

VI. $$d(u^n) = nu^{n-1}\,du.$$

* It is assumed that the functions u, v, w have derivatives. There exist continuous functions,

$$u = f(x),$$

18. Proof of I. — *The differential of a constant is zero.*

When a variable x takes an increment Δx, a constant does not vary. Consequently, $\Delta c = 0$, $\dfrac{\Delta c}{\Delta x} = 0$, and in the limit $\dfrac{dc}{dx} = 0$. Clearing of fractions,

$$dc = dx \cdot 0 = 0.$$

19. Proof of II. — *The differential of the sum of a finite number of functions is equal to the sum of their differentials.*

Let

$$y = u + v.$$

When x takes an increment Δx, u will change to $u + \Delta u$, v to $v + \Delta v$, and y to $y + \Delta y$. Consequently

$$y + \Delta y = u + \Delta u + v + \Delta v.$$

Subtraction of the two equations gives

$$\Delta y = \Delta u + \Delta v,$$

whence

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}.$$

As Δx approaches zero, $\dfrac{\Delta y}{\Delta x}$, $\dfrac{\Delta u}{\Delta x}$, $\dfrac{\Delta v}{\Delta x}$ approach $\dfrac{dy}{dx}$, $\dfrac{du}{dx}$, $\dfrac{dv}{dx}$ respectively. Therefore

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx},$$

and so

$$dy = du + dv.$$

By the same method we can prove

$$d\,(u \pm v \pm w \pm \cdots) = du \pm dv \pm dw \pm \cdots.$$

such that

$$\frac{\Delta u}{\Delta x}$$

does not approach a limit as Δx approaches zero. Such a function has no derivative $D_x u$ and therefore no differential

$$du = D_x u\, dx.$$

20. Proof of III. — *The differential of a constant times a function is equal to the constant times the differential of the function.*

Let
$$y = cu.$$
Then
$$y + \Delta y = c\,(u + \Delta u)$$
and so
$$\Delta y = c\,\Delta u,$$
$$\frac{\Delta y}{\Delta x} = c\frac{\Delta u}{\Delta x}.$$

As Δx approaches zero, $\dfrac{\Delta y}{\Delta x}$ and $c\dfrac{\Delta u}{\Delta x}$ approach $\dfrac{dy}{dx}$ and $c\dfrac{du}{dx}$.

Therefore
$$\frac{dy}{dx} = c\frac{du}{dx},$$
whence
$$dy = c\,du.$$

Fractions with a constant denominator should be differentiated by this formula. Thus
$$d\left(\frac{u}{c}\right) = d\left(\frac{1}{c}u\right) = \frac{1}{c}du.$$

21. Proof of IV. — *The differential of the product of two functions is equal to the first times the differential of the second plus the second times the differential of the first.*

Let
$$y = uv.$$
Then
$$\begin{aligned} y + \Delta y &= (u + \Delta u)\,(v + \Delta v) \\ &= uv + v\,\Delta u + (u + \Delta u)\,\Delta v. \end{aligned}$$

Subtraction gives
$$\Delta y = v\,\Delta u + (u + \Delta u)\,\Delta v,$$
whence
$$\frac{\Delta y}{\Delta x} = v\frac{\Delta u}{\Delta x} + (u + \Delta u)\frac{\Delta v}{\Delta x}.$$

Since u is a continuous function, Δu approaches zero as Δx approaches zero. Therefore, in the limit,
$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx},$$

and so

$$dy = v\,du + u\,dv.$$

In the same way we can show that

$$d\,(uvw) = uv\,dw + uw\,dv + vw\,du.$$

22. Proof of V. — *The differential of a fraction is equal to the denominator times the differential of the numerator minus the numerator times the differential of the denominator, all divided by the square of the denominator.*

Let

$$y = \frac{u}{v}.$$

Then

$$y + \Delta y = \frac{u + \Delta u}{v + \Delta v}$$

and

$$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{v\,\Delta u - u\,\Delta v}{v\,(v + \Delta v)}.$$

Dividing by Δx,

$$\frac{\Delta y}{\Delta x} = \frac{v\dfrac{\Delta u}{\Delta x} - u\dfrac{\Delta v}{\Delta x}}{v\,(v + \Delta v)}.$$

Since v is a continuous function of x, Δv approaches zero as Δx approaches zero. Therefore

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2},$$

whence

$$dy = \frac{v\,du - u\,dv}{v^2}.$$

23. Proof of VI. — *The differential of a variable raised to a constant power is equal to the product of the exponent, the variable raised to a power one less, and the differential of the variable.*

We consider three cases depending on whether the exponent is a positive whole number, a positive fraction, or a negative number. For the case of irrational exponent, see Ex. 25, page 61.

(1) Let n be a positive integer and $y = u^n$. Then

$$y+\Delta y=(u+\Delta u)^n=u^n+nu^{n-1}\Delta u+\frac{n(n-1)}{2}u^{n-2}(\Delta u)^2+\cdots$$

and

$$\Delta y = nu^{n-1}\,\Delta u + \frac{n\,(n-1)}{2}\,u^{n-2}\,(\Delta u)^2 + \cdots .$$

Dividing by Δu,

$$\frac{\Delta y}{\Delta u} = nu^{n-1} + \frac{n\,(n-1)}{2}\,u^{n-2}\,(\Delta u) + \cdots .$$

As Δu approaches zero, this approaches

$$\frac{dy}{du} = nu^{n-1}.$$

Consequently,

$$dy = nu^{n-1}\,du.$$

(2) Let n be a positive fraction $\frac{p}{q}$ and $y = u^n = u^{\frac{p}{q}}$. Then

$$y^q = u^p.$$

Since p and q are both positive integers, we can differentiate both sides of this equation by the formula just proved. Therefore

$$qy^{q-1}\,dy = pu^{p-1}\,du.$$

Solving for dy and substituting $u^{\frac{p}{q}}$ for y, we get

$$dy = \frac{pu^{p-1}}{qu^{p-\frac{p}{q}}}\,du = \frac{p}{q}\,u^{\frac{p}{q}-1}\,du = nu^{n-1}\,du.$$

(3) Let n be a negative number $-m$. Then

$$y = u^n = u^{-m} = \frac{1}{u^m}.$$

Since m is positive, we can find $d\,(u^m)$ by the formulas proved above. Therefore, by V,

$$dy = \frac{u^m d(1) - 1\,d(u^m)}{(u^m)^2} = \frac{-mu^{m-1}du}{u^{2m}} = -mu^{-m-1}\,du = nu^{n-1}\,du.$$

Therefore, whether n is an integer or fraction, positive or negative,

$$d\,(u^n) = nu^{n-1}\,du.$$

If the numerator of a fraction is constant, this formula can be used instead of V. Thus

$$d\left(\frac{c}{u}\right) = d\,(cu^{-1}) = -cu^{-2}\,du.$$

Example 1. $y = 4\,x^3.$

Using formulas III and VI,

$$dy = 4\,d\,(x^3) = 4 \cdot 3\,x^2\,dx = 12\,x^2\,dx.$$

Ex. 2. $y = \sqrt{x} + \dfrac{1}{\sqrt{x}} + 3.$

This can be written

$$y = x^{\frac{1}{2}} + x^{-\frac{1}{2}} + 3.$$

Consequently, by II and VI,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d\,(x^{\frac{1}{2}})}{dx} + \frac{d\,(x^{-\frac{1}{2}})}{dx} + \frac{d\,(3)}{dx}\\ &= \frac{1}{2}\,x^{-\frac{1}{2}}\frac{dx}{dx} - \frac{1}{2}\,x^{-\frac{3}{2}}\frac{dx}{dx} + 0\\ &= \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x^3}}.\end{aligned}$$

Ex. 3. $y = (x + a)\,(x^2 - b^2).$

Using IV, with $u = x + a$, $v = x^2 - b^2$,

$$\begin{aligned}\frac{dy}{dx} &= (x + a)\frac{d}{dx}(x^2 - b^2) + (x^2 - b^2)\frac{d}{dx}(x + a)\\ &= (x + a)\,(2\,x - 0) + (x^2 - b^2)\,(1 + 0)\\ &= 3\,x^2 + 2\,ax - b^2.\end{aligned}$$

Ex. 4. $y = \dfrac{x^2 + 1}{x^2 - 1}.$

Using V, with $u = x^2 + 1$, $v = x^2 - 1$,

$$dy = \frac{(x^2 - 1)\, d\,(x^2 + 1) - (x^2 + 1)\, d\,(x^2 - 1)}{(x^2 - 1)^2}$$

$$= \frac{(x^2 - 1)\, 2\,x\,dx - (x^2 + 1)\, 2\,x\,dx}{(x^2 - 1)^2}$$

$$= -\frac{4\,x\,dx}{(x^2 - 1)^2}.$$

Ex. 5. $y = \sqrt{x^2 - 1}$.

Using VI, with $u = x^2 - 1$,

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 - 1)^{\frac{1}{2}} = \tfrac{1}{2}(x^2 - 1)^{-\frac{1}{2}} \frac{d}{dx}(x^2 - 1)$$

$$= \tfrac{1}{2}(x^2 - 1)^{-\frac{1}{2}}(2\,x) = \frac{x}{\sqrt{x^2 - 1}}.$$

Ex. 6. $x^2 + xy - y^2 = 1$.

We can consider y a function of x determined by the equation. Then

$$d\,(x^2) + d\,(xy) - d\,(y^2) = d\,(1) = 0,$$

that is,

$$2\,x\,dx + x\,dy + y\,dx - 2\,y\,dy = 0,$$

$$(2\,x + y)\,dx + (x - 2\,y)\,dy = 0.$$

Consequently,

$$\frac{dy}{dx} = \frac{2\,x + y}{2\,y - x}.$$

Ex. 7. $x = t + \dfrac{1}{t}$, $y = t - \dfrac{1}{t}$.

In this case

$$dx = dt - \frac{dt}{t^2}, \quad dy = dt + \frac{dt}{t^2}.$$

Consequently,

$$\frac{dy}{dx} = \frac{1 + \dfrac{1}{t^2}}{1 - \dfrac{1}{t^2}} = \frac{t^2 + 1}{t^2 - 1}.$$

Ex. 8. Find an approximate value of $y = \left(\dfrac{1 - x}{1 + x}\right)^{\frac{1}{3}}$ when $x = 0.2$.

When $x = 0$, $y = 1$. Also

$$dy = -\frac{2\,dx}{3\,(1-x)^{\frac{2}{3}}\,(1+x)^{\frac{4}{3}}}.$$

When $x = 0$ this becomes

$$dy = -\tfrac{2}{3}\,dx.$$

If we assume that dy is approximately equal to Δy, the change in y when x changes from 0 to 0.2 is approximately

$$dy = -\tfrac{2}{3}\,(0.2) = -0.13.$$

The required value is then

$$y = 1 - 0.13 = .87.$$

EXERCISES

In the following exercises show that the differentials and derivatives have the values given:

1. $y = 3\,x^4 + 4\,x^3 - 6\,x^2 + 5,\quad dy = 12\,(x^3 + x^2 - x)\,dx.$

2. $y = 2\,x^{\frac{3}{2}} - 3\,x^{\frac{2}{3}} + 1,\quad \dfrac{dy}{dx} = \dfrac{3\,x^{\frac{5}{6}} - 2}{x^{\frac{1}{3}}}.$

3. $y = \dfrac{x^3 - x^2 + 1}{5},\quad \dfrac{dy}{dx} = \dfrac{3\,x^2 - 2\,x}{5}.$

4. $y = (x + 2\,a)\,(x - a)^2,\quad dy = 3\,(x^2 - a^2)\,dx.$

5. $y = x\,(2\,x - 1)\,(3\,x + 2),\quad \dfrac{dy}{dx} = 18\,x^2 + 2\,x - 2.$

6. $y = \dfrac{1}{x^2 + 1},\quad dy = \dfrac{-2\,x\,dx}{(x^2 + 1)^2}.$

7. $y = \dfrac{2\,x + 3}{4\,x - 5},\quad dy = \dfrac{-\,22\,dx}{(4\,x - 5)^2}.$

8. $\dfrac{d}{dx}\dfrac{1 - 2\,x}{(x - 1)^2} = \dfrac{2\,x}{(x - 1)^3}.$

9. $d\sqrt{1 + 2t - t^2} = \dfrac{(1 - t)\,dt}{\sqrt{1 + 2t - t^2}}.$

10. $\dfrac{d}{d\theta}\,\dfrac{1}{\theta + \sqrt{\theta^2 - 1}} = \dfrac{\sqrt{\theta^2 - 1} - \theta}{\sqrt{\theta^2 - 1}}.$

11. $\dfrac{d}{ds}\,s\,\sqrt{a^2 - s^2} = \dfrac{a^2 - 2\,s^2}{\sqrt{a^2 - s^2}}.$

12. $d\,\dfrac{y}{\sqrt{a^2 - y^2}} = \dfrac{a^2\,dy}{(a^2 - y^2)^{\frac{3}{2}}}.$

13. $\dfrac{d}{dx}\left(\dfrac{a}{x}\sqrt{a^2 - x^2}\right) = -\,\dfrac{a^3}{x^2\,\sqrt{a^2 - x^2}}.$

14. $\dfrac{d}{dx}\sqrt{\dfrac{x^2 - 1}{x^2 + 1}} = \dfrac{2\,x}{(x^2 + 1)\,\sqrt{x^4 - 1}}.$

15. $d\dfrac{(2 + 3\,x^6)^{\frac{5}{3}}}{x^{10}} = -\,\dfrac{20}{x^{11}}(2 + 3\,x^6)^{\frac{2}{3}}\,dx.$

16. $\dfrac{d}{dx}\,x^2y^2 = 2\,xy^2 + 2\,x^2y\,\dfrac{dy}{dx}.$

17. $d\,\sqrt{x^2 + y^2} = \dfrac{x\,dx + y\,dy}{\sqrt{x^2 + y^2}}.$

18. $y = (x+1)(2-3x)^2(2x-3)^3$,

$$\frac{dy}{dx} = (24 + 13x - 36x^2)(2-3x)(2x-3)^2.$$

19. $y = \dfrac{x^m}{(a+bx^n)^{\frac{m}{n}}}$, $\quad \dfrac{dy}{dx} = \dfrac{max^{m-1}}{(a+bx^n)^{\frac{m}{n}+1}}$.

20. $y = \dfrac{2x^2-1}{3x^3}\sqrt{x^2+1}$, $\quad \dfrac{dy}{dx} = \dfrac{1}{x^4\sqrt{x^2+1}}$.

21. $y = \dfrac{(x+\sqrt{1+x^2})^{n+1}}{n+1} + \dfrac{(x+\sqrt{1+x^2})^{n-1}}{n-1}$,

$$dy = 2\left(x + \sqrt{1+x^2}\right)^n dx.$$

22. $x^2 + y^2 = a^2$, $\quad \dfrac{dy}{dx} = -\dfrac{x}{y}$.

23. $x^3 + y^3 = 3axy$, $\quad \dfrac{dy}{dx} = \dfrac{ay - x^2}{y^2 - ax}$.

24. $2x^2 - 3xy + 4y^2 = 3x$, $\quad \dfrac{dy}{dx} = \dfrac{4x - 3y - 3}{3x - 8y}$.

25. $\dfrac{x}{y} + \dfrac{y}{x} = 1$, $\quad y\,dx - x\,dy = 0$.

26. $y = \dfrac{1}{x}$, $\quad \dfrac{dx}{\sqrt{1+x^4}} + \dfrac{dy}{\sqrt{1+y^4}} = 0$.

27. $y^{2n} + x^m y^n = x^{2m}$, $\quad my\,dx = nx\,dy$.

28. $x = \dfrac{t}{t-1}$, $\quad y = \dfrac{2t+3}{t-1}$, $\quad \dfrac{dy}{dx} = 5$.

29. $x = t - \sqrt{t^2-1}$, $\quad y = t + \sqrt{t^2-1}$, $\quad x\,dy + y\,dx = 0$.

30. $x = \dfrac{3at}{1+t^3}$, $\quad y = \dfrac{3at^2}{1+t^3}$, $\quad \dfrac{dy}{dx} = \dfrac{2t - t^4}{1 - 2t^3}$.

31. Given $y = \dfrac{x}{\sqrt{x^2+9}}$, find an approximate value for y when $x = 4.2$.

32. Find an approximate value of

$$\sqrt{\frac{x^2 - x + 1}{x^2 + x + 1}}$$

when $x = .3$.

33. Given $y = x^6$, find dy and Δy when x changes from 3 to 3.1. Is dy a satisfactory approximation for Δy? Express the difference as a percentage of Δy.

34. Find the slope of the curve

$$y = x(x^5 + 31)^{\frac{1}{3}}$$

at the point $x = 1$.

35. Find the points on the parabola $y^2 = 4ax$ where the tangent is inclined at an angle of 45° to the x-axis.

36. Given $y = (a + x)\sqrt{a - x}$, for what values of x does y increase as x increases and for what values does y decrease as x increases?

37. Find the points $P\ (x, y)$ on the curve

$$y = x + \frac{1}{x}$$

where the tangent is perpendicular to the line joining P to the origin.

38. Find the angle at which the circle

$$x^2 + y^2 = 2\,x - 3\,y$$

intersects the x-axis at the origin.

39. A line through the point (1, 2) cuts the x-axis at $(x, 0)$ and the y-axis at $(0, y)$. Find $\frac{dy}{dx}$.

40. If $x^2 - x + 2 = 0$, why is the equation

$$\frac{d}{dx}(x^2 - x + 2) = 0$$

not satisfied?

41. The distances x, x' of a point and its image from a lens are connected by the equation

$$\frac{1}{x} + \frac{1}{x'} = \frac{1}{f},$$

f being constant. If L is the length of a small object extending along the axis perpendicular to the lens and L' is the length of its image, show that

$$\frac{L'}{L} = \left(\frac{x'}{x}\right)^2$$

approximately, x and x' being the distances of the object and its image from the lens.

24. Higher Derivatives. — The first derivative $\frac{dy}{dx}$ is a function of x. Its derivative with respect to x, written $\frac{d^2y}{dx^2}$, is called the second derivative of y with respect to x. That is,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right).$$

Similarly,

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right),$$

$$\frac{d^4y}{dx^4} = \frac{d}{dx}\left(\frac{d^3y}{dx^3}\right), \text{ etc.}$$

The derivatives of $f(x)$ with respect to x are often written $f'(x), f''(x), f'''(x)$, etc. Thus, if $y = f(x)$,

$$\frac{dy}{dx} = f'(x),\quad \frac{d^2y}{dx^2} = f''(x),\quad \frac{d^3y}{dx^3} = f'''(x), \text{ etc.}$$

Example 1. $y = x^3$.

Differentiation with respect to x gives

$$\frac{dy}{dx} = 3x^2,$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(3x^2) = 6x,$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}(6x) = 6,$$

$$\frac{d^4y}{dx^4} = \frac{d}{dx}(6) = 0.$$

All higher derivatives are zero.

Ex. 2. $x^2 + xy + y^2 = 1$.

Differentiating with respect to x,

$$2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0,$$

whence

$$\frac{dy}{dx} = -\frac{2x + y}{x + 2y}.$$

The second derivative is

$$\frac{d^2y}{dx^2} = -\frac{d}{dx}\left(\frac{2x + y}{x + 2y}\right) = \frac{3x\frac{dy}{dx} - 3y}{(x + 2y)^2}.$$

Replacing $\frac{dy}{dx}$ by its value in terms of x and y and reducing,

$$\frac{d^2y}{dx^2} = -\frac{6(x^2 + xy + y^2)}{(x + 2y)^3} = -\frac{6}{(x + 2y)^3}.$$

The last expression is obtained by using the equation of the curve $x^2 + xy + y^2 = 1$. By differentiating this second derivative we could find the third derivative, etc.

25. Change of Variable. — We have represented the second derivative by $\frac{d^2y}{dx^2}$. This can be regarded as the quotient obtained by dividing a second differential

$$d^2y = d\,(dy)$$

by $(dx)^2$. The value of d^2y will however depend on the variable with respect to which y is differentiated.

Thus, suppose $y = x^2$, $x = t^3$. Then $\frac{d^2y}{dx^2} = 2$ and so

$$d^2y = 2\,(dx)^2 = 2\,(3\,t^2\,dt)^2 = 18\,t^4\,(dt)^2.$$

If, however, we differentiate with respect to t, since $y = t^6$, $\frac{d^2y}{dt^2} = 30\,t^4$ and

$$d^2y = 30\,t^4\,(dt)^2,$$

which is not equal to the value obtained when we differentiated y with respect to x.

For this reason we shall not use differentials of the second or higher orders except in the numerators of derivatives. Two derivatives like $\frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$ must not be combined like fractions because d^2y does not have the same value in the two cases.

If we have derivatives with respect to t and wish to find derivatives with respect to x, they can be found by using the identical relation

$$\frac{d}{dx}\,u = \frac{du}{dt}\,\frac{dt}{dx} = \frac{\dfrac{du}{dt}}{\dfrac{dx}{dt}}. \tag{25}$$

For example,

$$\frac{d}{dx}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx} = \frac{\dfrac{d^2y}{dt^2}}{\dfrac{dx}{dt}}.$$

Example. Given $x = t - \dfrac{1}{t}$, $y = t + \dfrac{1}{t}$, find $\dfrac{d^2y}{dx^2}$.

In this case

$$\frac{dy}{dx} = \frac{dt - \dfrac{dt}{t^2}}{dt + \dfrac{dt}{t^2}} = \frac{t^2 - 1}{t^2 + 1}.$$

Consequently,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{t^2 - 1}{t^2 + 1}\right) = \frac{4\,t}{(t^2 + 1)^2}\,\frac{dt}{dx} = \frac{4\,t}{(t^2 + 1)^2}\,\frac{1}{1 + \dfrac{1}{t^2}} = \frac{4\,t^3}{(t^2 + 1)^3}$$

EXERCISES

Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in each of the following exercises:

1. $y = \dfrac{x + 1}{x - 1}$.
2. $y = \sqrt{a^2 - x^2}$.
3. $y = (x - 1)^3\,(x + 2)^4$.
4. $y^2 = 4\,x$.
5. $x^2 + y^2 = a^2$.
6. $x^2 - 2\,y^2 = 1$.
7. $xy = x + y$.
8. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.
9. If a and b are constant and $y = ax^2 + bx$, show that
$$x^2\,\frac{d^2y}{dx^2} - x\,\frac{dy}{dx} + bx = 0.$$
10. If a, b, c, d are constant and $y = ax^3 + bx^2 + cx + d$, show that
$$\frac{d^4y}{dx^4} = 0.$$
11. Show that
$$\frac{d}{dt}\left(t\,\frac{dx}{dt} - x\right) = t\,\frac{d^2x}{dt^2}.$$
12. Show that
$$\frac{d}{dx}\left(x^3\,\frac{d^3y}{dx^3} - 3\,x^2\,\frac{d^2y}{dx^2} + 6\,x\,\frac{dy}{dx} - 6\,y\right) = x^3\,\frac{d^4y}{dx^4}.$$
13. Given $x = t^2 + t^3$, $y = t^2 - t^3$, find $\dfrac{d^2y}{dx^2}$ and $\dfrac{d^2x}{dy^2}$.
14. By differentiating the equation
$$\frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}}.$$
with respect to x, find $\dfrac{d^2y}{dx^2}$ in terms of derivatives of x with respect to y.

CHAPTER IV

RATES

26. Rate of Change. — If the change in a quantity z is proportional to the time in which it occurs, z is said to change at a constant rate. If Δz is the change occurring in an interval of time Δt, the rate of change of z is

$$\frac{\Delta z}{\Delta t}.$$

If the rate of change of z is not constant, it will be nearly constant if the interval Δt is very short. Then $\frac{\Delta z}{\Delta t}$ is approximately the rate of change, the approximation becoming greater as the increments become less. The exact rate of change at the time t is consequently defined as

$$\lim_{\Delta t \doteq 0} \frac{\Delta z}{\Delta t} = \frac{dz}{dt}, \tag{26}$$

that is, *the rate of change of any quantity is its derivative with respect to the time.*

If the quantity is increasing, its rate of change is positive; if decreasing, the rate is negative.

27. Velocity Along a Straight Line. — Let a particle P move along a straight line (Fig. 27). Let $s = OP$ be con-

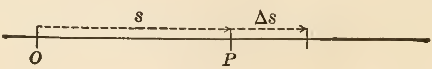


Fig. 27.

sidered positive on one side of O, negative on the other. If the particle moves with constant velocity the distance Δs in the time Δt, its velocity is

$$\frac{\Delta s}{\Delta t}.$$

If the velocity is not constant, it will be nearly so when Δt is very short. Therefore $\frac{\Delta s}{\Delta t}$ is approximately the velocity, the approximation becoming greater as Δt becomes less. The velocity at the time t is therefore defined as

$$v = \lim_{\Delta t \doteq 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}. \tag{27}$$

This equation shows that ds is the distance the particle would move in a time dt if the velocity remained constant. As a rule the velocity will not be constant and so ds will be different from the distance the particle does move in the time dt.

When s is increasing, the velocity is positive; when s is decreasing, the velocity is negative.

Example. A body starting from rest falls approximately

$$s = 16\,t^2$$

feet in t seconds. Find its velocity at the end of 10 seconds.

The velocity at any time t is

$$v = \frac{ds}{dt} = 32\,t \text{ ft./sec.*}$$

At the end of 10 seconds it is

$$v = 320 \text{ ft./sec.}$$

28. Acceleration Along a Straight Line. — The acceleration of a particle moving along a straight line is defined as the rate of change of its velocity. That is

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}. \tag{28}$$

This equation shows that dv is the amount v would increase in the time dt if the acceleration remained constant.

The acceleration is positive when the velocity is increasing, negative when it is decreasing.

Example. At the end of t seconds the vertical height of a ball thrown upward with a velocity of 100 ft./sec. is

$$h = 100\,t - 16\,t^2.$$

Find its velocity and acceleration. Also find when it is rising, when falling, and when it reaches the highest point.

* The notation ft./sec. means feet per second. Similarly, ft./sec.^{2}, used for acceleration, means feet per second per second.

The velocity and acceleration are

$$v = \frac{dh}{dt} = (100 - 32\,t) \text{ ft./sec.},$$

$$a = \frac{dv}{dt} = -32 \text{ ft./sec.}^2.$$

The ball will be rising while v is positive, that is, until $t = \frac{100}{32} = 3\frac{1}{8}$. It will be falling after $t = 3\frac{1}{8}$. It will be at the highest point when $t = 3\frac{1}{8}$.

29. Angular Velocity and Acceleration. — Consider a body rotating about a fixed axis. Let θ be the angle turned through at time t. The *angular velocity* is defined as the rate of change of θ, that is,

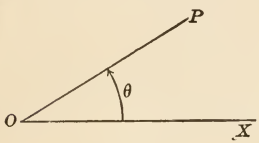


FIG. 29.

$$\text{angular velocity} = \omega = \frac{d\theta}{dt}.$$

The *angular acceleration* is the rate of change of angular velocity, that is,

$$\text{angular acceleration} = \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}.$$

Example 1. A wheel is turning 100 revolutions per minute about its axis. Find its angular velocity.

The angle turned through in one minute will be

$$\omega = 100 \cdot 2\,\pi = 200\,\pi \text{ radians/min.}$$

Ex. 2. A wheel, starting from rest under the action of a constant moment (or twist) about its axis, will turn in t seconds through an angle

$$\theta = kt^2,$$

k being constant. Find its angular velocity and acceleration at time t.

By definition

$$\omega = \frac{d\theta}{dt} = 2\,kt \text{ rad./sec.},$$

$$\alpha = \frac{d\omega}{dt} = 2\,k \text{ rad./sec.}^2.$$

30. Related Rates. — In many cases the rates of change of certain variables are known and the rates of others are to be calculated. This is done by expressing the quantities whose rates are wanted in terms of those whose rates are known and taking the derivatives with respect to t.

Example 1. The radius of a cylinder is increasing 2 ft./sec. and its altitude decreasing 3 ft./sec. Find the rate of change of its volume.

Let r be the radius and h the altitude. Then

$$v = \pi r^2 h.$$

The derivative with respect to t is

$$\frac{dv}{dt} = \pi r^2 \frac{dh}{dt} + 2\pi rh \frac{dr}{dt}.$$

By hypothesis

$$\frac{dr}{dt} = 2, \quad \frac{dh}{dt} = -3.$$

Hence

$$\frac{dv}{dt} = 4\pi rh - 3\pi r^2.$$

This is the rate of increase when the radius is r and altitude h. If $r = 10$ ft. and $h = 6$ ft.,

$$\frac{dv}{dt} = -60\pi \text{ cu. ft./sec.}$$

Ex. 2. A ship B sailing south at 16 miles per hour is northwest of a ship A sailing east at 10 miles per hour. At what rate are the ships approaching?

Let x and y be the distances of the ships A and B from the point where their paths cross. The distance between the ships is then

$$s = \sqrt{x^2 + y^2}.$$

This distance is changing at the rate

$$\frac{ds}{dt} = \frac{x\dfrac{dx}{dt} + y\dfrac{dy}{dt}}{\sqrt{x^2 + y^2}}.$$

By hypothesis,

$$\frac{dx}{dt} = 10,\ \frac{dy}{dt} = -16,\ \frac{x}{\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}} = \cos 45^\circ = \frac{1}{\sqrt{2}}.$$

Therefore

$$\frac{ds}{dt} = \frac{10-16}{\sqrt{2}} = -3\sqrt{2} \text{ mi./hr.}$$

The negative sign shows that s is decreasing, that is, the ships are approaching.

EXERCISES

1. From the roof of a house 50 ft. above the street a ball is thrown upward with a speed of 100 ft. per second. Its height above the ground t seconds later will be

$$h = 50 + 100\,t - 16\,t^2.$$

Find its velocity and acceleration when $t = 2$. How long does it continue to rise? What is the highest point reached?

2. A body moves in a straight line according to the law

$$s = \tfrac{1}{4}\,t^4 - 4\,t^3 + 16\,t^2.$$

Find its velocity and acceleration. During what interval is the velocity decreasing? When is it moving backward?

3. If v is the velocity and a the acceleration of a particle moving along the x-axis, show that

$$a\,dx = v\,dv.$$

4. If a particle moves along a line with the velocity

$$v^2 = 2\,gs,$$

where g is constant and s the distance from a fixed point in the line, show that the acceleration is constant.

5. When a particle moves with constant speed around a circle with center at the origin, its shadow on the x-axis moves with velocity v satisfying the equation

$$v^2 + n^2x^2 = n^2r^2,$$

n and r being constant. Show that the acceleration of the shadow is proportional to its distance from the origin.

6. A wheel is turning 500 revolutions per minute. What is its angular velocity? If the wheel is 4 ft. in diameter, with what speed does it drive a belt?

7. A rotating wheel is brought to rest by a brake. Assuming the friction between brake and wheel to be constant, the angle turned through in a time t will be

$$\theta = a + bt - ct^2,$$

a, b, c being constants. Find the angular velocity and acceleration. When will the wheel come to rest?

8. A wheel revolves according to the law $\omega = 30\,t + t^2$, where ω is the speed in radians per minute and t the time since the wheel started. A second wheel turns according to the law $\theta = \frac{1}{4}\,t^2$, where t is the time in seconds and θ the angle in degrees through which it has turned. Which wheel is turning faster at the end of one minute and how much?

9. A wheel of radius r rolls along a line. If v is the velocity and a the acceleration of its center, ω the angular velocity and α the angular acceleration about its axis, show that

$$v = r\omega, \qquad a = r\alpha.$$

10. The depth of water in a cylindrical tank, 6 feet in diameter, is increasing 1 foot per minute. Find the rate at which the water is flowing in.

11. A stone dropped into a pond sends out a series of concentric ripples. If the radius of the outer ripple increases steadily at the rate of 6 ft./sec., how rapidly is the area of disturbed water increasing at the end of 2 seconds?

12. At a certain instant the altitude of a cone is 7 ft. and the radius of its base 3 ft. If the altitude is increasing 2 ft./sec. and the radius of its base decreasing 1 ft./sec., how fast is the volume increasing or decreasing?

13. The top of a ladder 20 feet long slides down a vertical wall. Find the ratio of the speeds of the top and bottom when the ladder makes an angle of 30 degrees with the ground.

14. The cross section of a trough 10 ft. long is an equilateral triangle. If water flows in at the rate of 10 cu. ft./sec., find the rate at which the depth is increasing when the water is 18 inches deep.

15. A man 6 feet tall walks at the rate of 5 feet per second away from a lamp 10 feet from the ground. When he is 20 feet from the lamp post, find the rate at which the end of his shadow is moving and the rate at which his shadow is growing.

16. A boat moving 8 miles per hour is laying a cable. Assuming that the water is 1000 ft. deep, the cable is attached to the bottom and stretches in a straight line to the stern of the boat, at what rate is the cable leaving the boat when 2000 ft. have been paid out?

17. Sand when poured from a height on a level surface forms a cone with constant angle β at the vertex, depending on the material. If the sand is poured at the rate of c cu. ft./sec., at what rate is the radius increasing when it equals a?

18. Two straight railway tracks intersect at an angle of 60 degrees. On one a train is 8 miles from the junction and moving toward it at the rate of 40 miles per hour. On the other a train is 12 miles from the junction and moving from it at the rate of 10 miles per hour. Find the rate at which the trains are approaching or separating.

19. An elevated car running at a constant elevation of 50 ft. above the street passes over a surface car, the tracks crossing at right angles. If the speed of the elevated car is 16 miles per hour and that of the surface car 8 miles, at what rate are the cars separating 10 seconds after they meet?

20. The rays of the sun make an angle of 30 degrees with the horizontal. A ball drops from a height of 64 feet. How fast is its shadow moving just before the ball hits the ground?

CHAPTER V

MAXIMA AND MINIMA

31. A function of x is said to have a *maximum* at $x = a$, if when $x = a$ the function is greater than for any other value in the immediate neighborhood of a. It has a *minimum* if when $x = a$ the function is less than for any other value of x sufficiently near a.

If we represent the function by y and plot the curve $y = f(x)$, a maximum occurs at the top, a minimum at the bottom of a wave.

If the derivative is continuous, as in Fig. 31a, the tangent is horizontal at the highest and lowest points of a wave and the slope is zero. Hence in determining maxima and minima of a function $f(x)$ we first look for values of x such that

$$\frac{d}{dx} f(x) = f'(x) = 0.$$

If a is a root of this equation, $f(a)$ may be a maximum, a minimum, or neither.

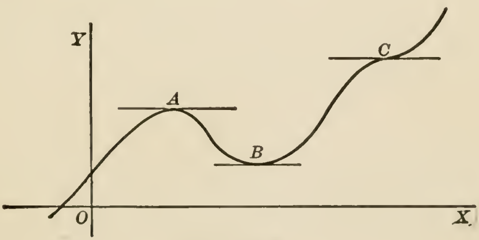


FIG. 31a.

If the slope is positive on the left of the point and negative on the right, as at A, the curve falls on both sides and the ordinate is a maximum. That is, $f(x)$ *has a maximum value* *at $x = a$, if $f'(x)$ is positive for values of x a little less and negative for values a little greater than a.*

If the slope is negative on the left and positive on the right, as at B, the curve rises on both sides and the ordinate is a minimum. That is, *$f(x)$ has a minimum at $x = a$, if $f'(x)$ is negative for values of x a little less and positive for values a little greater than a.*

If the slope has the same sign on both sides, as at C, the curve rises on one side and falls on the other and the ordinate is neither a maximum nor a minimum. That is, *$f(x)$ has neither a maximum nor a minimum at $x = a$ if $f'(x)$ has the same sign on both sides of a.*

Example 1. The sum of two numbers is 5. Find the maximum value of their product.

Let one of the numbers be x. The other is then $5 - x$. The value of x is to be found such that the product

$$y = x(5 - x) = 5x - x^2$$

is a maximum. The derivative is

$$\frac{dy}{dx} = 5 - 2x.$$

This is zero when $x = \frac{5}{2}$. If x is less than $\frac{5}{2}$, the derivative is positive. If x is greater than $\frac{5}{2}$ the derivative is negative. Near $x = \frac{5}{2}$ the graph then has the shape shown in Fig. 31b. At $x = \frac{5}{2}$ the function has its greatest value

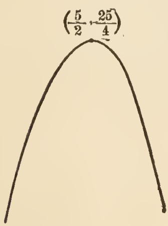


FIG. 31b.

$$\tfrac{5}{2}(5 - \tfrac{5}{2}) = \tfrac{25}{4}.$$

Ex. 2. Find the shape of the pint cup which requires for its construction the least amount of tin.

Let the radius of base be r and the depth h. The area of tin used is

$$A = \pi r^2 + 2\pi rh.$$

Let v be the number of cubic inches in a pint. Then

$$v = \pi r^2 h.$$

Consequently, $$h = \frac{v}{\pi r^2}$$

and $$A = \pi r^2 + \frac{2v}{r}.$$

Since π and v are constants,

$$\frac{dA}{dr} = 2\pi r - \frac{2v}{r^2} = 2\left(\frac{\pi r^3 - v}{r^2}\right).$$

This is zero if $\pi r^3 = v$. If there is a maximum or minimum it must then occur when

$$r = \sqrt[3]{\frac{v}{\pi}};$$

for, if r has any other value, $\frac{dA}{dr}$ will have the same sign on both sides of that value and A will be neither a maximum nor a minimum. Since the amount of tin used cannot be zero there must be a least amount. This must then be the value of A when $v = \pi r^3$. Also $v = \pi r^2 h$. We therefore conclude that $r = h$. The cup requiring the least tin thus has a depth equal to the radius of its base.

Ex. 3. The strength of a rectangular beam is proportional to the product of its width by the square of its depth. Find the strongest beam that can be cut from a circular log 24 inches in diameter.

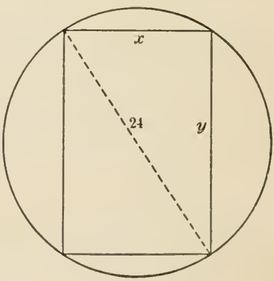


FIG. 31c.

In Fig. 31c is shown a section of the log and beam. Let x be the breadth and y the depth of the beam. Then

$$x^2 + y^2 = (24)^2.$$

The strength of the beam is

$$S = kxy^2 = kx(24^2 - x^2),$$

k being constant. The derivative of S is

$$\frac{dS}{dx} = k\,(24^2 - 3\,x^2).$$

If this is zero, $x = \pm 8\sqrt{3}$. Since x is the breadth of the beam, it cannot be negative. Hence

$$x = 8\sqrt{3}$$

is the only solution. Since the log cannot be infinitely strong, there must be a strongest beam. Since no other value can give either a maximum or a minimum, $x = 8\sqrt{3}$ must be the width of the strongest beam. The corresponding depth is $y = 8\sqrt{6}$.

Ex. 4. Find the dimensions of the largest right circular cylinder inscribed in a given right circular cone.

Let r be the radius and h the altitude of the cone. Let x be the radius and y the altitude of an inscribed cylinder (Fig. 31d). From the similar triangles DEC and ABC,

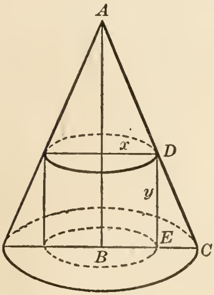


FIG. 31d.

$$\frac{DE}{EC} = \frac{AB}{BC},$$

that is,

$$\frac{y}{r - x} = \frac{h}{r}, \qquad y = \frac{h}{r}\,(r - x).$$

The volume of the cylinder is

$$v = \pi x^2 y = \frac{\pi h}{r}\,(rx^2 - x^3).$$

Equating its derivative to zero, we get

$$2\,rx - 3\,x^2 = 0.$$

Hence $x = 0$ or $x = \frac{2}{3}\,r$. The value $x = 0$ obviously does not give the maximum. Since there is a largest cylinder, its radius must then be $x = \frac{2}{3}\,r$. By substitution its altitude is then found to be $y = \frac{1}{3}\,h$.

32. Method of Finding Maxima and Minima. — The method used in solving these problems involves the following steps:

(1) Decide what is to be a maximum or minimum. Let it be y.

(2) Express y in terms of a *single* variable. Let it be x. It may be convenient to express y temporarily in terms of several variable quantities. If the problem can be solved by our present methods, there will be relations enough to eliminate all but one of these.

(3) Calculate $\frac{dy}{dx}$ and find for what values of x it is zero.

(4) It is usually easy to decide from the problem itself whether the corresponding values of y are maxima or minima. If not, determine the signs of $\frac{dy}{dx}$ when x is a little less and a little greater than the values in question and apply the criteria given in Art. 31.

EXERCISES

Find the maximum and minimum values of the following functions:

1. $2x^2 - 5x + 7$.

2. $6 + 12x - x^3$.

3. $x^4 - 2x^2 + 6$.

4. $\dfrac{x^2}{\sqrt{a^2 - x^2}}$.

Show that the following functions have no maxima or minima:

5. x^3.

6. $x^3 + 4x$.

7. $6x^5 - 15x^4 + 10x^3$.

8. $x\sqrt{a^2 + x^2}$.

9. Show that $x + \frac{1}{x}$ has a maximum and a minimum and that the maximum is less than the minimum.

10. The sum of the square and the reciprocal of a number is a minimum. Find the number.

11. Show that the largest rectangle with a given perimeter is a square.

12. Show that the largest rectangle that can be inscribed in a given circle is a square.

13. Find the altitude of the largest cylinder that can be inscribed in a sphere of radius a.

14. A rectangular box with square base and open at the top is to be made out of a given amount of material. If no allowance is made for thickness of material or waste in construction, what are the dimensions of the largest box that can be made?

15. A cylindrical tin can closed at both ends is to have a given capacity. Show that the amount of tin used will be a minimum when the height equals the diameter.

16. What are the most economical proportions for an open cylindrical water tank if the cost of the sides per square foot is two-thirds the cost of the bottom per square foot?

17. The top, bottom, and lateral surface of a closed tin can are to be cut from rectangles of tin, the scraps being a total loss. Find the most economical proportions for a can of given capacity.

18. Find the volume of the largest right cone that can be generated by revolving a right triangle of hypotenuse 2 ft. about one of its sides.

19. Four successive measurements of a distance gave a_1, a_2, a_3, a_4 as results. By the theory of least squares the most probable value of the distance is that which makes the sum of the squares of the four errors a minimum. What is that value?

20. If the sum of the length and girth of a parcel post package must not exceed 72 inches, find the dimensions of the largest cylindrical jug that can be sent by parcel post.

21. A circular filter paper of radius 6 inches is to be folded into a conical filter. Find the radius of the base of the filter if it has the maximum capacity.

22. Assuming that the intensity of light is inversely proportional to the square of the distance from the source, find the point on the line joining two sources, one of which is twice as intense as the other, at which the illumination is a minimum.

23. The sides of a trough of triangular section are planks 12 inches wide. Find the width at the top if the trough has the maximum capacity.

24. A fence 6 feet high runs parallel to and 5 feet from a wall. Find the shortest ladder that will reach from the ground over the fence to the wall.

25. A log has the form of a frustum of a cone 29 ft. long, the diameters of its ends being 2 ft. and 1 ft. A beam of square section is to be cut from the log. Find its length if the volume of the beam is a maximum.

26. A window has the form of a rectangle surmounted by a semicircle. If the perimeter is 30 ft., find the dimensions so that the greatest amount of light may be admitted.

27. A piece of wire 6 ft. long is to be cut into 6 pieces, two of one length and four of another. The two former are bent into circles which are held in parallel planes and fastened together by the four remaining pieces. The whole forms a model of a right cylinder. Calculate the lengths into which the wire must be divided to produce the cylinder of greatest volume.

28. Among all circular sectors with a given perimeter, find the one which has the greatest area.

29. A ship B is 75 miles due east of a ship A. If B sails west at 12 miles per hour and A south at 9 miles, find when the ships will be closest together.

30. A man on one side of a river $\frac{1}{2}$ mile wide wishes to reach a point on the opposite side 5 miles further along the bank. If he can walk 4 miles an hour and swim 2 miles an hour, find the route he should take to make the trip in the least time.

31. Find the length of the shortest line which will divide an equilateral triangle into parts of equal area.

32. A triangle is inscribed in an oval curve. If the area of the triangle is a maximum, show graphically that the tangents at the vertices of the triangle are parallel to the opposite sides.

33. A and C are points on the same side of a plane mirror. A ray of light passes from A to C by way of a point B on the mirror. Show that the length of the path ABC will be a minimum when the lines AB, CB make equal angles with the perpendicular to the mirror.

34. Let the velocity of light in air be v_1 and in water v_2. The path of a ray of light from a point A in the air to a point C below the surface of the water is bent at B where it enters the water. If θ_1 and θ_2 are the angles made by AB and BC with the perpendicular to the surface, show that the time required for light to pass from A to C will be least if B is so placed that

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{v_1}{v_2}.$$

35. The cost per hour of propelling a steamer is proportional to the cube of her speed through the water. Find the speed at which a boat should be run against a current of 5 miles per hour in order to make a given trip at least cost.

36. If the cost per hour for fuel required to run a steamer is proportional to the cube of her speed and is \$20 per hour for a speed of 10 knots, and if the other expenses amount to \$100 per hour, find the most economical speed in still water.

33. Other Types of Maxima and Minima. — The method given in Art. 31 is sufficient to determine maxima and minima if the function and its derivative are one-valued and continuous. In Figs. 33a and 33b are shown some types of maxima and minima that do not satisfy these conditions.

At B and C, Fig. 33a, the tangent is vertical and the derivative infinite. At D the slope on the left is different from that on the right. The derivative is discontinuous. At A and E the curve ends. This happens in problems where values beyond a certain range are impossible. According to

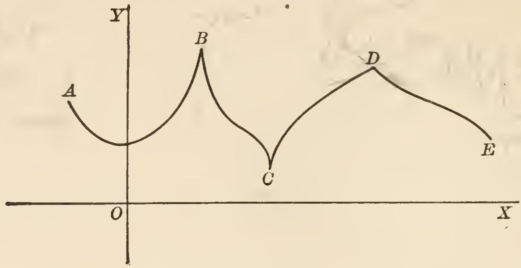


FIG. 33a.

our definition, y has maxima at A, B, D and minima at C and E.

If more than one value of the function corresponds to a single value of the variable, points like A and B, Fig. 33b, may occur. At such points two values of y coincide.

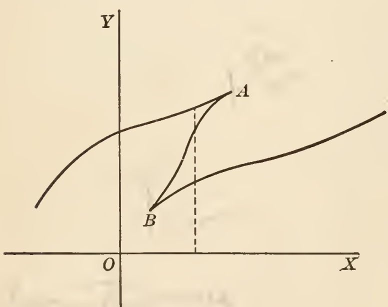


FIG. 33b.

These figures show that in determining maxima and minima special attention must be given to places where the derivative is discontinuous, the function ceases to exist, or two values of the function coincide.

Example 1. Find the maximum and minimum ordinates on the curve $y^3 = x^2$.

In this case, $y = x^{\frac{2}{3}}$ and

$$\frac{dy}{dx} = \frac{2}{3} x^{-\frac{1}{3}}.$$

No finite value of x makes the derivative zero, but $x = 0$

makes it infinite. Since y is never negative, the value 0 is a minimum (Fig. 33c).

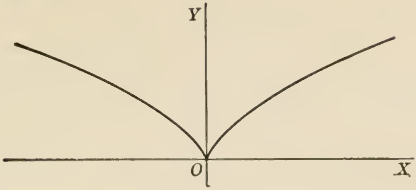
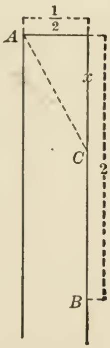


FIG. 33c.

Ex. 2. A man on one side of a river $\frac{1}{2}$ mile wide wishes to reach a point on the opposite side 2 miles down the river. If he can row 6 miles an hour and walk 4, find the route he should take to make the trip in the least time.



$2-x$

FIG. 33d.

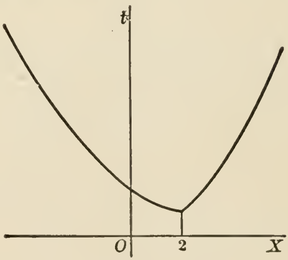


FIG. 33e.

Let A (Fig. 33d) be the starting point and B the destination. Suppose he rows to C, x miles down the river. The time of rowing will be $\frac{1}{6}\sqrt{x^2 + \frac{1}{4}}$ and the time of walking $\frac{1}{4}(2 - x)$. The total time is then

$$t = \tfrac{1}{6}\sqrt{x^2 + \tfrac{1}{4}} + \tfrac{1}{4}(2 - x).$$

Equating the derivative to zero, we get

$$\frac{x}{6\sqrt{x^2 + \frac{1}{4}}} - \frac{1}{4} = 0,$$

which reduces to $5x^2 + \frac{9}{4} = 0$. This has no real solution.

The trouble is that $\frac{1}{4}(2 - x)$ is the time of walking only if C is above B. If C is below B, the time is $\frac{1}{4}(x - 2)$. The complete value for t is then

$$t = \tfrac{1}{6}\sqrt{x^2 + \tfrac{1}{4}} \pm \tfrac{1}{4}(2 - x),$$

the sign being $+$ if $x < 2$ and $-$ if $x > 2$. The graph of the equation connecting x and t is shown in Fig. 33e. At $x = 2$ the derivative is discontinuous. Since he rows faster than he walks, the minimum obviously occurs when he rows all the way, that is, $x = 2$.

EXERCISES

Find the maximum and minimum values of y on the following curves:

1. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

2. $y^5 = x^2(x - 1)$.

3. $y^3 = x^4 - 1$.

4. $x = t^2 + t^3,\ y = t^2 - t^3$.

5. Find the rectangle of least area having a given perimeter.

6. Find the point on the parabola $y^2 = 4x$ nearest the point $(-1, 0)$.

7. A wire of length l is cut into two pieces, one of which is bent to form a circle, the other a square. Find the lengths of the pieces when the sum of the areas of the square and circle is greatest.

8. Find a point P on the line segment AB such that $PA^2 + PB^2$ is a maximum.

9. If the work per hour of moving a car along a horizontal track is proportional to the square of the velocity, what is the least work required to move the car one mile?

10. If 120 cells of electromotive force E volts and internal resistance 2 ohms are arranged in parallel rows with x cells in series in each row, the current which the resulting battery will send through an external resistance of $\frac{1}{3}$ ohm is

$$C = \frac{60\,xE}{x^2 + 20}.$$

How many cells should be placed in each row to give the maximum current?

CHAPTER VI

DIFFERENTIATION OF TRANSCENDENTAL FUNCTIONS

34. Formulas for Differentiating Trigonometric Functions. — Let u be the *circular measure* of an angle.

$$\text{VII.} \quad d \sin u = \cos u \, du.$$

$$\text{VIII.} \quad d \cos u = -\sin u \, du.$$

$$\text{IX.} \quad d \tan u = \sec^2 u \, du.$$

$$\text{X.} \quad d \cot u = -\csc^2 u \, du.$$

$$\text{XI.} \quad d \sec u = \sec u \tan u \, du.$$

$$\text{XII.} \quad d \csc u = -\csc u \cot u \, du.$$

The negative sign occurs in the differentials of all co-functions.

35. The Sine of a Small Angle. — Inspection of a table of natural sines will show that the sine of a small angle is very nearly equal to the circular measure of the angle. Thus

$$\sin 1° = 0.017452,$$

$$\frac{\pi}{180} = 0.017453.$$

We should then expect that

$$\lim_{\theta \doteq 0} \frac{\sin \theta}{\theta} = 1. \qquad (35)$$

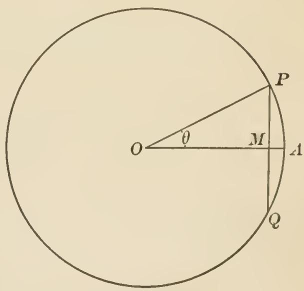


FIG. 35.

To show this graphically, let $\theta = AOP$ (Fig. 35). Draw PM perpendicular to OA. The circular measure of the angle is defined by the equation

$$\theta = \frac{\text{arc}}{\text{rad.}} = \frac{\text{arc } AP}{OP}.$$

Also $\sin\theta = \dfrac{MP}{OP}$. Hence

$$\frac{\sin\theta}{\theta} = \frac{MP}{\text{arc } AP} = \frac{\text{chord } QP}{\text{arc } QP}.$$

As θ approaches zero, the ratio of the arc to the chord approaches 1 (Art. 53). Therefore the limit of $\dfrac{\sin\theta}{\theta}$ is 1.

36. Proof of VII, the Differential of the Sine. — Let

$$y = \sin u.$$

Then

$$y + \Delta y = \sin(u + \Delta u)$$

and so

$$\Delta y = \sin(u + \Delta u) - \sin u.$$

It is shown in trigonometry that

$$\sin A - \sin B = 2\cos\tfrac{1}{2}(A + B)\sin\tfrac{1}{2}(A - B).$$

If then $A = u + \Delta u$, $B = u$,

$$\Delta y = 2\cos(u + \tfrac{1}{2}\Delta u)\sin\tfrac{1}{2}\Delta u,$$

whence

$$\frac{\Delta y}{\Delta u} = 2\cos(u + \tfrac{1}{2}\Delta u)\frac{\sin\frac{1}{2}\Delta u}{\Delta u} = \cos(u + \tfrac{1}{2}\Delta u)\frac{\sin\frac{1}{2}\Delta u}{\frac{1}{2}\Delta u}.$$

As Δu approaches zero

$$\frac{\sin\frac{1}{2}\Delta u}{\frac{1}{2}\Delta u} = \frac{\sin\theta}{\theta}$$

approaches 1 and $\cos(u + \frac{1}{2}\Delta u)$ approaches $\cos u$. Therefore

$$\frac{dy}{du} = \cos u.$$

Consequently,

$$dy = \cos u \, du.$$

37. Proof of VIII, the Differential of the Cosine. — By trigonometry

$$\cos u = \sin\left(\frac{\pi}{2} - u\right).$$

Using the formula just proved,

$$d \cos u = d \sin\left(\frac{\pi}{2} - u\right) = \cos\left(\frac{\pi}{2} - u\right) d\left(\frac{\pi}{2} - u\right) = -\sin u \, du.$$

38. Proof of IX, X, XI, and XII. — Differentiating both sides of the equation

$$\tan u = \frac{\sin u}{\cos u}$$

and using the formulas just proved for the differentials of $\sin u$ and $\cos u$,

$$d \tan u = \frac{\cos u \, d \sin u - \sin u \, d \cos u}{\cos^2 u} = \frac{\cos^2 u \, du + \sin^2 u \, du}{\cos^2 u}$$
$$= \sec^2 u \, du.$$

By differentiating both sides of the equations

$$\cot u = \frac{\cos u}{\sin u}, \qquad \sec u = \frac{1}{\cos u}, \qquad \csc u = \frac{1}{\sin u},$$

and using the formulas for the differentials of $\sin u$ and $\cos u$, we obtain the differentials of $\cot u$, $\sec u$ and $\csc u$.

Example 1. $y = \sin^2 (x^2 + 3)$.

Since

$$\sin^2 (x^2 + 3) = [\sin (x^2 + 3)]^2,$$

we use the formula for u^2 and so get

$$\begin{aligned} dy &= 2 \sin (x^2 + 3) \, d \sin (x^2 + 3) \\ &= 2 \sin (x^2 + 3) \cos (x^2 + 3) \, d (x^2 + 3) \\ &= 4 x \sin (x^2 + 3) \cos (x^2 + 3) \, dx. \end{aligned}$$

Ex. 2. $y = \sec 2x \tan 2x$.

$$\begin{aligned} \frac{dy}{dx} &= \sec 2x \frac{d}{dx} \tan 2x + \tan 2x \frac{d}{dx} \sec 2x \\ &= \sec 2x \sec^2 2x \, (2) + \tan 2x \sec 2x \tan 2x \, (2) \\ &= 2 \sec 2x (\sec^2 2x + \tan^2 2x). \end{aligned}$$

EXERCISES

In the following exercises show that the derivatives and differentials have the values given:

✓**1.** $y = 2 \sin 3x + 3 \cos 2x$, $\qquad \dfrac{dy}{dx} = 6 (\cos 3x - \sin 2x)$.

2. $y = \sin^2 \frac{x}{2}$, $\quad dy = \sin \frac{x}{2} \cos \frac{x}{2}\, dx.$

3. $y = 2 \cos x \sin 2x - \sin x \cos 2x$, $\quad dy = 3 \cos x \cos 2x\, dx.$

4. $y = \dfrac{1 - \cos \frac{1}{3} x}{\sin \frac{1}{3} x}$, $\quad \dfrac{dy}{dx} = \dfrac{1 - \cos \frac{1}{3} x}{3 \sin^2 \frac{1}{3} x}.$

5. $y = \tan 2x + \sec 2x$, $\quad \dfrac{dy}{dx} = 2 \sec 2x (\sec 2x + \tan 2x).$

6. $y = \cot^2 \frac{x}{2} \csc^2 \frac{x}{2}$, $\quad \dfrac{dy}{dx} = -\cot \frac{x}{2} \csc^2 \frac{x}{2} \left(\csc^2 \frac{x}{2} + \cot^2 \frac{x}{2} \right).$

7. $x = a \cos t,\ y = a \sin^3 t$, $\quad \dfrac{dy}{dx} = -3 \sin t \cos t.$

8. $x = a(\phi - \sin \phi),\ y = a(1 - \cos \phi)$, $\quad \dfrac{dy}{dx} = \cot \dfrac{\phi}{2}.$

9. $x = \cos t + t \sin t,\ y = \sin t - t \cos t$, $\quad \dfrac{d^2y}{dx^2} = \dfrac{1}{t \cos^3 t}.$

10. $y = \frac{1}{5} \cot^5 x - \frac{1}{3} \cot^3 x + \cot x + x$, $\quad dy = -\cot^6 x\, dx.$

11. $y = \frac{1}{5} \tan^5 x + \frac{2}{3} \tan^3 x + \tan x$, $\quad dy = \sec^6 x\, dx.$

12. $u = \frac{1}{7} \sec^7 \theta - \frac{2}{5} \sec^5 \theta + \frac{1}{3} \sec^3 \theta$, $\quad du = \tan^5 \theta \sec^3 \theta\, d\theta.$

13. $y = x\left(\dfrac{\cos^3 x}{3} - \cos x \right) + \dfrac{1}{9} \sin^3 x + \dfrac{2}{3} \sin x$, $\quad dy = x \sin^3 x\, dx.$

14. $y = -\dfrac{\cos x}{2}\left(\dfrac{1}{3} \sin^5 x + \dfrac{5}{12} \sin^3 x + \dfrac{5}{8} \sin x \right) + \dfrac{5}{16} x$, $\quad dy = \sin^6 x\, dx.$

15. $y = \dfrac{1 + \sin x}{1 - \sin x}$, $\quad \dfrac{dy}{dx} = \dfrac{2 \cos x}{(1 - \sin x)^2}.$

16. $y = \dfrac{\sec x - \tan x}{\sec x + \tan x}$, $\quad \dfrac{dy}{dx} = \dfrac{2 \sec x (\tan x - \sec x)}{\sec x + \tan x}.$

17. $y = (\cot x - 3 \tan x) \sqrt{\cot x}$, $\quad \dfrac{dy}{dx} = -\dfrac{3 \csc^4 x}{2 \cot^{\frac{3}{2}} x}.$

18. If $y = A \cos (nx) + B \sin (nx)$, where A and B are constant, show that

$$\frac{d^2y}{dx^2} + n^2 y = 0.$$

19. Find a constant A such that $y = A \sin 2x$ satisfies the equation

$$\frac{d^2y}{dx^2} + 5y = 3 \sin 2x.$$

20. Find constants A and B such that $y = A \sin 6x + B \cos 6x$ satisfies the equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 8y = 5 \sin 6x.$$

21. Find the slope of the curve $y = 2 \sin x + 3 \cos x$ at the point $x = \dfrac{\pi}{6}$.

22. Find the points on the curve $y = x + \sin 2x$ where the tangent is parallel to the line $y = 2x + 3$.

23. A weight supported by a spring hangs at rest at the origin. If the weight is lifted a distance A and let fall, its height at any subsequent time t will be

$$y = A \cos (2 \pi n t),$$

n being constant. Find its velocity and acceleration as it passes the origin. Where is the velocity greatest? Where is the acceleration greatest?

24. A revolving light 5 miles from a straight shore makes one complete revolution per minute. Find the velocity along the shore of the beam of light when it makes an angle of 60 degrees with the shore line.

25. In Ex. 24 with what velocity would the light be rotating if the spot of light is moving along the shore 15 miles per hour when the beam makes with the shore line an angle of 60 degrees?

26. Given that two sides and the included angle of a triangle have at a certain instant the values 6 ft., 10 ft., and 30 degrees respectively, and that these quantities are changing at the rates of 3 ft., -2 ft., and 10 degrees per second, how fast is the area of the triangle changing?

27. OA is a crank and AB a connecting rod attached to a piston B moving along a line through O. If OA revolves about O with angular velocity ω, prove that the velocity of B is ωOC, where C is the point in which the line BA cuts the line through O perpendicular to OB.

28. An alley 8 ft. wide runs perpendicular to a street 27 ft. wide. What is the longest beam that can be moved horizontally along the street into the alley?

29. A needle rests with one end in a smooth hemispherical bowl. The needle will sink to a position in which the center is as low as possible. If the length of the needle equals the diameter of the bowl, what will be the position of equilibrium?

30. A rope with a ring at one end is looped over two pegs in the same horizontal line and held taut by a weight fastened to the free end. If the rope slips freely, the weight will descend as far as possible. Find the angle formed at the bottom of the loop.

31. Find the angle at the bottom of the loop in Ex. 30 if the rope is looped over a circular pulley instead of the two pegs.

32. A gutter is to be made by bending into shape a strip of copper so that the cross section shall be an arc of a circle. If the width of the strip is a, find the radius of the cross section when the carrying capacity is a maximum.

33. A spoke in the front wheel of a bicycle is at a certain instant perpendicular to one in the rear wheel. If the bicycle rolls straight ahead, in what position will the outer ends of the two spokes be closest together?

39. Inverse Trigonometric Functions. — The symbol $\sin^{-1} x$ is used to represent the angle whose sine is x. Thus

$$y = \sin^{-1} x, \qquad x = \sin y$$

are equivalent equations. Similar definitions apply to $\cos^{-1} x$ $\tan^{-1} x$, $\cot^{-1} x$, $\sec^{-1} x$, and $\csc^{-1} x$.

Since supplementary angles and those differing by multiples of 2π have the same sine, an indefinite number of angles are represented by the same symbol $\sin^{-1} x$. The algebraic sign of the derivative depends on the angle differentiated. In the formulas given below it is assumed that $\sin^{-1} u$ and $\csc^{-1} u$ are angles in the first or fourth quadrant, $\cos^{-1} u$ and $\sec^{-1} u$ angles in the first or second quadrant. If angles in other quadrants are differentiated, the opposite sign must be used. The formulas for $\tan^{-1} u$ and $\cot^{-1} u$ are valid in all quadrants.

40. Formulas for Differentiating Inverse Trigonometric Functions. —

XIII. $$d \sin^{-1} u = \frac{du}{\sqrt{1-u^2}}.$$

XIV. $$d \cos^{-1} u = -\frac{du}{\sqrt{1-u^2}}.$$

XV. $$d \tan^{-1} u = \frac{du}{1+u^2}.$$

XVI. $$d \cot^{-1} u = -\frac{du}{1+u^2}.$$

XVII. $$d \sec^{-1} u = \frac{du}{u\sqrt{u^2-1}}.$$

XVIII. $$d \csc^{-1} u = -\frac{du}{u\sqrt{u^2-1}}.$$

41. Proof of the Formulas. — Let

$$y = \sin^{-1} u.$$

Then

$$\sin y = u.$$

Differentiation gives

$$\cos y \, dy = du,$$

whence
$$dy = \frac{du}{\cos y}.$$
But
$$\cos y = \pm\sqrt{1 - \sin^2 y} = \pm\sqrt{1 - u^2}.$$
If y is an angle in the first or fourth quadrant, $\cos y$ is positive. Hence
$$\cos y = \sqrt{1 - u^2}$$
and so
$$dy = \frac{du}{\sqrt{1 - u^2}}.$$

The other formulas are proved in a similar way.

EXERCISES

1. $y = \sin^{-1}(3x - 1)$, $\quad dy = \dfrac{3\,dx}{\sqrt{6x - 9x^2}}.$
2. $y = \cos^{-1}\left(1 - \dfrac{x}{a}\right)$, $\quad dy = \dfrac{dx}{\sqrt{2ax - x^2}}.$
3. $y = \tan^{-1}\dfrac{3x}{2}$, $\quad dy = \dfrac{6\,dx}{9x^2 + 4}.$
4. $y = \cot^{-1}\left(\dfrac{x}{2} - \dfrac{1}{2x}\right)$, $\quad \dfrac{dy}{dx} = \dfrac{-2}{x^2 + 1}.$
5. $y = \sec^{-1}\sqrt{4x + 1}$, $\quad dy = \dfrac{dx}{(4x + 1)\sqrt{x}}.$
6. $y = \dfrac{1}{2}\csc^{-1}\dfrac{3}{4x - 1}$, $\quad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{2 + 2x - 4x^2}}.$
7. $y = \tan^{-1}\dfrac{x - a}{x + a}$, $\quad \dfrac{dy}{dx} = \dfrac{a}{x^2 + a^2}.$
8. $x = \csc^{-1}(\sec\theta)$, $\quad \dfrac{dx}{d\theta} = -1.$
9. $y = \sin^{-1}\dfrac{x}{\sqrt{a^2 - x^2}}$, $\quad \dfrac{dy}{dx} = \dfrac{a^2}{(a^2 - x^2)\sqrt{a^2 - 2x^2}}.$
10. $y = \sec^{-1}\dfrac{1}{\sqrt{1 - x^2}}$, $\quad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1 - x^2}}.$
11. $y = \dfrac{x}{2}\sqrt{a^2 - x^2} + \dfrac{a^2}{2}\sin^{-1}\dfrac{x}{a}$, $\quad \dfrac{dy}{dx} = \sqrt{a^2 - x^2}.$
12. $y = \tan^{-1}\dfrac{4\sin x}{3 + 5\cos x}$, $\quad \dfrac{dy}{dx} = \dfrac{4}{5 + 3\cos x}.$
13. $y = \sec^{-1}\dfrac{1}{2x^2 - 1}$, $\quad \dfrac{dy}{dx} = -\dfrac{2}{\sqrt{1 - x^2}}.$

14. $y = a \sin^{-1}\frac{x}{a} + \sqrt{a^2 - x^2}$, $\qquad \frac{dy}{dx} = \sqrt{\frac{a-x}{a+x}}$.

15. $y = 2\,(3\,x+1)^{\frac{1}{6}} + 4\cot^{-1}\frac{(3\,x+1)^{\frac{1}{6}}}{2}$,

$$\frac{dy}{dx} = \frac{1}{(3\,x+1)^{\frac{5}{6}} + 4\,(3\,x+1)^{\frac{1}{2}}}.$$

16. $y = \frac{1}{6}\tan^{-1}\frac{3\,x}{2+2\,x^2}$, $\qquad \frac{dy}{dx} = \frac{1-x^2}{4\,x^4 + 17\,x^2 + 4}$.

17. $y = \cos^{-1}\frac{x+1}{2} - \frac{2}{\sqrt{3}}\cos^{-1}\frac{2\,x}{3-x}$, $\qquad \frac{dy}{dx} = \frac{x+1}{(3-x)\sqrt{3-2\,x-x^2}}$.

18. $y = \frac{\sqrt{x^2-a^2}}{2\,a^2x^2} - \frac{1}{2\,a^3}\csc^{-1}\frac{x}{a}$, $\qquad \frac{dy}{dx} = \frac{1}{x^3\sqrt{x^2-a^2}}$.

19. $y = \tan^{-1}\frac{x+\sqrt{x^2+4\,x-4}}{2}$, $\qquad \frac{dy}{dx} = \frac{1}{x\sqrt{x^2+4\,x-4}}$.

20. $y = x\sin^{-1}x + \sqrt{1-x^2}$, $\qquad \frac{d^2y}{dx^2} = \frac{1}{\sqrt{1-x^2}}$.

21. $y = x^2\sec^{-1}\frac{x}{2} - 2\sqrt{x^2-4}$, $\qquad \frac{dy}{dx} = 2\,x\sec^{-1}\frac{x}{2}$.

22. Let s be the arc from the x-axis to the point (x, y) on the circle $x^2 + y^2 = a^2$. Show that

$$\frac{ds}{dx} = -\frac{a}{y}, \quad \frac{ds}{dy} = \frac{a}{x}, \quad ds^2 = dx^2 + dy^2.$$

23. Let A be the area bounded by the circle $x^2 + y^2 = a^2$, the y-axis and the vertical line through (x, y). Show that

$$A = xy + a^2\tan^{-1}\frac{x}{y}, \quad dA = 2\,y\,dx.$$

24. The end of a string wound on a pulley moves with velocity v along a line perpendicular to the axis of the pulley. Find the angular velocity with which the pulley turns.

25. A tablet 8 ft. high is placed on a wall with its base 20 ft. above the level of an observer's eye. How far from the wall should the observer stand that the angle of vision subtended by the tablet be a maximum?

42. Exponential and Logarithmic Functions. — If a is a positive constant and u a variable, a^u is called an exponential function. If u is a fraction, it is understood that a^u is the positive root.

If $y = a^u$, then u is called the logarithm of y to base a. That is,

$$y = a^u, \qquad u = \log_a y \tag{42a}$$

are by definition equivalent equations. Elimination of u gives the important identity

$$a^{\log_a y} = y. \tag{42b}$$

This expresses symbolically that the logarithm is the power to which the base must be raised to equal the number.

43. The Curves $y = a^x$. — Let a be greater than 1. The graph of

$$y = a^x$$

has the general appearance of Fig. 43. If x receives a small increment h, the increment in y is

$$\Delta y = a^{x+h} - a^x = a^x (a^h - 1).$$

This increases as x increases. If then x increases by successive amounts h, the increments in y form steps of increasing height. The curve is thus concave upward and the arc lies below its chord.

The slope of the chord AP joining A (0, 1) and P (x, y) is

$$\frac{a^x - 1}{x}.$$

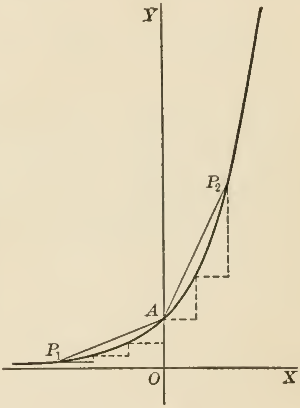


FIG. 43.

As P_1 moves toward A the slope of AP_1 increases. As P_2 moves toward A the slope of AP_2 decreases. Furthermore the slopes of AP_2 and AP_1 approach equality; for

$$\frac{a^{-k} - 1}{-k} = a^{-k}\left(\frac{a^k - 1}{k}\right),$$

and a^{-k} approaches 1 when k approaches zero. Now if two numbers, one always increasing, the other always decreasing, approach equality, they approach a common limit. Call this limit m. Then

$$\lim_{x \doteq 0} \frac{a^x - 1}{x} = m. \tag{43}$$

This is the slope of the curve $y = a^x$ at the point where it crosses the y-axis.

44. Definition of e. — We shall now show that there is a number e such that

$$\lim_{x \doteq 0} \frac{e^x - 1}{x} = 1. \tag{44}$$

In fact, let

$$e = a^{\frac{1}{m}}$$

where m is the slope found in Art. 43. Then

$$\lim_{x \doteq 0} \frac{e^x - 1}{x} = \lim_{x \doteq 0} \frac{a^{\frac{x}{m}} - 1}{x} = \frac{1}{m} \lim_{x \doteq 0} \frac{a^{\frac{x}{m}} - 1}{\frac{x}{m}} = \frac{1}{m} \cdot m = 1.$$

The curves $y = a^x$ all pass through the point A (0, 1). Equation (44) expresses that when $a = e$ the slope of the curve at A is 1. If $a > e$ the slope is greater than 1. If $a < e$, the slope is less than 1.

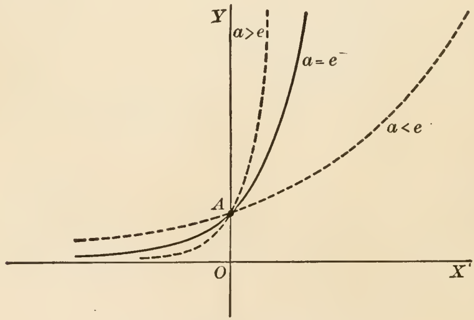


FIG. 44.

We shall find later that

$$e = 2.7183$$

approximately. Logarithms to base e are called *natural* logarithms. In this book we shall represent natural logarithms by the abbreviation ln. Thus ln u means the natural logarithm of u.

45. Differentials of Exponential and Logarithmic Functions. —

XIX. $$de^u = e^u\,du.$$

XX. $$da^u = a^u \ln a\,du.$$

XXI. $$d \ln u = \frac{du}{u}.$$

XXII. $$d \log_a u = \frac{\log_a e\,du}{u}.$$

46. Proof of XIX, the Differential of e^u. — Let

$$y = e^u.$$

Then

$$y + \Delta y = e^{u+\Delta u},$$

whence

$$\Delta y = e^{u+\Delta u} - e^u = e^u\,(e^{\Delta u} - 1)$$

and

$$\frac{\Delta y}{\Delta u} = e^u\,\frac{(e^{\Delta u} - 1)}{\Delta u}.$$

As Δu approaches zero, by (44),

$$\frac{e^{\Delta u} - 1}{\Delta u}$$

approaches 1. Consequently,

$$\frac{dy}{du} = e^u, \qquad dy = e^u\,du.$$

47. Proof of XX, the Differential of a^u. — The identity

$$a = e^{\ln a}$$

gives

$$a^u = e^{u \ln a}.$$

Consequently,

$$da^u = e^{u \ln a}\,d\,(u \ln a) = e^{u \ln a} \ln a\,du = a^u \ln a\,du.$$

48. Proof of XXI and XXII, the Differential of a Logarithm. — Let

$$y = \ln u.$$

Then

$$e^y = u.$$

Differentiating,

$$e^y\,dy = du.$$

Therefore

$$dy = \frac{du}{e^y} = \frac{du}{u}.$$

The derivative of $\log_a u$ is found in a similar way.

Example 1. $y = \ln(\sec^2 x)$.

$$dy = \frac{d \sec^2 x}{\sec^2 x} = \frac{2 \sec x \, (\sec x \tan x \, dx)}{\sec^2 x} = 2 \tan x \, dx.$$

Ex. 2. $y = 2^{\tan^{-1} x}$.

$$dy = 2^{\tan^{-1} x} \ln 2 \, d\,(\tan^{-1} x) = \frac{2^{\tan^{-1} x} \ln 2 \, dx}{1 + x^2}.$$

EXERCISES

1. $y = e^{\frac{1}{x}}$, $\qquad \dfrac{dy}{dx} = -\dfrac{1}{x^2} e^{\frac{1}{x}}$.

2. $y = a^{\tan 2x}$, $\qquad \dfrac{dy}{dx} = 2 a^{\tan 2x} \ln a \sec^2 2x$.

3. $y = e^{\frac{x-1}{x+1}}$, $\qquad \dfrac{dy}{dx} = \dfrac{2}{(x+1)^2} e^{\frac{x-1}{x+1}}$.

4. $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$, $\qquad \dfrac{dy}{dx} = \left(\dfrac{2}{e^x + e^{-x}}\right)^2$.

5. $y = x^n + n^x$, $\qquad \dfrac{dy}{dx} = nx^{n-1} + n^x \ln n$.

6. $y = a^x x^a$, $\qquad \dfrac{dy}{dx} = a^x x^{a-1} (a + x \ln a)$.

7. $y = \ln(3x^2 + 5x + 1)$, $\qquad \dfrac{dy}{dx} = \dfrac{6x + 5}{3x^2 + 5x + 1}$.

8. $y = \ln \sec^2 x$, $\qquad \dfrac{dy}{dx} = 2 \tan x$.

9. $y = \ln\left(x + \sqrt{x^2 - a^2}\right)$, $\qquad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{x^2 - a^2}}$.

10. $y = \ln(\sec ax + \tan ax)$, $\qquad \dfrac{dy}{dx} = a \sec ax$.

11. $y = \ln(a^x + b^x)$, $\qquad \dfrac{dy}{dx} = \dfrac{a^x \ln a + b^x \ln b}{a^x + b^x}$.

12. $y = \ln \sin x + \frac{1}{2} \cos^2 x$, $\qquad \dfrac{dy}{dx} = \dfrac{\cos^3 x}{\sin x}$.

13. $y = \dfrac{1}{2} \ln \tan \dfrac{x}{2} - \dfrac{1}{2} \dfrac{\cos x}{\sin^2 x}$, $\qquad \dfrac{dy}{dx} = \dfrac{1}{\sin^3 x}$.

14. $y = \frac{1}{4}\ln\frac{x^2}{x^2-4} - \frac{1}{x^2-4}$, $\quad \frac{dy}{dx} = \frac{8}{x(x^2-4)^2}$.

15. $y = \frac{1}{a}\ln\frac{x}{a+\sqrt{a^2-x^2}}$, $\quad \frac{dy}{dx} = \frac{1}{x\sqrt{a^2-x^2}}$.

16. $y = \ln(\sqrt{x+3}+\sqrt{x+2})+\sqrt{(x+3)(x+2)}$, $\quad \frac{dy}{dx} = \sqrt{\frac{x+3}{x+2}}$.

17. $y = \ln(\sqrt{x+a}+\sqrt{x})$, $\quad \frac{dy}{dx} = \frac{1}{2\sqrt{x^2+ax}}$.

18. $y = x\tan^{-1}\frac{x}{a} - \frac{a}{2}\ln(x^2+a^2)$, $\quad \frac{dy}{dx} = \tan^{-1}\frac{x}{a}$.

19. $y = e^{ax}(\sin ax - \cos ax)$, $\quad \frac{dy}{dx} = 2ae^{ax}\sin ax$.

20. $y = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln\cos x$, $\quad \frac{dy}{dx} = \tan^5 x$.

21. $x = a\ln t,\ y = \frac{a}{2}\left(t+\frac{1}{t}\right)$, $\quad \frac{dy}{dx} = \frac{1}{2}\left(t-\frac{1}{t}\right)$.

22. $x = e^t + e^{-t},\ y = e^t - e^{-t}$, $\quad \frac{d^2y}{dx^2} = -\frac{4}{y^3}$.

23. $y = \frac{1}{x}\ln x$, $\quad \frac{d^2y}{dx^2} = \frac{1}{x^3}(2\ln x - 3)$.

24. $y = xe^x$, $\quad \frac{d^ny}{dx^n} = (x+n)e^x$.

25. By taking logarithms of both sides of the equation $y = x^n$ and differentiating, show that the formula

$$\frac{d}{dx}x^n = nx^{n-1}$$

is true even when n is irrational.

26. Find the slope of the catenary

$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$$

at $x = 0$.

27. Find the points on the curve $y = e^{2x}\sin x$ where the tangent is parallel to the x-axis.

28. If $y = Ae^{nx} + Be^{-nx}$, where A and B are constant, show that

$$\frac{d^2y}{dx^2} - n^2y = 0.$$

29. If $y = ze^{-3x}$, where z is any function of x, show that

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = e^{-3x}\frac{d^2z}{dx^2}.$$

30. For what values of x does

$$y = 5\ln(x-2) + 3\ln(x+2) + 4x$$

increase as x increases?

31. From equation (44) show that

$$e = \lim_{x \doteq 0} (1 + x)^{\frac{1}{x}}.$$

32. If the space described by a point is $s = ae^t + be^{-t}$, show that the acceleration is equal to the space passed over.

33. Assuming the resistance encountered by a body sinking in water to be proportional to the velocity, the distance it descends in a time t is

$$s = \frac{g}{k} t + \frac{g}{k^2} (e^{-kt} - 1),$$

g and k being constants. Show that the velocity v and acceleration a satisfy the equation

$$a = g - kv.$$

Also show that for large values of t the velocity is approximately constant.

34. Assuming the resistance of air proportional to the square of the velocity, a body starting from rest will fall a distance

$$s = \frac{g}{k^2} \ln\left(\frac{e^{kt} + e^{-kt}}{2}\right)$$

in a time t. Show that the velocity and acceleration satisfy the equation

$$a = g - \frac{k^2v^2}{g}.$$

Also show that the velocity approaches a constant value.

CHAPTER VII

GEOMETRICAL APPLICATIONS

49. Tangent Line and Normal. — Let m_1 be the slope of a given curve at $P_1(x_1, y_1)$. It is shown in analytic geometry that a line through (x_1, y_1) with slope m_1 is represented by the equation

$$y - y_1 = m_1(x - x_1).$$

This equation then represents the tangent at (x_1, y_1) where the slope of the curve is m_1.

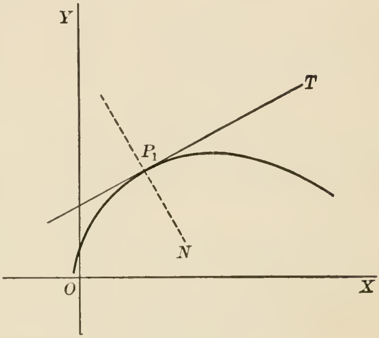


FIG. 49.

The line P_1N perpendicular to the tangent at its point of contact is called the *normal* to the curve at P_1. Since the slope of the tangent is m_1, the slope of a perpendicular line is $-\dfrac{1}{m_1}$ and so

$$y - y_1 = -\frac{1}{m_1}(x - x_1)$$

is the equation of the normal at (x_1, y_1).

Example 1. Find the equations of the tangent and normal to the ellipse $x^2 + 2y^2 = 9$ at the point $(1, 2)$.

The slope at any point of the curve is

$$\frac{dy}{dx} = -\frac{x}{2y}.$$

At $(1, 2)$ the slope is then

$$m_1 = -\tfrac{1}{4}.$$

The equation of the tangent is

$$y - 2 = -\tfrac{1}{4}(x - 1),$$

and the equation of the normal is

$$y - 2 = 4(x - 1).$$

Ex. 2. Find the equation of the tangent to $x^2 - y^2 = a^2$ at the point (x_1, y_1).

The slope at (x_1, y_1) is $\frac{x_1}{y_1}$. The equation of the tangent is then

$$y - y_1 = \frac{x_1}{y_1}(x - x_1)$$

which reduces to

$$x_1 x - y_1 y = x_1^2 - y_1^2.$$

Since (x_1, y_1) is on the curve, $x_1^2 - y_1^2 = a^2$. The equation of the tangent can therefore be reduced to the form

$$x_1 x - y_1 y = a^2.$$

50. Angle between Two Curves. — By the angle between two curves at a point of intersection we mean the angle between their tangents at that point.

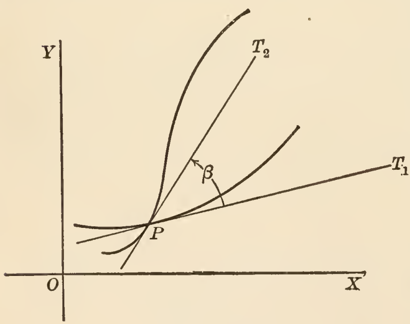


FIG. 50a.

Let m_1 and m_2 be the slopes of two curves at a point of intersection. It is shown in analytic geometry that the angle β from a line with slope m_1 to one with slope m_2 satisfies the equation

$$\tan \beta = \frac{m_2 - m_1}{1 + m_1 m_2}. \qquad (50)$$

This equation thus gives the angle β from a curve with slope m_1 to one with slope m_2, the angle being considered positive when measured in the counter-clockwise direction.

Example. Find the angles determined by the line $y = x$ and the parabola $y = x^2$.

Solving the equations simultaneously, we find that the line and parabola intersect at (1, 1) and (0, 0). The slope of the line is 1. The slope at any point of the parabola is

$$\frac{dy}{dx} = 2x.$$

At (1, 1) the slope of the parabola is then 2 and the angle from the line to the parabola is then given by

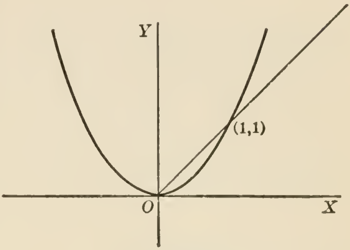


FIG. 50b.

$$\tan \beta_1 = \frac{2-1}{1+2} = \frac{1}{3},$$

whence

$$\beta_1 = \tan^{-1}\tfrac{1}{3} = 18° \, 26'.$$

At (0, 0) the slope of the parabola is 0 and so the angle from the line to the parabola is given by the equation

$$\tan \beta_2 = \frac{0-1}{1+0} = -1,$$

whence

$$\beta_2 = -45°.$$

The negative sign signifies that the angle is measured in the clockwise direction from the line to the parabola.

EXERCISES

Find the tangent and normal to each of the following curves at the point indicated:

1. The circle $x^2 + y^2 = 5$ at $(-1, 2)$.
2. The hyperbola $xy = 4$ at $(1, 4)$.
3. The parabola $y^2 = ax$ at $x = a$.
4. The exponential curve $y = ab^x$ at $x = 0$.
5. The sine curve $y = 3 \sin x$ at $x = \frac{\pi}{6}$.
6. The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, at (x_1, y_1).

7. The hyperbola $x^2 + xy - y^2 = 2x$, at $(2, 0)$.

8. The semicubical parabola $y^3 = x^2$, at $(-8, 4)$.

9. Find the equation of the normal to the cycloid

$$x = a(\phi - \sin\phi), \qquad y = a(1 - \cos\phi)$$

at the point $\phi = \phi_1$. Show that it passes through the point where the rolling circle touches the x-axis.

Find the angles at which the following pairs of curves intersect:

10. $y^2 = 4x, \quad x^2 = 4y.$

11. $x^2 + y^2 = 9, \; x^2 + y^2 - 6x = 9.$

12. $x^2 + y^2 + 2x = 7, \quad y^2 = 4x.$

13. $y = \sin x, \quad y = \cos x.$

14. $y = \log_{10} x, \quad y = \ln x.$

15. $y = \frac{1}{2}(e^x + e^{-x}), \quad y = 2e^x.$

16. Show that for all values of the constants a and b the curves

$$x^2 - y^2 = a^2, \qquad xy = b^2$$

intersect at right angles.

17. Show that the curves

$$y = e^{ax}, \qquad y = e^{ax}\sin(bx + c)$$

are tangent at each point of intersection.

18. Show that the part of the tangent to the hyperbola $xy = a^2$ intercepted between the coördinate axes is bisected at the point of tangency.

19. Let the normal to the parabola $y^2 = ax$ at P cut the x-axis at N. Show that the projection of PN on the x-axis has a constant length.

20. The focus F of the parabola $y^2 = ax$ is the point $(\frac{1}{4}a, 0)$. Show that the tangent at any point P of the parabola makes equal angles with FP and the line through P parallel to the axis.

21. The foci of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad a > b$$

are the points $F'\left(-\sqrt{a^2 - b^2}, 0\right)$ and $F\left(\sqrt{a^2 - b^2}, 0\right)$. Show that the tangent at any point P of the ellipse makes equal angles with FP and $F'P$.

22. Let P be any point on the catenary $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$, M the projection of P on the x-axis, and N the projection of M on the tangent at P. Show that MN is constant in length.

23. Show that the portion of the tangent to the tractrix

$$y = \frac{a}{2}\ln\left(\frac{a + \sqrt{a^2 - x^2}}{a - \sqrt{a^2 - x^2}}\right) - \sqrt{a^2 - x^2}$$

intercepted between the y-axis and the point of tangency is constant in length.

24. Show that the angle between the tangent at any point P and the line joining P to the origin is the same at all points of the curve

$$\ln \sqrt{x^2 + y^2} = k \tan^{-1} \frac{y}{x}.$$

25. A point at a constant distance along the normal from a given curve generates a curve which is called parallel to the first. Find the parametric equations of the parallel curve generated by the point at distance h along the normal drawn inside of the ellipse

$$x = a \cos \phi, \qquad y = b \sin \phi.$$

51. Direction of Curvature. — A curve is said to be *concave upward* at a point P if the part of the curve near P lies above the tangent at P. It is *concave downward* at Q if the part near Q lies below the tangent at Q.

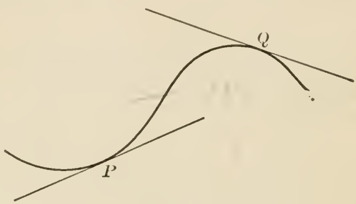


FIG. 51.

At points where $\dfrac{d^2y}{dx^2}$ *is positive, the curve is concave upward; where* $\dfrac{d^2y}{dx^2}$ *is negative, the curve is concave downward.*

For

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right).$$

If then $\dfrac{d^2y}{dx^2}$ is positive, by Art. 13, $\dfrac{dy}{dx}$, the slope, increases as x increases and decreases as x decreases. The curve therefore rises above the tangent on both sides of the point. If, however, $\dfrac{d^2y}{dx^2}$ is negative, the slope decreases as x increases and increases as x decreases, and so the curve falls below the tangent.

52. Point of Inflection. — A point like A (Fig. 52a), on one side of which the curve is concave upward, on the other concave downward, is called a *point of inflection.* It is assumed that there is a definite tangent at the point of inflection. A point like B is not called a point of inflection.

The second derivative is positive on one side of a point of inflection and negative on the other. Ordinary functions change sign only by passing through zero or infinity. Hence to find points of inflection we find where $\frac{d^2y}{dx^2}$ is zero or infinite.

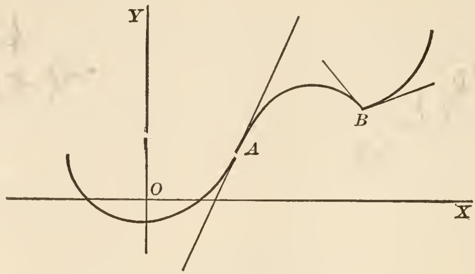


FIG. 52a.

If the second derivative changes sign at such a point, it is a point of inflection. If the second derivative has the same sign on both sides, it is not a point of inflection.

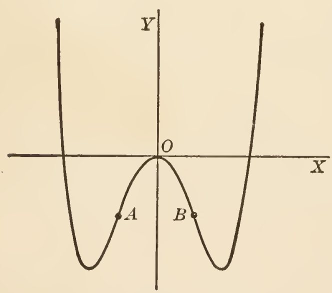


FIG. 52b.

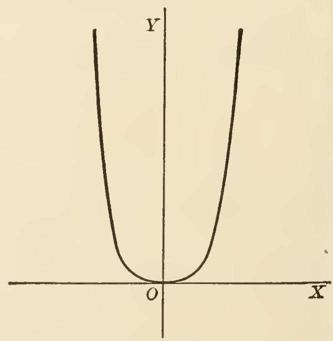


FIG. 52c.

Example 1. Examine the curve $3y = x^4 - 6x^2$ for direction of curvature and points of inflection.

The second derivative is

$$\frac{d^2y}{dx^2} = 4(x^2 - 1).$$

This is zero at $x = \pm 1$. It is positive and the curve concave upward on the left of $x = -1$ and on the right of $x = +1$. It is negative and the curve concave downward between $x = -1$ and $x = +1$. The second derivative changes sign at $A\,(-1, -\frac{5}{3})$ and $B\,(+1, -\frac{5}{3})$, which are therefore points of inflection (Fig. 52b).

Ex. 2. Examine the curve $y = x^4$ for points of inflection. In this case the second derivative is

$$\frac{d^2y}{dx^2} = 12\,x^2.$$

This is zero when x is zero but is positive for all other values of x. The second derivative does not change sign and there is consequently no point of inflection (Fig. 52c).

Ex. 3. If $x > 0$, show that $\sin x > x - \frac{x^3}{3!}$.*

Let

$$y = \sin x - x + \frac{x^3}{3!}.$$

We are to show that $y > 0$. Differentiation gives

$$\frac{dy}{dx} = \cos x - 1 + \frac{x^2}{2!}, \quad \frac{d^2y}{dx^2} = -\sin x + x.$$

When x is positive, $\sin x$ is less than x and so $\frac{d^2y}{dx^2}$ is positive. Therefore $\frac{dy}{dx}$ increases with x. Since $\frac{dy}{dx}$ is zero when x is zero, $\frac{dy}{dx}$ is then positive when $x > 0$, and so y increases with x. Since $y = 0$ when $x = 0$, y is therefore positive when $x > 0$, which was to be proved.

* If n is any positive integer $n!$ represents the product of the integers from 1 to n. Thus

$$3! = 1 \cdot 2 \cdot 3 = 6.$$

EXERCISES

Examine the following curves for direction of curvature and points of inflection:

1. $y = x^3 - 3x + 3.$

2. $y = 2x^3 - 3x^2 - 6x + 1.$

3. $y = x^4 - 4x^3 + 6x^2 + 12x.$

4. $y^3 = x - 1.$

5. $y = xe^x.$

6. $y = e^{-x^2}.$

7. $x^2y - 4x + 3y = 0.$

8. $x = \sin t, \quad y = \frac{1}{3}\sin 3t.$

Prove the following inequalities:

9. $x \ln x - x - \dfrac{x^2}{2} + \dfrac{3}{2} > 0, \quad$ if $0 < x < 1.$

10. $(x - 1)e^x + 1 > 0, \quad$ if $x > 0.$

11. $e^x < 1 + x + \dfrac{x^2}{2}e^a, \quad$ if $0 < x < a.$

12. $\ln \sec x > \dfrac{x^2}{2}, \quad$ if $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}.$

13. According to Van der Waal's equation, the pressure p and volume v of a gas at constant temperature T are connected by the equation

$$p = \frac{RT}{m(v - b)} - \frac{a}{v^2},$$

a, b, m, and R being constants. If p is taken as ordinate and v as abscissa, the curve represented by this equation has a point of inflection. The value of T for which the tangent at the point of inflection is horizontal is called the critical temperature. Show that the critical temperature is

$$T = \frac{8\,am}{27\,Rb}.$$

53. Length of a Curve. — The length of an arc PQ of a curve is defined as the limit (if there is a limit) approached by the length of a broken line with vertices on PQ as the number of its sides increases indefinitely, their lengths approaching zero.

We shall now show that if the slope of a curve is continuous the ratio of a chord to the arc it subtends approaches 1 as the chord approaches zero.

In the arc PQ (Fig. 53) inscribe a broken line $PABQ$. Projecting on PQ, we get

$$PQ = \text{proj. } PA + \text{proj. } AB + \text{proj. } BQ.$$

The projection of a chord, such as AB, is equal to the product of its length by the cosine of the angle it makes with PQ. On the arc AB is a tangent RS parallel to AB. Let α be the largest angle that any tangent on the arc PQ makes with the

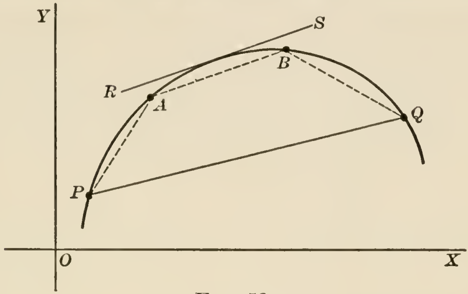


FIG. 53.

chord PQ. The angle between RS and PQ is not greater than α. Consequently, the angle between AB and PQ is not greater than α. Therefore

$$\text{proj. } AB \geqq AB \cos \alpha.$$

Similarly,

$$\text{proj. } PA \geqq PA \cos \alpha,$$
$$\text{proj. } BQ \geqq BQ \cos \alpha.$$

Adding these equations, we get

$$PQ \geqq (PA + AB + BQ) \cos \alpha.$$

It is evident that this result can be extended to a broken line with any number of sides. As the number of sides increases indefinitely, the expression in parenthesis approaches the length of the arc PQ. Therefore

$$PQ \geqq \text{arc } PQ \cos \alpha,$$

that is,

$$\frac{\text{chord } PQ}{\text{arc } PQ} \geqq \cos \alpha.$$

If the slope of the curve is continuous, the angle α approaches zero as Q approaches P. Hence $\cos \alpha$ approaches 1 and

$$\lim_{Q \doteq P} \frac{\text{chord } PQ}{\text{arc } PQ} \geqq 1.$$

Since the chord is always less than the arc, the limit cannot be greater than 1. Therefore, finally,

$$\lim_{Q \doteq P} \frac{\text{chord } PQ}{\text{arc } PQ} = 1. \tag{53}$$

54. Differential of Arc. — Let s be the distance measured along a curve from a fixed point A to a variable point P. Then s is a function of the coördinates of P. Let ϕ be the angle from the positive direction of the x-axis to the tangent PT drawn in the direction of increasing s.

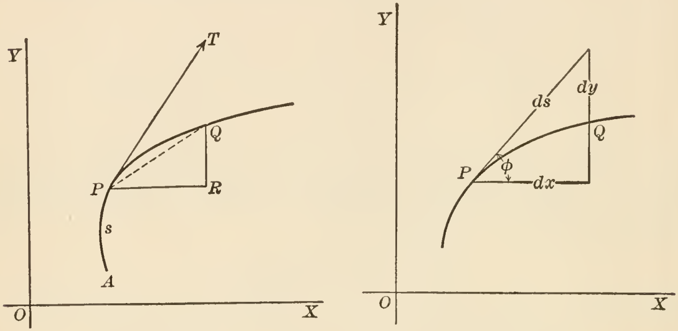


FIG. 54a. FIG. 54b.

If P moves to a neighboring position Q, the increments in x, y, and s are

$$\Delta x = PR, \qquad \Delta y = RQ, \qquad \Delta s = \text{arc } PQ.$$

From the figure it is seen that

$$\cos(RPQ) = \frac{\Delta x}{PQ} = \frac{\Delta x}{\Delta s}\frac{\Delta s}{PQ},$$

$$\sin(RPQ) = \frac{\Delta y}{PQ} = \frac{\Delta y}{\Delta s}\frac{\Delta s}{PQ}.$$

As Q approaches P, RPQ approaches ϕ and

$$\frac{\Delta s}{PQ} = \frac{\text{arc } PQ}{\text{chord } PQ}$$

approaches 1. The above equations then give in the limit

$$\cos\phi = \frac{dx}{ds}, \qquad \sin\phi = \frac{dy}{ds}. \tag{54a}$$

These equations express that dx and dy are the sides of a right triangle with hypotenuse ds extending along the tangent (Fig. 54b). All the equations connecting dx, dy, ds, and ϕ can be read off this triangle. One of particular importance is

$$ds^2 = dx^2 + dy^2. \tag{54b}$$

55. Curvature. — If an arc is everywhere concave toward its chord, the amount it is bent can be measured by the angle β between the tangents at its ends. The ratio

$$\frac{\beta}{\text{arc } PP'} = \frac{\phi' - \phi}{\Delta s} = \frac{\Delta\phi}{\Delta s}$$

is the average bending per unit length along PP'. The limit as P' approaches P,

$$\lim_{\Delta s \doteq 0} \frac{\Delta\phi}{\Delta s} = \frac{d\phi}{ds},$$

is called the *curvature* at P. It is greater where the curve bends more sharply, less where it is more nearly straight.

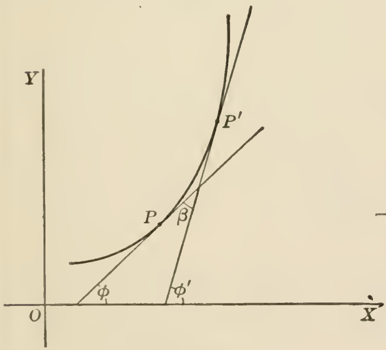


FIG. 55a.

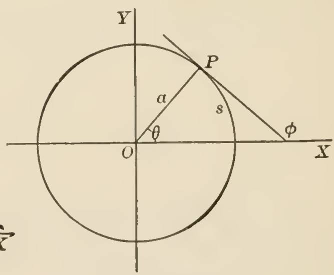


FIG. 55b.

In case of a circle (Fig. 55b)

$$\phi = \theta + \frac{\pi}{2}, \qquad s = a\theta.$$

Consequently,

$$\frac{d\phi}{ds} = \frac{d\theta}{a\,d\theta} = \frac{1}{a},$$

that is, *the curvature of a circle is constant and equal to the reciprocal of its radius.*

56. Radius of Curvature. — We have just seen that the radius of a circle is the reciprocal of its curvature. The *radius of curvature* of any curve is defined as the reciprocal of its curvature, that is,

$$\text{radius of curvature} = \rho = \frac{ds}{d\phi}. \tag{56a}$$

It is the radius of the circle which has the same curvature as the given curve at the given point.

To express ρ in terms of x and y we note that

$$\phi = \tan^{-1}\frac{dy}{dx}.$$

Consequently,

$$d\phi = \frac{1}{1+\left(\dfrac{dy}{dx}\right)^2}\, d\left(\frac{dy}{dx}\right) = \frac{\dfrac{d^2y}{dx^2}\,dx}{1+\left(\dfrac{dy}{dx}\right)^2}.$$

Also $$ds = \sqrt{dx^2 + dy^2}.$$

Substituting these values for ds and $d\phi$, we get

$$\rho = \frac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}. \tag{56b}$$

If the radical in the numerator is taken positive, ρ will have the same sign as $\dfrac{d^2y}{dx^2}$, that is, the radius will be positive when the curve is concave upward. If merely the numerical value is wanted, the sign can be omitted.

By a similar proof we could show that

$$\rho = \frac{\left[1+\left(\dfrac{dx}{dy}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2x}{dy^2}}. \tag{56c}$$

Example 1. Find the radius of curvature of the parabola $y^2 = 4x$ at the point (4, 4).

At the point (4, 4) the derivatives have the values

$$\frac{dy}{dx} = \frac{2}{y} = \frac{1}{2}, \qquad \frac{d^2y}{dx^2} = -\frac{4}{y^3} = -\frac{1}{16}.$$

Therefore

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left(1 + \frac{1}{4}\right)^{\frac{3}{2}}}{-\frac{1}{16}} = -10\sqrt{5}.$$

The negative sign shows that the curve is concave downward. The length of the radius is $10\sqrt{5}$.

Ex. 2. Find the radius of curvature of the curve represented by the polar equation $r = a\cos\theta$.

The expressions for x and y in terms of θ are

$$x = r\cos\theta = a\cos\theta\cos\theta = a\cos^2\theta,$$
$$y = r\sin\theta = a\cos\theta\sin\theta.$$

Consequently,

$$\frac{dy}{dx} = \frac{a(\cos^2\theta - \sin^2\theta)}{-2a\cos\theta\sin\theta} = \frac{a\cos 2\theta}{-a\sin 2\theta} = -\cot 2\theta,$$

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = -\frac{2\csc^2 2\theta\, d\theta}{a\sin 2\theta\, d\theta} = -\frac{2}{a}\csc^3 2\theta.$$

$$\rho = \frac{[1 + \cot^2 2\theta]^{\frac{3}{2}}}{-\frac{2}{a}\csc^3 2\theta} = -a\frac{(\csc^2 2\theta)^{\frac{3}{2}}}{2\csc^3 2\theta} = -\frac{a}{2}.$$

The radius is thus constant. The curve is in fact a circle.

57. Center and Circle of Curvature. — At each point of a curve is a circle on the concave side tangent at the point with radius equal to the radius of curvature. This circle is called the *circle of curvature*. Its center is called the *center of curvature*.

Since the circle and curve are tangent at P, they have the same slope $\frac{dy}{dx}$ at P. Since they have the same radius of curvature, the second derivatives will also be equal at P.

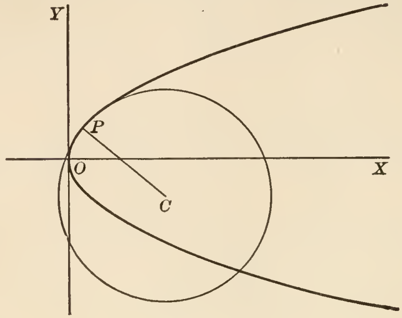


FIG. 57.

The circle of curvature is thus the circle through P such that $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ have the same values for the circle as for the curve at P.

EXERCISES

1. The length of arc measured from a fixed point on a certain curve is $s = x^2 + x$. Find the slope of the curve at $x = 2$.

2. Can $x = \cos s$, $y = \sin s$, represent a curve on which s is the length of arc measured from a fixed point? Can $x = \sec s$, $y = \tan s$, represent such a curve?

Find the radius of curvature on each of the following curves at the point indicated:

3. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, at $(0, b)$.

4. $x^2 + xy + y^2 = 3$, at $(1, 1)$.

5. $r = e^\theta$, at $\theta = \frac{\pi}{2}$.

6. $r = a(1 + \cos\theta)$, at $\theta = 0$.

Find an expression for the radius of curvature at any point of each of the following curves:

7. $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$.

8. $x = \ln \sec y$.

9. $x = \frac{1}{4}y^2 - \frac{1}{2}\ln y$.

10. $r = a\sec^2 \frac{1}{2}\theta$.

11. Show that the radius of curvature at a point of inflection is infinite.

12. A point on the circumference of a circle rolling along the x-axis generates the cycloid

$$x = a(\phi - \sin\phi), \qquad y = a(1 - \cos\phi),$$

a being the radius of the rolling circle and ϕ the angle through which it has turned. Show that the radius of the circle of curvature is bisected by the point where the rolling circle touches the x-axis.

13. A string held taut is unwound from a fixed circle. The end of the string generates a curve with parametric equations

$$x = a\cos\theta + a\theta\sin\theta, \qquad y = a\sin\theta - a\theta\cos\theta,$$

a being the radius of the circle and θ the angle subtended at the center by the arc unwound. Show that the center of curvature corresponding to any point of this path is the point where the string is tangent to the circle.

14. Show that the radius of curvature at any point (x, y) of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is three times the perpendicular from the origin to the tangent at (x, y).

58. Limit of $\dfrac{1-\cos x}{x}$. It is shown in trigonometry that

$$1 - \cos x = 2\sin^2\frac{x}{2}.$$

Consequently,

$$\frac{1-\cos x}{x} = \frac{2\sin^2\frac{x}{2}}{x} = \sin\frac{x}{2}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right).$$

As x approaches zero, $\dfrac{\sin\frac{x}{2}}{\frac{x}{2}}$ approaches 1. Therefore

$$\lim_{x \doteq 0}\frac{1-\cos x}{x} = 0 \cdot 1 = 0.$$

59. Derivatives of Arc in Polar Coördinates. — The angle from the outward drawn radius to the tangent drawn in the direction of increasing s is usually represented by the letter ψ.

Let r, θ be the polar coördinates of P, and $r + \Delta r$, $\theta + \Delta\theta$ those of Q (Fig. 59a). Draw QR perpendicular to PR and let Δs = arc PQ. Then

$$\sin(RPQ) = \frac{RQ}{PQ} = \frac{(r+\Delta r)\sin\Delta\theta}{PQ} = (r+\Delta r)\frac{\sin\Delta\theta}{\Delta\theta}\cdot\frac{\Delta\theta}{\Delta s}\cdot\frac{\Delta s}{PQ}.$$

$$\cos(RPQ) = \frac{PR}{PQ} = \frac{(r+\Delta r)\cos\Delta\theta - r}{PQ}$$

$$= \cos(\Delta\theta)\frac{\Delta r}{PQ} - \frac{r(1-\cos\Delta\theta)}{PQ}$$

$$= \cos(\Delta\theta)\frac{\Delta r}{\Delta s}\cdot\frac{\Delta s}{PQ} - \frac{r(1-\cos\Delta\theta)}{\Delta\theta}\frac{\Delta\theta}{\Delta s}\cdot\frac{\Delta s}{PQ}.$$

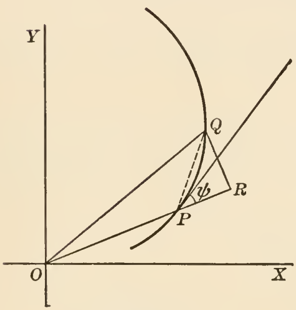


FIG. 59a.

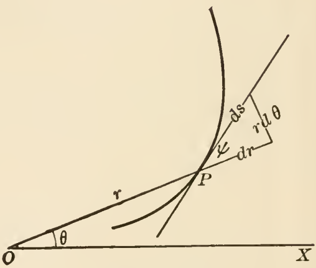


FIG. 59b.

As $\Delta\theta$ approaches zero,

$$\lim(RPQ) = \psi,\quad \lim\frac{\sin\Delta\theta}{\Delta\theta} = 1,\quad \lim\frac{1-\cos\Delta\theta}{\Delta\theta} = 0,\quad \lim\frac{\Delta s}{PQ} = 1.$$

The above equations then give in the limit,

$$\sin\psi = \frac{r d\theta}{ds},\qquad \cos\psi = \frac{dr}{ds}. \tag{59a}$$

These equations show that dr and $r\,d\theta$ are the sides of a right triangle with hypotenuse ds and base angle ψ. From this triangle all the equations connecting dr, $d\theta$, ds, and ψ can be obtained. The most important of these are

$$\tan\psi = \frac{r\,d\theta}{dr},\qquad ds^2 = dr^2 + r^2\,d\theta^2. \tag{59b}$$

Example. The logarithmic spiral $r = ae^\theta$.

In this case, $dr = ae^\theta \, d\theta$ and so

$$\tan \psi = \frac{r \, d\theta}{dr} = 1.$$

The angle ψ is therefore constant and equal to 45 degrees. The equation

$$\cos \psi = \frac{dr}{ds} = \frac{1}{\sqrt{2}}$$

shows that $\frac{dr}{ds}$ is also constant and so r and s increase proportionally.

EXERCISES

Find the angle ψ at the point indicated on each of the following curves:

1. The spiral $r = a\theta$, at $\theta = \frac{\pi}{3}$.

2. The circle $r = a \sin \theta$ at $\theta = \frac{\pi}{4}$.

3. The straight line $r = a \sec \theta$, at $\theta = \frac{\pi}{6}$.

4. The ellipse $r\,(2 - \cos \theta) = k$, at $\theta = \frac{\pi}{2}$.

5. The lemniscate $r^2 = 2\,a^2 \cos 2\,\theta$, at $\theta = \frac{5}{6}\,\pi$.

6. Show that the curves $r = ae^\theta$, $r = ae^{-\theta}$ are perpendicular at each of their points of intersection.

7. Find the angles at which the curves $r = a \cos \theta$, $r = a \sin 2\,\theta$ intersect.

8. Find the points on the cardioid $r = a\,(1 - \cos \theta)$ where the tangent is parallel to the initial line.

9. Let $P\,(r, \theta)$ be a point on the hyperbola $r^2 \sin 2\,\theta = c$. Show that the triangle formed by the radius OP, the tangent at P, and the x-axis is isosceles.

10. Find the slope of the curve $r = e^{2\theta}$ at the point where $\theta = \frac{\pi}{4}$.

60. Angle between Two Directed Lines in Space. — A directed line is one along which a positive direction is assigned. This direction is usually indicated by an arrow.

An angle between two directed lines is one along the sides of which the arrows point away from the vertex. There are two such angles less than 360 degrees, their sum being 360 degrees (Fig. 60). They have the same cosine.

If the lines do not intersect, the angle between them is defined as that between intersecting lines respectively parallel to the given lines.

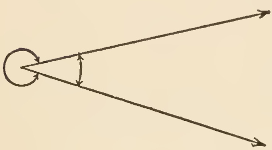


FIG. 60.

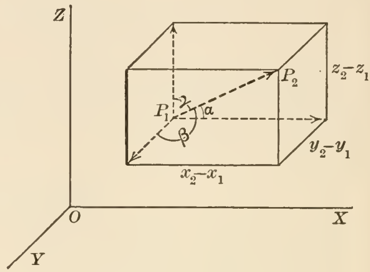


FIG. 61.

61. Direction Cosines. — It is shown in analytic geometry* that the angles α, β, γ between the coördinate axes and the line P_1P_2 (directed from P_1 to P_2) satisfy the equations

$$\cos\alpha = \frac{x_2 - x_1}{P_1P_2}, \quad \cos\beta = \frac{y_2 - y_1}{P_1P_2}, \quad \cos\gamma = \frac{z_2 - z_1}{P_1P_2}. \qquad (61a)$$

These cosines are called the *direction cosines* of the line. They satisfy the identity

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1. \qquad (61b)$$

If the direction cosines of two lines are $\cos\alpha_1$, $\cos\beta_1$, $\cos\gamma_1$ and $\cos\alpha_2$, $\cos\beta_2$, $\cos\gamma_2$, the angle θ between the lines is given by the equation

$$\cos\theta = \cos\alpha_1\cos\alpha_2 + \cos\beta_2\cos\beta_2 + \cos\gamma_1\cos\gamma_2. \qquad (61c)$$

In particular, if the lines are perpendicular, the angle θ is 90 degrees and

$$0 = \cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2. \qquad (61d)$$

* Cf. H. B. Phillips, *Analytic Geometry*, Art. 64, et seq.

62. Direction of the Tangent Line to a Curve. — The tangent line at a point P of a curve is defined as the limiting position PT approached by the secant PQ as Q approaches P along the curve.

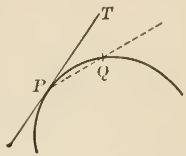


FIG. 62a.

Let s be the arc of the curve measured from some fixed point and $\cos\alpha$, $\cos\beta$, $\cos\gamma$ the direction cosines of the tangent drawn in the direction of increasing s.

If x, y, z are the coördinates of P, $x + \Delta x$, $y + \Delta y$, $z + \Delta z$, those of Q, the direction cosines of PQ are

$$\frac{\Delta x}{PQ}, \quad \frac{\Delta y}{PQ}, \quad \frac{\Delta z}{PQ}.$$

As Q approaches P, these approach the direction cosines of the tangent at P. Hence

$$\cos\alpha = \lim_{Q \doteq P} \frac{\Delta x}{PQ} = \lim \frac{\Delta x}{\Delta s}\cdot\frac{\Delta s}{PQ}.$$

On the curve, x, y, z are functions of s. Hence

$$\lim \frac{\Delta x}{\Delta s} = \frac{dx}{ds}, \qquad \lim \frac{\Delta s}{PQ} = \lim \frac{\text{arc}}{\text{chord}} = 1.*$$

Therefore

$$\cos\alpha = \frac{dx}{ds}. \tag{62a}$$

Similarly,

$$\cos\beta = \frac{dy}{ds}, \qquad \cos\gamma = \frac{dz}{ds}. \tag{62a}$$

These equations show that if a distance ds is measured along the tangent, dx, dy, dz are its projections on the coördinate axes (Fig. 62b). Since the square on the diagonal of a

* The proof that the limit of arc/chord is 1 was given in Art. 53 for the case of plane curves with continuous slope. A similar proof can be given for any curve, plane or space, that is continuous in direction.

rectangular parallelopiped is equal to the sum of the squares of its three edges,

$$ds^2 = dx^2 + dy^2 + dz^2. \tag{62b}$$

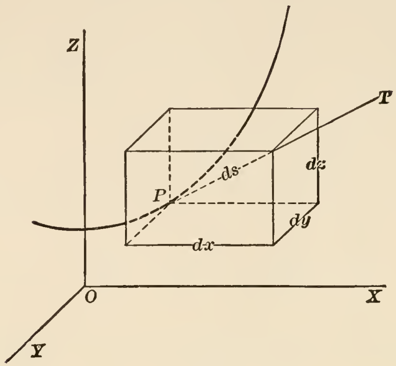


FIG. 62b.

Example. Find the direction cosines of the tangent to the parabola

$$x = at, \qquad y = bt, \qquad z = \tfrac{1}{4}\,ct^2$$

at the point where $t = 2$.

At $t = 2$ the differentials are

$$dx = a\,dt, \qquad dy = b\,dt, \qquad dz = \tfrac{1}{2}\,ct\,dt = c\,dt,$$

$$ds = \pm\sqrt{dx^2 + dy^2 + dz^2} = \pm\sqrt{a^2 + b^2 + c^2}\,dt.$$

There are two algebraic signs depending on the direction s is measured along the curve. If we take the positive sign, the direction cosines are

$$\frac{dx}{ds} = \frac{a}{\sqrt{a^2 + b^2 + c^2}}, \quad \frac{dy}{ds} = \frac{b}{\sqrt{a^2 + b^2 + c^2}},$$

$$\frac{dz}{ds} = \frac{c}{\sqrt{a^2 + b^2 + c^2}}.$$

63. Equations of the Tangent Line. — It is shown in analytic geometry that the equations of a straight line through a point $P_1\ (x_1,\ y_1,\ z_1)$ with direction cosines proportional to A, B, C are

$$\frac{x - x_1}{A} = \frac{y - y_1}{B} = \frac{z - z_1}{C}. \tag{63}$$

The direction cosines of the tangent line are proportional to dx, dy, dz. If then we replace A, B, C by numbers proportional to the values of dx, dy, dz at P_1, (63) will represent the tangent line at P_1.

Example 1. Find the equations of the tangent to the curve

$$x = t, \qquad y = t^2, \qquad z = t^3$$

at the point where $t = 1$.

The point of tangency is $t = 1$, $x_1 = 1$, $y_1 = 1$, $z_1 = 1$. At this point the differentials are

$$dx : dy : dz = dt : 2\,t\,dt : 3\,t^2\,dt = 1 : 2 : 3.$$

The equations of the tangent line are then

$$\frac{x - 1}{1} = \frac{y - 1}{2} = \frac{z - 1}{3}.$$

Ex. 2. Find the angle between the curve $3\,x + 2\,y - 2\,z = 3$, $4\,x^2 + y^2 = 2\,z^2$ and the line joining the origin to (1, 2, 2).

The curve and line intersect at (1, 2, 2). Along the curve y and z can be considered functions of x. The differentials satisfy the equations

$$3\,dx + 2\,dy - 2\,dz = 0, \quad 8\,x\,dx + 2\,y\,dy = 4\,z\,dz.$$

At the point of intersection these equations become

$$3\,dx + 2\,dy - 2\,dz = 0, \quad 8\,dx + 4\,dy = 8\,dz.$$

Solving for dx and dy in terms of dz, we get

$$dx = 2\,dz, \qquad dy = -2\,dz.$$

Consequently,

$$ds = \sqrt{dx^2 + dy^2 + dz^2} = 3\,dz$$

and

$$\cos\alpha = \frac{dx}{ds} = \frac{2}{3}, \quad \cos\beta = \frac{dy}{ds} = \frac{-2}{3}, \quad \cos\gamma = \frac{dz}{ds} = \frac{1}{3}.$$

The line joining the origin and (1, 2, 2) has direction cosines equal to

$$\tfrac{1}{3},\ \tfrac{2}{3},\ \tfrac{2}{3}.$$

The angle θ between the line and curve satisfies the equation

$$\cos\theta = \frac{2 - 4 + 2}{9} = 0.$$

The line and curve intersect at right angles.

EXERCISES

Find the equations of the tangent lines to the following curves at the points indicated:

1. $x = \sec t, \quad y = \tan t, \quad z = at, \quad \text{at } t = \frac{\pi}{4}.$

2. $x = e^t, \quad y = e^{-t}, \quad z = t^2, \quad \text{at } t = 1.$

3. $x = e^t \sin t, \quad y = e^t \cos t, \quad z = kt, \quad \text{at } t = \frac{\pi}{2}.$

4. On the circle

$$x = a\cos\theta, \quad y = a\cos\left(\theta + \frac{2}{3}\pi\right), \quad z = a\cos\left(\theta + \frac{4}{3}\pi\right)$$

show that ds is proportional to $d\theta$.

5. Find the angle at which the helix

$$x = a\cos\theta, \quad y = a\sin\theta, \quad z = k\theta$$

cuts the generators of the cylinder $x^2 + y^2 = a^2$ on which it lies.

6. Find the angle at which the conical helix

$$x = t\cos t, \quad y = t\sin t, \quad z = t$$

cuts the generators of the cone $x^2 + y^2 = z^2$ on which it lies.

7. Find the angle between the two circles cut from the sphere $x^2 + y^2 + z^2 = 14$ by the planes $x - y + z = 0$ and $x + y - z = 2$.

CHAPTER VIII

VELOCITY AND ACCELERATION IN A CURVED PATH

64. Speed of a Particle. — When a particle moves along a curve, its speed is the rate of change of distance along the path.

Let a particle P move along the curve AB, Fig. 64. Let s be the arc from a fixed point A to P. The speed of the particle is then

$$v = \frac{ds}{dt}. \tag{64}$$

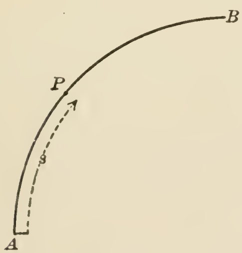


FIG. 64.

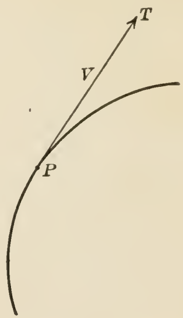


FIG. 65a.

65. Velocity of a Particle. — The velocity of a particle at the point P in its path is defined as the vector* PT tangent to the path at P, drawn in the direction of motion with length equal to the speed at P. To specify the velocity we must then give the speed and direction of motion.

* A vector is a quantity having length and direction. The direction is usually indicated by an arrow. Two vectors are called equal when they extend along the same line or along parallel lines and have the same length and direction.

The particle can be considered as moving instantaneously in the direction of the tangent. The velocity indicates in magnitude and direction the distance it would move in a unit of time if the speed and direction of motion did not change.

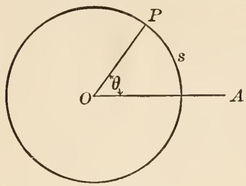


FIG. 65b.

Example. A wheel 4 ft. in diameter rotates at the rate of 500 revolutions per minute. Find the speed and velocity of a point on its rim.

Let OA be a fixed line through the center of the wheel and s the distance along the wheel from OA to a moving point P. Then

$$s = 2\,\theta \text{ ft.}$$

The speed of P is

$$\frac{ds}{dt} = 2\frac{d\theta}{dt} = 2\,(500)\,2\,\pi = 2000\,\pi \text{ ft./min.}$$

Its velocity is $2000\,\pi$ ft./min. *in the direction of the tangent at P.* The speeds of all points on the rim are the same. Their velocities differ in direction.

66. Components of Velocity in a Plane. — To specify a velocity in a plane it is customary to give its components, that is, its projections on the coördinate axes.

If PT is the velocity at P (Fig. 66), the x-component is

$$PQ = PT\cos\phi = \frac{ds}{dt}\cos\phi = \frac{ds}{dt}\frac{dx}{ds} = \frac{dx}{dt},$$

and the y-component is

$$QT = PT\sin\phi = \frac{ds}{dt}\sin\phi = \frac{ds}{dt}\frac{dy}{ds} = \frac{dy}{dt}.$$

The components are thus the rates of change of the coördinates.

Since

$$PT^2 = PQ^2 + QT^2,$$

the speed is expressed in terms of the components by the equation

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2.$$

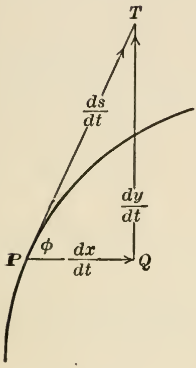


FIG. 66.

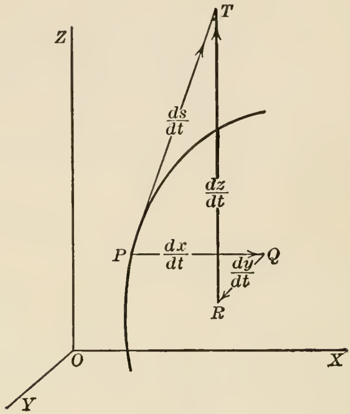


FIG. 67.

67. Components in Space. — If a particle is moving along a space curve, the projections of its velocity on the three coördinate axes are called components.

Thus, if PT (Fig. 67) represents the velocity of a point, its components are

$$PQ = PT \cos \alpha = \frac{ds}{dt}\frac{dx}{ds} = \frac{dx}{dt},$$

$$QR = PT \cos \beta = \frac{ds}{dt}\frac{dy}{ds} = \frac{dy}{dt},$$

$$RT = PT \cos \gamma = \frac{ds}{dt}\frac{dz}{ds} = \frac{dz}{dt}.$$

Since $PT^2 = PQ^2 + QR^2 + RT^2$, the speed and components are connected by the equation

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2.$$

68. Notation. — In this book we shall indicate a vector with given components by placing the components in brackets. Thus to indicate that a velocity has an x-component equal to 3 and a y-component equal to -2, we shall simply say that the velocity is $[3, -2]$. Similarly, a vector in space with x-component a, y-component b, and z-component c will be represented by the symbol $[a, b, c]$.

Example 1. Neglecting the resistance of the air a bullet fired with a velocity of 1000 ft. per second at an angle of 30 degrees with the horizontal plane will move a horizontal distance

$$x = 500\, t \sqrt{3}$$

and a vertical distance

$$y = 500\, t - 16.1\, t^2$$

in t seconds. Find its velocity and speed at the end of 10 seconds.

The components of velocity are

$$\frac{dx}{dt} = 500\sqrt{3}, \qquad \frac{dy}{dt} = 500 - 32.2\, t.$$

At the end of 10 seconds the velocity is then

$$V = [500\sqrt{3}, 178]$$

and the speed is

$$\frac{ds}{dt} = \sqrt{(500\sqrt{3})^2 + (178)^2} = 884 \text{ ft./sec.}$$

Ex. 2. A point on the thread of a screw which is turned into a fixed nut describes a helix with equations

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = k\theta,$$

θ being the angle through which the screw has turned, r the radius, and k the pitch of the screw. Find the velocity and speed of the point.

The components of velocity are

$$\frac{dx}{dt} = -r \sin\theta \frac{d\theta}{dt}, \quad \frac{dy}{dt} = r\cos\theta \frac{d\theta}{dt}, \quad \frac{dz}{dt} = k\frac{d\theta}{dt}.$$

Since $\frac{d\theta}{dt}$ is the angular velocity ω with which the screw is rotating, the velocity of the moving point is

$$V = [-r\omega \sin\theta, r\omega\cos\theta, k\omega]$$

and its speed is

$$\frac{ds}{dt} = \sqrt{r^2\omega^2 \sin^2\theta + r^2\omega^2\cos^2\theta + k^2\omega^2} = \omega\sqrt{r^2+k^2},$$

which is constant.

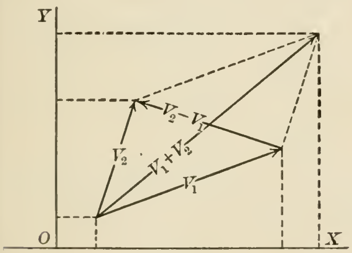


FIG. 69a.

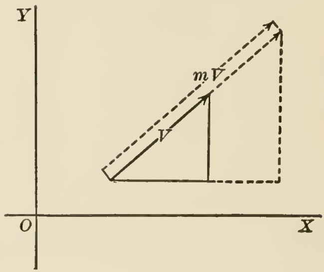


FIG. 69b.

69. Composition of Velocities. — By the sum of two velocities V_1 and V_2 is meant the velocity $V_1 + V_2$ whose components are obtained by adding corresponding components of V_1 and V_2. Similarly, the difference $V_2 - V_1$ is the velocity whose components are obtained by subtracting the components of V_1 from the corresponding ones of V_2.

Thus, if

$$V_1 = [a_1, b_1], \qquad V_2 = [a_2, b_2],$$
$$V_1 + V_2 = [a_1 + a_2, b_1 + b_2], \qquad V_2 - V_1 = [a_2 - a_1, b_2 - b_1].$$

If V_1 and V_2 extend from the same point (Fig. 69a), $V_1 + V_2$ is one diagonal of the parallelogram with V_1 and V_2 as adjacent sides and $V_2 - V_1$ is the other. In this case $V_2 - V_1$ extends from the end of V_1 to the end of V_2.

By the product mV of a vector by a number is meant a vector m times as long as V and extending in the same direction if m is positive but the opposite direction if m is negative. It is evident from Fig. 69b that the components of mV are m times those of V.

The quotient $\frac{V}{m}$ can be considered as a product $\frac{1}{m}V$. Its components are obtained by dividing those of V by m.

70. Acceleration. — The acceleration of a particle moving along a curved path is the rate of change of its velocity

$$A = \lim_{\Delta t \doteq 0} \frac{\Delta V}{\Delta t} = \frac{dV}{dt}.$$

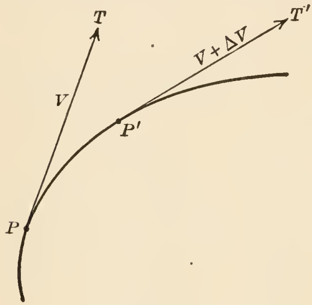


FIG. 70a.

In this equation ΔV is a vector and $\frac{\Delta V}{\Delta t}$ is obtained by dividing the components of ΔV by Δt.

Let the particle move from the point P where the velocity is V to an adjacent point P' where the velocity is $V + \Delta V$. The components of velocity will change from $\frac{dx}{dt}, \frac{dy}{dt}$ to

$$\frac{dx}{dt} + \Delta\frac{dx}{dt}, \quad \frac{dy}{dt} + \Delta\frac{dy}{dt}.$$

Consequently,

$$V = \left[\frac{dx}{dt}, \frac{dy}{dt}\right], \quad V + \Delta V = \left[\frac{dx}{dt} + \Delta\frac{dx}{dt}, \frac{dy}{dt} + \Delta\frac{dy}{dt}\right].$$

Subtraction and division by Δt give

$$\Delta V = \left[\Delta\frac{dx}{dt}, \Delta\frac{dy}{dt}\right], \quad \frac{\Delta V}{\Delta t} = \left[\frac{\Delta\frac{dx}{dt}}{\Delta t}, \frac{\Delta\frac{dy}{dt}}{\Delta t}\right].$$

As Δt approaches zero, the last equation approaches

$$A = \frac{dV}{dt} = \left[\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}\right]. \tag{70a}$$

In the same way the acceleration of a particle moving in space is found to be

$$A = \left[\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}\right]. \tag{70b}$$

Equations 70a and 70b express that *the components of the acceleration of a moving particle are the second derivatives of its coördinates with respect to the time.*

Example. A particle moves with a constant speed v around a circle of radius r. Find its velocity and acceleration at each point of the path.

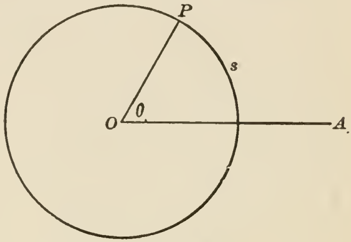


FIG. 70b.

Let $\theta = AOP$. The coördinates of P are

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

The velocity of P is

$$V = \left[-r\sin\theta\frac{d\theta}{dt}, r\cos\theta\frac{d\theta}{dt}\right].$$

Since $s = r\theta$, $\dfrac{ds}{dt} = v = r\dfrac{d\theta}{dt}$. The velocity can therefore be written

$$V = [-v\sin\theta, \quad v\cos\theta].$$

Since v is constant, the acceleration is

$$A = \frac{dV}{dt} = \left[\frac{d}{dt}(-v\sin\theta), \ \frac{d}{dt}(v\cos\theta)\right]$$

$$= \left[-v\cos\theta\frac{d\theta}{dt}, \ -v\sin\theta\frac{d\theta}{dt}\right].$$

Replacing $\dfrac{d\theta}{dt}$ by $\dfrac{v}{r}$, this reduces to

$$A = \left[-\frac{v^2}{r}\cos\theta, \quad -\frac{v^2}{r}\sin\theta\right] = \frac{v^2}{r}[-\cos\theta, \ -\sin\theta].$$

Now $[-\cos\theta, \ -\sin\theta]$ is a vector of unit length directed along PO toward the center. Hence the acceleration of P is directed toward the center of the circle and has a magnitude equal to $\frac{v^2}{r}$.

EXERCISES

1. A point P moves with constant speed v along the straight line $y = a$. Find the speed with which the line joining P to the origin rotates.

2. A rod of length a slides with its ends in the x- and y-axes. If the end in the x-axis moves with constant speed v, find the velocity and speed of the middle point of the rod.

3. A wheel of radius a rotates about its center with angular speed ω while the center moves along the x-axis with velocity v. Find the velocity and speed of a point on the perimeter of the wheel.

4. Two particles P_1 (x_1, y_1) and P_2 (x_2, y_2) move in such a way that

$$x_1 = 1 + 2t, \qquad y_1 = 2 - 3t^2,$$
$$x_2 = 3 + 2t^2, \qquad y_2 = -4t^3.$$

Find the two velocities and show that they are always parallel.

5. Two particles P_1 (x_1, y_1, z_1) and P_2 (x_2, y_2, z_2) move in such a way that

$$x_1 = a\cos\theta,\ y_1 = a\cos(\theta + \tfrac{1}{3}\pi),\ z_1 = a\cos(\theta + \tfrac{2}{3}\pi),$$
$$x_2 = a\sin\theta,\ y_2 = a\sin(\theta + \tfrac{1}{3}\pi),\ z_2 = a\sin(\theta + \tfrac{2}{3}\pi).$$

Find the two velocities and show that they are always at right angles.

6. A man can row 3 miles per hour and walk 4. He wishes to cross a river and arrive at a point 6 miles further up the river. If the river is $1\frac{2}{3}$ miles wide and the current flows 2 miles per hour, find the course he shall take to reach his destination in the least time.

7. Neglecting the resistance of the air a projectile fired with velocity $[a, b, c]$ moves in t seconds to a position

$$x = at, \quad y = bt, \quad z = ct - \tfrac{1}{2}gt^2.$$

Find its speed, velocity, and acceleration.

8. A particle moves along the parabola $x^2 = ay$ in such a way that $\frac{dx}{dt}$ is constant. Show that its acceleration is constant.

9. When a wheel rolls along a straight line, a point on its circumference describes a cycloid with parametric equations

$$x = a(\phi - \sin\phi), \qquad y = a(1 - \cos\phi),$$

a being the radius of the wheel and ϕ the angle through which it has rotated. Find the speed, velocity, and acceleration of the moving point.

10. Find the acceleration of a particle moving with constant speed v along the cardioid $r = a(1 - \cos\theta)$.

11. If a string is held taut while it is unwound from a fixed circle, its end describes the curve

$$x = a\cos\theta + a\theta\sin\theta,\ y = a\sin\theta - a\theta\cos\theta,$$

θ being the angle subtended at the center by the arc unwound. Show that the end moves at each instant with the same velocity it would have if the straight part of the string rotated with angular velocity $\frac{d\theta}{dt}$ about the point where it meets the fixed circle.

12. A piece of mechanism consists of a rod rotating in a plane with constant angular velocity ω about one end and a ring sliding along the rod with constant speed v. (1) If when $t = 0$ the ring is at the center of rotation, find its position, velocity, and acceleration as functions of the time. (2) Find the velocity and acceleration immediately after $t = t_1$, if at that instant the rod ceases to rotate but the ring continues sliding with unchanged speed along the rod. (3) Find the velocity and acceleration immediately after $t = t_1$ if at that instant the ring ceases sliding but the rod continues rotating. (4) How are the three velocities related? How are the three accelerations related?

13. Two rods AB, BC are hinged at B and lie in a plane. A is fixed, AB rotates with angular speed ω about A, and BC rotates with angular speed 2ω about B. (1) If when $t = 0$, C lies on AB produced, find the path, velocity, and acceleration of C. (2) Find the velocities and accelerations immediately after $t = t_1$ if at that instant one of the rotations ceases. (3) How are the actual velocity and acceleration related to these partial velocities and accelerations?

14. A hoop of radius a rolls with angular velocity ω_1 along a horizontal line, while an insect crawls along the rim with speed $a\omega_2$. If when $t = 0$ the insect is at the bottom of the hoop, find its path, velocity, and acceleration. The motion of the insect results from three simultaneous actions, the advance of the center of the hoop with speed $a\omega_1$, the rotation of the hoop about its center with angular speed ω_1, and the crawl of the insect advancing its radius with angular speed ω_2. Find the three velocities and accelerations which result if at the time $t = t_1$ two of these actions cease, the third continuing unchanged. How are the actual velocity and acceleration related to these partial velocities and accelerations?

Jan '25.

CHAPTER IX

ROLLE'S THEOREM AND INDETERMINATE FORMS

71. Rolle's Theorem. — *If $f'(x)$ is continuous, there is at least one real root of $f'(x) = 0$ between each pair of real roots of $f(x) = 0$.*

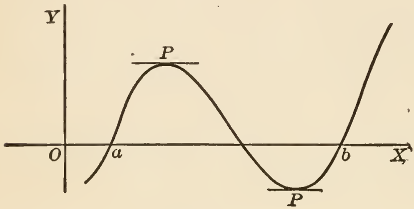


FIG. 71a.

To show this consider the curve

$$y = f(x).$$

Let $f(x)$ be zero at $x = a$ and $x = b$. Between a and b there must be one or more points P at maximum distance from the x-axis. At such a point the tangent is horizontal and so

$$\frac{dy}{dx} = f'(x) = 0.$$

That this theorem may not hold if $f'(x)$ is discontinuous is shown in Figs. 71b and 71c. In both cases the curve

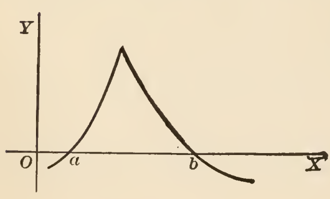


FIG. 71b.

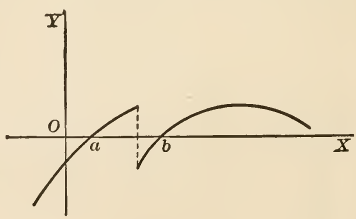


FIG. 71c.

crosses the x-axis at a and b but there is no intermediate point where the slope is zero.

Example. Show that the equation

$$x^3 + 3x - 6 = 0$$

cannot have more than one real root.

Let

$$f(x) = x^3 + 3x - 6.$$

Then

$$f'(x) = 3x^2 + 3 = 3(x^2 + 1).$$

Since $f'(x)$ does not vanish for any real value of x, $f(x) = 0$ cannot have more than one real root; for if there were two there would be a root of $f'(x) = 0$ between them.

72. Indeterminate Forms. — The expressions

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0 \cdot \infty, \quad \infty - \infty, \quad 1^\infty, \quad 0^0, \quad \infty^0$$

are called indeterminate forms. No definite values can be assigned to them.

If when $x = a$ a function $f(x)$ assumes an indeterminate form, there may however be a definite limit

$$\lim_{x \doteq a} f(x).$$

In such cases this limit is usually taken as the value of the function at $x = a$.

For example, when $x = 0$ the function

$$\frac{2x}{x} = \frac{0}{0}.$$

It is evident, however, that

$$\lim_{x \doteq 0} \frac{2x}{x} = \lim (2) = 2.$$

This example shows that an indeterminate form can often be made definite by an algebraic change of form.

73. The Forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$. — We shall now show that, if for a particular value of the variable a fraction $\dfrac{f(x)}{F(x)}$ assumes the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, numerator and denominator can be replaced by their derivatives without changing the value of the limit approached by the fraction as x approaches a.

1. Let $f'(x)$ and $F'(x)$ be continuous between a and b. *If* $f(a) = 0$, $F(a) = 0$, *and* $F(b)$ *is not zero, there is a number* x_1 *between* a *and* b *such that*

$$\frac{f(b)}{F(b)} = \frac{f'(x_1)}{F'(x_1)}. \tag{73a}$$

To show this let $\frac{f(b)}{F(b)} = R$. Then

$$f(b) - RF(b) = 0.$$

Consider the function

$$f(x) - RF(x).$$

This function vanishes when $x = b$. Since $f(a) = 0$, $F(a) = 0$, it also vanishes when $x = a$. By Rolle's Theorem there is then a value x_1 between a and b such that

$$f'(x_1) - RF'(x_1) = 0.$$

Consequently,

$$\frac{f(b)}{F(b)} = R = \frac{f'(x_1)}{F'(x_1)},$$

which was to be proved.

2. Let $f'(x)$ and $F'(x)$ be continuous near a. *If* $f(a) = 0$ *and* $F(a) = 0$, *then*

$$\lim_{x \doteq a} \frac{f(x)}{F(x)} = \lim_{x \doteq a} \frac{f'(x)}{F'(x)}. \tag{73b}$$

For, if we replace b by x, (73a) becomes

$$\frac{f(x)}{F(x)} = \frac{f'(x_1)}{F'(x_1)},$$

x_1 being between a and x. Since x_1 approaches a as x approaches a,

$$\lim_{x \doteq a} \frac{f(x)}{F(x)} = \lim_{x_1 \doteq a} \frac{f'(x_1)}{F'(x_1)} = \lim_{x \doteq a} \frac{f'(x)}{F'(x)}.$$

3. In the neighborhood of $x = a$, let $f'(x)$ and $F'(x)$ be

continuous at all points except $x = a$. *If $f(x)$ and $F(x)$ approach infinity as x approaches a,*

$$\lim_{x \doteq a} \frac{f(x)}{F(x)} = \lim_{x \doteq a} \frac{f'(x)}{F'(x)}.$$

To show this let c be near a and on the same side as x. Since $f(x) - f(c)$ and $F(x) - F(c)$ are zero when $x = c$, by Theorem 1,

$$\frac{f'(x_1)}{F'(x_1)} = \frac{f(x) - f(c)}{F(x) - F(c)} = \frac{f(x)}{F(x)} \frac{1 - \dfrac{f(c)}{f(x)}}{1 - \dfrac{F(c)}{F(x)}},$$

where x_1 is between x and c. As x approaches a, $f(x)$ and $F(x)$ increase indefinitely. The quantities $f(c)/f(x)$ and $F(c)/F(x)$ approach zero. The right side of this equation therefore approaches

$$\lim_{x \doteq a} \frac{f(x)}{F(x)}.$$

Since x_1 is between c and a, by taking c sufficiently near to a the left side of the equation can be made to approach

$$\lim_{x \doteq a} \frac{f'(x)}{F'(x)}.$$

Since the two sides are always equal, we therefore conclude that

$$\lim_{x \doteq a} \frac{f(x)}{F(x)} = \lim_{x \doteq a} \frac{f'(x)}{F'(x)}.$$

Example 1. Find the value approached by $\dfrac{\sin x}{x}$ as x approaches zero.

Since the numerator and denominator are zero when $x = 0$, we can apply Theorem 2 and so get

$$\lim_{x \doteq 0} \frac{\sin x}{x} = \lim_{x \doteq 0} \frac{\cos x}{1} = 1.$$

Ex. 2. Find the value of $\lim\limits_{x \doteq \pi} \dfrac{1 + \cos x}{(\pi - x)^2}$.

When $x = \pi$ the numerator and denominator are both zero. Hence

$$\lim_{x \doteq \pi} \frac{1 + \cos x}{(\pi - x)^2} = \lim_{x \doteq \pi} \frac{(-\sin x)}{-2(\pi - x)} = \frac{0}{0}.$$

Since this is indeterminate we apply the method a second time and so obtain

$$\lim_{x \doteq \pi} \frac{\sin x}{2(\pi - x)} = \lim_{x \doteq \pi} \frac{\cos x}{-2} = \frac{1}{2}.$$

The value required is therefore $\frac{1}{2}$.

Ex. 3. Find the value approached by $\dfrac{\tan 3x}{\tan x}$ as x approaches $\dfrac{\pi}{2}$.

When x approaches $\dfrac{\pi}{2}$ the numerator and denominator of this fraction approach ∞. Therefore, by Theorem 3,

$$\lim_{x \doteq \frac{\pi}{2}} \frac{\tan 3x}{\tan x} = \lim \frac{3 \sec^2 3x}{\sec^2 x} = \lim \frac{3 \cos^2 x}{\cos^2 3x}.$$

When x is replaced by $\dfrac{\pi}{2}$ the last expression takes the form $\dfrac{0}{0}$. Therefore

$$\lim \frac{3 \cos^2 x}{\cos^2 3x} = \lim \frac{6 \cos x \sin x}{6 \cos 3x \sin 3x}$$

$$= \lim \frac{\cos^2 x - \sin^2 x}{3(\cos^2 3x - \sin^2 3x)} = \frac{1}{3}.$$

74. The Forms $0 \cdot \infty$ and $\infty - \infty$. — By transforming the expression to a fraction it will take the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$.

For example,

$$x \ln x$$

has the form $0 \cdot \infty$ when $x = 0$. It can, however, be written

$$x \ln x = \frac{\ln x}{\frac{1}{x}},$$

which has the form $\frac{\infty}{\infty}\cdot$

The expression

$$\sec x - \tan x$$

has the form $\infty - \infty$ when $x = \frac{\pi}{2}\cdot$ It can, however, be written

$$\sec x - \tan x = \frac{1}{\cos x} - \frac{\sin x}{\cos x} = \frac{1 - \sin x}{\cos x},$$

which becomes $\frac{0}{0}$ when $x = \frac{\pi}{2}\cdot$

75. The Forms 0°, 1^∞, ∞°. — The logarithm of the given function has the form $0 \cdot \infty$. From the limit of the logarithm the limit of the function can be determined.

Example. Find the limit of $(1 + x)^{\frac{1}{x}}$ as x approaches zero.

Let $$y = (1 + x)^{\frac{1}{x}}.$$

Then

$$\ln y = \frac{1}{x} \ln (1 + x) = \frac{\ln (1 + x)}{x}\cdot$$

When x is zero this last expression becomes $\frac{0}{0}\cdot$ Therefore

$$\lim_{x \doteq 0} \frac{\ln (1 + x)}{x} = \lim \frac{1}{1 + x} = 1.$$

The limit of $\ln y$ being 1, the limit of y is e.

EXERCISES

1. Show by Rolle's Theorem that the equation

$$x^4 - 4x - 1 = 0$$

cannot have more than two real roots.

Determine the values of the following limits:

2. $\displaystyle \lim_{x \doteq 1} \frac{x^9 - 1}{x^{10} - 1}.$

3. $\displaystyle \lim_{x \doteq 1} \frac{x^n - 1}{x - 1}.$

4. $\displaystyle \lim_{x \doteq 0} \frac{1 - \cos x}{\sin x}.$

5. $\displaystyle \lim_{x \doteq a} \frac{e^x - e^a}{x - a}.$

6. $\displaystyle \lim_{x \doteq 0} \frac{\tan x - x}{x - \sin x}.$

7. $\displaystyle \lim_{x \doteq 0} \frac{x^2 \cos x}{\cos x - 1}.$

8. $\displaystyle \lim_{x \doteq 3} \frac{\ln (x - 2)}{x - 3}.$

9. $\displaystyle \lim_{x \doteq 0} \frac{\ln \cos x}{x}.$

10. $\displaystyle \lim_{x \doteq 2} \frac{\sin^2 \pi x}{(x - 2)^2}.$

11. $\displaystyle \lim_{x \doteq \frac{\pi}{2}} \frac{1 + \cos x - \sin x}{\cos x (2 \sin x - 1)}.$

12. $\displaystyle \lim_{x \doteq \alpha} \frac{\log_{10} (\sin x - \sin \alpha)}{\log_{10} (\tan x - \tan \alpha)}.$

13. $\displaystyle \lim_{x \doteq 0} \frac{6 \sin x - 6x + x^3}{x^2}.$

14. $\displaystyle \lim_{\phi \doteq \frac{\pi}{4}} \frac{\sec^2 \phi - 2 \tan \phi}{1 + \cos 4\phi}.$

15. $\displaystyle \lim_{x \doteq 0} \frac{\ln x}{\cot x}.$

16. $\displaystyle \lim_{x = \infty} \frac{\ln x}{x}.$

17. $\displaystyle \lim_{x \doteq \frac{\pi}{2}} \frac{\sec 3x - x}{\sec x - x}.$

18. $\displaystyle \lim_{x \doteq \frac{\pi}{4}} \frac{1 + \tan 2x}{\sec\left(x + \frac{\pi}{4}\right)}.$

19. $\displaystyle \lim_{x \doteq 0} x \cot x.$

20. $\displaystyle \lim_{x \doteq \frac{\pi}{2}} \tan x \cos 3x.$

21. $\displaystyle \lim_{x = \infty} (x + a) \ln \left(1 + \frac{a}{x}\right).$

22. $\displaystyle \lim_{x \doteq 3} (x - 3) \cot (\pi x).$

23. $\displaystyle \lim_{n = \infty} n \left[f\left(x + \frac{dx}{n}\right) - f(x)\right]$

24. $\displaystyle \lim_{x \doteq 0} x^2 e^{\frac{1}{x^2}}.$

25. $\displaystyle \lim_{x \doteq 0} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right).$

26. $\displaystyle \lim_{x \doteq 0} (\cot x - \ln x).$

27. $\displaystyle \lim_{x \doteq \frac{\pi}{2}} \left[\tan x - \frac{1}{\sin x - \sin^2 x}\right]$

28. $\displaystyle \lim_{x \doteq 0} x^x.$

29. $\displaystyle \lim_{x \doteq \frac{\pi}{2}} (\sin x)^{\tan x}.$

30. $\displaystyle \lim_{x \doteq 0} (1 + ax)^{\frac{1}{x}}.$

31. $\displaystyle \lim_{x = \infty} (x^m - a^m)^{\frac{1}{\ln x}}.$

CHAPTER X

SERIES AND APPROXIMATIONS

76. Mean Value Theorem. — *If* $f(x)$ and $f'(x)$ *are continuous from* $x = a$ *to* $x = b$, *there is a value* x_1 *between* a *and* b *such that*

$$\frac{f(b) - f(a)}{b - a} = f'(x_1). \tag{76}$$

To show this consider the curve $y = f(x)$. Since $f(a)$ and $f(b)$ are the ordinates at $x = a$ and $x = b$,

$$\frac{f(b) - f(a)}{b - a} = \text{slope of chord } AB.$$

On the arc AB let P_1 be a point at maximum distance from

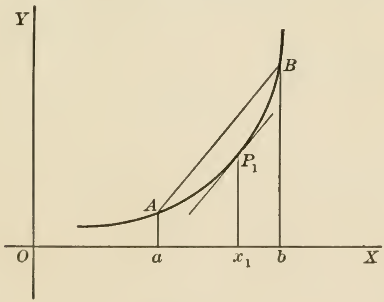


FIG. 76.

the chord. The tangent at P_1 will be parallel to the chord and so its slope $f'(x_1)$ will equal that of the chord. Therefore

$$\frac{f(b) - f(a)}{b - a} = f'(x_1),$$

which was to be proved.

Replacing b by x and solving for $f(x)$, equation (76) becomes

$$f(x) = f(a) + (x - a) f'(x_1),$$

x_1 being between a and x. This is a special case of a more general theorem which we shall now prove.

77. Taylor's Theorem. — *If $f(x)$ and all its derivatives used are continuous from a to x, there is a value x_1 between a and x such that*

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots + \frac{(x-a)^n}{n!} f^n(x_1).$$

To prove this let

$$\phi(x) = f(x) - f(a) - (x-a) f'(a) - \frac{(x-a)^2}{2!} f''(a) - \cdots - \frac{(x-a)^{n-1}}{(n-1)!} f^{n-1}(a).$$

It is easily seen that

$$\phi(a) = 0, \quad \phi'(a) = 0, \quad \phi''(a) = 0, \quad \ldots \phi^{n-1}(a) = 0, \quad \phi^n(x) = f^n(x).$$

When $x = a$ the function

$$\frac{\phi(x)}{(x-a)^n}$$

therefore assumes the form $\frac{0}{0}$. By Art. 73 there is then a value z_1 between a and x such that

$$\frac{\phi(x)}{(x-a)^n} = \frac{\phi'(z_1)}{n(z_1-a)^{n-1}}.$$

This new expression becomes $\frac{0}{0}$ when $z_1 = a$. There is consequently a value z_2 between z_1 and a (and so between x and a) such that

$$\frac{\phi'(z_1)}{n(z_1-a)^{n-1}} = \frac{\phi''(z_2)}{n(n-1)(z_2-a)^{n-2}}.$$

A continuation of this argument gives finally

$$\frac{\phi(x)}{(x-a)^n} = \frac{\phi^n(z_n)}{n!} = \frac{f^n(z_n)}{n!},$$

z_n being between x and a. If $x_1 = z_n$ we then have

$$\phi(x) = \frac{(x-a)^n}{n!} f^n(x_1).$$

Equating this to the original value of $\phi(x)$ and solving for $f(x)$, we get

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^n}{n!} f^n(x_1),$$

which was to be proved.

Example. Prove

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4\,x_1^4},$$

where x_1 is between 1 and x.

When $x = 1$ the values of $\ln x$ and its derivatives are

$$\begin{aligned}
f(x) &= \ln(x), & f(1) &= 0,\\
f'(x) &= \frac{1}{x}, & f'(1) &= 1,\\
f''(x) &= -\frac{1}{x^2}, & f''(1) &= -1,\\
f'''(x) &= \frac{2}{x^3}, & f'''(1) &= 2,\\
f''''(x) &= -\frac{6}{x^4}, & f''''(x_1) &= -\frac{6}{(x_1)^4}.
\end{aligned}$$

Taking $a = 1$, Taylor's Theorem gives

$$\ln x = 0 + 1(x-1) - \frac{1}{2}(x-1)^2 + \frac{2}{6}(x-1)^3 - \frac{6}{24}\,\frac{(x-1)^4}{x_1^4},$$

which is the result required.

78. Approximate Values of Functions. — The last term in Taylor's formula

$$\frac{(x-a)^n}{n!} f^n(x_1) = R_n$$

is called the *remainder*. If this is small, an approximate value of the function is

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a),$$

the error in the approximation being equal to the remainder.

To compute $f(x)$ by this formula, we must know the values of $f(a)$, $f'(a)$, etc. *We must then assign a value to a such that $f(a)$, $f'(a)$, etc., are known. Furthermore, a should be as close as possible to the value x at which $f(x)$ is wanted.* For, the smaller $x - a$, the fewer terms $(x-a)^2$, $(x-a)^3$, etc., need be computed to give a required approximation.

Example 1. Find tan 46° to four decimals.

The value closest to 46° for which $\tan x$ and its derivatives are known is 45°. Therefore we let $a = \frac{\pi}{4}$.

$$f(x) = \tan x, \qquad f\left(\frac{\pi}{4}\right) = 1,$$

$$f'(x) = \sec^2 x, \qquad f'\left(\frac{\pi}{4}\right) = 2,$$

$$f''(x) = 2\sec^2 x \tan x, \qquad f''\left(\frac{\pi}{4}\right) = 4,$$

$$f'''(x) = 2\sec^4 x + 4\sec^2 x \tan^2 x.$$

Using these values in Taylor's formula, we get

$$\tan x = 1 + 2\left(x - \frac{\pi}{4}\right) + \frac{4}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{f'''(x_1)}{3!}\left(x - \frac{\pi}{4}\right)^3$$

and

$$\tan 46° = 1 + 2\left(\frac{\pi}{180}\right) + 2\left(\frac{\pi}{180}\right)^2 = 1.0355$$

approximately. Since x_1 is between 45° and 46°, $f'''(x_1)$ does not differ much from

$$f'''(45°) = 8 + 8 = 16.$$

The error in the above approximation is thus very nearly

$$\frac{16}{6}\left(\frac{\pi}{180}\right)^3 < \frac{8}{3\,(50)^3} < \frac{1}{40{,}000} = 0.000025.$$

It is therefore correct to 4 decimals.

Ex. 2. Find the value of e to four decimals.

The only value of x for which e^x and its derivatives are known is $x = 0$. We therefore let a be zero.

$$f(x) = e^x,\quad f'(x) = e^x,\quad f''(x) = e^x,\quad \ldots\ldots,\quad f^n(x) = e^x,$$
$$f(0) = 1,\quad f'(0) = 1,\quad f''(0) = 1,\quad \ldots\ldots,\quad f^n(x_1) = e^{x_1}.$$

By Taylor's Theorem,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n e^{x_1}}{n!}.$$

Letting $x = 1$, this becomes

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!} + \frac{e^{x_1}}{n!}.$$

In particular, if $n = 2$,

$$e = 2 + \tfrac{1}{2}\,e^{x_1}.$$

Since x_1 is between 0 and 1, e is then between $2\frac{1}{2}$ and $2 + \frac{1}{2}\,e$, and therefore between $2\frac{1}{2}$ and 4. To get a better approximation let $n = 9$. Then

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{8!} = 2.7183$$

approximately, the error being

$$\frac{e^{x_1}}{9!} \leqq \frac{e}{9!} \leqq \frac{4}{9!} < .00002.$$

The value 2.7183 is therefore correct to four decimals.

EXERCISES

Determine the values of the following functions correct to four decimals:

1. $\sin 5°$.

2. $\cos 32°$.

3. $\cot 43°$.

4. $\tan 58°$.

5. $\sec (10°)$.

6. $\ln\left(\frac{9}{10}\right)$.

7. $\sqrt{e}$.

8. $\tan^{-1}\left(\frac{1}{10}\right)$.

9. Given $\ln 3 = 1.0986$, $\ln 5 = 1.6094$, find $\ln 17$.

79. Taylor's and Maclaurin's Series. — As n increases indefinitely, the remainder in Taylor's formula

$$R_n = \frac{(x-a)^n}{n} f^n(x_1)$$

often approaches zero. In that case

$$f(x) = \lim_{n=\infty}\left[f(a) + (x-a)f'(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a)\right].$$

This is usually written

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots,$$

the dots at the end signifying the limit of the sum as the number of terms is indefinitely increased. Such an infinite sum is called an *infinite series.* This one is called *Taylor's Series.*

In particular, if $a = 0$, Taylor's Series becomes

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots.$$

This is called *Maclaurin's Series.*

Example. Show that $\cos x$ is represented by the series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$$

The series given contains powers of x. This happens when $a = 0$, that is, when Taylor's Series reduces to Maclaurin's.

$$\begin{aligned} f(x) &= \cos x, & f(0) &= 1, \\ f'(x) &= -\sin x, & f'(0) &= 0, \\ f''(x) &= -\cos x, & f''(0) &= -1, \\ f'''(x) &= \sin x, & f'''(0) &= 0, \\ f''''(x) &= \cos x, & f''''(0) &= 1. \end{aligned}$$

These values give

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \pm \frac{x^n}{n!} f^n(x_1).$$

The nth derivative of $\cos x$ is $\pm\cos x$ or $\pm\sin x$, depending on whether n is even or odd. Since $\sin x$ and $\cos x$ are never greater than 1, $f_n(x_1)$ is not greater than 1. Furthermore

$$\frac{x^n}{n!} = \frac{x}{1}\cdot\frac{x}{2}\cdot\frac{x}{3}\cdot \;\cdots\; \cdot\frac{x}{n}$$

can be made as small as you please by taking n sufficiently large. Hence the remainder approaches zero and so

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots,$$

which was to be proved.

EXERCISES

1. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots.$

2. $\cos x = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}}\left(x - \frac{\pi}{4}\right)^2 + \frac{1}{6\sqrt{2}}\left(x - \frac{\pi}{4}\right)^3 + \cdots$

3. $2^x = 1 + x \ln 2 + \frac{(x \ln 2)^2}{2!} + \frac{(x \ln 2)^3}{3!} + \cdots.$

4. $e^x \sin x = x + 2 \cdot \frac{x^2}{2!} + 2\,\frac{x^3}{3!} - 4\,\frac{x^5}{5!} - 8\,\frac{x^6}{6!} - \frac{8\,x^7}{7!} + \cdots.$

5. $e^x \cos x = 1 + x - \frac{2\,x^3}{3!} - \frac{4\,x^4}{4!} - \frac{4\,x^5}{5!} + \frac{8\,x^7}{7!} + \cdots.$

6. $(a + x)^n = a^n + na^{n-1}x + \frac{n\,(n-1)}{2!}a^{n-2}x^2 + \cdots$, if $|x|$* $< |a|$.

7. $\sqrt{x} = 2 + \frac{x-4}{4} - \frac{(x-4)^2}{64} + \frac{(x-4)^3}{512} - \cdots$, if $|x - 4| < 1$.

8. $\ln x = \ln 3 + \frac{x-3}{3} - \frac{(x-3)^2}{2\cdot 3^2} + \frac{(x-3)^3}{3\cdot 3^3} - \cdots$, if $|x - 3| < 1$.

9. $\ln(x+5) = \ln 6 + \frac{x-1}{6} - \frac{(x-1)^2}{2\cdot 6^2} + \frac{(x-1)^3}{3\cdot 6^3} - \cdots$, if $|x-1| < 1$.

80. Convergence and Divergence of Series.— An infinite series is said to *converge* if the sum of the first n terms approaches a limit as n increases indefinitely. If this sum does not approach a limit, the series is said to *diverge*.

The series for $\sin x$ and $\cos x$ converge for all values of x. The geometrical series

$$a + ar + ar^2 + ar^3 + ar^4 + \cdots$$

* The symbol $|x|$ is used to represent the numerical value of x without its algebraic sign. Thus, $|-3| = |3| = 3$.

converges when r is numerically less than 1. For the sum of the first n terms is

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} = a\,\frac{1 - r^n}{1 - r}.$$

If r is numerically less than 1, r^n approaches zero and S_n approaches

$$S = \frac{a}{1 - r}$$

as n increases indefinitely.

The series

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

is divergent, for the sum oscillates between 0 and 1 and does not approach a limit. The geometrical series

$$1 + 2 + 4 + 8 + 16 + \cdots$$

diverges because the sum increases indefinitely and so does not approach a limit.

81. Tests for Convergence. — The convergence of a series can often be determined from the problem in which it occurs. Thus the series

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

converges because the sum of n terms approaches $\cos x$ as n increases indefinitely.

The terms near the beginning of a series (if they are all finite) have no influence on the convergence or divergence of the series. This is determined by terms indefinitely far out in the series.

82. General Test. — *For the series*

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

to converge it is necessary and sufficient that the sum of terms beyond u_n approach zero as n increases indefinitely.

For, if the series converges, the sum of n terms must approach a limit and so the sum of terms beyond the nth must approach zero.

83. Comparison Test. — *A series is convergent if beyond a certain point its terms are in numerical value respectively less than those of a convergent series whose terms are all positive.*

For, if a series converges, the sum of terms beyond the nth will approach zero as n increases indefinitely. If then another series has lesser corresponding terms, their sum will approach zero and the series will converge.

84. Ratio Test. — *If the ratio* $\frac{u_{n+1}}{u_n}$ *of consecutive terms approaches a limit* r *as* n *increases indefinitely, the series*

$$u_1 + u_2 + u_3 + \cdots + u_n + u_{n+1} + \cdots$$

is convergent if r *is numerically less than 1 and divergent if* r *is numerically greater than 1.*

Since the limit is r, by taking n sufficiently large the ratio of consecutive terms can be made as nearly r as we please. If $r < 1$, let r_1 be a fixed number between r and 1. We can take n so large that the ratio of consecutive terms is less than r_1. Then

$$u_{n+1} < r_1 u_n, \quad u_{n+2} < r_1 u_{n+1} < r_1^2 u_n, \text{ etc.}$$

Beyond u_n the terms of the given series are therefore less than those of the geometrical progression

$$u_n + r_1 u_n + r_1^2 u_n + \cdots$$

which converges since r_1 is numerically less than 1. Consequently the given series converges.

If, however, r is greater than 1, the terms of the series must ultimately increase. The terms do not then approach zero and their sum cannot approach a limit.

Example. Find for what values of x the series

$$x + 2x^2 + 3x^3 + 4x^4 + \cdots$$

converges.

The ratio of consecutive terms is

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)x^{n+1}}{nx^n} = \left(1 + \frac{1}{n}\right)x.$$

The limit of this ratio is

$$r = \lim_{n=\infty}\left(1+\frac{1}{n}\right)x = x.$$

The series will converge if x is numerically less than 1.

85. Power Series. — A series of powers of $(x - a)$ of the form

$$P(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots,$$

where a, a_0, a_1, a_2, etc., are constants, is called a *power series.*

If a power series converges when $x = b$, it will converge for all values of x nearer to a than b is, that is, such that

$$|x - a| < |b - a|.$$

In fact, if the series converges when $x = b$, each term of

$$a_0 + a_1(b-a) + a_2(b-a)^2 + a_3(b-a)^3 + \cdots$$

will be less than a maximum value M, that is,

$$|a_n(b-a)^n| < M.$$

Consequently,

$$|a_n| < \frac{M}{|(b-a)^n|}.$$

The terms of the series

$$a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots$$

are then respectively less than those of the geometrical series

$$M + \frac{M}{|b-a|}|x-a| + \frac{M}{|b-a|^2}|x-a|^2 + \frac{M}{|b-a|^3}|x-a|^3 + \cdots$$

in which the ratio is

$$\frac{|x-a|}{|b-a|}.$$

If then $|x - a| < |b - a|$, the progression and consequently the given series will converge.

If a power series diverges when $x = b$, it will diverge for all values of x further from a than b is, that is, such that

$$|x - a| > |b - a|.$$

For it could not converge beyond b, since by the proof just given it would then converge at b.

This theorem shows in certain cases why a Taylor's Series is not convergent. Take, for example, the series

$$\ln (1 + x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots.$$

As x approaches -1, $\ln (1 + x)$ approaches infinity. Since a convergent series cannot have an infinite value, we should expect the series to diverge when $x = -1$. It must then diverge when x is at a distance greater than 1 from $a = 0$. The series in fact converges between $x = -1$ and $x = 1$ and diverges for values of x numerically greater than 1.

86. Operations with Power Series. — It is shown in more advanced treatises that convergent series can be added, subtracted, multiplied and divided like polynomials. In case of division, however, the resulting series will not usually converge beyond a point where the denominator is zero.

Example. Express $\tan x$ as a series in powers of x.

We could use Maclaurin's series with $f(x) = \tan x$. It is easier, however, to expand $\sin x$ and $\cos x$ and divide the one by the other to get $\tan x$. Thus

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \dfrac{x^3}{6} + \dfrac{x^5}{120} - \cdots}{1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} - \cdots} = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots.$$

EXERCISES

1. Show that

$$\ln \left(\frac{1+x}{1-x}\right) = \ln (1 + x) - \ln (1 - x) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right),$$

and that the series converges when $|x| < 1$.

2. By expanding $\cos 2x$, show that

$$\sin^2 x = \frac{1 - \cos 2x}{2} = 2\frac{x^2}{2!} - 2^3\frac{x^4}{4!} + 2^5\frac{x^6}{6!} - \cdots.$$

Prove that the series converges for all values of x.

3. Show that

$$(1 + e^x)^2 = 1 + 2e^x + e^{2x} = 4 + 4x + 3x^2 + \tfrac{5}{3}x^3 + \tfrac{3}{4}x^4 + \cdots$$

and that the series converges for all values of x.

4. Given $f(x) = \sin^{-1} x$, show that

$$f'(x) = \frac{1}{\sqrt{1 - x^2}} = (1 - x^2)^{-\frac{1}{2}}.$$

Expand this by the binomial theorem and determine $f''(x)$, etc., by differentiating the result. Hence show that

$$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\frac{x^5}{5} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\frac{x^7}{7} + \cdots$$

and that the series converges when $|x| < 1$.

5. By a method similar to that used in Ex. 4, show that

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

and that the series converges when $|x| < 1$.

6. Prove

$$\sec x = \frac{1}{\cos x} = 1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \cdots.$$

For what values of x do you think the series converges?

CHAPTER XI

PARTIAL DIFFERENTIATION

87. Functions of Two or More Variables. — A quantity u is called a function of two independent variables x and y,

$$u = f(x, y),$$

if u is determined when arbitrary values (or values arbitrary within certain limits) are assigned to x and y.

For example,

$$u = \sqrt{1 - x^2 - y^2}$$

is a function of x and y. If u is to be real, x and y must be so chosen that $x^2 + y^2$ is not greater than 1. Within that limit, however, x and y can be chosen independently and a value of u will then be determined.

In a similar way we define a function of three or more independent variables. An illustration of a function of variables that are not independent is furnished by the area of a triangle. It is a function of the sides a, b, c and angles A, B, C of the triangle, but is not a function of these six quantities considered as independent variables; for, if values not belonging to the same triangle are given to them, no triangle and consequently no area will be determined.

The increment of a function of several variables is its increase when all the variables change. Thus, if

$$u = f(x, y),$$
$$u + \Delta u = f(x + \Delta x, y + \Delta y)$$

and so

$$\Delta u = f(x + \Delta x, y + \Delta y) - f(x, y).$$

A function is called *continuous* if its increment approaches zero when all the increments of the variables approach zero.

88. Partial Derivatives. — Let

$$u = f(x, y)$$

be a function of two independent variables x and y. If we keep y constant, u is a function of x. The derivative of this function with respect to x is called the *partial derivative* of u with respect to x and is denoted by

$$\frac{\partial u}{\partial x} \quad \text{or} \quad f_x(x, y).$$

Similarly, if we differentiate with respect to y with x constant, we get the partial derivative with respect to y denoted by

$$\frac{\partial u}{\partial y} \quad \text{or} \quad f_y(x, y).$$

For example, if

$$u = x^2 + xy - y^2,$$

then

$$\frac{\partial u}{\partial x} = 2x + y, \quad \frac{\partial u}{\partial y} = x - 2y.$$

Likewise, if u is a function of any number of independent variables, the partial derivative with respect to one of them is obtained by differentiating with the others constant.

89. Higher Derivatives. — The first partial derivatives are functions of the variables. By differentiating these functions partially, we get higher partial derivatives.

For example, the derivatives of $\frac{\partial u}{\partial x}$ with respect to x and y are

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y\, \partial x}.$$

Similarly,

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x\, \partial y}, \quad \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2}.$$

It can be shown that

$$\frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial^2 u}{\partial y\, \partial x},$$

if both derivatives are continuous, that is, *partial derivatives are independent of the order in which the differentiations are performed.**

Example. $u = x^2y + xy^2$.

$$\frac{\partial u}{\partial x} = 2\,xy + y^2, \quad \frac{\partial u}{\partial y} = x^2 + 2\,xy,$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(2\,xy + y^2) = 2\,y, \quad \frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial}{\partial y}(2\,xy + y^2) = 2\,x + 2\,y,$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial}{\partial x}(x^2 + 2\,xy) = 2\,x + 2\,y, \quad \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(x^2 + 2\,xy) = 2\,x.$$

90. Dependent Variables. — It often happens that some of the variables are functions of others. For example, let

$$u = x^2 + y^2 + z^2$$

and let z be a function of x and y. When y is constant, z will be a function of x and the partial derivative of u with respect to x will be

$$\frac{\partial u}{\partial x} = 2\,x + 2\,z\frac{\partial z}{\partial x}\cdot$$

Similarly, the partial derivative with respect to y with x constant is

$$\frac{\partial u}{\partial y} = 2\,y + 2\,z\frac{\partial z}{\partial y}\cdot$$

If, however, we consider z constant, the partial derivatives are

$$\frac{\partial u}{\partial x} = 2\,x, \quad \frac{\partial u}{\partial y} = 2\,y.$$

The value of a partial derivative thus depends on what quantities are kept constant during the differentiation.

The quantities kept constant are sometimes indicated by subscripts. Thus, in the above example

$$\left(\frac{\partial u}{\partial x}\right)_{y,z} = 2\,x, \quad \left(\frac{\partial u}{\partial x}\right)_y = 2\,x + 2\,z\frac{\partial z}{\partial x}, \quad \left(\frac{\partial u}{\partial x}\right)_z = 2\,x + 2\,y\frac{\partial y}{\partial x}\cdot$$

* For a proof see Wilson, *Advanced Calculus*, § 50.

It will usually be clear from the context what independent variables u is considered a function of. Then $\frac{\partial u}{\partial x}$ will represent the derivative with all those variables except x constant.

Example. If a is a side and A the opposite angle of a right triangle with hypotenuse c, find $\left(\frac{\partial a}{\partial c}\right)_A$.

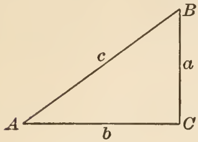


FIG. 90.

From the triangle it is seen that

$$a = c \sin A.$$

Differentiating with A constant, we get

$$\frac{\partial a}{\partial c} = \sin A,$$

which is the value required.

91. Geometrical Representation. — Let $z = f(x, y)$ be the equation of a surface. The points with constant y-coördinate form the curve AB (Fig. 91a) in which the plane $y =$ constant intersects the surface. In this plane z is the vertical and x the horizontal coördinate. Consequently,

$$\frac{\partial z}{\partial x}$$

is the slope of the curve AB at P.

Similarly, the locus of points with given x is the curve CD and

$$\frac{\partial z}{\partial y}$$

is the slope of this curve at P.

Example. Find the lowest point on the paraboloid

$$z = x^2 + y^2 - 2x - 4y + 6.$$

At the lowest point, the curves AB and CD (Fig. 91b) will have horizontal tangents. Hence

$$\frac{\partial z}{\partial x} = 2x - 2 = 0, \qquad \frac{\partial z}{\partial y} = 2y - 4 = 0.$$

Consequently, $x = 1$, $y = 2$. These values substituted in the equation of the surface give $z = 1$. The point required is then $(1, 2, 1)$. That this is really the lowest point is shown by the graph.

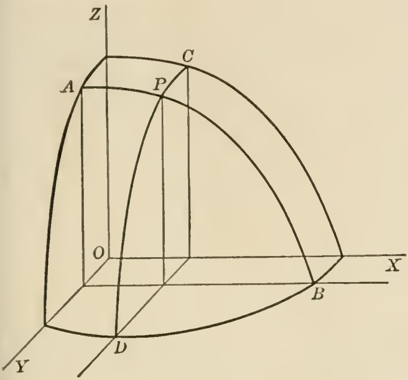


FIG. 91a.

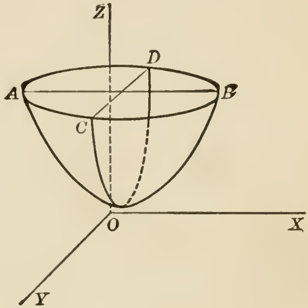


FIG. 91b.

EXERCISES

In each of the following exercises show that the partial derivatives satisfy the equation given:

1. $u = \dfrac{x^2 + y^2}{x + y}$, $\qquad x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = u.$

2. $z = (x + a)(y + b)$, $\qquad \dfrac{\partial z}{\partial x}\dfrac{\partial z}{\partial y} = z.$

3. $z = (x^2 + y^2)^n$, $\qquad y\dfrac{\partial z}{\partial x} = x\dfrac{\partial z}{\partial y}.$

4. $u = \ln(x^2 + xy + y^2)$, $\qquad x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2.$

5. $u = \dfrac{y}{z} + \dfrac{z}{x} + \dfrac{x}{y}$, $\qquad x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0.$

6. $u = \tan^{-1}\left(\dfrac{y}{x}\right)$, $\qquad \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0.$

7. $u = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$, $\qquad \dfrac{\partial^2 u}{dx^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0.$

In each of the following exercises verify that

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}.$$

8. $u = \dfrac{y}{x}$. **10.** $u = \sin(x + y)$.

9. $u = \ln(x^2 + y^2)$. **11.** $u = xyz$.

12. Given $v = \sqrt{x^2 + y^2 + z^2}$, verify that

$$\frac{\partial^3 v}{\partial x\,\partial y\,\partial z} = \frac{\partial^3 v}{\partial z\,\partial y\,\partial x}.$$

Prove the following relations assuming that z is a function of x and y:

13. $u = (x + z)\,e^{y+z}$, $\quad \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = (1 + x + z)\left(1 + \dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y}\right) e^{y+z}$.

14. $u = xyz$, $\quad z\left(x\dfrac{\partial u}{\partial x} - y\dfrac{\partial u}{\partial y}\right) = u\left(x\dfrac{\partial z}{\partial x} - y\dfrac{\partial z}{\partial y}\right)$.

15. $u = e^x + e^y + e^z$, $\quad \dfrac{\partial^2 u}{\partial x\,\partial y} = e^z\left(\dfrac{\partial^2 z}{\partial x\,\partial y} + \dfrac{\partial z}{\partial x}\dfrac{\partial z}{\partial y}\right)$.

16. $\dfrac{\partial}{\partial x}\left(z\dfrac{\partial u}{\partial x} - u\dfrac{\partial z}{\partial x}\right) = z\dfrac{\partial^2 u}{\partial x^2} - u\dfrac{\partial^2 z}{\partial x^2}$.

17. If $x = r\cos\theta$, $y = r\sin\theta$, show that

$$\left(\frac{\partial x}{\partial r}\right)_\theta = \left(\frac{\partial r}{\partial x}\right)_y.$$

18. Let a and b be the sides of a right triangle with hypotenuse c and opposite angles A and B. Let p be the perpendicular from the vertex of the right angle to the hypotenuse. Show that

$$\left(\frac{\partial p}{\partial a}\right)_b = \frac{b^3}{c^3}, \qquad \left(\frac{\partial p}{\partial a}\right)_A = \frac{b}{c}.$$

19. If K is the area of a triangle, a side and two adjacent angles of which are c, A, B, show that

$$\left(\frac{\partial K}{\partial a}\right)_{c,B} = \frac{b^2}{2}, \qquad \left(\frac{\partial K}{\partial B}\right)_{c,A} = \frac{a^2}{2}.$$

20. If K is the area of a triangle with sides a, b, c, show that

$$\left(\frac{\partial K}{\partial a}\right)_{b,c} = \frac{a}{2}\cot A.$$

21. Find the lowest point on the surface

$$z = 2x^2 + y^2 + 8x - 2y + 9.$$

22. Find the highest point on the surface

$$z = 2y - x^2 + 2xy - 2y^2 + 1.$$

92. Increment. — Let $u = f(x, y)$ be a function of two independent variables x and y. When x changes to $x + \Delta x$ and y to $y + \Delta y$, the increment of u is

$$\Delta u = f(x + \Delta x, y + \Delta y) - f(x, y). \qquad (92a)$$

By the mean value theorem, Art. 76,

$$f(x + \Delta x, y + \Delta y) = f(x, y + \Delta y) + \Delta x f_x(x_1, y + \Delta y),$$

x_1 lying between x and $x + \Delta x$. Similarly

$$f(x, y + \Delta y) = f(x, y) + \Delta y f_y(x, y_1),$$

y_1 being between y and $y + \Delta y$. Using these values in (92a), we get

$$\Delta u = \Delta x f_x(x_1, y + \Delta y) + \Delta y f_y(x, y_1). \qquad (92b)$$

As Δx and Δy approach zero, x_1 approaches x and y_1 approaches y. If $f_x(x, y)$ and $f_y(x, y)$ are continuous,

$$f_x(x_1, y + \Delta y) = f_x(x, y) + \epsilon_1 = \frac{\partial u}{\partial x} + \epsilon_1,$$

$$f_y(x, y_1) = f_y(x, y) + \epsilon_2 = \frac{\partial u}{\partial y} + \epsilon_2,$$

ϵ_1 and ϵ_2 approaching zero as Δx and Δy approach zero. These values substituted in (92b) give

$$\Delta u = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y. \qquad (92c)$$

The quantity

$$\frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y$$

is called the *principal part* of Δu. It differs from Δu by an amount $\epsilon_1 \Delta x + \epsilon_2 \Delta y$. As Δx and Δy approach zero, ϵ_1 and ϵ_2 approach zero and so this difference becomes an indefinitely small fraction of the larger of the increments Δx and Δy. We express this by saying the principal part differs from Δu by an infinitesimal of higher order than Δx and Δy (Art. 9). When Δx and Δy are sufficiently small this principal part then gives a satisfactory approximation for Δu.

Analogous results can be obtained for any number of independent variables. For example, if there are three independent variables x, y, z, the principal part of Δu is

$$\frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \frac{\partial u}{\partial z}\Delta z.$$

In each case, if the partial derivatives are continuous, the principal part differs from Δu by an amount which becomes indefinitely small in comparison with the largest of the increments of the independent variables as those increments all approach zero.

Example. Find the change in the volume of a cylinder when its length increases from 6 ft. to 6 ft. 1 in. and its diameter decreases from 2 ft. to 23 in.

Since the volume is $v = \pi r^2 h$, the exact change is

$$\Delta v = \pi \left(1 - \tfrac{1}{24}\right)^2 \left(6 + \tfrac{1}{12}\right) - \pi \cdot 1^2 \cdot 6 = -0.413\,\pi \text{ cu. ft.}$$

The principal part of this increment is

$$\frac{\partial v}{\partial r}\,\Delta r + \frac{\partial v}{\partial h}\,\Delta h = 2\,\pi\,rh\left(-\frac{1}{24}\right) + \pi\,r^2\left(\frac{1}{12}\right) = -0.417\,\pi \text{ cu. ft.}$$

93. Total Differential. — If u is a function of two independent variables x and y, the *total differential* of u is the principal part of Δu, that is,

$$du = \frac{\partial u}{\partial x}\,\Delta x + \frac{\partial u}{\partial y}\,\Delta y. \tag{93a}$$

This definition applies to any function of x and y. The particular values $u = x$ and $u = y$ give

$$dx = \Delta x, \qquad dy = \Delta y, \tag{93b}$$

that is, *the differentials of the independent variables are equal to their increments.*

Combining (93a) and (93b), we get

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy. \tag{93c}$$

We shall show later (Art. 97) that this equation is valid even if x and y are not the independent variables.

The quantities

$$d_x u = \frac{\partial u}{\partial x}\,dx, \qquad d_y u = \frac{\partial u}{\partial y}\,dy$$

are called *partial differentials.* Equation (93c) expresses that *the total differential of a function is equal to the sum of the partial differentials obtained by letting the variables change one at a time.*

Similar results can be obtained for functions of any number of variables. For instance, if u is a function of three independent variables x, y, z,

$$du = \frac{\partial u}{\partial x}\,\Delta x + \frac{\partial u}{\partial y}\,\Delta y + \frac{\partial u}{\partial z}\,\Delta z.$$

The particular values $u = x$, $u = y$, $u = z$ give

$$dx = \Delta x, \qquad dy = \Delta y, \qquad dz = \Delta z.$$

The previous equation can then be written

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy + \frac{\partial u}{\partial z}\,dz \tag{93d}$$

and in this form it can be proved valid even when x, y, z are not the independent variables.

Example 1. Find the total differential of the function

$$u = x^2y + xy^2.$$

By equation (93c)

$$\begin{aligned} du &= \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \\ &= (2\,xy + y^2)\,dx + (x^2 + 2\,xy)\,dy. \end{aligned}$$

Ex. 2. Find the error in the volume of a rectangular box due to small errors in its three edges.

Let the edges be x, y, z. The volume is then

$$v = xyz.$$

The error in v, due to small errors Δx, Δy, Δz in x, y, z, is Δv. If the increments are sufficiently small, this will be approximately

$$dv = yz\,dx + xz\,dy + xy\,dz.$$

Dividing by v, we get

$$\begin{aligned} \frac{dv}{v} &= \frac{yz\,dx + xz\,dy + xy\,dz}{xyz} \\ &= \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}. \end{aligned}$$

Now $\frac{dx}{x}$ expresses the error dx as a fraction or percentage of x.

The equation just obtained expresses that the percentage error in the volume is equal to the sum of the percentage errors in the edges. If, for example, the error in each edge is not more than one per cent, the error in the volume is not more than three per cent.

94. Calculation of Differentials. — In proving the formulas of differentiation it was assumed that u, v, etc., were functions of a single variable. It is easy to show that the same formulas are valid when those quantities are functions of two or more variables and du, dv, etc., are their total differentials.

Take, for example, the differential of uv. By (93c) the result is

$$d\,(uv) = \frac{\partial}{\partial u}\,(uv)\,du + \frac{\partial}{\partial v}\,(uv)\,dv = v\,du + u\,dv,$$

which is the formula IV of Art. 17.

Example. $u = ye^x + ze^y$.

Differentiating term by term, we get

$$du = ye^x\,dx + e^x\,dy + ze^y\,dy + e^y\,dz.$$

We obtain the same result by using (93d); for that formula gives

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy + \frac{\partial u}{\partial z}\,dz = ye^x\,dx + (e^x + ze^y)\,dy + e^y\,dz.$$

95. Partial Derivatives as Ratios of Differentials. — The equation

$$d_x u = \frac{\partial u}{\partial x}\,dx$$

shows that the partial derivative $\frac{\partial u}{\partial x}$ is the ratio of two differentials $d_x u$ and dx. Now $d_x u$ is the value of du when the same quantities are kept constant that are constant in the calculation of $\frac{\partial u}{\partial x}$. Therefore, *the partial derivative* $\frac{\partial u}{\partial x}$ *is the* *value to which* $\frac{du}{dx}$ *reduces when du and dx are determined with the same quantities constant that are constant in the calculation of* $\frac{\partial u}{\partial x}$.

Example. Given $u = x^2 + y^2 + z^2$, $v = xyz$, find $\left(\frac{\partial u}{\partial x}\right)_{v,z}$.

Differentiating the two equations with v and z constant, we get

$$du = 2\,x\,dx + 2\,y\,dy, \qquad 0 = yz\,dx + xz\,dy.$$

Eliminating dy,

$$du = 2\,x\,dx - 2\,\frac{y^2}{x}\,dx = 2\left(\frac{x^2 - y^2}{x}\right)dx.$$

Under the given conditions the ratio of du to dx is then

$$\frac{du}{dx} = \frac{2\,(x^2 - y^2)}{x}.$$

Since v and z were kept constant, this ratio represents $\left(\frac{\partial u}{\partial x}\right)_{v,z}$; that is,

$$\left(\frac{\partial u}{\partial x}\right)_{v,z} = \frac{2\,(x^2 - y^2)}{x}.$$

EXERCISES

1. One side of a right triangle increases from 5 to 5.2 while the other decreases from 12 to 11.75. Find the increment of the hypotenuse and its principal part.

2. A closed box, 12 in. long, 8 in. wide, and 6 in. deep, is made of material $\frac{1}{4}$ inch thick. Find approximately the volume of material used.

3. Two sides and the included angle of a triangle are $b = 20$, $c = 30$, and $A = 45°$. By using the formula

$$a^2 = b^2 + c^2 - 2\,bc \cos A,$$ .68

find approximately the change in a when b increases 1 unit, c decreases $\frac{1}{2}$ unit, and A increases 1 degree.

4. The period of a simple pendulum is

$$T = 2\,\pi\sqrt{\frac{l}{g}}.$$

Find the error in T due to small errors in l and g.

5. If g is computed by the formula,

$$s = \tfrac{1}{2}gt^2,$$

find the error in g due to small errors in s and t.

6. The area of a triangle is determined by the formula

$$K = \tfrac{1}{2}ab\sin C.$$

Find the error in K due to small errors in a, b, C.

Find the total differentials of the following functions:

7. xy^2z^3.

8. $xy\sin(x+y)$.

9. $\dfrac{x}{y}+\dfrac{y}{z}+\dfrac{z}{x}$.

10. $\tan^{-1}\dfrac{y}{x}+\tan^{-1}\dfrac{x}{y}$.

11. The pressure, volume, and temperature of a perfect gas are connected by the equation $pv = kt$, k being constant. Find dp in terms of dv and dt.

12. If x, y are rectangular and r, θ polar coördinates of the same point, show that

$$x\,dy - y\,dx = r^2\,d\theta,\quad dx^2+dy^2 = dr^2 + r^2\,d\theta^2.$$

13. If $x = u - v$, $y = u^2+v^2$, find $\left(\dfrac{\partial x}{\partial v}\right)_y$.

14. If $u = xy+yz+zx$, $x^2+z^2 = 2yz$, find $\left(\dfrac{\partial u}{\partial z}\right)_y$.

15. If $yz = ux + v^2$, $vx = uy + z^2$, find $\left(\dfrac{\partial v}{\partial z}\right)_{u,x}$.

16. A variable triangle with sides a, b, c and opposite angles A, B, C is inscribed in a fixed circle. Show that

$$\frac{da}{\cos A}+\frac{db}{\cos B}+\frac{dc}{\cos C} = 0.$$

96. Derivative of a Function of Several Variables. — Let $u = f(x, y)$ and let x and y be functions of two variables s and t. When t changes to $t+\Delta t$, x and y will change to $x+\Delta x$ and $y+\Delta y$. The resulting increment in u will be

$$\Delta u = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y.$$

Consequently,

$$\frac{\Delta u}{\Delta t} = \frac{\partial u}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial u}{\partial y}\frac{\Delta y}{\Delta t} + \epsilon_1\frac{\Delta x}{\Delta t} + \epsilon_2\frac{\Delta y}{\Delta t}.$$

As Δt approaches zero, Δx and Δy will approach zero and so

ϵ_1 and ϵ_2 will approach zero. Taking the limit of both sides,

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}. \tag{96a}$$

If x or y is a function of t only, the partial derivative $\frac{\partial x}{\partial t}$ or $\frac{\partial y}{\partial t}$ is replaced by a total derivative $\frac{dx}{dt}$ or $\frac{dy}{dt}$. If both x and y are functions of t, u is a function of t with total derivative

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}. \tag{96b}$$

Likewise, if u is a function of three variables x, y, z, that depend on t,

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}. \tag{96c}$$

As before, if a variable is a function of t only, its partial derivative is replaced by a total one. Similar results hold for any number of variables.

The term

$$\frac{\partial u}{\partial x}\frac{\partial x}{\partial t}$$

is the result of differentiating u with respect to t, leaving all the variables in u except x constant. Equations (96a) and (96c) express that *if u is a function of several variable quantities, $\frac{\partial u}{\partial t}$ can be obtained by differentiating with respect to t as if only one of those quantities were variable at a time and adding the results.*

Example 1. Given $y = x^x$, find $\frac{dy}{dx}$.

The function x^x can be considered a function of two variables, the lower x and the upper x. If the upper x is held constant and the lower allowed to vary, the derivative (as in case of x^n) is

$$x \cdot x^{x-1} = x^x.$$

If the lower x is held constant while the upper varies, the derivative (as in case of a^x) is

$$x^x \ln x.$$

The actual derivative of y is then the sum

$$\frac{dy}{dx} = x^x + x^x \ln x.$$

Ex. 2. Given $u = f(x, y, z)$, y and z being functions of x, find $\frac{du}{dx}$.

By equation (96c) the result is

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} + \frac{\partial u}{\partial z}\frac{dz}{dx}.$$

In this equation there are two derivatives of u with respect to x. If y and z are replaced by their values in terms of x, u will be a function of x only. The derivative of that function is $\frac{du}{dx}$. If y and z are replaced by constants, u will be a second function of x. Its derivative is $\frac{\partial u}{\partial x}$.

Ex. 3. Given $u = f(x, y, z)$, z being a function of x and y. Find the partial derivative of u with respect to x.

It is understood that y is to be constant in this partial differentiation. Equation (96c) then gives

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}.$$

In this equation appear two partial derivatives of u with respect to x. If z is replaced by its value in terms of x and y, u will be expressed as a function of x and y only. Its partial derivative is the one on the left side of the equation. If z is kept constant, u is again a function of x and y. Its partial derivative appears on the right side of the equation. We must not of course use the same symbol for both of these derivatives. A way to avoid the confusion is to use the letter f instead of u on the right side of the equation. It then becomes

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x}.$$

It is understood that $f(x, y, z)$ is a definite function of x, y, z and that $\frac{\partial f}{\partial x}$ is the derivative obtained with all the variables but x constant.

97. Change of Variable. — If u is a function of x and y we have said that the equation

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy$$

is true whether x and y are the independent variables or not. To show this let s and t be the independent variables and x and y functions of them. Then, by definition,

$$du = \frac{\partial u}{\partial s}\,ds + \frac{\partial u}{\partial t}\,dt.$$

Since u is a function of x and y which are functions of s and t, by equation (96a),

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}, \qquad \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}.$$

Consequently,

$$\begin{aligned} du &= \left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}\right)ds + \left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}\right)dt \\ &= \frac{\partial u}{\partial x}\left(\frac{\partial x}{\partial s}\,ds + \frac{\partial x}{\partial t}\,dt\right) + \frac{\partial u}{\partial y}\left(\frac{\partial y}{\partial s}\,ds + \frac{\partial y}{\partial t}\,dt\right) = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy, \end{aligned}$$

which was to be proved.

A similar proof can be given in case of three or more variables.

98. Implicit Functions. — If two or more variables are connected by an equation, a differential relation can be obtained by equating the total differentials of the two sides of the equation.

Example 1. $f(x, y) = 0$.

In this case

$$d \cdot f(x, y) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = d \cdot 0 = 0.$$

Consequently,

$$\frac{dy}{dx} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}}.$$

Ex. 2. $f(x, y, z) = 0$.

Differentiation gives

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = 0.$$

If z is considered a function of x and y, its partial derivative with respect to x is found by keeping y constant. Then $dy = 0$ and

$$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial z}}.$$

Similarly, if x is constant, $dx = 0$ and

$$\frac{\partial z}{\partial y} = -\frac{\dfrac{\partial f}{\partial y}}{\dfrac{\partial f}{\partial z}}.$$

Ex. 3. $f_1(x, y, z) = 0$, $f_2(x, y, z) = 0$.

We have two differential relations

$$\frac{\partial f_1}{\partial x} dx + \frac{\partial f_1}{\partial y} dy + \frac{\partial f_1}{\partial z} dz = 0,$$

$$\frac{\partial f_2}{\partial x} dx + \frac{\partial f_2}{\partial y} dy + \frac{\partial f_2}{\partial z} dz = 0.$$

We could eliminate y from the two equations $f_1 = 0$, $f_2 = 0$. We should then obtain z as a function of x. The total derivative of this function is found by eliminating dy and solving for the ratio $\frac{dz}{dx}$. The result is

$$\frac{dz}{dx}=\frac{\frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial y}}{\frac{\partial f_1}{\partial z}\frac{\partial f_2}{\partial y}-\frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial z}}.$$

99. Directional Derivative. — Let $u = f(x, y)$. At each point $P(x, y)$ in the xy-plane, u has a definite value. If we move away from P in any definite direction PQ, x and y will

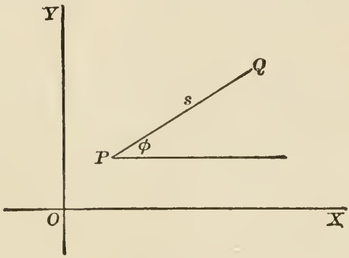


FIG. 99.

be functions of the distance moved. The derivative of u with respect to s is

$$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\frac{dx}{ds}+\frac{\partial u}{\partial y}\frac{dy}{ds}=\frac{\partial u}{\partial x}\cos\phi+\frac{\partial u}{\partial y}\sin\phi.$$

This is called the derivative of u in the direction PQ. The partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are special values of $\frac{\partial u}{\partial s}$ which result when PQ is drawn in the direction of OX or OY.

Similarly, if $u = f(x, y, z)$,

$$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\frac{dx}{ds}+\frac{\partial u}{\partial y}\frac{dy}{ds}+\frac{\partial u}{\partial z}\frac{dz}{ds}=\frac{\partial u}{\partial x}\cos\alpha+\frac{\partial u}{\partial y}\cos\beta+\frac{\partial u}{\partial z}\cos\gamma$$

is the rate of change of u with respect to s as we move along a line with direction cosines $\cos\alpha$, $\cos\beta$, $\cos\gamma$. The partial derivatives of u are the values to which $\frac{\partial u}{\partial s}$ reduces when s is measured in the direction of a coördinate axis.

Example. Find the derivative of $x^2 + y^2$ in the direction $\phi = 45°$ at the point (1, 2).

The result is

$$\frac{\partial}{\partial s}(x^2 + y^2) = 2\,x\frac{\partial x}{\partial s} + 2\,y\frac{\partial y}{\partial s} = 2\,x\cos\phi + 2\,y\sin\phi$$

$$= 2\cdot\frac{1}{\sqrt{2}} + 4\cdot\frac{1}{\sqrt{2}} = 3\sqrt{2}.$$

100. Exact Differentials. — If P and Q are functions of two independent variables x and y,

$$P\,dx + Q\,dy$$

may or may not be the total differential of a function u of x and y. If it is the total differential of such a function,

$$P\,dx + Q\,dy = du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy.$$

Since dx and dy are arbitrary, this requires

$$P = \frac{\partial u}{\partial x}, \qquad Q = \frac{\partial u}{\partial y}.$$

Consequently,

$$\frac{\partial P}{\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}, \qquad \frac{\partial Q}{\partial x} = \frac{\partial^2 u}{\partial x\,\partial y}.$$

Since the two second derivatives of u with respect to x and y are equal,

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}. \tag{100a}$$

An expression $P\,dx + Q\,dy$ is called an *exact differential* if it is the total differential of a function of x and y. We have just shown that (100a) must then be satisfied. Conversely, it can be shown that if this equation is satisfied $P\,dx + Q\,dy$ is an exact differential.*

* See Wilson, *Advanced Calculus*, § 92.

Similarly, if

$$P\,dx + Q\,dy + R\,dz$$

is the differential of a function u of x, y, z,

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \qquad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}, \tag{100b}$$

and conversely.

Example 1. Show that

$$(x^2 + 2\,xy)\,dx + (x^2 + y^2)\,dy$$

is an exact differential.

In this case

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(x^2 + 2\,xy) = 2\,x, \qquad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2) = 2\,x.$$

The two partial derivatives being equal, the expression is exact.

Ex. 2. In thermodynamics it is shown that

$$dU = T\,dS - p\,dv,$$

U being the internal energy, T the absolute temperature, S the entropy, p the pressure, and v the volume of a homogeneous substance. Any two of these five quantities can be assigned independently and the others are then determined. Show that

$$\left(\frac{\partial T}{\partial p}\right)_S = \left(\frac{\partial v}{\partial S}\right)_p.$$

The result to be proved expresses that

$$T\,dS + v\,dp$$

is an exact differential. That such is the case is shown by replacing $T\,dS$ by its value $dU + p\,dv$. We thus get

$$T\,dS + v\,dp = dU + p\,dv + v\,dp = d\,(U + pv).$$

EXERCISES

1. If $u = f(x, y)$, $y = \phi(x)$, find $\dfrac{du}{dx}$.

2. If $u = f(x, y, z)$, $z = \phi(x)$, find $\left(\dfrac{\partial u}{\partial x}\right)_y$.

3. If $u = f(x, y, z)$, $z = \phi(x, y)$, $y = \psi(x)$, find $\frac{du}{dx}$.

4. If $u = f(x, y)$, $y = \phi(x, r)$, $r = \psi(x, s)$, find $\left(\frac{\partial u}{\partial x}\right)_y$, $\left(\frac{\partial u}{\partial x}\right)_r$, and $\left(\frac{\partial u}{\partial x}\right)_s$.

5. If $f(x, y, z) = 0$, $z = F(x, y)$, find $\frac{dz}{dx}$.

6. If $F(x, y, z) = 0$, show that

$$\frac{\partial x}{\partial y}\,\frac{\partial y}{\partial z}\,\frac{\partial z}{\partial x} = -1.$$

7. If $u = xf(z)$, $z = \frac{y}{x}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$.

8. If $u = f(r, s)$, $r = x + at$, $s = y + bt$, show that $\frac{\partial u}{\partial t} = a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y}$.

9. If $z = f(x + ay)$, show that $\frac{\partial z}{\partial y} = a\frac{\partial z}{\partial x}$.

10. If $u = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$, show that

$$\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2.$$

11. The position of a pair of rectangular axes moving in a plane is determined by the coördinates h, k of the moving origin and the angle ϕ between the moving x-axis and a fixed one. A variable point P has coördinates x', y' with respect to the moving axes and x, y with respect to the fixed ones. Then

$$x = f(x', y', h, k, \phi), \qquad y = F(x', y', h, k, \phi).$$

Find the velocity of P. Show that it is the sum of two parts, one representing the velocity the point would have if it were rigidly connected with the moving axes, the other representing its velocity with respect to those axes conceived as fixed.

12. Find the directional derivatives of the rectangular coördinates x, y and the polar coördinates r, θ of a point in a plane. Show that they are identical with the derivatives with respect to s given in Arts. 54 and 59.

13. Find the derivative of $x^2 - y^2$ in the direction $\phi = 30°$ at the point (3, 4).

14. At a distance r in space the potential due to an electric charge e is $V = \frac{e}{r}$. Find its directional derivative.

15. Show that the derivative of xy along the normal at any point of the curve $x^2 - y^2 = a^2$ is zero.

16. Given $u = f(x, y)$, show that

$$\left(\frac{\partial u}{\partial s_1}\right)^2 + \left(\frac{\partial u}{\partial s_2}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2,$$

if s_1 and s_2 are measured along perpendicular directions.

Determine which of the following expressions are exact differentials:

17. $y\,dx - x\,dy$.

18. $(2x + y)\,dx + (x - 2y)\,dy$.

19. $ex\,dx + ey\,dy + (x + y)\,ez\,dz$.

20. $yz\,dx - xz\,dy + y^2\,dz$.

21. Under the conditions of Ex. 2, page 131, show that

$$\left(\frac{\partial v}{\partial T}\right)_p = -\left(\frac{\partial S}{\partial p}\right)_T, \qquad \left(\frac{\partial p}{\partial T}\right)_v = \left(\frac{\partial S}{\partial v}\right)_T.$$

22. In case of a perfect gas, $pv = kT$. Using this and the equation

$$dU = T\,dS - p\,dv,$$

show that

$$\frac{\partial U}{\partial p} = 0.$$

Since U is always a function of p and T, this last equation expresses that U is a function of T only.

101. Direction of the Normal at a Point of a Surface. — Let the equation of a surface be

$$F(x, y, z) = 0.$$

Differentiation gives

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz = 0. \tag{101a}$$

Let PN be the line through $P\ (x, y, z)$ with direction cosines proportional to

$$\frac{\partial F}{\partial x} : \frac{\partial F}{\partial y} : \frac{\partial F}{\partial z}.$$

If P moves along a curve on the surface, the direction cosines of its tangent PT are proportional to

$$dx : dy : dz.$$

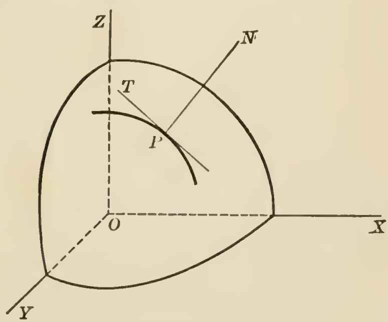


FIG. 101.

Equation (101a) expresses that PN and PT are perpendicular to each other (Art. 61). Consequently PN is perpendicular to all the tangent lines through P. This is expressed by saying PN is the *normal* to the surface at P. We conclude that *the normal to the surface* $F(x, y, z) = 0$ *at* $P(x, y, z)$ *has direction cosines proportional to*

$$\frac{\partial F}{\partial x} : \frac{\partial F}{\partial y} : \frac{\partial F}{\partial z}. \tag{101}$$

102. Equations of the Normal at $P_1(x_1, y_1, z_1)$. — Let A, B, C be proportional to the direction cosines of the normal to a surface at $P_1(x_1, y_1, z_1)$. The equations of the normal are (Art. 63)

$$\frac{x - x_1}{A} = \frac{y - y_1}{B} = \frac{z - z_1}{C}. \tag{102}$$

103. Equation of the Tangent Plane at $P_1(x_1, y_1, z_1)$. — All the tangent lines at P_1 on the surface are perpendicular

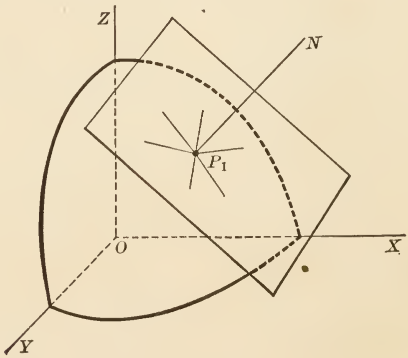


FIG. 103.

to the normal at that point. All these lines therefore lie in a plane perpendicular to the normal, called the *tangent plane* at P_1.

It is shown in analytical geometry that if A, B, C are proportional to the direction cosines of the normal to a plane passing through (x_1, y_1, z_1), the equation of the plane is

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0.^* \tag{103}$$

* See Phillips, *Analytic Geometry*, Art. 68.

If A, B, C are proportional to the direction cosines of the normal to a surface at P_1, this is then the equation of the tangent plane at P_1.

Example. Find the equations of the normal line and tangent plane at the point $(1, -1, 2)$ of the ellipsoid

$$x^2 + 2y^2 + 3z^2 = 3x + 12.$$

The equation given is equivalent to

$$x^2 + 2y^2 + 3z^2 - 3x - 12 = 0.$$

The direction cosines of its normal are proportional to the partial derivatives

$$2x - 3 : 4y : 6z.$$

At the point $(1, -1, 2)$, these are proportional to

$$A : B : C = -1 : -4 : 12 = 1 : 4 : -12.$$

The equations of the normal are

$$\frac{x-1}{1} = \frac{y+1}{4} = \frac{z-2}{-12}.$$

The equation of the tangent plane is

$$x - 1 + 4(y+1) - 12(z-2) = 0.$$

EXERCISES

Find the equations of the normal and tangent plane to each of the following surfaces at the point indicated:

1. Sphere, $x^2 + y^2 + z^2 = 9$, at $(1, 2, 2)$.

2. Cylinder, $x^2 + xy + y^2 = 7$, at $(2, -3, 3)$.

3. Cone, $z^2 = x^2 + y^2$, at $(3, 4, 5)$.

4. Hyperbolic paraboloid, $xy = 3z - 4$, at $(5, 1, 3)$.

5. Elliptic paraboloid, $x = 2y^2 + 3z^2$, at $(5, 1, 1)$.

6. Find the locus of points on the cylinder

$$(x+z)^2 + (y-z)^2 = 4$$

where the normal is parallel to the xy-plane.

7. Show that the normal at any point $P(x, y, z)$ of the surface $y^2 + z^2 = 4x$ makes equal angles with the x-axis and the line joining P and $A(1, 0, 0)$.

8. Show that the normal to the spheroid

$$\frac{x^2+z^2}{9} + \frac{y^2}{25} = 1$$

at $P(x, y, z)$ determines equal angles with the lines joining P with $A'(0, -4, 0)$ and $A(0, 4, 0)$.

104. Maxima and Minima of Functions of Several Variables. — A *maximum* value of a function u is a value greater than any given by neighboring values of the variables. In passing from a maximum to a neighboring value, the function decreases, that is

$$\Delta u < 0. \tag{104a}$$

A *minimum* value is a value less than any given by neighboring values of the variables. In passing from a minimum to a neighboring value

$$\Delta u > 0. \tag{104b}$$

If the condition (104a) or (104b) is satisfied for all small changes of the variables, it must be satisfied when a single variable changes. If then all the independent variables but x are kept constant, u must be a maximum or minimum in x. If $\frac{\partial u}{\partial x}$ is continuous, by Art. 31,

$$\frac{\partial u}{\partial x} = 0. \tag{104c}$$

Therefore, *if the first partial derivatives of u with respect to the independent variables are continuous, those derivatives must be zero when u is a maximum or minimum.*

When the partial derivatives are zero, the total differential is zero. For example, if x and y are the independent variables,

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy = 0 \cdot dx + 0 \cdot dy = 0. \tag{104d}$$

Therefore, *if the first partial derivatives are continuous, the total differential of u is zero when u is either a maximum or a minimum.*

To find the maximum and minimum values of a function, we equate its differential or the partial derivatives with respect to the *independent* variables to zero and solve the resulting equations. It is usually possible to decide from the problem whether a value thus found is a maximum, minimum, or neither.

Example 1. Show that the maximum rectangular parallelopiped with a given area of surface is a cube.

Let x, y, z be the edges of the parallelopiped. If V is the volume and A the area of its surface

$$V = xyz, \qquad A = 2\,xy + 2\,xz + 2\,yz.$$

Two of the variables x, y, z are independent. Let them be x, y. Then

$$z = \frac{A - 2\,xy}{2\,(x + y)}.$$

Therefore

$$V = \frac{xy\,(A - 2\,xy)}{2\,(x + y)},$$

$$\frac{\partial V}{\partial x} = \frac{y^2}{2}\left[\frac{A - 2\,x^2 - 4\,xy}{(x + y)^2}\right] = 0,$$

$$\frac{\partial V}{\partial y} = \frac{x^2}{2}\left[\frac{A - 2\,y^2 - 4\,xy}{(x + y)^2}\right] = 0.$$

The values $x = 0$, $y = 0$ cannot give maxima. Hence

$$A - 2\,x^2 - 4\,xy = 0, \qquad A - 2\,y^2 - 4\,xy = 0.$$

Solving these equations simultaneously with

$$A = 2\,xy + 2\,xz + 2\,yz,$$

we get

$$x = y = z = \sqrt{\frac{A}{6}}.$$

We know there is a maximum. Since the equations give only one solution it must be the maximum.

Ex. 2. Find the point in the plane

$$x + 2\,y + 3\,z = 14$$

nearest to the origin.

The distance from any point (x, y, z) of the plane to the origin is

$$D = \sqrt{x^2 + y^2 + z^2}.$$

If this is a minimum

$$d \cdot D \doteq \frac{x\,dx + y\,dy + z\,dz}{\sqrt{x^2 + y^2 + z^2}} = 0,$$

that is,

$$x\,dx + y\,dy + z\,dz = 0. \tag{104e}$$

From the equation of the plane we get

$$dx + 2\,dy + 3\,dz = 0. \tag{104f}$$

The only equation connecting x, y, z is that of the plane. Consequently, dx, dy, dz can have any values satisfying this last equation. If x, y, z are so chosen that D is a minimum (104e) must be satisfied by all of these values. If two linear equations have the same solutions, one is a multiple of the other. Corresponding coefficients are proportional. The coefficients of dx, dy, dz in (104e) are x, y, z. Those in (104f) are 1, 2, 3. Hence

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3}.$$

Solving these simultaneously with the equation of the plane, we get $x = 1$, $y = 2$, $z = 3$. There is a minimum. Since we get only one solution, it is the minimum.

EXERCISES

1. An open rectangular box is to have a given capacity. Find the dimensions of the box requiring the least material. $x = y = 2z$

2. A tent having the form of a cylinder surmounted by a cone is to contain a given volume. Find its dimensions if the canvas required is a minimum.

3. When an electric current of strength I flows through a wire of resistance R the heat produced is proportional to I^2R. Two terminals are connected by three wires of resistances R_1, R_2, R_3 respectively. A given current flowing between the terminals will divide between the wires in such a way that the heat produced is a minimum. Show that the currents I_1, I_2, I_3 in the three wires will satisfy the equations

$$I_1R_1 = I_2R_2 = I_3R_3.$$

4. A particle attracted toward each of three points A, B, C with a force proportional to the distance will be in equilibrium when the sum of the squares of the distances from the points is least. Find the position of equilibrium.

5. Show that the triangle of greatest area with a given perimeter is equilateral.

6. Two adjacent sides of a room are plane mirrors. A ray of light starting at P strikes one of the mirrors at Q, is reflected to a point R on the second mirror, and is there reflected to S. If P and S are in the same horizontal plane find the positions of Q and R so that the path $PQRS$ may be as short as possible.

7. A table has four legs attached to the top at the corners A_1, A_2, A_3, A_4 of a square. A weight W placed upon the table at a point of the diagonal A_1A_3, two-thirds of the way from A_1 to A_3, will cause the legs to shorten the amounts s_1, s_2, s_3, s_4, while the weight itself sinks a distance h. The increase in potential energy due to the contraction of a leg is ks^2, where k is constant and s the contraction. The decrease in potential energy due to the sinking of the weight is Wh. The whole system will settle to a position such that the potential energy is a minimum. Assuming that the top of the table remains plane, find the ratios of s_1, s_2, s_3, s_4.

SUPPLEMENTARY EXERCISES

CHAPTER III

Find the differentials of the following functions:

1. $\dfrac{\sqrt{ax^2+b}}{bx}$.
2. $\dfrac{x}{b\sqrt{ax^2+b}}$.
3. $\dfrac{2\sqrt{ax^2+bx}}{bx}$.
4. $\dfrac{2ax+b}{\sqrt{ax^2+bx+c}}$.
5. $\dfrac{(ax+b)^{n+2}}{a^2(n+2)}-\dfrac{b(ax+b)^{n+1}}{a^2(n+1)}$.
6. $x(a^2+x^2)\sqrt{a^2-x^2}$.
7. $\dfrac{(2x+1)(2x+7)^2}{(2x+5)^3}$.
8. $\dfrac{(x+2)^6(x+4)^2}{(x+1)^2(x+3)^6}$.
9. $\dfrac{(2x^2-1)\sqrt{x^2+1}}{x^3}$.
10. $x(x^n+n)^{\frac{n-1}{n}}$.

Find $\dfrac{dy}{dx}$ in each of the following cases:

11. $2x^2-4xy+3y^2=6x-4y+18$.
12. $x^3+3x^2y=y^3$.
13. $x=3y^2+2y^3$.
14. $(x^2+y^2)^2=2a^2(x^2-y^2)$.
15. $x=t+\dfrac{1}{t-1},\quad y=2t-\dfrac{1}{(t-1)^2}$.
16. $x=\dfrac{t}{\sqrt{1+t^2}},\quad y=\dfrac{1}{\sqrt{1-t^2}}$.
17. $x=t(t^2+a^2)^{\frac{1}{2}}\quad y=t(t^2+a^2)^{\frac{3}{2}}$.
18. $x=z^2+2z,\quad z=y^2+2y$.
19. $x^2+z^2=a^2,\quad yz=b^2$.

20. The volume elasticity of a fluid is $e=-v\dfrac{dp}{dv}$. If a gas expands according to Boyle's law, $pv=$ constant, show that $e=p$.

21. When a gas expands without receiving or giving out heat, the pressure, volume, and temperature satisfy the equations

$$pv=RT,\quad pv^n=C,$$

R, n, and C being constants. Find $\dfrac{dp}{dT}$ and $\dfrac{dv}{dT}$.

22. If v is the volume of a spherical segment of altitude h, show that $\frac{dv}{dh}$ is equal to the area of the circle forming the plane face of the segment.

23. If a polynomial equation

$$f(x) = 0$$

has two roots equal to r, $f(x)$ has $(x - r)^2$ as a factor, that is,

$$f(x) = (x - r)^2 f_1(x),$$

where $f_1(x)$ is a polynomial in x. Hence show that r is a root of

$$f'(x) = 0,$$

where $f'(x)$ is the derivative of $f(x)$.

Show by the method of Ex. 23 that each of the following equations has a double root and find it:

24. $x^3 - 3x^2 + 4 = 0.$

25. $x^3 - x^2 - 5x - 3 = 0.$

26. $4x^3 - 8x^2 - 3x + 9 = 0.$

27. $4x^4 - 12x^3 + x^2 + 12x + 4 = 0.$

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in each of the following cases.

28. $y = x\sqrt{a^2 - x^2}.$

29. $y = \frac{x^2}{(x+1)^2}.$

30. $xy = a^2.$

31. $ax + by + c = 0.$

32. $x = 2 + 3t,\ y = 4 - 5t.$

33. $x = \frac{t}{t+1},\quad y = \frac{t^2}{t+1}.$

34. If $y = x^2$, find $\frac{d^2y}{dx^2}$ and $\frac{d^2x}{dy^2}.$

35. Given $x^2 - y^2 = 1$, verify that

$$\frac{d^2y}{dx^2} = -\frac{d^2x}{dy^2}\left(\frac{dy}{dx}\right)^3.$$

36. If n is a positive integer, show that

$$\frac{d^n}{dx^n}x^n = \text{constant}.$$

37. If u and v are functions of x, show that

$$\frac{d^4}{dx^4}(uv) = \frac{d^4u}{dx^4}\cdot v + 4\frac{d^3u}{dx^3}\cdot\frac{dv}{dx} + 6\frac{d^2u}{dx^2}\cdot\frac{d^2v}{dx^2} + 4\frac{du}{dx}\cdot\frac{d^3v}{dx^3} + u\frac{d^4v}{dx^4}.$$

Compare this with the binomial expansion for $(u + v)^4$.

38. If $f(x) = (x - r)^3 f_1(x)$, where $f_1(x)$ is a polynomial, show that

$$f'(r) = f''(r) = 0.$$

CHAPTER IV

39. A particle moves along a straight line the distance

$$s = 4t^3 - 21t^2 + 36t + 1$$

feet in t seconds. Find its velocity and acceleration. When is the particle moving forward? When backward? When is the velocity increasing? When decreasing?

40. Two trains start from different points and move along the same track in the same direction. If the train in front moves a distance $6t^3$ in t hours and the rear one $12t^2$, how fast will they be approaching or separating at the end of one hour? At the end of two hours? When will they be closest together?

41. If $s = \sqrt{t}$, show that the acceleration is negative and proportional to the cube of the velocity.

42. The velocity of a particle moving along a straight line is

$$v = 2t^2 - 3t.$$

Find its acceleration when $t = 2$.

43. If $v^2 = \frac{k}{s}$, where k is constant, find the acceleration.

44. Two wheels, diameters 3 and 5 ft., are connected by a belt. What is the ratio of their angular velocities and which is greater? What is the ratio of their angular accelerations?

45. Find the angular velocity of the earth about its axis assuming that there are $365\frac{1}{4}$ days in a year.

46. A wheel rolls down an inclined plane, its center moving the distance $s = 5t^2$ in t seconds. Show that the acceleration of the wheel about its axis is constant.

47. An amount of money is drawing interest at 6 per cent. If the interest is immediately added to the principal, what is the rate of change of the principal?

48. If water flows from a conical funnel at a rate proportional to the square root of the depth, at what rate does the depth change?

49. A kite is 300 ft. high and there are 300 ft. of cord out. If the kite moves horizontally at the rate of 5 miles an hour directly away from the person flying it, how fast is the cord being paid out?

50. A particle moves along the parabola

$$100y = 16x^2$$

in such a way that its abscissa changes at the rate of 10 ft./ sec. Find the velocity and acceleration of its projection on the y-axis.

51. The side of an equilateral triangle is increasing at the rate of 10 ft. per minute and its area at the rate of 100 sq. ft. per minute. How large is the triangle?

CHAPTER V

52. The velocity of waves of length λ in deep water is proportional to

$$\sqrt{\frac{\lambda}{a}+\frac{a}{\lambda}}$$

when a is a constant. Show that the velocity is a minimum when $\lambda = a$.

53. The sum of the surfaces of a sphere and cube is given. Show that the sum of the volumes is least when the diameter of the sphere equals the edge of the cube.

54. A box is to be made out of a piece of cardboard, 6 inches square, by cutting equal squares from the corners and turning up the sides. Find the dimensions of the largest box that can be made in this way.

55. A gutter of trapezoidal section is made by joining 3 pieces of material each 4 inches wide, the middle one being horizontal. How wide should the gutter be at the top to have the maximum capacity?

56. A gutter of rectangular section is to be made by bending into shape a strip of copper. Show that the capacity of the gutter will be greatest if its width is twice its depth.

57. If the top and bottom margins of a printed page are each of width a, the side margins of width b, and the text covers an area c, what should be the dimensions of the page to use the least paper?

58. Find the dimensions of the largest cone that can be inscribed in a sphere of radius a.

59. Find the dimensions of the smallest cone that can contain a sphere of radius a.

60. To reduce the friction of a liquid against the walls of a channel, the channel should be so designed that the area of wetted surface is as small as possible. Show that the best form for an open rectangular channel with given cross section is that in which the width equals twice the depth.

61. Find the dimensions of the best trapezoidal channel, the banks making an angle θ with the vertical.

62. Find the least area of canvas that can be used to make a conical tent of 1000 cu. ft. capacity.

63. Find the maximum capacity of a conical tent made of 100 sq. ft. of canvas.

64. Find the height of a light above the center of a table of radius a, so as best to illuminate a point at the edge of the table; assuming that the illumination varies inversely as the square of the distance from the light and directly as the sine of the angle between the rays and the surface of the table.

65. A weight of 100 lbs., hanging 2 ft. from one end of a lever, is to be raised by an upward force applied at the other end. If the lever weighs 3 lbs. to the foot, find its length so that the force may be a minimum.

66. A vertical telegraph pole at a bend in the line is to be supported from tipping over by a stay 40 ft. long fastened to the pole and to a stake in the ground. How far from the pole should the stake be driven to make the tension in the stay as small as possible?

67. The lower corner of a leaf of a book is folded over so as just to reach the inner edge of the page. If the width of the page is 6 inches, find the width of the part folded over when the length of the crease is a minimum.

68. If the cost of fuel for running a train is proportional to the square of the speed and \$10 per hour for a speed of 12 mi./hr., and the fixed charges on \$90 per hour, find the most economical speed.

69. If the cost of fuel for running a steamboat is proportional to the cube of the speed and \$10 per hour for a speed of 10 mi./hr., and the fixed charges are \$14 per hour, find the most economical speed against a current of 2 mi./hr.

CHAPTER VI

Differentiate the following functions:

70. $\dfrac{\sin x}{x}$.

71. $\dfrac{\sin\theta}{1-\cos\theta}$.

72. $\dfrac{1+\cos\theta}{\sin\theta}$.

73. $\sin ax \cos ax$.

74. $\cot\dfrac{\theta}{2} - \csc\dfrac{\theta}{2}$.

75. $\tan 2x - \cot 2x$.

76. $\sec^2 x - \tan^2 x$.

77. $\sin^3\dfrac{3}{x}\sec\dfrac{x}{3}$.

78. $\tan\dfrac{x}{1-x}$.

79. $\dfrac{2\tan x}{1-\tan^2 x}$.

80. $5\sec^7\theta - 7\sec^5\theta$.

81. $\sec x\csc x - 2\cot x$.

Differentiate both sides of each of the following equations and show that the resulting derivatives are equal.

82. $\sec^2 x + \csc^2 x = \sec^2 x \csc^2 x$.

83. $\sin 2x = 2\sin x\cos x$.

84. $\sin 3x = 3\sin x - 4\sin^3 x$.

85. $\sin(x+a) = \sin x\cos a + \cos x\sin a$.

86. $\sec^2 x = 1 + \tan^2 x$.

87. $\sin x + \sin a = 2\sin\frac{1}{2}(x+a)\cos\frac{1}{2}(x-a)$.

88. $\cos a - \cos x = 2 \sin \frac{1}{2}(x + a) \sin \frac{1}{2}(x - a)$.

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in each of the following cases:

89. $x = a\cos^2\theta, \quad y = a\sin^2\theta$.
90. $x = a\cos^5\theta, \quad y = a\sin^5\theta$.
91. $x = \tan\theta - \theta, \quad y = \cos\theta$.
92. $x = \sec^2\theta, \quad y = \tan^2\theta$.
93. $x = \sec\theta, \quad y = \tan\theta$.
94. $x = \csc\theta - \cot\theta, \quad y = \csc\theta + \cot\theta$.

Differentiate the following functions:

95. $\sin^{-1}\sqrt{\frac{x}{2}}$.
96. $\cos^{-1}\left(\frac{1}{x}\right)$.
97. $\tan^{-1}\left(\frac{1-2x}{3}\right)$.
98. $\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}$.
99. $\cos^{-1}\frac{x}{\sqrt{x^2+1}}$.
100. $\csc^{-1}\frac{\sqrt{5}}{2x-1}$.
101. $\sec^{-1}\frac{1}{2}\left(x+\frac{1}{x}\right)$.
102. $a\csc^{-1}\frac{a}{x} + \sqrt{a^2 - x^2}$.
103. $\frac{x}{1+x^2} - \cot^{-1}x$.
104. $\sqrt{1-x}\sin^{-1}x - \sqrt{x}$.
105. $\sec^{-1}\frac{x+1}{x-1} + \sin^{-1}\frac{x-1}{x+1}$.
106. $\sin^{-1}\frac{a+b\cos x}{b+a\cos x}$.
107. $\frac{1}{2}\cos^{-1}x + \frac{b}{2}\sqrt{1-x^2}$.
108. $\sqrt{x^2-a^2} - a\sec^{-1}\frac{x}{a}$.

109. $e^{\sqrt{x}}$.
110. $\sqrt{e^x}$.
111. $(\sqrt{e})^x$.
112. $5^{t\ln t}$.
113. $7^{\frac{1}{x}}$.
114. $a^x \ln x$.
115. $\ln \sin^n x$.
116. $\ln\ln x$.
117. $\ln\left(\frac{1-e^x}{e^x}\right)$.
118. $\frac{1}{a}\tan^{-1}\frac{1}{a} + \frac{1}{2}\ln(a^2+x^2)$.
119. $e^{-kt}\cos(a+bt)$.
120. $\ln\left(a+\sqrt{a^2+x^2}\right)$.
121. $\left(x+\frac{1}{x}\right)\ln\left(x+\frac{1}{x}\right) - x - \frac{1}{x}$.
122. $\ln\frac{\sqrt{x+a}+\sqrt{x-a}}{\sqrt{x+a}-\sqrt{x-a}}$.
123. $\tan^{-1}\frac{1}{2}(e^x + e^{-x})$.
124. $\ln\left(\sqrt{x}+\sqrt{x+2}\right)$.

125. $(x+1)\ln(x^2+2x+5) + \frac{3}{2}\tan^{-1}\frac{x+1}{2}$.
126. $\sec\frac{1}{2}x\tan\frac{1}{2}x + \ln(\sec\frac{1}{2}x + \tan\frac{1}{2}x)$.

127. $x \sec^{-1} \frac{1}{2}\left(x + \frac{1}{x}\right) - \ln (x^2 + 1)$.

128. $\frac{x}{3} \ln \left(\frac{4}{9} x^2 + 1\right) - \frac{2}{3} x + \tan^{-1} \frac{2}{3} x$.

CHAPTER VII

Find the equations of the tangent and normal to each of the following curves at the point indicated:

129. $y^2 = 2x + y$, at $(1, 2)$.
130. $x^2 - y^2 = 5$, at $(3, 2)$.
131. $x^2 + y^2 = x + 3y$, at $(-1, 1)$.
132. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 2$, at $(1, 1)$.
133. $y = \ln x$, at $(1, 0)$.
134. $x^2 (x + y) = a^2 (x - y)$, at $(0, 0)$.
135. $x = 2 \cos \theta$, $y = 3 \sin \theta$, at $\theta = \frac{\pi}{2}$.
136. $r = a (1 + \cos \theta)$, at $\theta = \frac{\pi}{4}$.

Find the angles at which the following pairs of curves intersect:

137. $x^2 + y^2 = 8x$, $y^2 (2 - x) = x^3$.
138. $y^2 = 2ax + a^2$, $x^2 = 2by + b^2$.
139. $x^2 = 4ay$, $(x^2 + 4a^2) y = 8a^3$.
140. $y^2 = 6x$, $x^2 + y^2 = 16$.
141. $y = \frac{1}{2} (e^x + e^{-x})$, $y = 1$.
142. $y = \sin x$, $y = \sin 2x$.
143. Show that all the curves obtained by giving different values to n in the equation

$$\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2,$$

are tangent at (a, b).

144. Show that for all values of a and b the curves

$$x^3 - 3xy^2 = a, \quad 3x^2y - y^3 = b,$$

intersect at right angles.

Examine each of the following curves for direction of curvature and points of inflection:

145. $y = \frac{1 - x}{1 + x^2}$.
146. $y = \tan x$.
147. $x = 6y^2 - 2y^3$.

148. $x = 2t - \frac{1}{t^2},\ y = 2t + \frac{1}{t^2}.$

149. Clausius's equation connecting the pressure, volume, and temperature of a gas is

$$p = \frac{RT}{v - a} - \frac{c}{T(v + b)^3},$$

R, a, b, c being constants. If T is constant and p, v the coördinates of a point, this equation represents an isothermal. Find the value of T for which the tangent at the point of inflection is horizontal.

150. If two curves $y = f(x)$, $y = F(x)$ intersect at $x = a$, and $f'(a) = F'(a)$, but $f''(a)$ is not equal to $F''(a)$, show that the curves are tangent and do not cross at $x = a$. Apply to the curves $y = x^2$ and $y = x^3$ at $x = 0$.

151. If two curves $y = f(x)$, $y = F(x)$ intersect at $x = a$, and $f'(a) = F'(a)$, $f''(a) = F''(a)$, but $f'''(a)$ is not equal to $F'''(a)$, show that the curves are tangent and cross at $x = a$. Apply to the curves $y = x^2$ and $y = x^2 + (x - 1)^3$ at $x = 1$.

Find the radius of curvature on each of the following curves at the point indicated:

152. Parabola $y^2 = ax$ at its vertex.

153. Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at its vertices.

154. Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $x = \sqrt{a^2 + b^2}$.

155. $y = \ln \csc x$, at $\left(\frac{\pi}{2}, 0\right)$.

156. $x = \frac{1}{2} \sin y - \frac{1}{2} \ln (\sec y + \tan y)$, at any point (x, y).

157. $x = a \cos^3 \theta$, $y = a \sin^3 \theta$, at any point.

158. Find the center of curvature of $y = \ln (x - 2)$ at $(3, 0)$.

Find the angle ψ at the point indicated on each of the following curves:

159. $r = 2^\theta$, at $\theta = 0$.

160. $r = a + b \cos \theta$, at $\theta = \frac{\pi}{2}$.

161. $r(1 - \cos \theta) = k$, at $\theta = \frac{\pi}{3}$.

162. $r = a \sin 2\theta$, at $\theta = \frac{\pi}{4}$.

Find the angles at which the following pairs of curves intersect:

163. $r(1 - \cos \theta) = a, \quad r = a(1 - \cos \theta)$.

164. $r = a \sec^2 \frac{\theta}{2}, \qquad r = b \csc^2 \frac{\theta}{2}$.

165. $r = a\cos\theta,\qquad r = a\cos 2\theta.$
166. $r = a\sec\theta,\qquad r = 2a\sin\theta.$

Find the equations of the tangent lines to the following curves at the points indicated:

167. $x = 2t, y = \frac{2}{t}, z = t^2$, at $t = 2$.
168. $x = \sin t, y = \cos t, z = \sec t$, at $t = 0$.
169. $x^2 + y^2 + z^2 = 6, x + y + z = 2$, at $(1, 2, -1)$.
170. $z = x^2 + y^2, z^2 = 2x - 2y$, at $(1, -1, 2)$.

CHAPTER VIII

171. A point describes a circle with constant speed. Show that its projection on a fixed diameter moves with a speed proportional to the distance of the point from that diameter.

172. The motion of a point (x, y) is given by the equations

$$x = \frac{t}{2}\sqrt{a^2 - t^2} + \frac{a^2}{2}\sin^{-1}\frac{t}{a},$$

$$y = \frac{t}{2}\sqrt{a^2 + t^2} + \frac{a^2}{2}\ln\left(t + \sqrt{a^2 + t^2}\right).$$

Show that its speed is constant.

Find the speed, velocity, and acceleration in each of the following cases:

173. $x = 2 + 3t,\ y = 4 - 9t.$
174. $x = a\cos(\omega t + \alpha),\ y = a\sin(\omega t + \alpha).$
175. $x = a + \alpha t,\ y = b + \beta t,\ z = c + \gamma t.$
176. $x = e^t\sin t,\ y = e^t\cos t,\ z = kt.$
177. The motion of a point $P(x, y)$ is determined by the equations

$$x = a\cos(nt + \alpha),\ y = b\sin(nt + \alpha).$$

Show that its acceleration is directed toward the origin and has a magnitude proportional to the distance from the origin.

178. A particle moves with constant acceleration along the parabola $y^2 = 2cx$. Show that the acceleration is parallel to the x-axis.

179. A particle moves with acceleration $[a, o]$ along the parabola $y^2 = 2cx$. Find its velocity.

180. Show that the vector $\left[\frac{d^2x}{ds^2}, \frac{d^2y}{ds^2}\right]$ extends along the normal at (x, y) and is in magnitude equal to the curvature at (x, y).

CHAPTER IX

181. Show that the function

$$x^{\frac{2}{3}} - 1$$

vanishes at $x = -1$ and $x = 1$, but that its derivative does not vanish between these values. Is this an exception to Rolle's theorem?

182. Show that the equation

$$x^5 - 5x + 4 = 0$$

has only two distinct real roots.

183. Show that

$$\operatorname*{Lim}_{x \doteq 0} \frac{x^2 \sin \frac{1}{x}}{\sin x} = 0,$$

but that this value cannot be found by the methods of Art. 73. Explain.

184. Show that

$$\operatorname*{Lim}_{x \doteq 0} \frac{1 - \cos x}{\cos x} = 0.$$

Why cannot this result be obtained by the methods of Art. 73?

Find the values of the following limits:

185. $\operatorname*{Lim}_{x \doteq 0} \dfrac{xe^{3x} - x}{1 - \cos 2x}$.

186. $\operatorname*{Lim}_{x \doteq 3} \dfrac{\sqrt{3x} - \sqrt{12 - x}}{2x - 3\sqrt{19 - 5x}}$.

187. $\operatorname*{Lim}_{x \doteq 1} \dfrac{\tan \frac{\pi x}{2}}{1 + \csc(x - 1)}$.

188. $\operatorname*{Lim}_{x \doteq 1} \dfrac{\ln(1 - x)}{\cot(\pi x)}$.

189. $\operatorname*{Lim}_{x \doteq 0} \left(\dfrac{1}{x^2} - \dfrac{\cot x}{x}\right)$.

190. $\operatorname*{Lim}_{x \doteq \infty} x^n e^{-x^2}$.

191. $\operatorname*{Lim}_{x \doteq 0} \dfrac{x \ln x}{\sin^2 x - x \cot x}$.

192. $\operatorname*{Lim}_{x \doteq 0} (\sec x)^{\frac{1}{x^2}}$.

193. The area of a regular polygon of n sides inscribed in a circle of radius a is

$$na^2 \sin \frac{\pi}{n} \cos \frac{\pi}{n}.$$

Show that this approaches the area of the circle when n increases indefinitely.

194. Show that the curve

$$x^3 + y^3 = 3xy$$

is tangent to both coördinate axes at the origin.

CHAPTER X

Determine the values of the following functions correct to four decimals:

195. $\cos 62°$.

196. $\sin 33°$.

197. $\ln(1.2)$.

198. $\sqrt[5]{1.1}$.

199. $\tan^{-1}(\frac{1}{10})$.

200. $\csc(31°)$.

201. Calculate π by expanding $\tan^{-1} x$ and using the formula

$$\frac{\pi}{4} = \tan^{-1}(1).$$

202. Given $\ln 5 = 1.6094$, calculate $\ln 24$.

203. Prove that

$$D = \sqrt{\tfrac{3}{2} h}$$

is an approximate formula for the distance of the horizon, D being the distance in miles and h the altitude of the observer in feet.

Prove the following expansions indicating if possible the values of x for which they converge:

204. $\ln(1 + x^2) = \ln 10 + \frac{3}{5}(x - 3) - \frac{2}{25}(x - 3)^2 + \cdots$.

205. $\ln(e^x + e^{-x}) = \frac{x^2}{2} - \frac{x^4}{12} + \frac{x^6}{45} + \cdots$.

206. $\ln(1 + \sin x) = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \cdots$.

207. $e^x \sec x = 1 + x + x^2 + \frac{2}{3}x^3 + \frac{1}{2}x^4 + \cdots$.

208. $\ln\left(x + \sqrt{1 + x^2}\right) = x - \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \cdots$.

209. $\ln \frac{x + 1}{x - 1} = 2\left[\frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \cdots\right]$.

210. $\ln \tan x = \ln x + \frac{x^2}{3} + \frac{7x^4}{90} + \cdots$.

211. $e^{\sin x} = 1 + x + \frac{x^2}{2!} - \frac{3x^4}{4!} - \cdots$.

212. $e^{\tan x} = 1 + x + \frac{x^2}{2!} + \frac{3x^3}{3!} + \frac{9x^4}{4!} + \cdots$.

Determine the values of x for which the following series converge:

213. $1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$.

214. $(x - 1) + \frac{(x - 1)^2}{2^2} + \frac{(x - 1)^3}{3^3} + \frac{(x - 1)^4}{4^4} + \cdots$.

215. $1 + 2x + 3x^2 + 4x^3 + \cdots$.

216. $2 + \frac{x + 2}{1 \cdot 2} + \frac{(x + 2)^2}{2 \cdot 3} + \frac{(x + 2)^3}{3 \cdot 4} + \cdots$.

CHAPTER XI

In each of the following exercises show that the partial derivatives satisfy the equation given:

217. $u = xy + y^2z^2, \quad x\dfrac{\partial u}{\partial x} + z\dfrac{\partial u}{\partial z} = y\dfrac{\partial u}{\partial y}.$

218. $z = x^4 - 2x^2y^2 + y^4, \quad y\dfrac{\partial z}{\partial x} + x\dfrac{\partial z}{\partial y} = 0.$

219. $u = (x+y)\ln xz, \quad x\left(\dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial y}\right) = z\dfrac{\partial u}{\partial z}.$

220. $u = \left(x + \dfrac{1}{y}\right)\tan^{-1}\left(y - \dfrac{1}{z}\right), \quad \dfrac{\partial u}{\partial x} + y^2\dfrac{\partial u}{\partial y} = y^2z^2\dfrac{\partial u}{\partial z}.$

221. $u = xy + \dfrac{z}{x},$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xz\frac{\partial^2 u}{\partial x \partial z} + z^2\frac{\partial^2 u}{\partial z^2} = y^2\frac{\partial^2 u}{\partial y^2}.$$

222. $z = \ln(x^2 + y^2), \quad \dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0.$

223. $u = \dfrac{y + x}{y - z},$

$$\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial x^2}.$$

Prove the following relations assuming that z is a function of x and y:

224. $u = (x + y - z)^2,$

$$\frac{\partial u}{\partial y} - \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}\frac{\partial z}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial z}{\partial y}.$$

225. $u = z + e^{xy},$

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$

226. $u = z(x^2 - y^2),$

$$y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = (x^2 - y^2)\left(y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y}\right)$$

227. If $x = \dfrac{r}{2}(e^\theta + e^{-\theta}), \quad y = \dfrac{r}{2}(e^\theta - e^{-\theta})$, show that

$$\left(\frac{\partial x}{\partial r}\right)_\theta = \left(\frac{\partial r}{\partial x}\right)_y.$$

228. If $xyz = a^3$, show that

$$\left(\frac{\partial y}{\partial x}\right)_z\left(\frac{\partial z}{\partial y}\right)_x\left(\frac{\partial x}{\partial z}\right)_y = -1.$$

In each of the following exercises find Δz and its principal part, assuming that x and y are the independent variables. When Δx and Δy approach zero, show that the difference of Δz and its principal part is an infinitesimal of higher order than Δx and Δy.

229. $z = xy$.
230. $z = x^2 - y^2 + 2x$.
231. $z = \dfrac{y}{x^2+1}$.
232. $z = \sqrt{x^2+y^2}$.

Find the total differentials of the following functions:

233. $ax^4 + bx^2y^2 + cy^4$.
234. $\ln(x^2+y^2+z^2)$.
235. $x^2 \tan^{-1}\dfrac{y}{x} - y^2 \tan^{-1}\dfrac{x}{y}$.
236. $yze^x + zxe^y + xye^z$.

237. If $u = x^n f(z)$, $z = \dfrac{y}{x}$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

238. If $u = f(r, s)$, $r = x + y$, $s = x - y$, show that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2\frac{\partial u}{\partial r}.$$

239. If $u = f(r, s, t)$, $r = \dfrac{x}{y}$, $s = \dfrac{y}{z}$, $t = \dfrac{z}{x}$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.$$

240. If α is the angle between the x-axis and the line OP from the origin to $P(x, y, z)$, find the derivatives of α in the directions parallel to the coördinate axes.

241. Show that

$$(\cot y - y\sec x\tan x)\,dx - (x\csc^2 y + \sec x)\,dy$$

is an exact differential.

Find the equations of the normal and tangent plane to each of the following surfaces at the point indicated:

242. $x^2 + 2y^2 - z^2 = 16$, at $(3, 2, -1)$.
243. $2x + 3y - 4z = 4$, at $(1, 2, 1)$.
244. $z^2 = 8xy$, at $(2, 1, -4)$.
245. $y = z^2 - x^2 + 1$, at $(3, 1, -3)$.
246. Show that the largest rectangular parallelopiped with a given surface is a cube.

247. An open rectangular box is to be constructed of a given amount of material. Find the dimensions if the capacity is a maximum.

248. A body has the shape of a hollow cylinder with conical ends. Find the dimensions of the largest body that can be constructed from a given amount of material.

249. Find the volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

250. Show that the triangle of greatest area inscribed in a given circle is equilateral.

251. Find the point so situated that the sum of its distances from the vertices of an acute angled triangle is a minimum.

252. At the point (x, y, z) of space find the direction along which a given function $F(x, y, z)$ has the largest directional derivative.

ANSWERS TO EXERCISES

Page 8

1. $-\frac{3}{5}$.
2. $\sqrt{2}$.
3. -1.
4. -1.
5. 1.
6. $\frac{1}{2}$.

Page 14

3. 2.
5. The tangents are parallel to the x-axis at $(-1, -1)$, $(0, 0)$, and $(1, -1)$. The slope is positive between $(-1, -1)$ and $(0, 0)$ and on the right of $(1, -1)$.
10. Negative.
11. Positive in 1st and 4th quadrants, negative in 2nd and 3rd.

Pages 27, 28

31. When $x = 4$, $y = \frac{4}{5}$ and $dy = 0.072\,dx$. When x changes to 4.2, $dx = 0.2$ and an approximate value for y is $y + dy = 0.814$. This agrees to 3 decimals with the exact value.
32. When $x = 0$, the function is equal to 1 and its differential is $-dx$. When $x = 0.3$, an approximate value is then $1 - dx = 0.7$. The exact value is 0.754.
34. 18.
35. $(a, 2\,a)$.
36. Increases when $x < \dfrac{a}{3}$, decreases when $x > \dfrac{a}{3}$.
37. $x = \pm\dfrac{1}{\sqrt[4]{2}}$.
38. $\tan^{-1}\frac{2}{3}$.
39. $-\dfrac{2}{(x-1)^2} = -\dfrac{(y-2)^2}{2}$.

Page 31

1. $-\dfrac{2}{(x-1)^2}$, $\dfrac{4}{(x-1)^3}$.
2. $-\dfrac{x}{\sqrt{a^2 - x^2}}$, $-\dfrac{a^2}{(a^2 - x^2)^{\frac{3}{2}}}$.
3. $(x-1)^2(x+2)^3(7\,x+2)$, $(x-1)(x+2)^2(42\,x^2 + 24\,x - 12)$.
4. $\dfrac{2}{y}$, $-\dfrac{4}{y^3}$.
5. $-\dfrac{x}{y}$, $-\dfrac{a^2}{y^3}$.
6. $\dfrac{x}{2\,y}$, $-\dfrac{1}{4\,y^3}$.
7. $\dfrac{1-y}{x-1}$, $\dfrac{2}{(x-1)^3}$.
8. $-\dfrac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$, $\dfrac{a^{\frac{2}{3}}}{3\,x^{\frac{4}{3}}\,y^{\frac{2}{3}}}$.

13. $\dfrac{d^2y}{dx^2} = -\dfrac{12}{t(2+3t)^3}, \quad \dfrac{d^2x}{dy^2} = \dfrac{12}{t(2-3t)^3}.$

Pages 36–38

1. $v = 100 - 32t$, $a = -32$. Rises until $t = 3\frac{1}{8}$. Highest point $h = 206.25$.

2. $v = t^3 - 12t^2 + 32t$, $a = 3t^2 - 24t + 32$. Velocity decreasing between $t = 1.691$ and $t = 6.309$. Moving backward when t is negative or between 4 and 8.

7. $\omega = b - 2ct$, $\alpha = -2c$. Wheel comes to rest when $t = \dfrac{b}{2c}$.

10. 9π cu. ft./min.

11. 144π sq. ft./sec.

12. Decreasing 8π cu. ft./sec.

13. $\sqrt{3} : 1$.

14. $\frac{1}{3}\sqrt{3}$ in./sec.

15. $12\frac{1}{2}$ ft./sec., $7\frac{1}{2}$ ft./sec.

16. $4\sqrt{3}$ mi./hr.

17. $\dfrac{c \tan\beta}{\pi a^2}$ ft./sec.

18. Neither approaching nor separating.

19. 25.8 ft./sec.

20. $64\sqrt{3}$ ft./sec.

Pages 43–45

1. Minimum $3\frac{7}{8}$.

2. Minimum -10, maximum 22.

3. Maximum at $x = 0$, minima at $x = -1$ and $x = +1$.

4. Minimum when $x = 0$.

10. $\frac{3}{2}\sqrt[3]{2}$.

13. $\frac{2}{3}a\sqrt{3}$.

14. Length of base equals twice the depth of the box.

16. Radius of base equals two-thirds of the altitude.

17. Altitude equals $\dfrac{4}{\pi}$ times diameter of base.

18. $\dfrac{16\pi\sqrt{3}}{27}$.

19. $\frac{1}{4}(a_1 + a_2 + a_3 + a_4)$.

20. Girth equals twice length.

21. Radius equals $2\sqrt{6}$ inches.

22. The distance from the more intense source is $\sqrt[3]{2}$ times the distance from the other source.

23. $12\sqrt{2}$.

24. $[5^{\frac{2}{3}} + 6^{\frac{2}{3}}]^{\frac{3}{2}}$.

25. $19\frac{1}{3}$ ft.

26. Radius of semicircle equals height of rectangle.

27. 4 pieces 6 inches long and 2 pieces 2 ft. long.

28. The angle of the sector is two radians.

29. At the end of 4 hours.

30. He should land 4.71 miles from his destination.

31. $\frac{a\sqrt{2}}{2}$, a being the length of side.

35. $2\frac{1}{2}$ mi./hr.

36. 13.6 knots.

Page 48

1. Maximum $= a$, minimum $= -a$.
2. Maximum $= 0$, minimum $= -(\frac{4}{27})^{\frac{1}{5}}$.
3. Minimum $= -1$.
4. Minimum $= 0$, maximum $= \frac{4}{27}$.
10. Either 4 or 5.

Pages 52, 53

19. $A = 3$.
20. $A = -\frac{7}{68}$, $B = -\frac{3}{34}$.
21. $\sqrt{3} - \frac{3}{2}$.
22. $n\pi \pm \frac{\pi}{6}$, n being any integer.
23. Velocity $= -2\pi nA$, acceleration $= 0$.
24. $\frac{40\pi}{3}$ miles per minute.
25. $\frac{9}{4}$ radians per hour.
26. $\frac{9}{2} + \frac{5}{6}\pi\sqrt{3}$.
28. $13\sqrt{13}$.
29. The needle will be inclined to the horizontal at an angle of about 32° 30′
30. 120°.
31. 120°.
32. $\frac{a}{\pi}$.
33. If the spokes are extended outward, they will form the sides of an isosceles triangle.

Page 56

24. $\omega = \frac{v}{r}\cos\phi$, r being the radius of pulley and ϕ the angle formed by the string and line along which its end moves.
25. $4\sqrt{35}$.

Page 61

27. $x = n\pi + \cot^{-1} 2$, n being any integer.
30. $x < -3$, $x > 2$, or $-2 < x < 1$.

Pages 65, 66

1. $2y - x = 5$, $y + 2x = 0$.
2. $y + 4x = 8$, $4y - x = 15$.
3. $2y \mp x = \pm a$, $y \pm 2x = \pm 3a$.

4. $y = a(x \ln b + 1), \quad x + ay \ln b = a^2 \ln b.$

5. $y - \frac{3}{2} = \frac{3}{2}\sqrt{3}\left(x - \frac{\pi}{6}\right), \quad y - \frac{3}{2} + \frac{2}{9}\sqrt{3}\left(x - \frac{\pi}{6}\right) = 0.$

6. $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1, \quad b^2x_1y - a^2y_1x = (b^2 - a^2)x_1y_1.$

7. $x + y = 2, \quad x - y = 2.$

8. $x + 3y = 4, \quad y - 3x = 28.$

9. $y + x \tan \frac{1}{2}\phi_1 = a\phi_1 \tan \frac{1}{2}\phi_1.$

10. $90°, \quad \tan^{-1} \frac{3}{4}.$

11. $45°.$

12. $90°.$

13. $\tan^{-1} 2\sqrt{2}.$

14. $\tan^{-1} \frac{\ln 10 - 1}{\ln 10 + 1}.$

15. $\tan^{-1} 3\sqrt{3}.$

Page 70

1. Point of inflection (0, 3). Concave upward on the right of this point, downward on the left.
2. Point of inflection $(\frac{1}{2}, -\frac{5}{2})$. Concave upward on the right of this point, downward on the left.
3. The curve is everywhere concave upward. There is no point of inflection.
4. Point of inflection (1, 0). Concave on the left of this point, downward on the right.
5. Point of inflection $\left(-2, -\frac{2}{e^2}\right)$. Concave upward on the right, downward on the left.
6. Points of inflection at $x = \pm\frac{1}{\sqrt{2}}$. Concave downward between the points of inflection, upward outside.
7. Points of inflection (0, 0), $(\pm 3, \pm 1)$. Concave upward when $-3 < x < 0$ or $x > 3$.
8. Point of inflection at the origin. Concave upward on the left of the origin.

Pages 76, 77

1. $\pm 2\sqrt{6}.$

3. $\frac{a^2}{b}.$

4. $3\sqrt{2}.$

5. $e^{\frac{\pi}{2}}\sqrt{2}.$

6. $\frac{4}{3}a.$

7. $\frac{y^2}{a}.$

8. $\sec y.$

9. $\frac{(y^2 + 1)^2}{4y}.$

10. $2a\sec^3\frac{\theta}{2}.$

Page 79

There are two angles ψ depending on the direction in which s is measured along the curve. In the following answers only one of these angles is given.

1. $\tan^{-1}\left(\frac{\pi}{3}\right)$.
2. $\frac{\pi}{4}$.
3. $\frac{\pi}{3}$.
4. $\tan^{-1}(-2)$.
5. $\frac{1}{6}\pi$.
7. 0°, 90°, and $\tan^{-1} 3\sqrt{3}$.
8. $\theta = \pm\frac{2}{3}\pi$.
10. 3.

Page 84

1. $\dfrac{x-\sqrt{2}}{\sqrt{2}} = \dfrac{y-1}{2} = \dfrac{z - \frac{\pi a}{4}}{a}$.
2. $\dfrac{x-e}{e} = 1 - ye = \dfrac{z-1}{2}$.
3. $\dfrac{x - e^{\frac{\pi}{2}}}{e^{\frac{\pi}{2}}} = -\dfrac{y}{e^{\frac{\pi}{2}}} = \dfrac{z - \frac{\pi k}{2}}{k}$.
5. $\tan^{-1}\dfrac{a}{k}$.
6. $\tan^{-1}\dfrac{t}{\sqrt{2}}$.
7. $69^\circ\, 29'$.

Pages 92, 93

1. The angular speed is $\dfrac{av}{a^2+x^2}$, where x is the abscissa of the moving point.
2. If x_1 is the abscissa of the end in the x-axis and y_1 the ordinate of the end in the y-axis, the velocity of the middle point is

$$\left[\pm\tfrac{1}{2}v,\quad \mp\frac{vx_1}{2y_1}\right],$$

the upper signs being used if the end in the x-axis moves to the right, the lower signs if it moves to the left. The speed is $\dfrac{av}{2y_1}$.

3. The velocity is

$$[v - a\omega\sin\theta,\quad a\omega\cos\theta],$$

where θ is the angle from the x-axis to the radius through the moving point. The speed is

$$\sqrt{v^2 + a^2\omega^2 - 2a\omega v\sin\theta}.$$

6. The boat should be pointed 30° up the river.
7. Velocity $= [a, b, c - gt]$, Acceleration $= [0, 0, -g]$,

$$\text{Speed} = \sqrt{a^2 + b^2 + (c-gt)^2}.$$

9. $$\text{Velocity} = [a\omega\,(1 - \cos\phi),\quad a\omega\sin\phi],$$
$$\text{Speed} = a\omega\sqrt{2 - 2\cos\phi} = 2\,a\omega\sin\tfrac{1}{2}\phi,$$
$$\text{Acceleration} = [a\omega^2\sin\phi,\quad a\omega^2\cos\phi].$$

10. $$\left[-\frac{3\,v^2}{4\,a}\,\frac{\sin\frac{3}{2}\theta}{\sin\frac{1}{2}\theta},\quad \frac{3\,v^2}{4\,a}\,\frac{\cos\frac{3}{2}\theta}{\sin\frac{1}{2}\theta}\right].$$

12. $x = vt\cos\omega t$, $y = vt\sin\omega t$. The velocity is the sum of the partial velocities, but the acceleration is not.

13. $x = a\cos\omega t + b\cos 2\,\omega t$, $y = a\sin\omega t + b\sin 2\,\omega t$. The velocity is the sum of the partial velocities and the acceleration the sum of the partial accelerations.

14. $x = a\omega_1 t - a\sin(\omega_1 + \omega_2)\,t$, $y = a\cos(\omega_1 + \omega_2)\,t$. The velocity is the sum of the partial velocities and the acceleration the sum of the partial accelerations.

Page 100

2. $\frac{9}{10}$.
3. n.
4. 0.
5. e^a.
6. 2.
7. -2.
8. 1.
9. 0.
10. π^2. ?
11. 1.
12. 1.
13. 0.
14. $-\frac{1}{2}$.
15. 0.
16. 0.
17. $-\frac{1}{3}$.
18. $\frac{1}{2}$.
19. 1.
20. -3.
21. a.
22. $\frac{1}{\pi}$.
23. $f'(x)\,dx$.
24. ∞.
25. $\frac{1}{2}$.
26. ∞.
27. ∞.
28. 1.
29. 1.
30. a.
31. e^m

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1. 0.0872.
2. 0.8480.
3. 1.0724.
4. 1.6003.
5. 1.0154.
6. 0.1054.
7. 1.6487.
8. 0.0997.
9. 2.833.

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21. $(-2, 1, 0)$.
22. $(1, 1, 2)$.

Pages 123, 124

1. Increment = -0.151, principal part = -0.154.
4. $\dfrac{dT}{T} = \dfrac{1}{2}\left(\dfrac{dl}{l} - \dfrac{dg}{g}\right)$. Since dl and dg may be either positive or negative, the percentage error in T may be $\frac{1}{2}$ the sum of the percentage errors in l and g.
5. The percentage error in g may be as great as that in s plus twice that in T.

13. $-\dfrac{u+v}{u}$.

15. $\dfrac{2z^2 - uy}{zx - 2uv}$.

14. $\dfrac{1}{x}(x^2 + y^2 + xy - z^2)$.

Pages 131, 132

1. $\dfrac{du}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{d\phi}{dx}$.

2. $\left(\dfrac{\partial u}{\partial x}\right)_y = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial z}\dfrac{d\phi}{dx}$.

3. $\dfrac{du}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{d\psi}{\partial x} + \dfrac{\partial f}{\partial z}\left(\dfrac{\partial \phi}{\partial x} + \dfrac{\partial \phi}{\partial y}\dfrac{d\psi}{dx}\right)$.

5. $\dfrac{dz}{dx} = \dfrac{\dfrac{\partial f}{\partial y}\dfrac{\partial F}{\partial x} - \dfrac{\partial f}{\partial x}\dfrac{\partial F}{\partial y}}{\dfrac{\partial f}{\partial y} + \dfrac{\partial f}{\partial z}\dfrac{\partial F}{\partial y}}$.

13. $3\sqrt{3} - 4$.

14. $-\dfrac{e}{r^3}(x\cos\alpha + y\cos\beta + z\cos\gamma)$.

Page 135

1. $\dfrac{x-1}{1} = \dfrac{y-2}{2} = \dfrac{z-2}{2}$, $(x-1) + 2(y-2) + 2(z-2) = 0$.
2. Normal, $y + 4x = 5$, $z = 3$.
   Tangent plane, $x - 4y - 14 = 0$.
3. $\dfrac{x-3}{3} = \dfrac{y-4}{4} = \dfrac{z-5}{-5}$, $3x + 4y - 5z = 0$.
4. $\dfrac{x-5}{1} = \dfrac{y-1}{5} = \dfrac{z-3}{-3}$, $x + 5y - 3z - 1 = 0$.
5. $\dfrac{x-5}{-1} = \dfrac{y-1}{4} = \dfrac{z-1}{6}$, $x - 4y - 6z + 5 = 0$.
6. $x + z = y - z = \pm\sqrt{2}$.

Pages 138, 139

1. The box should have a square base with side equal to twice the depth.
2. The cylinder and cone have volumes in the ratio 3 : 2 and lateral surfaces in the ratio 2 : 3.
4. The center of gravity of the triangle ABC.

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INTEGRAL CALCULUS

BY

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PREFACE

THIS text on Integral Calculus completes the course in mathematics begun in the Analytic Geometry and continued in the Differential Calculus. Throughout this course I have endeavored to encourage individual work and to this end have presented the detailed methods and formulas rather as suggestions than as rules necessarily to be followed.

The book contains more exercises than are ordinarily needed. As material for review, however, a supplementary list of exercises is placed at the end of the text.

The appendix contains a short table of integrals which includes most of the forms occurring in the exercises. Through the courtesy of Prof. R. G. Hudson I have taken a two-page table of natural logarithms from his Engineers' Manual.

I am indebted to Professors H. W. Tyler, C. L. E. Moore, and Joseph Lipka for suggestions and assistance in preparing the manuscript.

H. B. PHILLIPS.

CAMBRIDGE, MASS.
June, 1917.

CONTENTS

INTEGRAL CALCULUS

CHAPTER I

INTEGRATION

1. Integral. — A function $F(x)$ whose differential is equal to $f(x)\,dx$ is called an *integral* of $f(x)\,dx$. Such a function is represented by the notation $\int f(x)\,dx$. Thus

$$F(x) = \int f(x)\,dx, \qquad dF(x) = f(x)\,dx,$$

are by definition equivalent equations. The process of finding an integral of a given differential is called *integration.*

For example, since $d(x^2) = 2x\,dx$,

$$\int 2x\,dx = x^2.$$

Similarly,

$$\int \cos x\,dx = \sin x, \qquad \int e^x\,dx = e^x.$$

The test of integration is to differentiate the integral. If it is correct, its differential must be the expression integrated.

2. Constant of Integration. — If C is any constant,

$$d[F(x) + C] = d\,F(x).$$

If then $F(x)$ is one integral of a given differential, $F(x) + C$ is another. For example,

$$\int 2x\,dx = x^2 + C, \qquad \int \cos x\,dx = \sin x + C,$$

where C is any constant.

We shall now prove that, *if two continuous functions of one variable have the same differential, their difference is constant.*

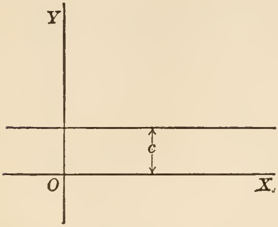

FIG. 2.

Suppose $F_1(x)$ and $F_2(x)$ are functions having the same differential. Then

$$dF_1(x) = dF_2(x).$$

Let $y = F_2(x) - F_1(x)$ and plot the locus representing y as a function of x. The slope of this locus is

$$\frac{dy}{dx} = \frac{dF_2(x) - d\,F_1(x)}{dx} = 0.$$

Since the slope is everywhere zero, the locus is a horizontal line. The equation of such a line is $y = C$. Therefore,

$$F_2(x) - F_1(x) = C,$$

which was to be proved.

If then $F(x)$ is one continuous integral of $f(x)\,dx$, any other continuous integral has the form

$$\int f(x)\,dx = F(x) + C. \tag{2}$$

Any value can be assigned to C. It is called an *arbitrary* constant.

3. Formulas. — Let a and n be constants, u, v, w, variables.

I. $$\int du \pm dv \pm dw = \int du \pm \int dv \pm \int dw.$$

II. $$\int a\,du = a \int du.$$

III. $$\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \text{ if } n \text{ is not } -1.$$

IV. $$\int u^{-1}\,du = \int \frac{du}{u} = \ln u + C.$$

These formulas are proved by showing that the differential of the right member is equal to the expression under the integral sign. Thus to prove III we differentiate the right side and so obtain

$$d\left(\frac{u^{n+1}}{n+1}+C\right)=\frac{(n+1)\,u^n\,du}{n+1}=u^n\,du.$$

Formula I expresses that the integral of an algebraic sum of differentials is obtained by integrating them separately and adding the results.

Formula II expresses that a constant factor can be transferred from one side of the symbol $\int$ to the other without changing the result. A variable cannot be transferred in this way. Thus it is not correct to write

$$\int x\,dx = x\int dx = x^2.$$

Example 1. $\int x^5\,dx.$

Apply Formula III, letting $u = x$ and $n = 5$. Then $dx = du$ and

$$\int x^5\,dx=\frac{x^{5+1}}{5+1}+C=\frac{x^6}{6}+C.$$

Ex. 2. $\int 3\sqrt{x}\,dx.$

By Formula II we have

$$\int 3\sqrt{x}\,dx = 3\int x^{\frac{1}{2}}\,dx=\frac{3\,x^{\frac{3}{2}}}{\frac{3}{2}}+C=2\,x^{\frac{3}{2}}+C.$$

Ex. 3. $\int (x-1)\,(x+2)\,dx.$

We expand and integrate term by term.

$$\begin{aligned}\int (x-1)\,(x+2)\,dx &= \int (x^2+x-2)\,dx\\ &= \tfrac{1}{3}\,x^3+\tfrac{1}{2}\,x^2-2\,x+C.\end{aligned}$$

Ex. 4. $\displaystyle\int \frac{x^2 - 2x + 1}{x^3}\,dx.$

Dividing by x^3 and using negative exponents, we get

$$\begin{aligned}\int \frac{x^2 - 2x + 1}{x^3}\,dx &= \int (x^{-1} - 2x^{-2} + x^{-3})\,dx \\ &= \ln x + 2x^{-1} - \tfrac{1}{2}x^{-2} + C \\ &= \ln x + \frac{2}{x} - \frac{1}{2x^2} + C.\end{aligned}$$

Ex. 5. $\displaystyle\int \sqrt{2x+1}\,dx.$

If $u = 2x + 1$, $du = 2\,dx$. We therefore place a factor 2 before dx and $\frac{1}{2}$ outside the integral sign to compensate for it.

$$\begin{aligned}\int \sqrt{2x+1}\,dx &= \tfrac{1}{2}\int (2x+1)^{\frac{1}{2}}\,2\,dx = \tfrac{1}{2}\int u^{\frac{1}{2}}\,du \\ &= \frac{1}{2}\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{1}{3}(2x+1)^{\frac{3}{2}} + C.\end{aligned}$$

Ex. 6. $\displaystyle\int \frac{x\,dx}{x^2+1}.$

Apply IV with $u = x^2 + 1$. Then $du = 2x\,dx$ and

$$\int \frac{x\,dx}{x^2+1} = \frac{1}{2}\int \frac{2x\,dx}{x^2+1} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln u + C = \ln\sqrt{x^2+1} + C.$$

Ex. 7. $\displaystyle\int \frac{4x+2}{2x-1}\,dx.$

By division, we find

$$\frac{4x+2}{2x-1} = 2 + \frac{4}{2x-1}.$$

Therefore

$$\int \frac{4x+2}{2x-1}\,dx = \int \left(2 + \frac{4}{2x-1}\right)dx = 2x + 2\ln(2x-1) + C.$$

EXERCISES

Find the values of the following integrals:

1. $\int (x^4 - 3x^3 + 5x^2)\,dx.$
2. $\int \left(x^2 - \frac{1}{x^2}\right) dx.$
3. $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx.$
4. $\int \left(\sqrt{2x} - \frac{1}{\sqrt{2x}}\right) dx.$
5. $\int \sqrt{x}\,(x^2 + 2x + 1)\,dx.$
6. $\int (\sqrt{a} - \sqrt{x})^3\,dx.$
7. $\int x(x + a)(x + b)\,dx.$
8. $\int \frac{2x + 3}{x}\,dx.$
9. $\int \frac{(y + 2)^2}{y}\,dy.$
10. $\int \frac{(x^2 + 1)(x^2 - 2)}{x^{\frac{2}{3}}}\,dx.$
11. $\int \frac{dx}{x + 1}.$
12. $\int \frac{dx}{(x + 1)^2}.$
13. $\int \frac{dx}{\sqrt{2x + 1}}.$
14. $\int \frac{x\,dx}{x^2 + 2}.$
15. $\int \frac{x\,dx}{\sqrt{x^2 - 1}}.$
16. $\int \frac{x\,dx}{(a + bx^2)^3}.$
17. $\int x\sqrt{a^2 - x^2}\,dx.$
18. $\int \frac{x^2\,dx}{a^3 + x^3}.$
19. $\int x^2\sqrt{x^3 - 1}\,dx.$
20. $\int \frac{2x + a}{x^2 + ax + b}\,dx.$
21. $\int \frac{(2x + a)\,dx}{\sqrt{x^2 + ax + b}}.$
22. $\int \frac{t^4\,dt}{1 - at^5}.$
23. $\int t\,(a^2 - t^2)^{\frac{3}{2}}\,dt.$
24. $\int \frac{x + 1}{x - 2}\,dx.$
25. $\int \left(2 + \frac{1}{2x^2 + 1}\right) \frac{x\,dx}{2x^2 + 1}.$
26. $\int \left(1 - \frac{1}{x}\right)^8 \frac{dx}{x^2}.$
27. $\int \frac{x^{n-1}\,dx}{(x^n + a)^n}.$
28. $\int (\sqrt{2x} - \sqrt{2a})^{10} \frac{dx}{\sqrt{x}}.$
29. $\int \frac{x^3 - 2}{x^3 + 2}\,x^2\,dx.$
30. $\int (x^3 - 1)^2\,x\,dx.$

4. Motion of a Particle. — Let the acceleration of a particle moving along a straight line be a, the velocity v, and the distance passed over s. Then,

$$a = \frac{dv}{dt}, \qquad v = \frac{ds}{dt}.$$

Consequently,

$$dv = a\,dt, \qquad ds = v\,dt.$$

If then a is a known function of the time or a constant,

$$v = \int a\,dt + C_1, \qquad s = \int v\,dt + C_2. \tag{4}$$

If the particle moves along a curve and the components of velocity or acceleration are known, each coördinate can be found in a similar way.

Example 1. A body falls from rest under the constant acceleration of gravity g. Find its velocity and the distance traversed as functions of the time t.

In this case

$$a = \frac{dv}{dt} = g.$$

Hence

$$v = \int g\,dt = gt + C.$$

Since the body starts from rest, $v = 0$ when $t = 0$. These values of v and t must satisfy the equation $v = gt + C$. Hence

$$0 = g \cdot 0 + C,$$

whence $C = 0$ and $v = gt$. Since $v = \dfrac{ds}{dt}$, $ds = gt\,dt$ and

$$s = \int gt\,dt + C = \tfrac{1}{2}\,gt^2 + C.$$

When $t = 0$, $s = 0$. Consequently, $C = 0$ and $s = \frac{1}{2}\,gt^2$.

Ex. 2. A projectile is fired with a velocity v_0 in a direction making an angle α with the horizontal plane. Neglecting the resistance of the air, find its motion.

Pass a vertical plane through the line along which the particle starts. In this plane take the starting point as origin, the horizontal line as x-axis, and the vertical line as y-axis. The only acceleration is that of gravity acting downward and equal to g. Hence

$$\frac{d^2x}{dt^2} = 0, \qquad \frac{d^2y}{dt^2} = -g.$$

Integration gives,

$$\frac{dx}{dt} = C_1, \qquad \frac{dy}{dt} = -gt + C_2.$$

When $t = 0$, $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are the components of v_0. Hence $C_1 = v_0 \cos\alpha, C_2 = v_0 \sin\alpha$, and

$$\frac{dx}{dt} = v_0 \cos\alpha,$$

$$\frac{dy}{dt} = v_0 \sin\alpha - gt.$$

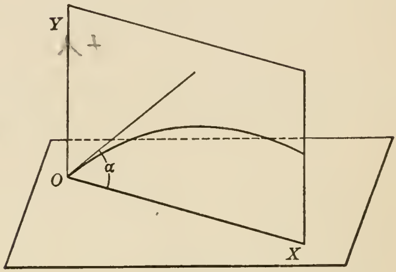

FIG. 4.

Integrating again, we get

$$x = v_0 t \cos\alpha,$$

$$y = v_0 t \sin\alpha - \tfrac{1}{2} gt^2,$$

the constants being zero because x and y are zero when $t = 0$.

5. Curves with a Given Slope. — If the slope of a curve is a given function of x,

$$\frac{dy}{dx} = f(x),$$

then

$$dy = f(x)\, dx$$

and

$$y = \int f(x)\, dx + C$$

is the equation of the curve.

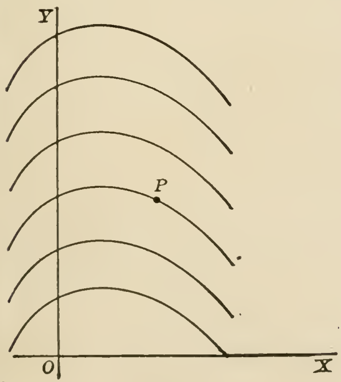

FIG. 5.

Since the constant can have any value, there are an infinite number of curves having the given slope. If the curve is required to pass through a given point P, the value of C can be found by substituting the coördinates of P in the equation after integration.

Example 1. Find the curve passing through (1, 2) with slope equal to $2x$.

In this case

$$\frac{dy}{dx} = 2x.$$

Hence

$$y = \int 2x\,dx = x^2 + C.$$

Since the curve passes through (1, 2), the values $x = 1$, $y = 2$ must satisfy the equation, that is

$$2 = 1 + C.$$

Consequently, $C = 1$ and $y = x^2 + 1$ is the equation of the curve.

Ex. 2. On a certain curve

$$\frac{d^2y}{dx^2} = x.$$

If the curve passes through $(-2, 1)$ and has at that point the slope -2, find its equation.

By integration we get

$$\frac{dy}{dx} = \int \frac{d^2y}{dx^2}\,dx = \int x\,dx = \frac{1}{2}x^2 + C.$$

At $(-2, 1)$, $x = -2$ and $\frac{dy}{dx} = -2$. Hence

$$-2 = 2 + C,$$

or $C = -4$. Consequently,

$$y = \int (\tfrac{1}{2}x^2 - 4)\,dx = \tfrac{1}{6}x^3 - 4x + C.$$

Since the curve passes through $(-2, 1)$,

$$1 = -\tfrac{8}{6} + 8 + C.$$

Consequently, $C = -5\tfrac{2}{3}$, and

$$y = \tfrac{1}{6}x^3 - 4x - 5\tfrac{2}{3}$$

is the equation of the curve.

6. Separation of the Variables. — The integration formulas contain only one variable. If a differential contains two or more variables, it must be reduced to a form in which each term contains a single variable. If this cannot be done, we cannot integrate the differential by our present methods.

Example 1. Find the curves such that the part of the tangent included between the coördinate axes is bisected at the point of tangency.

Let $P(x, y)$ be the point at which AB is tangent to the curve. Since P is the middle point of AB,

$$OA = 2y, \qquad OB = 2x.$$

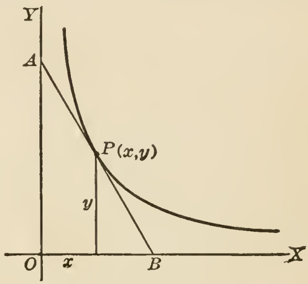

FIG. 6.

The slope of the curve at P is

$$\frac{dy}{dx} = -\frac{OA}{OB} = -\frac{y}{x}.$$

This can be written

$$\frac{dy}{y} + \frac{dx}{x} = 0.$$

Since each term contains a single variable, we can integrate and so get

$$\ln y + \ln x = C.$$

This is equivalent to

$$\ln xy = C.$$

Hence

$$xy = e^C = k.$$

C, and consequently k, can have any value. The curves are rectangular hyperbolas with the coördinate axes as asymptotes.

Ex. 2. According to Newton's law of cooling,

$$\frac{d\theta}{dt} = -k(\theta - a),$$

when k is constant, a the temperature of the air, and θ the temperature at the time t of a body cooling in the air. Find θ as a function of t.

Multiplying by dt and dividing by $\theta - a$, Newton's equation becomes

$$\frac{d\theta}{\theta - a} = -k\,dt.$$

Integrating both sides, we get

$$\ln(\theta - a) = -kt + C.$$

Hence

$$\theta - a = e^{-kt+C} = e^C e^{-kt}.$$

When $t = 0$, let $\theta = \theta_0$. Then

$$\theta_0 - a = e^C e^0 = e^C,$$

and so

$$\theta - a = (\theta_0 - a)\,e^{-kt}$$

is the equation required.

Ex. 3. The retarding effect of fluid friction on a rotating disk is proportional to the angular speed ω. Find ω as a function of the time t.

The statement means that the rate of change of ω is proportional to ω, that is,

$$\frac{d\omega}{dt} = k\omega,$$

where k is constant. Separating the variables, we get

$$\frac{d\omega}{\omega} = k\,dt,$$

whence

$$\ln \omega = kt + C,$$

and

$$\omega = e^{kt+C} = e^C e^{kt}.$$

Let ω_0 be the value of ω when $t = 0$. Then

$$\omega_0 = e^{k \cdot 0} e^C = e^C.$$

Replacing e^C by ω_0, the previous equation becomes

$$\omega = \omega_0 e^{kt},$$

which is the result required.

Ex. 4. A cylindrical tank full of water has a leak at the bottom. Assuming that the water escapes at a rate proportional to the depth and that $\frac{1}{10}$ of it escapes the first day, how long will it take to half empty?

Let the radius of the tank be a, its height h and the depth of the water after t days x. The volume of the water at any time is $\pi a^2 x$ and its rate of change

$$-\pi a^2 \frac{dx}{dt}.$$

This is assumed to be proportional to x, that is,

$$\pi a^2 \frac{dx}{dt} = kx,$$

where k is constant. Separating the variables,

$$\frac{\pi a^2\, dx}{x} = k\, dt.$$

Integration gives

$$\pi a^2 \ln x = kt + C.$$

When $t = 0$ the tank is full and $x = h$. Hence

$$\pi a^2 \ln h = C.$$

Subtracting this from the preceding equation, we get

$$\pi a^2 \ln \frac{x}{h} = kt.$$

When $t = 1$, $x = \frac{9}{10} h$. Consequently,

$$\pi a^2 \ln \tfrac{9}{10} = k.$$

When $x = \frac{1}{2} h$,

$$t = \frac{\pi a^2 \ln \frac{x}{h}}{k} = \frac{\ln \frac{1}{2}}{\ln \frac{9}{10}} = 6.57 \text{ days}.$$

EXERCISES

1. If the velocity of a body moving along a line is $v = 2t + 3t^2$, find the distance traversed between $t = 2$ and $t = 5$.

2. Find the distance a body started vertically downward with a velocity of 30 ft./sec. will fall in the time t.

3. From a point 60 ft. above the street a ball is thrown vertically upward with a speed of 100 ft./sec. Find its height as a function of the time. Also find the highest point reached.

4. A rifle ball is fired through a 3-inch plank the resistance of which causes a negative constant acceleration. If its velocity on entering the plank is 1000 ft./sec. and on leaving it 500 ft./sec., how long does it take the ball to pass through?

5. A particle starts at (1, 2). After t seconds the component of its velocity parallel to the x-axis is $2t - 1$ and that parallel to the y-axis is $1 - t$. Find its coördinates as functions of the time. Also find the equation of its path.

6 A bullet is fired at a velocity of 3000 ft./sec. at an angle of 45° from a point 100 ft. above the ground. Neglecting the resistance of the air, find where the bullet will strike the ground.

7. Find the motion of a particle started from the origin with velocity v_0 in the vertical direction, if its acceleration is a constant K in a direction making 30° with the horizontal plane.

8. Find the equation of the curve with slope $2 - x$ passing through (1, 0).

9. Find the equation of the curve with slope equal to y passing through (0, 1).

10. On a certain curve

$$\frac{dy}{dx} = 2x + 3.$$

If the curve passes through (1, 2), find its lowest point.

11. On a certain curve

$$\frac{d^2y}{dx^2} = x - 1.$$

If the curve passes through (− 1, 1) and has at that point the slope 2, find its equation.

12. On a certain curve

$$\frac{d^2y}{dx^2} = 2 - 3x.$$

If the slope is − 1 at $x = 0$, find the difference of the ordinates at $x = 3$ and $x = 4$.

13. The pressure of the air p and altitude above sea level h are connected by the equation

$$\frac{dp}{dh} = -kp,$$

where k is constant. Show that $p = p_0e^{-kh}$, when p_0 is the pressure at sea level.

14. Radium decomposes at a rate proportional to the amount present. If half the original quantity disappears in 1800 years, what percentage disappears in 100 years?

15. When bacteria grow in the presence of unlimited food, they increase at a rate proportional to the number present. Express that number as a function of the time.

16. Cane sugar is decomposed into other substances through the presence of acids. The rate at which the process takes place is proportional to the mass x of sugar still unchanged. Show that $x = ce^{-kt}$. What does c represent?

17. The rate at which water flows from a small opening at the bottom of a tank is proportional to the square root of the depth of the water. If half the water flows from a cylindrical tank in 5 minutes, find the time required to empty the tank.

18. Solve Ex. 17, when the cylindrical tank is replaced by a conical funnel.

19. A sum of money is placed at compound interest at 6 per cent per annum, the interest being added to the principal at each instant. How many years will be required for the sum to double?

20. The amount of light absorbed in penetrating a thin sheet of water is proportional to the amount falling on the surface and approximately proportional to the thickness of the sheet, the approximation increasing as the thickness approaches zero. Show that the rate of change of illumination is proportional to the depth and so find the illumination as a function of the depth.

CHAPTER II

FORMULAS AND METHODS OF INTEGRATION

7. Formulas. — The following is a short list of integration formulas. In these u is any variable or function of a single variable and du is its differential. The constant is omitted but it should be added to each function determined by integration. A more extended list of formulas is given in the Appendix.

$$\text{I.} \quad \int u^n\, du = \frac{u^{n+1}}{n+1}, \text{ if } n \text{ is not } -1.$$

$$\text{II.} \quad \int \frac{du}{u} = \ln u.$$

$$\text{III.} \quad \int \cos u\, du = \sin u.$$

$$\text{IV.} \quad \int \sin u\, du = -\cos u.$$

$$\text{V.} \quad \int \sec^2 u\, du = \tan u.$$

$$\text{VI.} \quad \int \csc^2 u\, du = -\cot u.$$

$$\text{VII.} \quad \int \sec u \tan u\, du = \sec u.$$

$$\text{VIII.} \quad \int \csc u \cot u\, du = -\csc u.$$

$$\text{IX.} \quad \int \tan u\, du = -\ln \cos u.$$

$$\text{X.} \quad \int \cot u\, du = \ln \sin u.$$

$$\text{XI.} \quad \int \sec u\, du = \ln (\sec u + \tan u).$$

XII. $$\int \csc u\, du = \ln (\csc u - \cot u).$$

XIII. $$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a}.^*$$

XIV. $$\int \frac{du}{u^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{u}{a}.$$

XV. $$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{u}{a}.^*$$

XVI. $$\int \frac{du}{\sqrt{u^2 \pm a^2}} = \ln \left(u + \sqrt{u^2 \pm a^2}\right).$$

XVII. $$\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \ln \frac{u - a}{u + a}.$$ or $= \frac{1}{2a} \ln \frac{a-u}{a+u}$

XVIII. $$\int e^u\, du = e^u.$$

Any one of these formulas can be proved by showing that the differential of the right member is equal to the expression under the integral sign. Thus to show that

$$\int \sec u\, du = \ln (\sec u + \tan u),$$

we note that

$$d \ln (\sec u + \tan u) = \frac{(\sec u \tan u + \sec^2 u)\, du}{\sec u + \tan u} = \sec u\, du.$$

8. Integration by Substitution. — When some function of the variable is taken as u, a given differential may assume the form of the differential in one of the integration formulas or differ from such form only by a constant factor. Integration accomplished in this way is called integration by substitution.

Each differential is the product of a function of u by du. More errors result from failing to pay attention to the du

* In Formulas XIII and XV it is assumed that $\sin^{-1} \frac{u}{a}$ is an angle in the 1st or 4th quadrant, and $\sec^{-1} \frac{u}{a}$ an angle in the 1st or 2nd quadrant. In other cases the algebraic sign of the result must be changed.

than from any other one cause. Thus the student may carelessly conclude from Formula III that the integral of a cosine is a sine and so write

$$\int \cos 2x\, dx = \sin 2x.$$

If, however, we let $2x = u$, dx is not du but $\frac{1}{2}\, du$ and so

$$\int \cos 2x\, dx = \tfrac{1}{2} \int \cos u\, du = \tfrac{1}{2} \sin u = \tfrac{1}{2} \sin 2x.$$

Example 1. $\displaystyle\int \sin^3 x \cos x\, dx.$

If we let $u = \sin x$, $du = \cos x\, dx$ and

$$\int \sin^3 x \cos x\, dx = \int u^3\, du = \tfrac{1}{4} u^4 + C = \tfrac{1}{4} \sin^4 x + C.$$

Ex. 2. $\displaystyle\int \frac{\sin \frac{1}{3} x\, dx}{1 + \cos \frac{1}{3} x}.$

We observe that $\sin \frac{1}{3} x\, dx$ differs only by a constant fa tor from the differential of $1 + \cos \frac{1}{3} x$. Hence we let

$$u = 1 + \cos \tfrac{1}{3} x.$$

Then $\qquad du = -\frac{1}{3} \sin \frac{1}{3} x\, dx, \qquad \sin \frac{1}{3} x\, dx = -3\, du,$

and
$$\int \frac{\sin \frac{1}{3} x\, dx}{1 + \cos \frac{1}{3} x} = -3 \int \frac{du}{u} = -3 \ln u + C$$
$$= -3 \ln (1 + \cos \tfrac{1}{3} x) + C.$$

Ex. 3. $\displaystyle\int (\tan x + \sec x) \sec x\, dx.$

Expanding we get

$$\int (\tan x + \sec x) \sec x\, dx = \int \tan x \sec x\, dx + \int \sec^2 x\, dx$$
$$= \sec x + \tan x + C.$$

Ex. 4. $\displaystyle\int \frac{3\, dx}{\sqrt{2 - 3x^2}}.$

This resembles the integral in formula XIII. Let $u = x\sqrt{3}$, $a = \sqrt{2}$. Then $du = \sqrt{3}\,dx$ and

$$\int \frac{3\,dx}{\sqrt{2 - 3x^2}} = \int \frac{3\dfrac{du}{\sqrt{3}}}{\sqrt{a^2 - u^2}} = \sqrt{3}\int \frac{du}{\sqrt{a^2 - u^2}}$$

$$= \sqrt{3}\sin^{-1}\frac{u}{a} + C = \sqrt{3}\sin^{-1}\frac{x\sqrt{3}}{\sqrt{2}} + C.$$

Ex. 5. $\displaystyle\int \frac{dt}{t\sqrt{4t^2 - 9}}\cdot$

This suggests the integral in formula XV. Let $u = 2t$, $a = 3$. Then

$$\int \frac{dt}{t\sqrt{4t^2 - 9}} = \int \frac{2\,dt}{2t\sqrt{4t^2 - 9}} = \int \frac{du}{u\sqrt{u^2 - a^2}}$$

$$= \frac{1}{a}\sec^{-1}\frac{u}{a} + C = \frac{1}{3}\sec^{-1}\frac{2t}{3} + C.$$

Ex. 6. $\displaystyle\int \frac{x\,dx}{\sqrt{2x^2 + 1}}\cdot$

This may suggest formula XVI. If, however, we let $u = x\sqrt{2}$, $du = \sqrt{2}\,dx$, which is not a constant times $x\,dx$. We should let

$$u = 2x^2 + 1.$$

Then $x\,dx = \frac{1}{4}\,du$ and

$$\int \frac{x\,dx}{\sqrt{2x^2 + 1}} = \frac{1}{4}\int \frac{du}{\sqrt{u}} = \frac{1}{4}\int u^{-\frac{1}{2}}\,du$$

$$= \tfrac{1}{2}\sqrt{u} + C = \tfrac{1}{2}\sqrt{2x^2 + 1} + C.$$

Ex. 7. $\displaystyle\int e^{\tan x}\sec^2 x\,dx.$

If $u = \tan x$, by formula XVIII

$$\int e^{\tan x}\sec^2 x\,dx = \int e^u\,du = e^u + C = e^{\tan x} + C.$$

EXERCISES

Determine the values of the following integrals:

1. $\int (\sin 2x - \cos 3x)\, dx.$
2. $\int \cos\left(\frac{2x-3}{5}\right) dx.$
3. $\int \sin(nt + \alpha)\, dt.$
4. $\int \sec^2 \frac{1}{3}\theta\, d\theta.$
5. $\int \csc\frac{\theta}{4}\cot\frac{\theta}{4}\, d\theta.$
6. $\int \cos\theta \sin\theta\, d\theta.$
7. $\int \frac{dx}{\cos^2 x}.$
8. $\int \frac{dx}{\sin^2 2x}.$
9. $\int \frac{\cos x\, dx}{\sin^2 x}.$
10. $\int \frac{\sin x\, dx}{\cos^5 x}.$
11. $\int \left(\csc\frac{\theta}{2} - \cot\frac{\theta}{2}\right)\csc\frac{\theta}{2}\, d\theta.$
12. $\int \cos(x^2 - 1)\, x\, dx.$
13. $\int \frac{1 + \sin 3x}{\cos^2 3x}\, dx.$
14. $\int (\sec x - 1)^2\, dx.$
15. $\int \frac{1 - \sin x}{\cos x}\, dx.$
16. $\int (\cos\theta - \sin\theta)^2\, d\theta.$
17. $\int \frac{(\cos x + \sin x)^2}{\sin x}\, dx.$
18. $\int \sin^2 x \cos x\, dx.$
19. $\int \tan^3 x \sec^2 x\, dx.$
20. $\int \sec^2 x \tan x\, dx.$
21. $\int \cos^5 x \sin x\, dx.$
22. $\int \frac{\sec^2 x\, dx}{1 + 2\tan x}.$
23. $\int \frac{\cos 2x\, dx}{1 - \sin 2x}.$
24. $\int \frac{\sec^2 (ax)\, dx}{1 + \tan (ax)}.$
25. $\int \frac{dx}{\sqrt{3 - 2x^2}}.$
26. $\int \frac{2\, dx}{3x^2 + 4}.$
27. $\int \frac{dx}{x\sqrt{3x^2 - 4}}.$
28. $\int \frac{dy}{12y^2 + 3}.$
29. $\int \frac{dx}{\sqrt{7x^2 + 1}}.$
30. $\int \frac{dx}{x\sqrt{a^2x^2 - 9}}.$
31. $\int \frac{dx}{3 - 4x^2}.$
32. $\int \frac{dx}{\sqrt{4x^2 - 3}}.$
33. $\int \frac{(3x - 2)\, dx}{\sqrt{4 - x^2}}.$
34. $\int \frac{2x + 3}{\sqrt{x^2 + 4}}\, dx.$
35. $\int \frac{x + 4}{4x^2 - 5}\, dx.$
36. $\int \frac{5x - 2}{\sqrt{3x^2 - 9}}\, dx.$
37. $\int \frac{\cos x\, dx}{\sqrt{2 - \sin^2 x}}.$
38. $\int \frac{\sin x \cos x\, dx}{\sqrt{2 - \sin^2 x}}.$
39. $\int \frac{\cos x\, dx}{1 + \sin^2 x}.$

40. $\int \frac{\sec^2 x\,dx}{\tan x\sqrt{\tan^2 x - 1}}$.

41. $\int \frac{\sec x \tan x\,dx}{\sqrt{\sec^2 x + 1}}$.

42. $\int \frac{\sin\theta\,d\theta}{\sqrt{1-\cos\theta}}$.

43. $\int \frac{dx}{x\,[4 - (\ln x)^2]}$.

44. $\int \frac{\sin x \cos x\,dx}{\sqrt{\cos^2 x - \sin^2 x}}$.

45. $\int \frac{x\,dx}{\sqrt{a^4 - x^4}}$.

46. $\int e^{-k^2 x}\,dx$.

47. $\int (e^{ax} + e^{-ax})^2\,dx$.

48. $\int \frac{e^{3x}\,dx}{1 + e^{3x}}$.

49. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$.

50. $\int e^{-\frac{1}{x}}\frac{dx}{x^2}$.

51. $\int \frac{e^x\,dx}{1 + e^{2x}}$.

52. $\int \frac{e^{-x}\,dx}{1 - e^{-2x}}$.

53. $\int \frac{e^{ax}\,dx}{\sqrt{1 - e^{2ax}}}$.

54. $\int \frac{dx}{e^x + e^{-x}}$.

9. Integrals Containing $ax^2 + bx + c$. — Integrals containing a quadratic expression $ax^2 + bx + C$ can often be reduced to manageable form by completing the square of $ax^2 + bx$.

Example 1. $\int \frac{dx}{3x^2 + 6x + 5}$.

Completing the square, we get

$$3x^2 + 6x + 5 = 3(x^2 + 2x + 1) + 2 = 3(x+1)^2 + 2.$$

If then $u = (x+1)\sqrt{3}$,

$$\int \frac{dx}{3x^2 + 6x + 5} = \int \frac{d(x+1)}{3(x+1)^2 + 2} = \frac{1}{\sqrt{3}}\int \frac{du}{u^2 + 2}$$

$$= \frac{1}{\sqrt{6}}\tan^{-1}\frac{(x+1)\sqrt{3}}{\sqrt{2}} + C.$$

Ex. 2. $\int \frac{2\,dx}{\sqrt{2 - 3x - x^2}}$.

The coefficient of x^2 being negative, we place the terms x^2 and $3x$ in a parenthesis preceded by a minus sign. Thus

$$2 - 3x - x^2 = 2 - (x^2 + 3x) = \tfrac{17}{4} - (x + \tfrac{3}{2})^2.$$

If then, $u = x + \frac{3}{2}$, we have

$$\int \frac{2\,dx}{\sqrt{2 - 3x - x^2}} = 2\int \frac{du}{\sqrt{\frac{17}{4} - u^2}} = 2\sin^{-1}\frac{x + \frac{3}{2}}{\frac{1}{2}\sqrt{17}} + C.$$

Ex. 3. $\displaystyle\int \frac{(2x - 1)\,dx.}{\sqrt{4x^2 + 4x + 2}}.$

Since the numerator contains the first power of x, we resolve the integral into two parts,

$$\int \frac{(2x-1)\,dx}{\sqrt{4x^2+4x+2}} = \frac{1}{4}\int \frac{(8x+4)\,dx}{\sqrt{4x^2+4x+2}} - 2\int \frac{dx}{\sqrt{4x^2+4x+2}}.$$

In the first integral on the right the numerator is taken equal to the differential of $4x^2 + 4x + 2$. In the second the numerator is dx. The outside factors $\frac{1}{4}$ and -2 are chosen so that the two sides of the equation are equal. The first integral has the form

$$\frac{1}{4}\int \frac{du}{\sqrt{u}} = \frac{1}{2}\sqrt{u} = \frac{1}{2}\sqrt{4x^2 + 4x + 2}.$$

The second integral is evaluated by completing the square. The final result is

$$\int \frac{(2x - 1)\,dx}{\sqrt{4x^2 + 4x + 2}} = \frac{1}{2}\sqrt{4x^2 + 4x + 2} - \ln\left(2x + 1 + \sqrt{4x^2 + 4x + 2}\right) + C.$$

EXERCISES

1. $\displaystyle\int \frac{dx}{x^2 + 6x + 13}.$

2. $\displaystyle\int \frac{dx}{\sqrt{2 + 4x - 4x^2}}.$

3. $\displaystyle\int \frac{dx}{\sqrt{3x^2 + 4x + 2}}.$

4. $\displaystyle\int \frac{dx\cdot}{\sqrt{1 + 5x - 5x^2}}.$

5. $\displaystyle\int \frac{dx}{(x-3)\sqrt{2x^2 - 12x + 15}}.$

6. $\displaystyle\int \frac{dx}{(x + a)(x + b)}.$

7. $\displaystyle\int \frac{(2x + 5)\,dx}{4x^2 - 4x - 2}.$

8. $\displaystyle\int \frac{(2x - 1)\,dx}{\sqrt{3x^2 - 6x + 1}}.$

9. $\displaystyle\int \frac{x\,dx}{3x^2 + 2x + 2}.$

10. $\displaystyle\int \frac{(2x + 3)\,dx.}{(2x + 1)\sqrt{4x^2 + 4x - 1}}.$

11. $\displaystyle\int \frac{(3x - 3)\,dx}{(x^2 - 2x + 3)^{\frac{3}{2}}}.$

12. $\displaystyle\int \sqrt{\frac{x + 1}{x - 2}}\,dx.$

13. $\displaystyle\int \frac{e^x\,dx}{2e^{2x} + 3e^x - 1}.$

10. Integrals of Trigonometric Functions. — A power of a trigonometric function multiplied by its differential can be integrated by Formula I. Thus, if $u = \tan x$,

$$\int \tan^4 x \cdot \sec^2 x\, dx = \int u^4\, du = \tfrac{1}{5} \tan^5 x + C.$$

Differentials can often be reduced to the above form by trigonometric transformations. This is illustrated by the following examples.

Example 1. $\int \sin^4 x \cos^3 x\, dx.$

If we take $\cos x\, dx$ as du and use the relation $\cos^2 x = 1 - \sin^2 x$, the other factors can be expressed in terms of $\sin x$ *without introducing radicals.* Thus

$$\int \sin^4 x \cos^3 x\, dx = \int \sin^4 x \cos^2 x \cdot \cos x\, dx$$

$$= \int \sin^4 x\, (1 - \sin^2 x)\, d \sin x = \tfrac{1}{5} \sin^5 x - \tfrac{1}{7} \sin^7 x + C.$$

Ex. 2. $\int \tan^3 x \sec^4 x\, dx.$

If we take $\sec^2 x\, dx$ as du and use the relation $\sec^2 x = 1 + \tan^2 x$, the other factors can be expressed in terms of $u = \tan x$ without introducing radicals. Thus

$$\int \tan^3 x \sec^4 x\, dx = \int \tan^3 x \cdot \sec^2 x \cdot \sec^2 x\, dx$$

$$= \int \tan^3 x\, (1 + \tan^2 x)\, d \tan x$$

$$= \tfrac{1}{4} \tan^4 x + \tfrac{1}{6} \tan^6 x + C.$$

Ex. 3. $\int \tan^3 x \sec^3 x\, dx.$

If we take $\tan x \sec x\, dx = d \sec x$ as du, and use the relation $\tan^2 x = \sec^2 x - 1$, the integral takes the form

$$\begin{aligned}\int \tan^3 x \sec^3 x\, dx &= \int \tan^2 x \cdot \sec^2 x \cdot \tan x \sec x\, dx \\ &= \int (\sec^2 x - 1) \sec^2 x \cdot d \sec x \\ &= \tfrac{1}{5} \sec^5 x - \tfrac{1}{3} \sec^3 x + C.\end{aligned}$$

Ex. 4. $\int \sin 2\,x \cos 3\,x\, dx.$

This is the product of the sine of one angle and the cosine of another. This product can be resolved into a sum or difference by the formula

$$\sin A \cos B = \tfrac{1}{2}\, [\sin (A + B) + \sin (A - B)].$$

Thus

$$\begin{aligned}\sin 2\,x \cos 3\,x &= \tfrac{1}{2}\, [\sin 5\,x + \sin (-x)] \\ &= \tfrac{1}{2}\, [\sin 5\,x - \sin x]\end{aligned}$$

Consequently,

$$\begin{aligned}\int \sin 2\,x \cos 3\,x\, dx &= \tfrac{1}{2} \int (\sin 5\,x - \sin x)\, dx \\ &= -\tfrac{1}{10} \cos 5\,x + \tfrac{1}{2} \cos x + C.\end{aligned}$$

Ex. 5. $\int \tan^5 x\, dx.$

If we replace $\tan^2 x$ by $\sec^2 x - 1$, the integral becomes

$$\int \tan^5 x\, dx = \int \tan^3 x\, (\sec^2 x - 1)\, dx = \tfrac{1}{4} \tan^4 x - \int \tan^3 x\, dx.$$

The integral is thus made to depend on a simpler one $\int \tan^3 x\, dx$. Similarly,

$$\int \tan^3 x\, dx = \int \tan x\, (\sec^2 x - 1)\, dx = \tfrac{1}{2} \tan^2 x + \ln \cos x.$$

Hence finally

$$\int \tan^5 x\, dx = \tfrac{1}{4} \tan^4 x - \tfrac{1}{2} \tan^2 x - \ln \cos x + C.$$

11. Even Powers of Sines and Cosines. — Integrals of the form

$$\int \sin^m x \cos^n x \, dx,$$

where m or n is odd can be evaluated by the methods of Art. 10. If both m and n are even, however, those methods fail. In that case we can evaluate the integral by the use of the formulas

$$\left.\begin{aligned} \sin^2 u &= \frac{1 - \cos 2u}{2}, \\ \cos^2 u &= \frac{1 + \cos 2u}{2}, \\ \sin u \cos u &= \frac{\sin 2u}{2}. \end{aligned}\right\} \qquad (11)$$

Example 1. $\int \cos^4 x \, dx.$

By the above formulas

$$\int \cos^4 x \, dx = \int (\cos^2 x)^2 \, dx = \int \left(\frac{1 + \cos 2x}{2}\right)^2 dx$$

$$= \int (\tfrac{1}{4} + \tfrac{1}{2} \cos 2x + \tfrac{1}{4} \cos^2 2x)$$

$$= \int [\tfrac{1}{4} + \tfrac{1}{2} \cos 2x + \tfrac{1}{8}(1 + \cos 4x)] \, dx$$

$$= \tfrac{3}{8} x + \tfrac{1}{4} \sin 2x + \tfrac{1}{32} \sin 4x + C.$$

Ex. 2. $\int \cos^2 x \sin^2 x \, dx.$

$$\int \cos^2 x \sin^2 x \, dx = \int \tfrac{1}{4} \sin^2 2x \, dx = \int \tfrac{1}{8}(1 - \cos 4x) \, dx$$

$$= \tfrac{1}{8} x - \tfrac{1}{32} \sin 4x + C.$$

12. Trigonometric Substitutions. — Differentials containing $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$, which are not reduced to manageable form by taking the radical as a new variable, can often be integrated by one of the following substitutions:

For $\sqrt{a^2 - x^2}$, let $x = a \sin \theta$.

For $\sqrt{a^2 + x^2}$, let $x = a \tan \theta$.

For $\sqrt{x^2 - a^2}$, let $x = a \sec \theta$.

Example 1. $\int \sqrt{a^2 - x^2}\, dx$.

Let $x = a \sin \theta$. Then

$$\sqrt{a^2 - x^2} = a \cos \theta, \quad dx = a \cos \theta \, d\theta.$$

Consequently,

$$\int \sqrt{a^2 - x^2}\, dx = a^2 \int \cos^2 \theta \, d\theta = \frac{a^2}{2}\left(\theta + \frac{1}{2} \sin 2\,\theta\right) + C.$$

Since $x = a \sin \theta$,

$$\theta = \sin^{-1} \frac{x}{a}, \qquad \frac{1}{2} \sin 2\,\theta = \sin \theta \cos \theta = \frac{x\sqrt{a^2 - x^2}}{a^2}.$$

Hence finally

$$\int \sqrt{a^2 - x^2}\, dx = \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2} + C.$$

Ex. 2. $\int \frac{dx}{(x^2 + a^2)^2}$.

If we let $x = a \tan \theta$, $x^2 + a^2 = a^2 \sec^2 \theta$, $dx = a \sec^2 \theta \, d\theta$, and

$$\int \frac{dx}{(x^2 + a^2)^2} = \frac{1}{a^3} \int \frac{d\theta}{\sec^2 \theta} = \frac{1}{a^3} \int \cos^2 \theta \, d\theta$$
$$= \frac{1}{2\,a^3} (\theta + \sin \theta \cos \theta) + C.$$

Since

$$x = a \tan \theta, \quad \theta = \tan^{-1} \frac{x}{a}, \quad \sin \theta \cos \theta = \frac{ax}{a^2 + x^2}.$$

Hence

$$\int \frac{dx}{(x^2 + a^2)^2} = \frac{1}{2\,a^3}\left[\tan^{-1} \frac{x}{a} + \frac{ax}{a^2 + x^2}\right] + C.$$

EXERCISES

1. $\int \sin^3 x \, dx.$

2. $\int \cos^5 x \, dx.$

3. $\int (\cos x + \sin x)^3 \, dx.$

4. $\int \cos^2 x \sin^3 x \, dx.$

5. $\int \sin^4 \tfrac{1}{2} x \cos^5 \tfrac{1}{2} x \, dx.$

6. $\int \sin^5 3\theta \cos^3 3\theta \, d\theta.$

7. $\int (\cos^2 \theta - \sin^2 \theta) \sin \theta \, d\theta.$

8. $\int \frac{\cos^3 x \, dx}{1 - \sin x}.$

9. $\int \frac{\cos^2 x \, dx}{\sin x}.$

10. $\int \frac{\sin^5 \theta \, d\theta}{\cos \theta}.$

11. $\int \sec^4 x \, dx.$

12. $\int \csc^{10} y \, dy.$

13. $\int \tan^2 x \, dx.$

14. $\int \frac{\sec^3 \theta + \tan^3 \theta}{\sec \theta + \tan \theta} \, d\theta.$

15. $\int \tan \tfrac{1}{2} x \sec^3 \tfrac{1}{2} x \, dx.$

16. $\int \tan^5 2x \sec^3 2x \, dx.$

17. $\int \cot^3 x \, dx.$

18. $\int \tan^7 x \, dx.$

19. $\int \frac{\cos^2 x \, dx}{\sin^6 x}.$

20. $\int \sec^3 x \csc x \, dx.$

21. $\int \sin^2 ax \, dx.$

22. $\int \cos^2 ax \, dx.$

23. $\int \cos^2 x \sin^4 x \, dx.$

24. $\int \cos^4 \tfrac{1}{2} x \sin^2 \tfrac{1}{2} x \, dx.$

25. $\int \sin^6 x \, dx.$

26. $\int \frac{dx}{1 - \sin x}.$

27. $\int \frac{dx}{1 + \cos x}.$

28. $\sqrt{1 + \sin \theta} \, d\theta.$

29. $\int \sqrt{x^2 - a^2} \, dx.$

30. $\int \sqrt{x^2 + a^2} \, dx.$

31. $\int \frac{x^2 \, dx}{\sqrt{x^2 + a^2}}.$

32. $\int \frac{dx}{(x^2 - a^2)^{\frac{3}{2}}}.$

33. $\int \frac{dx}{x \sqrt{a^2 - x^2}}.$

34. $\int \frac{dx}{x \sqrt{2ax - x^2}}.$

35. $\int \frac{x \, dx}{(a^2 - x^2)^{\frac{3}{2}}}.$

36. $\int x^3 \sqrt{x^2 + a^2} \, dx.$

37. $\int \frac{dx}{x^2 \sqrt{x^2 + a^2}}.$

38. $\int \sqrt{x^2 - 4x + 5} \, dx.$

39. $\int \frac{(x^2 - x) \, dx}{\sqrt{2 - 2x - 4x^2}}.$

13. Integration of Rational Fractions. — A fraction, such as

$$\frac{x^3 + 3x}{x^2 - 2x - 3},$$

whose numerator and denominator are polynomials is called a *rational fraction*.

If the degree of the numerator is equal to or greater than that of the denominator, the fraction should be reduced by division. Thus

$$\frac{x^3 + 9x + 12}{x^2 - 2x - 3} = x + 2 + \frac{10x + 6}{x^2 - 2x + 3}.$$

A fraction with numerator of lower degree than its denominator can be resolved into a sum of *partial* fractions with denominators that are factors of the original denominator. Thus

$$\frac{10x + 6}{x^2 - 2x - 3} = \frac{10x + 6}{(x - 3)(x + 1)} = \frac{9}{x - 3} + \frac{1}{x + 1}.$$

These fractions can often be found by trial. If not, proceed as in the following examples.

CASE 1. Factors of the denominator all of the first degree and none repeated.

Ex. 1. $\displaystyle\int \frac{x^4 + 2x + 6}{x^3 + x^2 - 2x}\,dx.$

Dividing numerator by denominator, we get

$$\frac{x^4 + 2x + 6}{x^3 + x^2 - 2x} = x - 1 + \frac{3x^2 + 6}{x^3 + x^2 - 2x}$$

$$= x - 1 + \frac{3x^2 + 6}{x(x - 1)(x + 2)}.$$

Assume

$$\frac{3x^2 + 6}{x(x - 1)(x + 2)} = \frac{A}{x} + \frac{B}{x - 1} + \frac{C}{x + 2}.$$

The two sides of this equation are merely different ways of writing the same function. If then we clear of fractions, the two sides of the resulting equation

$$3\,x^2 + 6 = A\,(x-1)\,(x+2) + Bx\,(x+2) + Cx\,(x-1)$$
$$= (A+B+C)\,x^2 + (A+2\,B-C)\,x - 2\,A$$

are identical. That is

$$A+B+C=3, \quad A+2\,B-C=0, \quad -2\,A=6.$$

Solving these equations, we get

$$A=-3, \quad B=3, \quad C=3.$$

Conversely, if A, B, C, have these values, the above equations are identically satisfied. Therefore

$$\int\frac{x^4+2\,x+6}{x^3+x^2-2\,x}\,dx = \int\left(x-1-\frac{3}{x}+\frac{3}{x-1}+\frac{3}{x+2}\right)dx$$
$$= \tfrac{1}{2}\,x^2 - x - 3\ln x + 3\ln(x-1) + 3\ln(x+2) + C$$
$$= \frac{1}{2}\,x^2 - x + 3\ln\frac{(x-1)\,(x+2)}{x} + C.$$

The constants can often be determined more easily by substituting particular values for x on the two sides of the equation. Thus, the equation above,

$$3\,x^2+6 = A\,(x-1)\,(x+2) + Bx\,(x+2) + Cx\,(x-1)$$

is an identity, that is, it is satisfied by all values of x. In particular, if $x=0$, it becomes

$$6=-2\,A,$$

whence $A=-3$. Similarly, by substituting $x=1$ and $x=-2$, we get

$$9=3\,B, \quad 18=6\,C,$$

whence $\qquad B=3, \qquad C=3.$

CASE 2. Factors of the denominator all of first degree but some repeated.

Ex. 2. $\displaystyle\int\frac{(8\,x^3+7)\,dx}{(x+1)\,(2\,x+1)^3}.$

Assume

$$\frac{8x^3+7}{(x+1)(2x+1)^3} = \frac{A}{x+1} + \frac{B}{(2x+1)^3} + \frac{C}{(2x+1)^2} + \frac{D}{2x+1}.$$

Corresponding to the repeated factor $(2x+1)^3$, we thus introduce fractions with $(2x+1)^3$ and all lower powers as denominators. Clearing and solving as before, we find

$$A = 1, \qquad B = 12, \qquad C = -6, \qquad D = 0.$$

Hence

$$\int \frac{8x^3+7}{(x+1)(2x+1)^3}\,dx = \int \left[\frac{1}{x+1} + \frac{12}{(2x+1)^3} - \frac{6}{(2x+1)^2}\right] dx$$

$$= \ln(x+1) - \frac{3}{(2x+1)^2} + \frac{3}{2x+1} + C.$$

CASE 3. Denominator containing factors of the second degree but none repeated.

Ex. 3. $\displaystyle\int \frac{4x^2+x+1}{x^3-1}\,dx.$

The factors of the denominator are $x-1$ and x^2+x+1. Assume

$$\frac{4x^2+x+1}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}.$$

With the quadratic denominator x^2+x+1, we thus use a numerator that is not a single constant but a linear function $Bx+C$. Clearing fractions and solving for A, B, C, we find

$$A = 2, \qquad B = 2, \qquad C = 1.$$

Therefore

$$\int \frac{4x^2+x+1}{x^3-1}\,dx = \int \left(\frac{2}{x-1} + \frac{2x+1}{x^2+x+1}\right) dx$$

$$= 2\ln(x-1) + \ln(x^2+x+1) + C.$$

CASE 4. Denominator containing factors of the second degree, some being repeated.

Ex. 4. $\displaystyle\int \frac{x^3+1}{x(x^2+1)^2}\,dx.$

Assume

$$\frac{x^3+1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{(x^2+1)^2} + \frac{Dx+E}{x^2+1}.$$

Corresponding to the repeated second degree factor $(x^2+1)^2$, we introduce partial fractions having as denominators $(x^2+1)^2$ and all lower powers of x^2+1, the numerators being all of first degree. Clearing fractions and solving for A, B, C, D, E, we find

$$A = 1, \quad B = -1, \quad C = -1, \quad D = -1, \quad E = 1.$$

Hence

$$\int \frac{x^3+1}{x(x^2+1)^2}\,dx = \int \left[\frac{1}{x} - \frac{x+1}{(x^2+1)^2} - \frac{x-1}{x^2+1}\right] dx$$

$$= \ln \frac{x}{\sqrt{x^2+1}} + \frac{1}{2}\tan^{-1} x - \frac{x-1}{2(x^2+1)} + C.$$

14. Integrals Containing $(ax+b)^{\frac{p}{q}}$. — Integrals containing $(ax+b)^{\frac{p}{q}}$ can be rationalized by the substitution

$$ax + b = z^q.$$

If several fractional powers of the same linear function $ax+b$ occur, the substitution

$$ax + b = z^n$$

may be used, n being so chosen that all the roots can be extracted.

Example 1. $\displaystyle\int \frac{dx}{1+\sqrt{x}}.$

Let $x = z^2$. Then $dx = 2\,z\,dz$ and

$$\int \frac{dx}{1+\sqrt{x}} = \int \frac{2\,z\,dz}{1+z} = \int \left(2 - \frac{2}{1+z}\right) dz$$

$$= 2\,z - 2\ln(1+z) + C$$

$$= 2\sqrt{x} - 2\ln(1+\sqrt{x}) + C.$$

Ex. 2. $\displaystyle\int \frac{(2\,x-3)^{\frac{1}{2}}\,dx}{(2\,x-3)^{\frac{1}{3}}+1}.$

To rationalize both $(2x-3)^{\frac{1}{2}}$ and $(2x-3)^{\frac{1}{3}}$, let $2x-3=z^6$. Then

$$\int\frac{(2x-3)^{\frac{1}{2}}\,dx}{(2x-3)^{\frac{1}{3}}+1}=\int\frac{3z^8\,dz}{z^2+1}=3\int\left(z^6-z^4+z^2-1+\frac{1}{z^2+1}\right)dz$$

$$=3\left(\frac{z^7}{7}-\frac{z^5}{5}+\frac{z^3}{3}-z+\tan^{-1}z\right)+C$$

$$=\tfrac{3}{7}(2x-3)^{\frac{7}{6}}-\tfrac{3}{5}(2x-3)^{\frac{5}{6}}+(2x-3)^{\frac{1}{2}}$$
$$-3(2x-3)^{\frac{1}{6}}+\tan^{-1}(2x-3)^{\frac{1}{6}}+C.$$

EXERCISES

1. $\int\frac{x^3+x^2}{x^2-3x+2}\,dx.$
2. $\int\frac{2x+3}{x^2+x}\,dx.$
3. $\int\frac{x^2+1}{x(x^2-1)}\,dx.$
4. $\int\frac{x^3-1}{4x^3-x}\,dx.$
5. $\int\frac{x\,dx}{(x+1)(x+3)(x+5)}.$
6. $\int\frac{16x\,dx}{(2x-1)(2x-3)(2x-5)}.$
7. $\int\frac{x^3+1}{x^3-x^2}\,dx.$
8. $\int\frac{x^2\,dx}{(x+1)(x-1)^2}.$
9. $\int\frac{dx}{(x^2-1)^2}.$
10. $\int\left(\frac{x-1}{x+1}\right)^4dx$
11. $\int\frac{dx}{x^5-x^4}.$
12. $\int\frac{x^2\,dx}{(x^2-4)^2}.$
13. $\int\frac{x\,dx}{(x^2-4)^2}.$
14. $\int\frac{x^4\,dx}{x^4-1}.$
15. $\int\frac{dx}{x^3+1}.$
16. $\int\frac{x^2\,dx}{x^3+1}.$
17. $\int\frac{dx}{x^5-x^3+x^2-1}.$
18. $\int\frac{2x^2+x-2}{(x^2-1)^2}\,dx.$
19. $\int\frac{x^4+24x^2-8x}{(x^3-8)^2}\,dx.$
20. $\int\frac{(x+1)^{\frac{1}{3}}\,dx}{x}.$
21. $\int\frac{x^{\frac{1}{2}}-x^{\frac{1}{6}}+1}{x^{\frac{2}{3}}-x^{\frac{5}{6}}}\,dx.$
22. $\int x\sqrt{ax+b}\,dx.$
23. $\int\frac{\sqrt{x+2}-1}{x+3}\,dx.$
24. $\int\frac{dx}{(x^{\frac{1}{2}}-1)(x^{\frac{1}{3}}+1)}.$
25. $\int\frac{dx}{\sqrt{x+1}-\sqrt{x-1}}.$

15. Integration by Parts. — From the formula

$$d\,(uv) = u\,dv + u\,dv$$

re get

$$u\,dv = d\,(uv) - v\,du,$$

whence

$$\int u\,dv = uv - \int v\,du. \qquad (15)$$

If $\int v\,du$ is known this gives $\int v\,du$. Integration by the use of this formula is called *integration by parts.*

Example 1. $\int \ln x\,dx.$

Let $u = \ln x$, $dv = dx$. Then $du = \dfrac{dx}{x}$, $v = x$, and

$$\int \ln x\,dx = \ln x \cdot x - \int x \cdot \frac{dx}{x}$$
$$= x\,(\ln x - 1) + C.$$

Ex. 2. $\int x^2 \sin x\,dx.$

Let $u = x^2$ and $dv = \sin x\,dx$. Then $du = 2\,x\,dx$, $v = -\cos x$, and

$$\int x^2 \sin x\,dx = -x^2 \cos x + \int 2\,x \cos x\,dx.$$

A second integration by parts with $u = 2\,x$, $dv = \cos x\,dx$ gives

$$\int 2\,x \cos x\,dx = 2\,x \sin x - \int 2 \sin x\,dx$$
$$= 2\,x \sin x + 2 \cos x + C.$$

Hence finally

$$\int x^2 \sin x\,dx = -x^2 \cos x + 2\,x \sin x + 2 \cos x + C.$$

The method of integration by parts applies particularly to functions that are simplified by differentiation, like $\ln x$, or to products of functions of different classes, like $x \sin x$. In applying the method the given differential must be resolved into a product $u \cdot dv$. The part called dv must have a known integral and the part called u should usually be simplified by differentiation.

Sometimes after integration by parts a multiple of the original differential appears on the right side of the equation. It can be transposed to the other side and the integral can be solved for algebraically. This is shown in the following examples.

Ex. 3. $\int \sqrt{a^2 - x^2}\, dx.$

Integrating by parts with $u = \sqrt{a^2 - x^2}$, $dv = dx$, we get

$$\int \sqrt{a^2 - x^2}\, dx = x\sqrt{a^2 - x^2} - \int \frac{-x^2\, dx}{\sqrt{a^2 - x^2}}.$$

Adding a^2 to the numerator of the integral and subtracting an equivalent integral, this becomes

$$\begin{aligned}\int \sqrt{a^2 - x^2}\, dx &= x\sqrt{a^2 - x^2} - \int \frac{a^2 - x^2}{\sqrt{a^2 - x^2}}\, dx + a^2 \int \frac{dx}{\sqrt{a^2 - x^2}} \\ &= x\sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2}\, dx + a^2 \int \frac{dx}{\sqrt{a^2 - x^2}}.\end{aligned}$$

Transposing $\int \sqrt{a^2 - x^2}\, dx$ and dividing by 2, we get

$$\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C.$$

Ex. 4. $\int e^{ax} \cos bx\, dx.$

Integrating by parts with $u = e^{ax}$, $dv = \cos bx\, dx$, we get

$$\int e^{ax} \cos bx\, dx = \frac{e^{ax} \sin bx}{b} - \frac{a}{b} \int e^{ax} \sin bx\, dx.$$

Integrating by parts again with $u = e^{ax}$, $dv = \sin bx\,dx$, this becomes

$$\int e^{ax} \cos bx\,dx = \frac{e^{ax} \sin bx}{b} - \frac{a}{b}\left[-\frac{e^{ax}\cos bx}{b} + \frac{a}{b}\int e^{ax}\sin bx\,dx\right]$$
$$= e^{ax}\left(\frac{b \sin bx + a \cos bx}{b^2}\right) - \frac{a^2}{b^2}\int e^{ax} \sin bx\,dx.$$

Transposing the last integral and dividing by $1 + \frac{b^2}{a^2}$, this gives

$$\int e^{ax} \cos bx\,dx = e^{ax}\left(\frac{b \sin bx + a \cos bx}{a^2 + b^2}\right).$$

16. Reduction Formulas. — Integration by parts is often used to make an integral depend on a simpler one and so to obtain a formula by repeated application of which the given integral can be determined.

To illustrate this take the integral

$$\int \sin^n x\,dx,$$

where n is a positive integer. Integrating by parts with $u = \sin^{n-1} x$, $dv = \sin x\,dx$, we get

$$\int \sin^n x\,dx = -\sin^{n-1} x \cos x + \int (n-1) \sin^{n-2} x \cos^2 x\,dx$$
$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\,(1 - \sin^2 x)\,dx$$
$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\,dx$$
$$- (n-1)\int \sin^n x\,dx.$$

Transposing the last integral and dividing by n, we get

$$\int \sin^n x\,dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx.$$

By successive application of this formula we can make $\int \sin^n x\,dx$ depend on $\int dx$ or $\int \sin x\,dx$ according as n is even or odd.

Example. $\int \sin^6 x\, dx.$

By the formula just proved

$$\int \sin^6 x\, dx = -\frac{\sin^5 x \cos x}{6} + \frac{5}{6}\int \sin^4 x\, dx$$

$$= -\frac{\sin^5 x \cos x}{6} + \frac{5}{6}\left[-\frac{\sin^3 x \cos x}{4} + \frac{3}{4}\int \sin^2 x\, dx\right]$$

$$= -\frac{\sin^5 x \cos x}{6} - \frac{5}{24}\sin^3 x \cos x - \frac{5}{16}\sin x \cos x + \frac{5}{16}x + C.$$

EXERCISES

1. $\int x \cos 2x\, dx.$
2. $\int \ln x \cdot x\, dx.$
3. $\int \sin^{-1} x\, dx.$
4. $\int x \tan^{-1} x\, dx.$
5. $\int \ln\left(x + \sqrt{a^2 + x^2}\right) dx.$
6. $\int \frac{\ln x\, dx}{\sqrt{x-1}}.$
7. $\int \ln(\ln x)\frac{dx}{x}.$
8. $\int x^2 \sec^{-1} x\, dx.$
9. $\int e^{-x} \ln(e^x + 1)\, dx.$
10. $\int x^2 e^x\, dx.$
11. $\int x^3 e^{-x}\, dx.$
12. $\int (x-1)^2 \sin(2x)\, dx.$
13. $\int \sqrt{x^2 - a^2}\, dx.$
14. $\int \sqrt{a^2 + x^2}\, dx.$
15. $\int e^{2x} \sin 3x\, dx.$
16. $\int e^x \cos x\, dx.$
17. $\int e^{-x} \sin 2x\, dx.$
18. $\int \sec^3 \theta\, d\theta.$
19. $\int \sin 2x \cos 3x\, dx.$
20. Prove the formula

$$\int \sec^n(x)\, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}(x)\, dx.$$

and use it to integrate $\int \sec^5 x\, dx.$

21. Prove the formula

$$\int (a^2 - x^2)^n\, dx = \frac{x(a^2 - x^n)^n}{2n+1} + \frac{2na^2}{2n+1}\int (a^2 - x^2)^{n-1}\, dx$$

and use it to integrate $\int (a^2 - x^2)^{\frac{5}{2}}\, dx.$

CHAPTER III

DEFINITE INTEGRALS

17. Summation. — Between $x = a$ and $x = b$ let $f(x)$ be a continuous function of x. Divide the interval between a and b into any number of equal parts Δx and let $x_1, x_2, \ldots x_n$, be the points of division. Form the sum

$$f(a)\,\Delta x + f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x.$$

This sum is represented by the notation

$$\sum\nolimits_a^b f(x)\,\Delta x.$$

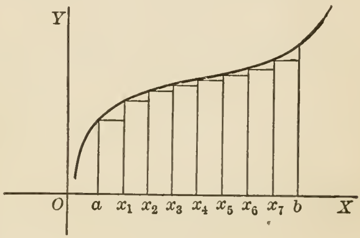

FIG. 17a.

Since $f(a)$, $f(x_1)$, $f(x_2)$, etc., are the ordinates of the curve $y = f(x)$ at $x = x_1, x_2$, etc., the terms $f(a)\,\Delta x, f(x_1)\,\Delta x, f(x_2)\,\Delta x$, etc., represent the areas of the rectangles in Fig. 17*a*, and $\sum_a^b f(x)\,\Delta x$ is the sum of those rectangles.

Example 1. Find the value of $\sum_1^2 x^2\,\Delta x$ when $\Delta x = \frac{1}{4}$.

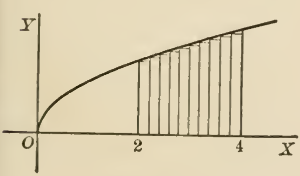

FIG. 17b.

The interval between 1 and 2 is divided into parts of length $\Delta x = \frac{1}{4}$. The points of division are $1\frac{1}{4}$, $1\frac{1}{2}$, $1\frac{3}{4}$. Therefore

$$\sum\nolimits_1^2 x^2\,\Delta x = 1^2 \cdot \Delta x + (\tfrac{5}{4})^2\,\Delta x + (\tfrac{3}{2})^2\,\Delta x + (\tfrac{7}{4})^2\,\Delta x$$
$$= \tfrac{63}{8}\,\Delta x = \tfrac{63}{8} \cdot \tfrac{1}{4} = 1.97.$$

Ex. 2. Find approximately the area bounded by the x-axis, the curve $y = \sqrt{x}$, and the ordinates $x = 2$, $x = 4$.

From Fig. 17*b* it appears that a fairly good approximation will be obtained by dividing the interval between 2 and 4 into 10 parts each of length 0.2. The value of the area thus obtained is

$$\sum_2^4 \sqrt{x}\,\Delta x = (\sqrt{2}+\sqrt{2.2}+\sqrt{2.4}+\cdots+\sqrt{3.8})(0.2) = 3.39.$$

The area correct to two decimals (given by the method of Art. 20) is 3.45.

18. Definite and Indefinite Integrals. — If we increase indefinitely the number of parts into which $b - a$ is divided, the intervals Δx approach zero and $\sum_a^b f(x)\,\Delta x$ usually approaches a limit. This limit is called the *definite* integral of $f(x)\,dx$ between $x = a$ and $x = b$. It is represented by the notation $\int_a^b f(x)\,dx$. That is

$$\int_a^b f(x)\,dx = \lim_{\Delta x \doteq 0} \sum_a^b f(x)\,\Delta x. \tag{18}$$

The number a is called the *lower limit*, b the *upper limit* of the integral.

In contradistinction to the definite integral (which has a definite value), the integral that we have previously used (which contains an undetermined constant) is called an *indefinite* integral. The connection between the two integrals will be shown in Art. 21.

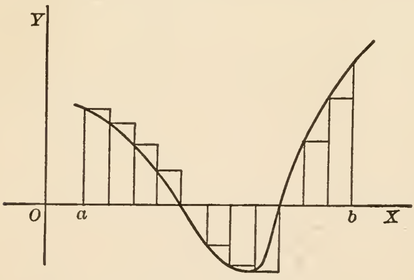

FIG. 19*a*.

19. Geometrical Representation.—If the curve $y = f(x)$ lies above the x-axis and $a < b$, as in Fig. 17*a*, $\int_a^b f(x)\,dx$ represents the limit approached by the sum of the inscribed rectangles and that limit is the area between $x = a$ and $x = b$ bounded by the curve and the x-axis.

At a point below the x-axis the ordinate $f(x)$ is negative and so the product $f(x)\,\Delta x$ is the negative of the area of the corresponding rectangle. Therefore (Fig. 19a)

$$\sum_a^b f(x)\,\Delta x = (\text{sum of rectangles above } OX) - (\text{sum of rectangles below } OX),$$

and in the limit

$$\int_a^b f(x)\,dx = (\text{area above } OX) - (\text{area below } OX) \qquad (19a)$$

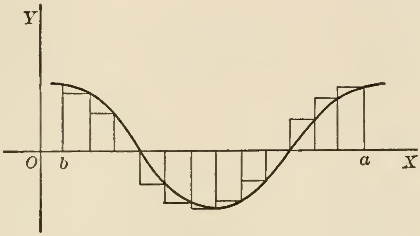

FIG. 19b.

If, however, $a > b$, as in Fig. 19b, x decreases as we pass from a to b, Δx is negative and instead of the above equation we have

$$\int_a^b f(x)\,dx = (\text{area below } OX) - (\text{area above } OX). \qquad (19b)$$

Example 1. Show graphically that $\int_0^{2\pi} \sin^3 x\,dx = 0$.

The curve $y = \sin^3 x$ is shown in Fig. 19c. Between $x = 0$ and $x = 2\pi$ the areas above and below the x-axis are equal. Hence

$$\int_0^{2\pi} \sin^3 x\,dx = A_1 - A_2 = 0.$$

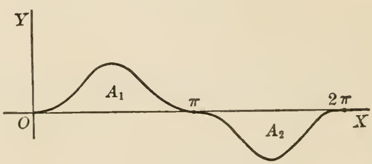

FIG. 19c.

Ex. 2. Show that

$$\int_{-1}^1 e^{-x^2}\,dx = 2\int_0^1 e^{-x^2}\,dx.$$

The curve $y = e^{-x^2}$ is shown in Fig. 19a. It is symmetrica with respect to the y-axis The area between $x = -1$ and $x = 0$ is therefore equa to that between $x = 0$ and $x = 1$. Consequently

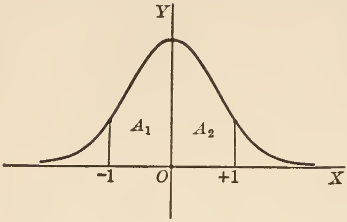

FIG. 19d.

$$\int_{-1}^{1} e^{-x^2}\,dx = A_1 + A_2 = 2A_2$$

$$= 2\int_{0}^{1} e^{-x^2}\,dx.$$

EXERCISES

Find the values of the following sums:

1. $\sum_0^2 x\,\Delta x, \qquad \Delta x = \frac{1}{3}.$

2. $\sum_1^{10} \frac{\Delta x}{x}, \qquad \Delta x = 1.$

3. $\sum_{-2}^{2} \sqrt[3]{x}\,\Delta x, \qquad \Delta x = \frac{1}{2}.$

4. Show that

$$\sum_0^{\frac{\pi}{6}} \sin x\,\Delta x = 1 - \cos\frac{\pi}{6}$$

approximately. Use a table of natural sines and take $\Delta x = \frac{\pi}{60}$.

5. Calculate π approximately by the formula

$$\pi = 4\sum_0^1 \frac{\Delta x}{1 + x^2}, \qquad \Delta x = 0.1.$$

6. Find correct to one decimal the area bounded by the parabola $y = x^2$, the x-axis, and the ordinates $x = 0$, $x = 2$. The exact area is $\frac{8}{3}$.

7. Find correct to one decimal the area of the circle $x^2 + y^2 = 4$.

By representing the integrals as areas prove graphically the following equations:

8. $\int_0^{\pi} \sin(2x)\,dx = 0.$

9. $\int_0^{2\pi} \cos^7 x\,dx = 0.$

10. $\int_0^{\pi} \sin^5 x\,dx = 2\int_0^{\frac{\pi}{2}} \sin^5 x\,dx.$

11. $\int_{-a}^{+a} \frac{x\,dx}{1+x^4} = 0.$

12. $\int_{-1}^{1} \frac{dx}{1+x^4} = 2\int_{0}^{1} \frac{dx}{1+x^4}.$

13. $\int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(a-x)\,dx.$

20. Derivative of Area. — The area A bounded by a curve

$$y = f(x),$$

a fixed ordinate $x = a$, and a movable ordinate MP, is a function of the abscissa x of the movable ordinate.

Let x change to $x + \Delta x$. The increment of area is

$$\Delta A = MPQN.$$

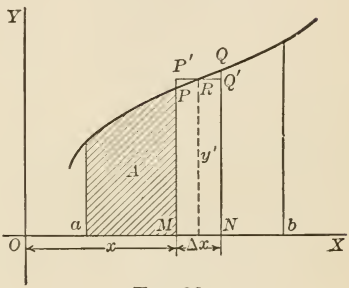

FIG. 20.

Construct the rectangle $MP'Q'N$ equal in area to $MPQN$. If some of the points of the arc PQ are above $P'Q'$, others must be below to make $MPQN$ and $MP'Q'N$ equal. Hence $P'Q'$ intersects PQ at some point R. Let y' be the ordinate of R. Then y' is the altitude of $MP'Q'N$ and so

$$\Delta A = MPQN = MP'Q'N = y'\,\Delta x.$$

Consequently

$$\frac{\Delta A}{\Delta x} = y'.$$

When Δx approaches zero, if the curve is continuous, y' approaches y. Therefore in the limit

$$\frac{dA}{dx} = y = f(x). \tag{20a}$$

Let the indefinite integral of $f(x)\,dx$ be

$$\int f(x)\,dx = F(x) + C.$$

From equation (20a) we then have

$$A = \int f(x)\,dx = F(x) + C.$$

The area is zero when $x = a$. Consequently

$$0 = F(a) + C,$$

whence $C = -F(a)$ and

$$A = F(x) - F(a).$$

This is the area from $x = a$ to the ordinate MP with abscissa x. The area between $x = a$ and $x = b$ is then

$$A = F(b) - F(a). \tag{20b}$$

The difference $F(b) - F(a)$ is often represented by the notation $F(x)\Big|_a^b$, that is,

$$F(x)\Big|_a^b = F(b) - F(a). \tag{20c}$$

21. Relation of the Definite and Indefinite Integrals. — The definite integral $\int_a^b f(x)\,dx$ is equal to the area bounded by the curve $y = f(x)$, the x-axis, and the ordinates $x = a$, $x = b$. If

$$\int f(x)\,dx = F(x) + C,$$

by equation (20b) this area is $F(b) - F(a)$. We therefore conclude that

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a), \tag{21}$$

that is, *to find the value of the definite integral* $\int_a^b f(x)\,dx$, *substitute* $x = a$, *and* $x = b$ *in the indefinite integral* $\int f(x)\,dx$ *and subtract the former from the latter result.*

Example. Find the value of the integral

$$\int_0^1 \frac{dx}{1+x^2}.$$

The value required is

$$\int_0^1 \frac{dx}{1+x^2} = \tan^{-1} x \Big|_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4}.$$

22. Properties of Definite Integrals. — A definite integral has the following simple properties:

$$\text{I.} \quad \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

$$\text{II.} \quad \int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

$$\text{III.} \quad \int_a^b f(x)\,dx = (b-a)f(x_1), \qquad a \gtreqless x_1 \gtreqless b.$$

The first of these is due to the fact that if Δx is positive when x varies from a to b, it is negative when x varies from b to a. The two integrals thus represent the same area with different algebraic signs.

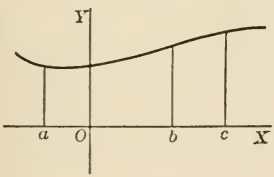

FIG. 22*a*.

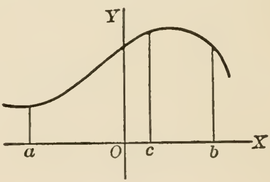

FIG. 22*b*.

The second property expresses that the area from a to c is equal to the sum of the areas from a to b and b to c. This is the case not only when b is between a and c, as in Fig. 22*a*, but also when b is beyond c, as in Fig. 22*b*. In the latter case $\int_b^c f(x)\,dx$ is negative and the sum

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

is equal to the difference of the two areas.

Equation III expresses that the area $PQMN$ is equal to that of a rectangle $P'Q'MN$ with altitude between MP and NQ.

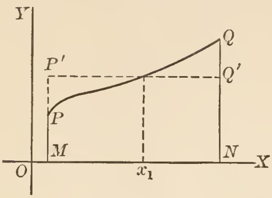

FIG. 22c.

23. Infinite Limits. — It has been assumed that the limits a and b were finite. If the integral

$$\int_a^b f(x)\,dx$$

approaches a limit when b increases indefinitely, that limit is defined as the value of $\int_a^\infty f(x)\,dx$. That is,

$$\int_a^\infty f(x)\,dx = \lim_{b=\infty} \int_a^b f(x)\,dx. \qquad (23)$$

If the indefinite integral

$$\int f(x)\,dx = F(x)$$

approaches a limit when x increases indefinitely,

$$\int_a^\infty f(x)\,dx = \lim_{b=\infty} [F(b) - F(a)] = F(\infty) - F(a).$$

The value is thus obtained by equation (21) just as if the limits were finite.

Example 1. $\displaystyle\int_0^\infty \frac{dx}{1+x^2}.$

The indefinite integral is

$$\int \frac{dx}{1+x^2} = \tan^{-1} x.$$

When x approaches infinity, this approaches $\frac{\pi}{2}$. Hence

$$\int_0^\infty \frac{dx}{1+x^2} = \tan^{-1} x \Big|_0^\infty = \frac{\pi}{2}.$$

Ex. 2. $\int_0^\infty \cos x\,dx.$

The indefinite integral $\sin x$ does not approach a limit when x increases indefinitely. Hence

$$\int_0^\infty \cos x\,dx$$

has no definite value.

24. Infinite Values of the Function. — If the function $f(x)$ becomes infinite when $x = b$, $\int_a^b f(x)\,dx$ is defined as the limit

$$\int_a^b f(x)\,dx = \lim_{z=b} \int_a^z f(x)\,dx,$$

z being between a and b.

Similarly, if $f(a)$ is infinite,

$$\int_a^b f(x)\,dx = \lim_{z=a} \int_z^b f(x)\,dx,$$

z being between a and b.

If the function becomes infinite at a point c between a and b, $\int_a^b f(x)\,dx$ is defined by the equation

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx. \tag{24}$$

Example 1. $\int_{-1}^1 \frac{dx}{\sqrt[3]{x}}.$

When $x = 0$, $\frac{1}{\sqrt[3]{x}}$ is infinite. We therefore divide the integral into two parts:

$$\int_{-1}^1 \frac{dx}{\sqrt[3]{x}} = \int_{-1}^0 \frac{dx}{\sqrt[3]{x}} + \int_0^1 \frac{dx}{\sqrt[3]{x}} = -\frac{3}{2} + \frac{3}{2} = 0.$$

Ex. 2. $\int_{-1}^{1} \frac{dx}{x^2}$.

If we use equation (21), we get

$$\int_{-1}^{1} \frac{dx}{x^2} = -\frac{1}{x}\Big|_{-1}^{1} = -2.$$

Since the integral is obviously positive, the result -2 is absurd. This is due to the fact that $\frac{1}{x^2}$ becomes infinite when $x = 0$. Resolving the integral into two parts, we get

$$\int_{-1}^{1} \frac{dx}{x^2} = \int_{-1}^{0} \frac{dx}{x^2} + \int_{0}^{1} \frac{dx}{x^2} = \infty + \infty = \infty.$$

25. Change of Variable. — If a change of variable is made in evaluating an integral, the limits can be replaced by the corresponding values of the new variable. To see this, suppose that when x is expressed in terms of t,

$$\int f(x)\,dx = F(x)$$

is changed into

$$\int \phi(t)\,dt = \Phi(t).$$

If t_0, t_1, are the values of t, corresponding to x_0, x_1,

$$F(x_0) = \Phi(t_0), \qquad F(x_1) = \Phi(t_1),$$

and so

$$F(x_1) - F(x_0) = \Phi(t_1) - \Phi(t_0),$$

that is

$$\int_{x_0}^{x_1} f(x)\,dx = \int_{t_0}^{t_1} \phi(t)\,dt.$$

If more than one value of t corresponds to the same value of x, care should be taken to see that when t varies from t_0 to t_1, x varies from x_0 to x_1, and that for all intermediate values, $f(x)\,dx = \phi(t)\,dt$.

Example. $\int_{-a}^{a} \sqrt{a^2 - x^2}\,dx$.

Substituting $x = a \sin \theta$, we find

$$\int \sqrt{a^2 - x^2}\, dx = a^2 \int \cos^2 \theta\, d\theta = \frac{a^2}{2}\left[\theta + \frac{\sin 2\,\theta}{2}\right].$$

When $x = a$, $\sin \theta = 1$, and $\theta = \frac{\pi}{2}$. When $x = -a$, $\sin \theta = -1$ and $\theta = -\frac{\pi}{2}$. Therefore

$$\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx = a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 \theta\, d\theta = \frac{a^2}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi a^2}{2}.$$

Since $\sin \frac{3}{2}\pi = -1$, it might seem that we could use $\frac{3}{2}\pi$ as the lower limit. We should then get

$$a^2 \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \cos^2 \theta\, d\theta = -\frac{\pi a^2}{2}.$$

This is not correct because in passing from $\frac{3}{2}\pi$ to $\frac{1}{2}\pi$, θ crosses the third and second quadrants. There $\cos \theta$ is negative and

$$\sqrt{a^2 - x^2}\, dx = (-a \cos \theta) \cos \theta\, d\theta,$$

and not $a^2 \cos^2 \theta\, d\theta$ as assumed above.

EXERCISES

Find the values of the following definite integrals:

1. $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sec^2 x\, dx.$

2. $\displaystyle\int_{\frac{a}{2}}^{a} \frac{dx}{\sqrt{a^2 - x^2}}.$

3. $\displaystyle\int_{-2}^{2} (x-1)^3\, dx.$

4. $\displaystyle\int_{-5}^{9} \frac{x\, dx}{\sqrt{x^2 + 144}}.$

5. $\displaystyle\int_{-\pi}^{\frac{3}{2}\pi} \sin^3 \theta\, d\theta.$

6. $\displaystyle\int_{2}^{3} x \ln x\, dx.$

7. $\displaystyle\int_{3}^{4} \frac{dx}{x^2 - 3x + 2}.$

8. $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan x\, dx.$

9. $\displaystyle\int_{0}^{a \ln 2} \left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right) dx.$

10. $\displaystyle\int_{0}^{1} \frac{dx}{\sqrt{x}}.$

11. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \csc^2 x\, dx.$

12. $\int_1^{\infty} \frac{dx}{x\sqrt{x^2 - 1}}.$

13. $\int_0^{\infty} e^{-k^2x^2}\, x\, dx.$

14. $\int_2^{\infty} \frac{dx}{x^2 - 1}.$

Evaluate the following definite integrals by making the change of variable indicated:

15. $\int_{-1}^{1} \frac{dx}{(1 + x^2)^2}, \qquad x = \tan\theta.$

16. $\int_1^5 \frac{\sqrt{x - 1}}{x}\, dx, \qquad x - 1 = z^2.$

17. $\int_{\frac{3}{4}}^{\frac{4}{3}} \frac{dz}{z\sqrt{z^2 + 1}}, \qquad z = \frac{1}{x}.$

18. $\int_0^{\frac{\pi}{2}} \frac{\cos\theta\, d\theta}{6 - 5\sin\theta + \sin^2\theta}, \qquad \sin\theta = z.$

19. $\int_0^a \frac{x^3\, dx}{a^2 + x^2}, \qquad a^2 + x^2 = z^2.$

CHAPTER IV

SIMPLE AREAS AND VOLUMES

26. Area Bounded by a Plane Curve. Rectangular Coördinates. — The area bounded by the curve $y = f(x)$, the x-axis and two ordinates $x = a$, $x = b$, is the limit approached by the sum of rectangles $y\,\Delta x$. That is,

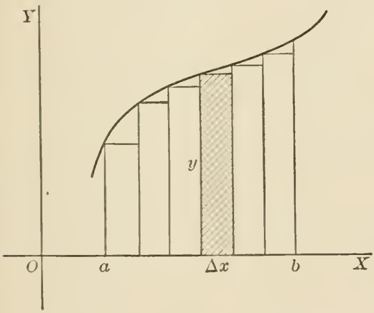

FIG. 26*a*.

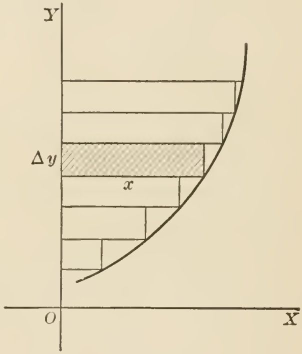

FIG. 26*b*.

$$A = \lim_{\Delta x \doteq 0} \sum_a^b y\,\Delta x = \int_a^b y\,dx = \int_a^b f(x)\,dx. \qquad (26a)$$

Similarly, the area bounded by a curve, the abscissas $y = a$, $y = b$, and the y-axis is

$$A = \lim_{\Delta y \doteq 0} \Sigma\, x\,\Delta y = \int_a^b x\,dy. \qquad (26b)$$

Example 1. Find the area bounded by the curve $x = 2 + y - y^2$ and the y-axis.

The curve (Fig. 26c) crosses the y-axis at $y = -1$ and $y = 2$. The area required is, therefore,

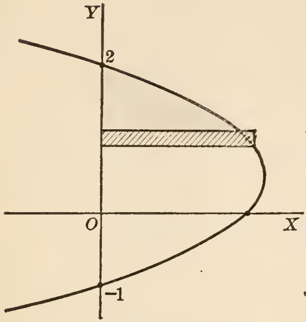

FIG. 26c.

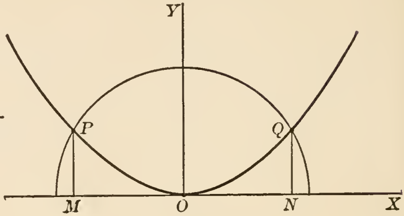

FIG. 26d.

$$A = \int_{-1}^{2} x\,dy = \int_{-1}^{2} (2 + y - y^2)\,dy = 2y + \frac{y^2}{2} - \frac{y^3}{3}\Bigg|_{-1}^{2} = 4\tfrac{1}{2}.$$

Ex. 2. Find the area within the circle $x^2 + y^2 = 16$ and parabola $x^2 = 6y$.

Solving the equations simultaneously, we find that the parabola and circle intersect at $P(-2\sqrt{3}, 2)$ and $Q(2\sqrt{3}, 2)$. The area $MPQN$ (Fig. 26d) under the circle is

$$\int_{-2\sqrt{3}}^{2\sqrt{3}} y\,dx = \int_{-2\sqrt{3}}^{2\sqrt{3}} \sqrt{16 - x^2}\,dx = \frac{16}{3}\pi + 4\sqrt{3}.$$

The area $MPO + OQN$ under the parabola is

$$\int_{-2\sqrt{3}}^{2\sqrt{3}} \frac{x^2}{6}\,dx = \frac{8}{3}\sqrt{3}.$$

The area between the curves is the difference

$$MPQN - MPO - OQN = \tfrac{16}{3}\pi + \tfrac{4}{3}\sqrt{3}.$$

Ex. 3. Find the area within the hypocycloid $x = a \sin^3 \phi$, $y = a \cos^3 \phi$.

The area OAB in the first quadrant is

$$\int_0^a y\,dx = \int_0^{\frac{\pi}{2}} a\cos^3\phi \cdot 3\,a\sin^2\phi\cos\phi\,d\phi$$

$$= 3\,a^2\int_0^{\frac{\pi}{2}} \cos^4\phi\sin^2\phi\,d\phi = \tfrac{3}{32}\,\pi\,a^2.$$

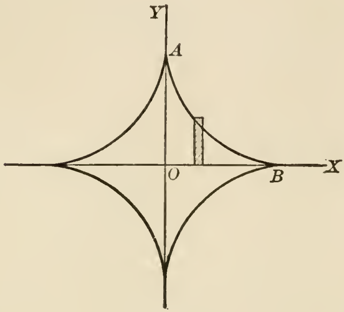

FIG. 26*e*.

The entire area is then

$$4\cdot OAB = \tfrac{3}{8}\,\pi\,a^2.$$

EXERCISES

1. Find the area bounded by the line $2\,y - 3\,x - 5 = 0$, the x-axis, and the ordinates $x = 1$, $x = 3$.

2. Find the area bounded by the parabola $y = 3\,x^2$, the y-axis, and the abscissas $y = 2$, $y = 4$.

3. Find the area bounded by $y^3 = x$, the line $y = -2$, and the ordinates $x = 0$, $x = 3$.

4. Find the area bounded by the parabola $y = 2\,x - x^2$ and the x-axis.

5. Find the area bounded by $y = \ln x$, the x-axis, and the ordinates $x = 2$, $x = 8$.

6. Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

7. Find the area bounded by the coördinate axes and the curve $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$.

8. Find the area within a loop of the curve $x^2 = y^2 (4 - y^2)$.

9. Find the area within the loop of the curve $y^2 = (x - 1)(x - 2)^2$.

10. Show that the area bounded by an arc of the hyperbola $xy = k^2$, the x-axis and the ordinates at its ends, is equal to the area bounded by the same arc, the y-axis and the abscissas at its ends.

11. Find the area bounded by the curves $y^2 = 4\,ax$, $x^2 = 4\,ay$.]

12. Find the area bounded by the parabola $y = 2\,x - x^2$ and the line $y = -x$.

13. Find the areas of the two parts into which the circle $x^2 + y^2 = 8$ is divided by the parabola $y^2 = 2\,x$.

14. Find the area within the parabola $x^2 = 4\,y + 4$ and the circle $x^2 + y^2 = 16$.

15. Find the area bounded by $y^2 = 4\,x$, $x^2 = 4\,y$, and $x^2 + y^2 = 5$.

16. Find the area of a circle by using the parametric equations $x = a \cos\theta$, $y = a \sin\theta$.

17. Find the area bounded by the x-axis and one arch of the cycloid.

$$x = a(\phi - \sin\phi),\ y = a(1 - \cos\phi).$$

18. Find the area within the cardioid

$$x = a\cos\theta\,(1 - \cos\theta),\ \ y = a\sin\theta\,(1 - \cos\theta).$$

19. Find the area bounded by an arch of the trochoid,

$$x = a\phi - b\sin\phi, \qquad y = a - b\cos\phi,$$

and the tangent at the lowest points of the curve.

20. Find the area of the ellipse $x^2 - xy + y^2 = 3$.

21. Find the area bounded by the curve $y^2 = \dfrac{x^3}{2\,a - x}$ and its asymptote $x = 2\,a$.

22. Find the area within the curve

$$\frac{x^2}{a^2} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1.$$

27. Area Bounded by a Plane Curve. Polar Coördinates. — To find the area of the sector POQ bounded by two radii OP, OQ and the arc PQ of a given curve.

Divide the angle POQ into any number of equal parts $\Delta\theta$ and construct the circular sectors shown in Fig. 27*a*. One of these sectors ORS has the area

$$\tfrac{1}{2}\,OR^2\,\Delta\theta = \tfrac{1}{2}\,r^2\,\Delta\theta.$$

If α and β are the limiting values of θ, the sum of all the sectors is then

$$\sum_{\alpha}^{\beta} \tfrac{1}{2} r^2 \Delta\theta.$$

As $\Delta\theta$ approaches zero, this sum approaches the area A of the sector POQ. Therefore

$$A = \lim_{\Delta\theta \doteq 0} \sum_{\alpha}^{\beta} \tfrac{1}{2} r^2 \Delta\theta = \int_{\alpha}^{\beta} \tfrac{1}{2} r^2 \, d\theta. \qquad (27)$$

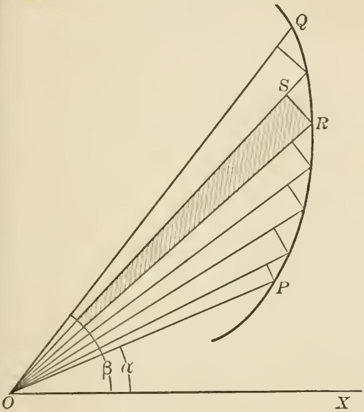

FIG. 27*a*.

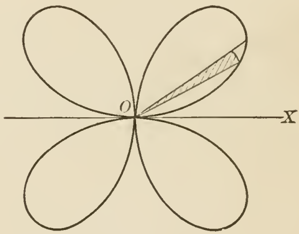

FIG. 27*b*.

In this equation r must be replaced by its value in terms of θ from the equation of the curve.

Example. Find the area of one loop of the curve $r = a \sin 2\,\theta$ (Fig. 27*b*).

A loop of the curve extends from $\theta = 0$ to $\theta = \frac{\pi}{2}$. Its area is

$$A = \int_0^{\frac{\pi}{2}} \frac{1}{2} r^2 \, d\theta = \int_0^{\frac{\pi}{2}} \frac{a^2}{2} \sin^2 (2\,\theta) \, d\theta$$

$$= \frac{a^2}{4} \int_0^{\frac{\pi}{2}} (1 - \cos 4\,\theta) \, d\theta = \frac{\pi a^2}{8}.$$

EXERCISES

1. Find the area of the circle $r = a$.

2. Find the area of the circle $r = a\cos\theta$.

3. Find the area bounded by the coördinate axes and the line $r = a\sec\left(\theta - \frac{\pi}{3}\right)$.

4. Find the area bounded by the initial line and the first turn of the spiral $r = ae^{\theta}$.

5. Find the area of one loop of the curve $r^2 = a^2 \cos 2\,\theta$.

6. Find the area enclosed by the curve $r = \cos\theta + 2$.

7. Find the area within the cardioid $r = a\,(1 + \cos\theta)$.

8. Find the area bounded by the parabola $r = a\sec^2\frac{\theta}{2}$ and the y-axis.

9. Find the area bounded by the parabola

$$r = \frac{2\,a}{1 - \cos\theta}$$

and the radii $\theta = \frac{\pi}{4}$, $\theta = \frac{\pi}{2}$.

10. Find the area bounded by the initial line and the second and third turns of the spiral $r = a\theta$.

11. Find the area of the curve $r = 2\,a\cos 3\,\theta$ outside the circle $r = a$.

12. Show that the area of the sector bounded by any two radii of the spiral $r\theta = a$ is proportional to the difference of those radii.

13. Find the area common to the two circles $r = a\cos\theta$, $r = a\cos\theta + a\sin\theta$.

14. Find the entire area enclosed by the curve $r = a\cos^3\frac{\theta}{3}$.

15. Find the area within the curve $(r - a)^2 = a^2\,(1 - \theta^2)$.

16. Through a point within a closed curve a chord is drawn. Show that, if either of the areas determined by the chord and curve is a maximum or minimum, the chord is bisected by the fixed point.

28. Volume of a Solid of Revolution. — To find the volume generated by revolving the area $ABCD$ about the x-axis.

Inscribe in the area a series of rectangles as shown in Fig. 28a. One of these rectangles $PQSR$ generates a circular cylinder with radius y and altitude Δx. The volume of this cylinder is

$$\pi y^2 \Delta x.$$

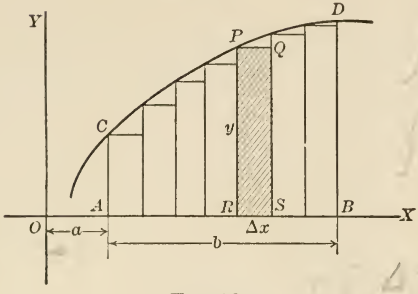

FIG. 28a.

If a and b are the limiting values of x, the sum of the cylinders is

$$\sum_a^b \pi y^2 \Delta x.$$

The volume generated by the area is the limit of this sum

$$v = \lim_{\Delta x \doteq 0} \sum_a^b \pi y^2 \Delta x = \int_a^b \pi y^2 \, dx. \tag{28}$$

If the area does not reach the axis, as in Fig. 28b, let y_1 and y_2 be the distances from the axis to the bottom and top

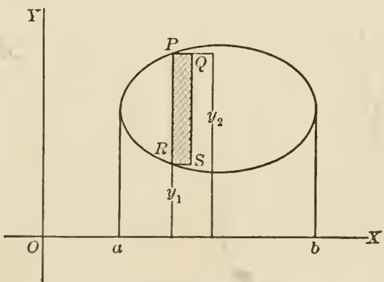

FIG. 28b.

of the rectangle $PQRS$. When revolved about the axis, it generates a hollow cylinder, or washer, of volume

$$\pi (y_2^2 - y_1^2) \Delta x.$$

The volume generated by the area is then

$$v = \lim_{\Delta x \doteq 0} \sum_a^b \pi (y_2^2 - y_1^2) \Delta x = \int_a^b \pi (y_2^2 - y_1^2)\, dx.$$

If the area is revolved about some other axis, y in these formulas must be replaced by the perpendicular from a point of the curve to the axis and x by the distance along the axis to that perpendicular.

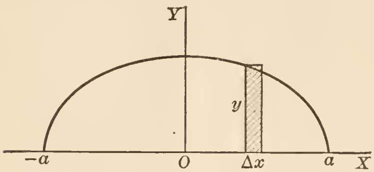

FIG. 28c.

Example 1. Find the volume generated by revolving the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

about the x-axis.

From the equation of the curve we get

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2).$$

The volume required is, therefore,

$$v = \int_{-a}^{a} \pi y^2\, dx = \frac{\pi b^2}{a^2} \int_{-a}^{a} (a^2 - x^2)\, dx = \frac{4}{3} \pi a b^2.$$

Ex. 2. A circle of radius a is revolved about an axis in its plane at the distance b (greater than a) from its center. Find the volume generated.

Revolve the circle, Fig. 28d, about the line CD. The rectangle MN generates a washer with radii

$$R_1 = b - x = b - \sqrt{a^2 - y^2},$$
$$R_2 = b + x = b + \sqrt{a^2 - y^2}.$$

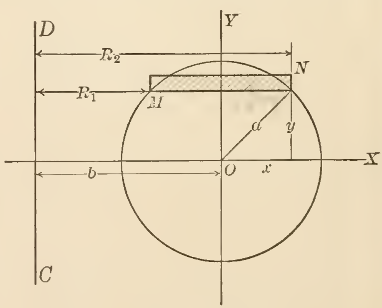

FIG. 28d.

The volume of the washer is

$$\pi (R_2^2 - R_1^2) = 4\pi b \sqrt{a^2 - y^2}\, \Delta y.$$

The volume required is then

$$v = \int_{-a}^{a} 4\,\pi b\,\sqrt{a^2 - y^2}\,dy = 2\,\pi^2 a^2 b.$$

Ex. 3. Find the volume generated by revolving the circle $r = a \sin\theta$ about the x-axis.

In this case

$$y = r\sin\theta = a\sin^2\theta,$$
$$x = r\cos\theta = a\cos\theta\sin\theta.$$

The volume required is

$$v = \int \pi y^2\,dx = \int_{\pi}^{0} \pi a^3 \sin^4\theta\,(\cos^2\theta - \sin^2\theta)\,d\theta = \frac{\pi^2 a^3}{4}.$$

The reason for using π as the lower limit and 0 as the upper is to make dx positive along the upper part ABC of the curve.

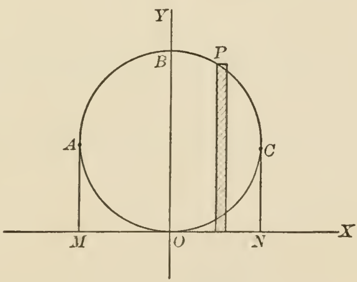

FIG. 28c.

As θ varies from π to 0, the point P describes the path $OABCO$. Along OA and CO dx is negative. The integral thus gives the volume generated by $MABCN$ minus that generated by OAM and OCN.

EXERCISES

1. Find the volume of a sphere by integration.

2. Find the volume of a right cone by integration.

3. Find the volume generated by revolving about the x-axis the area bounded by the x-axis and the parabola $y = 2\,x - x^2$.

4. Find the volume generated by revolving about OY the area bounded by the coördinate axes and the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$.

5. Find the volume generated by revolving about the x-axis the area bounded by the catenary $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{\frac{-x}{a}}\right)$, the x-axis and the lines $x = \pm a$.

6. Find the volume generated by revolving one arch of the sine curve $y = \sin x$ about OX.

7. A cone has its vertex on the surface of a sphere and its axis coincides with a diameter of the sphere. Find the common volume.

8. Find the volume generated by revolving about the y-axis, the part of the parabola $y^2 = 4\,ax$ cut off by the line $x = a$.

9. Find the volume generated by revolving about $x = a$ the part of the parabola $y^2 = 4\,ax$ cut off by the line $x = a$.

10. Find the volume generated by revolving about $y = -2\,a$ the part of the parabola $y^2 = 4\,ax$ cut off by the line $x = a$.

11. Find the volume generated by revolving one arch of the cycloid

$$x = a\,(\phi - \sin\phi), \qquad y = a\,(1 - \cos\phi)$$

about the x-axis.

12. Find the volume generated by revolving the curve

$$x = a\cos^3\phi, \qquad y = a\sin^3\phi$$

about the y-axis.

13. Find the volume generated by revolving the cardioid $r = a\,(1 + \cos\theta)$ about the initial line.

14. Find the volume generated by revolving the cardioid $r = a\,(1 + \cos\theta)$ about the line $x = -\frac{a}{4}$.

15. Find the volume generated by revolving the ellipse

$$x^2 + xy + y^2 = 3$$

about the x-axis.

16. Find the volume generated by revolving about the line $y = x$ the part of the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ cut off by the line $x + y = a$.

29. Volume of a Solid with Given Area of Section. — Divide the solid into slices by parallel planes. Let X be the area of section at distance x from a fixed point. The plate $PQRS$ with lateral surface perpendicular to PQR has the volume

$$PQR \cdot \Delta x = X\,\Delta x.$$

If a and b are the limiting values of x, the sum of such plates is

$$\sum_a^b X\,\Delta x.$$

The volume required is the limit of this sum

$$v = \lim_{\Delta x \doteq 0} \sum_a^b X\,\Delta x = \int_a^b X\,dx. \qquad (29)$$

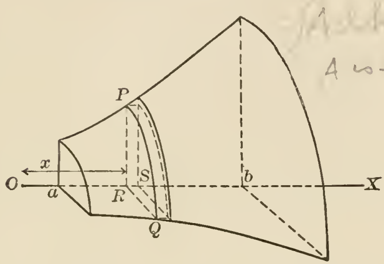

FIG. 29a.

Example 1. Find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

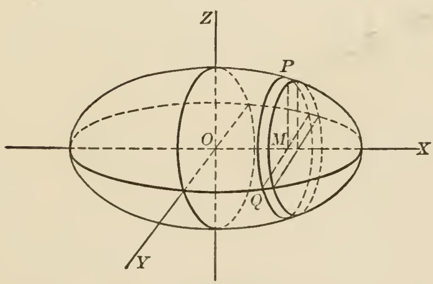

FIG. 29b.

The section perpendicular to the x-axis at the distance x from the center is an ellipse

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 - \frac{x^2}{a^2}.$$

The semi-axes of this ellipse are

$$MP = c\sqrt{1 - \frac{x^2}{a^2}}, \qquad MQ = b\sqrt{1 - \frac{x^2}{a^2}}.$$

By exercise 6, page 49, the area of this ellipse is

$$\pi \cdot MP \cdot MQ = \pi bc\left(1 - \frac{x^2}{a^2}\right).$$

The volume of the ellipsoid is, therefore,

$$\int_{-a}^{a} \pi bc\left(1 - \frac{x^2}{a^2}\right) dx = \frac{4}{3}\pi abc.$$

Ex. 2. The axes of two equal right circular cylinders intersect at right angles. Find the common volume.

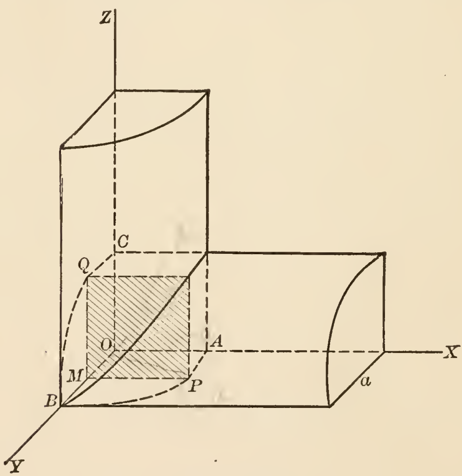

FIG. 29c.

In Fig. 29c, the axes of the cylinders are OX and OZ and $OABC$ is $\frac{1}{8}$ of the common volume. The section of $OABC$ by a plane perpendicular to OY is a square of side

$$MP = MQ = \sqrt{a^2 - y^2}.$$

The area of the section is therefore

$$MP \cdot MQ = a^2 - y^2,$$

and the required volume is

$$v = 8 \int_0^a (a^2 - y^2)\, dy = \frac{16\, a^3}{3}.$$

EXERCISES

1. Find the volume of a pyramid by integration.

2. A wedge is cut from the base of a right circular cylinder by a plane passing through a diameter of the base and inclined at an angle α to the base. Find the volume of the wedge.

3. Two circles have a diameter in common and lie in perpendicular planes. A square moves in such a way that its plane is perpendicular to the common diameter and its diagonals are chords of the circles. Find the volume generated.

4. The plane of a moving circle is perpendicular to that of an ellipse and the radius of the circle is an ordinate of the ellipse. Find the volume generated when the circle moves from one vertex of the ellipse to the other.

5. The plane of a moving triangle is perpendicular to a fixed diameter of a circle, its base is a chord of the circle, and its vertex lies on a line parallel to the fixed diameter at distance h from the plane of the circle. Find the volume generated by the triangle in moving from one end of the diameter to the other.

6. A triangle of constant area A rotates about a line perpendicular to its plane while advancing along the line. Find the volume swept out in advancing a distance h.

7. Show that if two solids are so related that every plane parallel to a fixed plane cuts from them sections of equal area, the volumes of the solids are equal.

8. A cylindrical surface passes through two great circles of a sphere which are at right angles. Find the volume within the cylindrical surface and sphere.

9. Two cylinders of equal altitude h have a common upper base and their lower bases are tangent. Find the volume common to the two cylinders.

10. A circle moves with its center on the z-axis and its plane parallel to a fixed plane inclined at 45° to the z-axis. If the radius of the circle is always $r = \sqrt{a^2 - z^2}$, where z is the coördinate of its center, find the volume described.

CHAPTER V

OTHER GEOMETRICAL APPLICATIONS

30. Infinitesimals of Higher Order. — In the applications of the definite integral that we have previously made, the quantity desired has in each case been a limit of the form

$$\lim_{\Delta x \doteq 0} \sum_a^b f(x)\, \Delta x.$$

We shall now consider cases involving limits of the form

$$\lim_{\Delta x \doteq 0} \sum_a^b F(x, \Delta x)$$

when $F(x, \Delta x)$ is only approximately expressible in the form $f(x)\, \Delta x$. Such cases are usually handled by neglecting infinitesimals of higher order than Δx. That such neglect does not change the limit is indicated by the following theorem:

If for values of x between a and b, $F(x, \Delta x)$ differs from $f(x)\, \Delta x$ by an infinitesimal of higher order than Δx,

$$\lim_{\Delta x \doteq 0} \sum_a^b F(x, \Delta x) = \lim_{\Delta x \doteq 0} \sum_a^b f(x)\, \Delta x.$$

To show this let ϵ be a number so chosen that

$$F(x, \Delta x) = f(x)\, \Delta x + \epsilon\, \Delta x.$$

If $F(x, \Delta x)$ and $f(x)\, \Delta x$ differ by an infinitesimal of higher order than Δx, $\epsilon\, \Delta x$ is of higher order than Δx and so ϵ approaches zero as Δx approaches zero (Differential Calculus, Art. 9). The difference

$$\sum_a^b F(x, \Delta x) - \sum_a^b f(x)\, \Delta x = \sum_a^b \epsilon\, \Delta x$$

is graphically represented by a sum of rectangles (Fig. 30), whose altitudes are the various values of ϵ. Since all these values approach zero * with Δx, the total area approaches zero and so

$$\lim_{\Delta x \doteq 0} \sum_a^b F(x, \Delta x) = \lim_{\Delta x \doteq 0} \sum_a^b f(x) \Delta x,$$

FIG. 30.

which was to be proved.

31. Length of a Curve. Rectangular Coördinates. — In the arc AB of a curve inscribe a series of chords. The length of one of these chords PQ is

$$\sqrt{\Delta x^2 + \Delta y^2} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \Delta x,$$

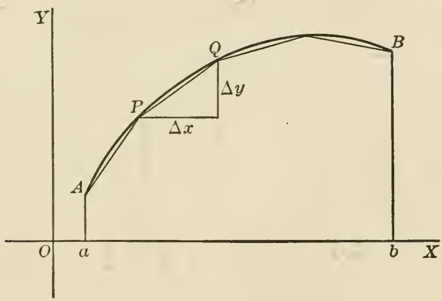

FIG. 31*a*.

and the sum of their lengths is

$$\sum_a^b \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \Delta x.$$

The length of the arc AB is defined as the limit approached by this sum when the number of chords is increased indefinitely, their lengths approaching zero.

* For the discussion to be strictly accurate it must be shown that there is a number larger than any of the ϵ's which approaches zero. In the language of higher mathematics, the approach to the limit must be uniform. In ordinary cases that certainly would be true. A similar remark applies to all the applications of the above theorem.

The quantity $\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$ is not a function of x alone. When Δx approaches zero, however, the difference of $\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$ and $\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ approaches zero. If then we replace $\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x$ by $\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,\Delta x$, the error is an infinitesimal of higher order than Δx. Therefore the length of arc is

$$s = \lim_{\Delta x \doteq 0} \sum \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,\Delta x = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$$

In applying this formula $\frac{dy}{dx}$ must be determined from the equation of the curve. The result can also be written

$$s = \int_A^B \sqrt{dx^2 + dy^2}. \tag{31}$$

In this formula, y may be expressed in terms of x, or x in terms of y, or both may be expressed in terms of a parameter. In any case the limits are the values at A and B of the variable that remains.

Example 1. Find the length of the arc of the parabola $y^2 = 4\,x$ between $x = 0$ and $x = 1$.

In this case $\frac{dx}{dy} = \frac{y}{2}$. The limiting values of y are 0 and 2. Hence

$$s = \int^2 \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy = \int_0^2 \frac{1}{2}\sqrt{y^2+4}\,dy = \sqrt{2} + \ln\left(1+\sqrt{2}\right).$$

Ex. 2. Find the perimeter of the curve

$$x = a\cos^3\phi, \quad y = a\sin^3\phi.$$

In this case

$$\begin{aligned} ds = \sqrt{dx^2 + dy^2} &= \sqrt{9\,a^2\cos^4\phi\sin^2\phi + 9\,a^2\sin^4\phi\cos^2\phi}\;d\phi \\ &= 3\,a\cos\phi\sin\phi\;d\phi. \end{aligned}$$

One-fourth of the curve is described when ϕ varies from 0 to $\frac{\pi}{2}$. Hence the perimeter is

$$s = 4\int_0^{\frac{\pi}{2}} 3\,a\cos\phi\sin\phi\,d\phi = 6\,a.$$

EXERCISES

1. Find the circumference of a circle by integration.
2. Find the length of $y^2 = x^3$ between $(0, 0)$ and $(4, 8)$.
3. Find the length of $x = \ln \sec y$ between $y = 0$ and $y = \frac{\pi}{3}$.
4. Find the length of $x = \frac{1}{4}y^2 - \frac{1}{2}\ln y$ between $y = 1$ and $y = 2$.
5. Find the length of $y = e^x$ between $(0, 1)$ and $(1, e)$.
6. Find the perimeter of the curve

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

7. Find the length of the catenary

$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$$

between $x = -a$ and $x = a$.

8. Find the length of one arch of the cycloid

$$x = a(\phi - \sin\phi), \qquad y = a(1 - \cos\phi).$$

9. Find the length of the involute of the circle

$$x = a(\cos\theta + \theta\sin\theta), \quad y = a(\sin\theta - \theta\cos\theta),$$

between $\theta = 0$ and $\theta = 2\pi$.

10. Find the length of an arc of the cycloid

$$x = a(\theta + \sin\theta), \quad y = a(1 - \cos\theta).$$

If s is the length of arc between the origin and any point (x, y) of the same arch, show that

$$s^2 = 8\,ay.$$

32. Length of a Curve. Polar Coördinates. — The differential of arc of a curve is (Differential Calculus, Arts. 54, 59)

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{dr^2 + r^2\,d\theta^2}.$$

Equation (31) is, therefore, equivalent to

$$s = \int_A^B \sqrt{dr^2 + r^2\, d\theta^2}. \tag{32}$$

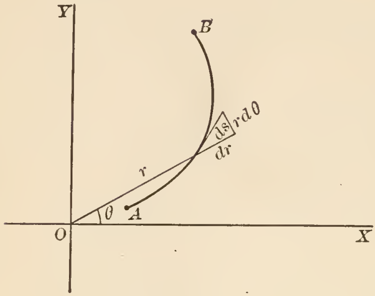

FIG. 32.

In using this formula, r must be expressed in terms of θ or θ in terms of r from the equation of the curve. The limits are the values at A and B of the variable that remains.

Example. Find the length of the first turn of the spiral $r = a\,\theta$.

In this case $dr = a\, d\theta$ and

$$s = \int_0^{2\pi} \sqrt{a^2\, d\theta^2 + a^2\theta^2\, d\theta^2} = a \int_0^{2\pi} \sqrt{1 + \theta^2}\, d\theta$$

$$= \pi a \sqrt{1 + 4\pi^2} + \frac{a}{2} \ln \left(2\pi + \sqrt{1 + 4\pi^2}\right).$$

EXERCISES

1. Find the circumference of the circle $r = a$.

2. Find the circumference of the circle $r = 2a\cos\theta$.

3. Find the length of the spiral $r = e^{a\theta}$ between $\theta = 0$ and $\theta = \frac{1}{a}$.

4. Find the distance along the straight line $r = a\sec\left(\theta - \frac{\pi}{3}\right)$ from $\theta = 0$ to $\theta = \frac{\pi}{2}$.

5. Find the arc of the parabola $r = a\sec^2 \frac{1}{2}\theta$ cut off by the y-axis.

6. Find the length of one loop of the curve

$$r = a\cos^4 \frac{\theta}{4}.$$

7. Find the perimeter of the cardioid

$$r = a(1 + \cos\theta).$$

8. Find the complete perimeter of the curve $r = a\sin^3 \frac{\theta}{3}$.

33. Area of a Surface of Revolution. — To find the area generated by revolving the arc AB about the x-axis.

Join A and B by a broken line with vertices on the arc. Let x, y be the coördinates of P and $x + \Delta x$, $y + \Delta y$ those of Q. The chord PQ generates a frustum of a cone whose area is

$$\pi (2y + \Delta y) PQ = \pi (2y + \Delta y) \sqrt{\Delta x^2 + \Delta y^2}.$$

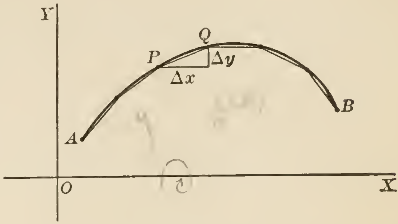

FIG. 33a.

The area generated by the broken line is then

$$\sum_A^B \pi (2y + \Delta y) \sqrt{\Delta x^2 + \Delta y^2}.$$

The area S generated by the arc AB is the limit approached by this sum when Δx and Δy approach zero. Neglecting infinitesimals of higher order, $(2y + \Delta y) \sqrt{\Delta x^2 + \Delta y^2}$ can be replaced by $2y\sqrt{dx^2 + dy^2} = 2y\,ds$. Hence the area generated is

$$S = \int_A^B 2\pi y\,ds. \tag{33a}$$

In this formula y and ds must be calculated from the equation of the curve. The limits are the values at A and B of the variable in terms of which they are expressed.

Similarly, the area generated by revolving about the y-axis is

$$S = \int_A^B 2\pi x\,ds. \tag{33b}$$

Example. Find the area of the surface generated by revolving about the y-axis the part of the curve $y = 1 - x^2$ above the x-axis.

In this case

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + 4\,x^2}\, dx.$$

The area required is generated by the part AB of the curve between $x = 0$ and $x = 1$. Hence

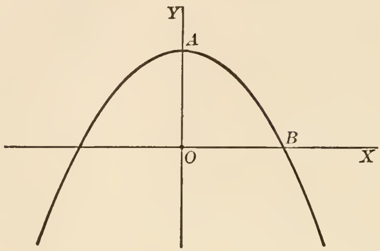

FIG. 33b.

$$S = \int_A^B 2\,\pi\,x\,ds = \int_0^1 2\,\pi x\,\sqrt{1 + 4\,x^2}\,dx$$

$$= \frac{\pi}{6}(1 + 4\,x^2)^{\frac{3}{2}}\Big|_0^1 = \frac{\pi}{6}(5\sqrt{5} - 1).$$

EXERCISES

1. Find the area of the surface of a sphere.

2. Find the area of the surface of a right circular cone.

3. Find the area of the spheroid generated by revolving an ellipse about its major axis.

4. Find the area generated by revolving the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ about the y-axis.

5. Find the area generated by revolving about OX, the part of the catenary

$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$$

between $x = -a$ and $x = a$.

6. Find the area generated by revolving one arch of the cycloid

$$x = a\,(\phi - \sin\phi), \qquad y = a\,(1 - \cos\phi)$$

about OX.

7. Find the area generated by revolving the cardioid $r = a(1 + \cos\theta)$ about the initial line.

8. The arc of the circle

$$x^2 + y^2 = a^2$$

between $(a, 0)$ and $(0, a)$ is revolved about the line $x + y = a$. Find the area of the surface generated.

9. The arc of the parabola $y^2 = 4x$ between $x = 0$ and $x = 1$ is revolved about the line $y = -2$. Find the area generated.

10. Find the area of the surface generated by revolving the lemniscate $r^2 = 2a^2 \cos 2\theta$ about the line $\theta = \frac{\pi}{4}$.

34. Unconventional Methods. — The methods that have been given for finding lengths, areas, and volumes are the ones most generally applicable. In particular cases other methods may give the results more easily. To solve a problem by integration, it is merely necessary to express the required quantity in any way as a limit of the form used in defining the definite integral.

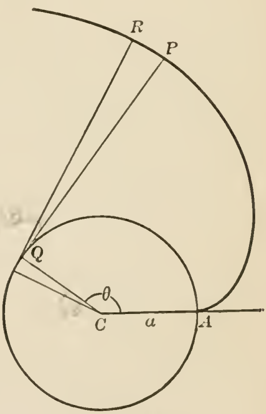

FIG. 34a.

Example 1. When a string held taut is unwound from a fixed circle, its end describes a curve called the involute of the circle. Find the length of the part described when the first turn of the string is unwound.

Let the string begin to unwind at A. When the end reaches P the part unwound QP is equal to the arc AQ. Hence

$$QP = AQ = a\theta.$$

When P moves to R the arc PR is approximately the arc of a circle with center at Q and central angle $\Delta\theta$. Hence

$$PR = a\theta\,\Delta\theta$$

approximately. The length of the curve described when θ varies from 0 to 2π is then

$$s = \lim_{\Delta\theta \doteq 0} \sum_0^{2\pi} a\theta\,\Delta\theta = \int_0^{2\pi} a\theta\,d\theta = 2\pi^2 a.$$

Ex. 2. Find the volume generated by rotating about the y-axis the area bounded by the parabola $x^2 = y - 1$, the x-axis, and the ordinates $x = \pm 1$.

Resolve the area into slices by ordinates at distances Δx apart. When revolved about the y-axis, the rectangle PM between the ordinates x, $x + \Delta x$ generates a hollow cylinder whose volume is

$$\pi\,(x + \Delta x)^2\,y - \pi\,x^2 y = 2\,\pi x y\,\Delta x + \pi y\,(\Delta x)^2.$$

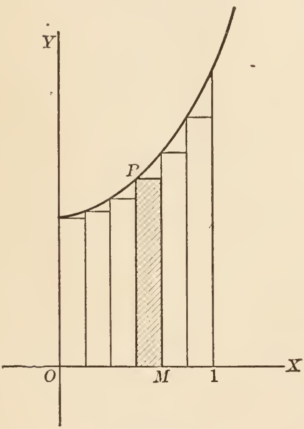

FIG. 34*b*.

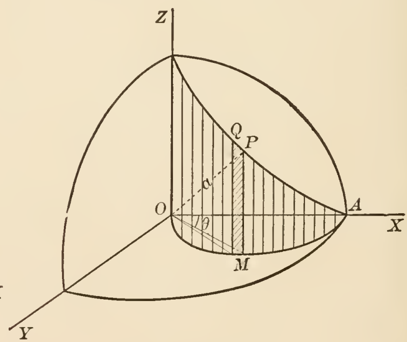

FIG. 34*c*.

Since $\pi y\,(\Delta x)^2$ is an infinitesimal of higher order than Δx, the required volume is

$$\lim_{\Delta x \doteq 0} \sum_0^1 2\,\pi x y\,\Delta x = \int_0^1 2\,\pi x\,(1 + x^2)\,dx = \tfrac{3}{2}\,\pi.$$

Ex. 3. Find the area of the cylinder $x^2 + y^2 = ax$ within the sphere $x^2 + y^2 + z^2 = a^2$.

Fig. 34*c* shows one-fourth of the required area. Divide the circle OA into equal arcs Δs. The generators through the points of division cut the surface of the cylinder into strips. Neglecting infinitesimals of higher order, the area of the strip MPQ is $MP \cdot \Delta s$. If r, θ are the polar coördinates of M, $r = a \cos\theta$ and

$$\Delta s = a\,\Delta\theta, \qquad MP = \sqrt{a^2 - r^2} = a \sin\theta.$$

The required area is therefore given by

$$\frac{S}{4} = \lim_{\Delta\theta \doteq 0} \sum_0^{\frac{\pi}{2}} a^2 \sin\theta\,\Delta\theta = \int_0^{\frac{\pi}{2}} a^2 \sin\theta\, d\theta.$$

Consequently

$$S = 4\,a^2 \int_0^{\frac{\pi}{2}} \sin\theta\, d\theta = 4\,a^2.$$

EXERCISES

1. Find the area swept over by the string in example 1, page 67.

2. Find the area of surface cut from a right circular cylinder by a plane passing through a diameter of the base and inclined 45° to the base.

3. The axes of two right circular cylinders of equal radius intersect at right angles. Find the area of the solid common to the two cylinders (Fig. 29c).

4. An equilateral triangle of side a is revolved about a line parallel to the base at distance b below the base. Find the volume generated.

5. The area bounded by the hyperbola $x^2 - y^2 = a^2$ and the lines $y = \pm a$ is revolved about the x-axis. Find the volume generated.

6. The vertex of a cone of vertical angle $2\,\alpha$ is the center of a sphere of radius a. Find the volume common to the cone and sphere.

7. The axis of a cone of altitude h and radius of base $2\,a$ is a generator of a cylinder of radius a. Find the area of the surface of the cylinder within the cone.

8. Find the area of the surface of the cone in Ex. 7 within the cylinder.

9. Find the volume of the cylinder in Ex. 7 within the cone.

CHAPTER VI

MECHANICAL AND PHYSICAL APPLICATIONS

35. Pressure. — The pressure of a liquid upon a horizontal area is equal to the weight of a vertical column of the liquid having the area as base and reaching to the surface. By the pressure at a point P in the liquid is meant the pressure upon a horizontal surface of unit area at that point. The

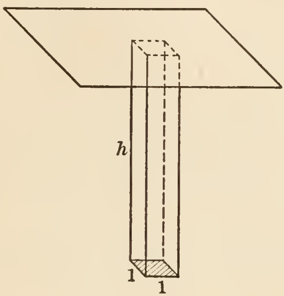

FIG. 35*a*.

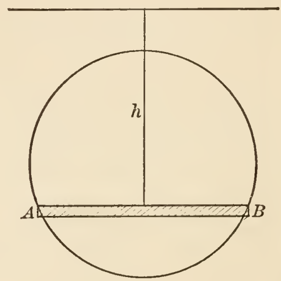

FIG. 35*b*.

volume of a column of unit section and height h is h. Hence the pressure at depth h is

$$p = wh, \tag{35a}$$

w being the weight of a cubic unit of the liquid.

To find the pressure upon a vertical plane area (Fig. 35*b*), we make use of the fact that the pressure at a point is the same in all directions. The pressure upon the strip AB parallel to the surface is then approximately

$$p\,\Delta A,$$

p being the pressure at any point of the strip and ΔA its area. The reason for this not being exact is that the pressure at the top of the strip is a little less than at the bottom. This difference is, however, infinitesimal, and, since it multiplies ΔA, the error is an infinitesimal of higher order than ΔA. The total pressure is, therefore,

$$P = \lim_{\Delta A \doteq 0} \sum p\,\Delta A = \int p\,dA = w \int h\,dA. \qquad (35b)$$

Before integration dA must be expressed in terms of h. The limits are the values of h at the top and bottom of the submerged area. In case of water the value of w is about 62.5 lbs. per cubic foot.

Example. Find the water pressure upon a semicircle of

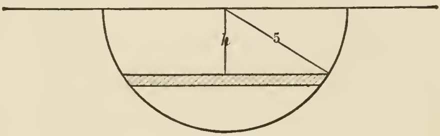

FIG. 35c.

radius 5 ft., if its plane is vertical and its diameter in the surface of the water.

In this case the element of area is

$$dA = 2\sqrt{25 - h^2}\,dh.$$

Hence

$$P = w \int h\,dA = 2w \int_0^5 h\sqrt{25 - h^2}\,dh$$
$$= \tfrac{250}{3} w = \tfrac{250}{3}(62.5) = 5208.3 \text{ lbs.}$$

EXERCISES

1. Find the pressure sustained by a rectangular floodgate 10 ft. broad and 12 ft. deep, the upper edge being in the surface of the water.

2. Find the pressure on the lower half of the floodgate in the preceding problem.

3. Find the pressure on a triangle of base b and altitude h, submerged so that its vertex is in the surface of the water, and its altitude vertical.

4. Find the pressure upon a triangle of base b and altitude h, submerged so that its base is in the surface of the liquid and its altitude vertical.

5. Find the pressure upon a semi-ellipse submerged with one axis in the surface of the liquid and the other vertical.

6. A vertical masonry dam in the form of a trapezoid is 200 ft. long at the surface of the water, 150 ft. long at the bottom, and 60 ft. high. What pressure must it withstand?

7. One end of a water main, 2 ft. in diameter, is closed by a vertical bulkhead. Find the pressure on the bulkhead if its center is 40 ft. below the surface of the water.

8. A rectangular tank is filled with equal parts of water and oil. If the oil is half as heavy as water, show that the pressure on the sides is one-fourth greater than it would be if the tank were filled with oil.

36. Moment. — Divide a plane area or length into small parts such that the points of each part differ only infinitesimally in distance from a given axis. Multiply each part by the distance of one of its points from the axis, the distance being considered positive for points on one side of the axis and negative for points on the other. The limit approached by the sum of these products when the parts are taken smaller and smaller is called the *moment* of the area or length with respect to the axis.

Similarly, to find the moment of a length, area, volume, or mass in space with respect to a plane, we divide it into elements whose points differ only infinitesimally in distance from the plane and multiply each element by the distance of one of its points from the plane (considered positive for points on one side of the plane and negative on the other). The moment with respect to the plane is the limit approached by the sum of these products when the elements are taken smaller and smaller.

Example. Find the moment of a rectangle about an axis parallel to one of its sides at distance c.

Divide the rectangle into strips parallel to the axis (Fig. 36). Let y be the distance from the axis to a strip. The area of the strip is $b\,\Delta y$. Hence the moment is

$$\lim_{\Delta y \doteq 0} \sum_{c}^{c+a} yb\,\Delta y = \int_{c}^{c+a} by\,dy = ab\left(c + \frac{a}{2}\right).$$

Since ab is the area of the rectangle and $c + \frac{a}{2}$ is the distance from the axis to its center, the moment is equal to the product of the area and the distance from the axis to the center of the rectangle.

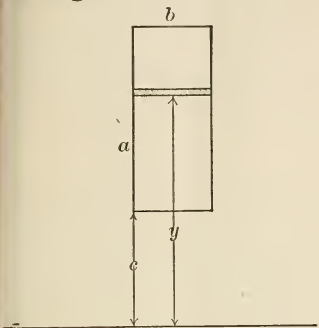

FIG. 36.

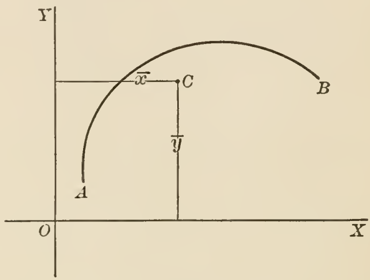

FIG. 37*a*.

37. The Center of Gravity of a Length or Area in a Plane. — The center of gravity of a length or area in a plane is the point at which it could be concentrated without changing its moment with respect to any axis in the plane.

Let C $(\bar{x}, \bar{y})$ be the center of gravity of the arc AB (Fig. 37*a*), and let s be the length of the arc. The moment of AB with respect to the x-axis is

$$\int_A^B y\,ds.$$

If the length s were concentrated at C, its moment would be $s\bar{y}$. By the definition of center of gravity

$$s\bar{y} = \int_A^B y\,ds,$$

whence

$$\bar{y} = \frac{\int_A^B y\,ds}{s}.$$

Similarly,

$$\bar{x} = \frac{\int_A^B x\,ds}{s}.$$

The limits are the values at A and B of the variable in terms of which the integral is expressed.

Let C $(\bar{x}, \bar{y})$ be the center of gravity of an area (Figs. 37b, 37c). Divide the area into strips dA and let (x, y) be the center of gravity of the strip dA. The moment of the area with respect to the x-axis is

$$\int y\, dA.$$

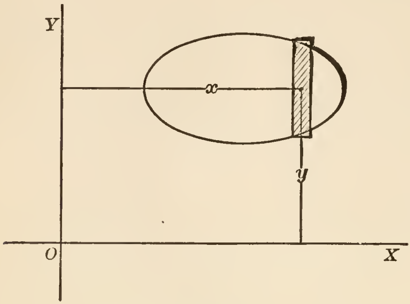

FIG. 37b.

If the area were concentrated at C, the moment would be $A\bar{y}$, where A is the total area. Hence

$$A\bar{y} = \int y\, dA,$$

or

$$\bar{y} = \frac{\int y\, dA}{A}.$$

Similarly,

$$\bar{x} = \frac{\int x\, dA}{A}.$$

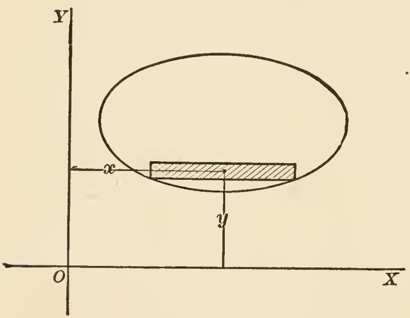

FIG. 37c.

The strip is usually taken parallel to a coördinate axis. The area can, however, be divided into strips of any other kind if convenient.

Example 1. Find the center of gravity of a quadrant of the circle $x^2 + y^2 = a^2$.

In this case

$$ds = \sqrt{dx^2 + dy^2} = \frac{a}{y}\, dx$$

and

$$\int y\,ds = \int_0^a y \cdot \frac{a}{y}\,dx = a^2.$$

The length of the arc is

$$s = \frac{1}{4}(2\,\pi a) = \frac{\pi}{2}a.$$

Hence

$$\bar{y} = \frac{\int y\,ds}{s} = \frac{2\,a}{\pi}.$$

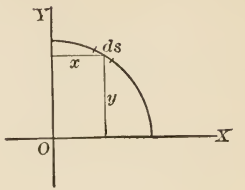

FIG. 37d.

It is evident from the symmetry of the figure that $\bar{x}$ has the same value.

Ex. 2. Find the center of gravity of the area of a semicircle.

From symmetry it is evident that the center of gravity is in the y-axis (Fig. 37*e*). Take the element of area parallel to OX. Then $dA = 2\,x\,dy$ and

$$\int y\,dA = \int 2\,xy\,dy = 2\int_0^a y\sqrt{a^2 - y^2}\,dy = \tfrac{2}{3}\,a^3.$$

The area is $A = \frac{\pi}{2}a^2$. Hence

$$\bar{y} = \int \frac{y\,dA}{A} = \frac{4\,a}{3\,\pi}.$$

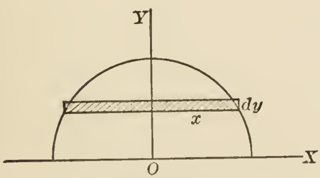

FIG. 37*e*.

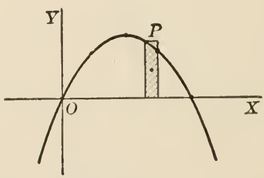

FIG. 37*f*.

Ex. 3. Find the center of gravity of the area bounded by the x-axis and the parabola $y = 2\,x - x^2$.

Take the element of area perpendicular to OX. If (x, y)

are the coördinates of the top of the strip, its center of gravity is $\left(x, \frac{y}{2}\right)$. Hence its moment with respect to the x-axis is

$$\frac{y}{2} \cdot dA = \frac{1}{2} y^2 \, dx.$$

The moment of the whole area about OX is then

$$\int \frac{y^2}{2} dx = \int_0^2 \frac{1}{2} (2x - x^2)^2 \, dx = \frac{16}{15}.$$

The area is

$$A = \int y \, dx = \int_0^2 (2x - x^2) \, dx = \frac{4}{3}.$$

Hence $\bar{y} = \frac{4}{5}$. Similarly,

$$\bar{x} = \frac{\int x \, dA}{A} = \frac{\int_0^2 (2x^2 - x^3) \, dx}{A} = 1.$$

38. Center of Gravity of a Length, Area, Volume, or Mass in Space. — The center of gravity is defined as the point at which the mass, area, length, or volume can be concentrated without changing its moment with respect to any plane.

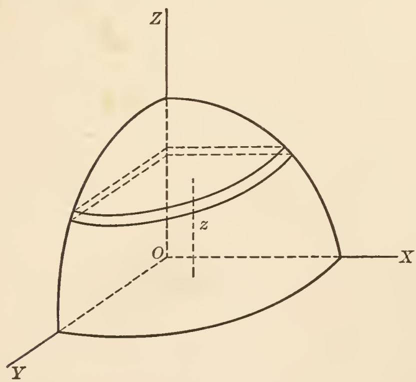

FIG. 38a.

Thus to find the center of gravity of a solid mass (Fig. 38a) cut it into slices of mass Δm. If (x, y, z) is the center of gravity of the slice, its moment with respect to the xy-plane is $z \, \Delta m$ and the moment of the whole mass is

$$\lim_{\Delta m \doteq 0} \sum z \, \Delta m = \int z \, dm.$$

If the whole mass M were concentrated at its center of gravity $(\bar{x}, \bar{y}, \bar{z})$, the moment with respect to the xy-plane would be $\bar{z}M$. Hence

$$\bar{z}M = \int z\,dm,$$

or

$$\bar{z} = \frac{\int z\,dm}{M}. \tag{38}$$

Similarly,

$$\bar{x} = \frac{\int x\,dm}{M}, \qquad \bar{y} = \frac{\int y\,dm}{M}. \tag{38}$$

The mass of a unit volume is called the density. If then dv is the volume of the element dm and ρ its density,

$$dm = \rho\,dv.$$

To find the center of gravity of a length, area, or volume it is merely necessary to replace M in these formulas by s, S, or v.

Example 1. Find the center of gravity of the volume of an octant of a sphere of radius a.

The volume of the slice (Fig. 38a) is

$$dv = \tfrac{1}{4}\pi x^2\,dz = \tfrac{1}{4}\pi\,(a^2 - z^2)\,dz.$$

Hence

$$\int z\,dv = \int_0^a \frac{1}{4}\pi\,(a^2 - z^2)\,z\,dz = \frac{\pi}{16}a^4.$$

The volume of an octant of a sphere is $\frac{1}{6}\pi a^3$. Hence

$$\bar{z} = \frac{\int z\,dv}{v} = \frac{\frac{\pi}{16}a^4}{\frac{\pi}{6}a^3} = \frac{3}{8}a.$$

From symmetry it is evident that $\bar{x}$ and $\bar{y}$ have the same value.

Ex. 2. Find the center of gravity of a right circular cone whose density is proportional to the distance from its base.

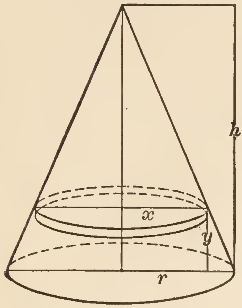

FIG. 38b.

Cut the cone into slices parallel to the base. Let y be the distance of a slice from the base. Except for infinitesimals of higher order, its volume is $\pi x^2\,dy$, and its density is ky where K is constant. Hence its mass is

$$\Delta m = k\pi x^2 y\,dy.$$

By similar triangles $x = \frac{r}{h}(h - y)$. Hence

$$\int y\,dm = \int_0^h \frac{k\pi r^2}{h^2}(h-y)^2 y^2\,dy = \frac{k\pi r^2 h^3}{30}$$

$$M = \int dm = \int_0^h \frac{k\pi r^2}{h^2}(h-y)^2 y\,dy = \frac{k\pi r^2 h^2}{12}.$$

Therefore, finally,

$$\bar{y} = \frac{\int y\,dm}{M} = \frac{2}{5}h.$$

EXERCISES

1. The wind produces a uniform pressure upon a rectangular door. Find the moment tending to turn the door on its hinges.

2. Find the moment of the pressure upon a rectangular floodgate about a horizontal line through its center, when the water is level with the top of the gate.

3. A triangle of base b and altitude h is submerged with its base horizontal, altitude vertical, and vertex c feet below the surface of the water. Find the moment of the pressure upon the triangle about a horizontal line through the vertex.

4. Find the center of gravity of the area of a triangle.

5. Find the center of gravity of the segment of the parabola $y^2 = ax$, cut off by the line $x = a$.

6. Find the center of gravity of the area of a quadrant of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

7. Find the center of gravity of the area bounded by the coördinate axes and the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$.

8. Find the center of gravity of the area above the x-axis bounded by the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

9. Find the center of gravity of the area bounded by the x-axis and one arch of the curve $y = \sin x$.

10. Find the center of gravity of the area bounded by the two parabolas $y^2 = ax$, $x^2 = ay$.

11. Find the center of gravity of the area of the upper half of the cardioid $r = a\,(1 + \cos\theta)$.

12. Find the center of gravity of the area bounded by the x-axis and one arch of the cycloid,

$$x = a\,(\phi - \sin\phi), \qquad y = a\,(1 - \cos\phi).$$

13. Find the center of gravity of the area within a loop of the lemniscate $r^2 = a^2 \cos 2\,\theta$.

14. Find the center of gravity of the arc of a semicircle of radius a.

15. Find the center of gravity of the arc of the catenary

$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$$

between $x = -a$ and $x = a$.

16. Find the center of gravity of the arc of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ in the first quadrant.

17. Find the center of gravity of the arc of the curve

$$x = \tfrac{1}{4}\,y^2 - \tfrac{1}{2}\ln y$$

between $y = 1$ and $y = 2$.

18. Find the center of gravity of an arch of the cycloid

$$x = a\,(\phi - \sin\phi), \qquad y = a\,(1 - \cos\phi).$$

19. Find the center of gravity of a right circular cone of constant density.

20. Find the center of gravity of a hemisphere of constant density.

21. Find the center of gravity of the solid generated by revolving about OX the area bounded by the parabola $y^2 = 4\,x$ and the line $x = 4$.

22. Find the center of gravity of a hemisphere whose density is proportional to the distance from the plane face.

23. Find the center of gravity of the solid generated by rotating a sector of a circle about one of its bounding radii.

24. Find the center of gravity of the solid generated by revolving the cardioid $r = a(1 + \cos\theta)$ about the initial line.

25. Find the center of gravity of the wedge cut from a right circular cylinder by a plane passing through a diameter of the base and making with the base the angle α.

26. Find the center of gravity of a hemispherical surface.

27. Show that the center of gravity of a zone of a sphere is midway between the bases of the zone.

28. The segment of the parabola $y^2 = 2ax$ cut off by the line $x = a$ is revolved about the x-axis. Find the center of gravity of the surface generated.

39. Theorems of Pappus. *Theorem I.* — If the arc of a plane curve is revolved about an axis in its plane, and not crossing the arc, the area generated is equal to the product of the length of the arc and the length of the path described by its center of gravity.

Theorem II. If a plane area is revolved about an axis in its plane and not crossing the area, the volume generated is equal to the product of the area and the length of the path described by its center of gravity.

To prove the first theorem, let the arc be rotated about the x-axis. The ordinate of its center of gravity is

$$\bar{y} = \frac{\int y\,ds}{s},$$

whence

$$2\pi \int y\,ds = 2\pi\bar{y}s.$$

The left side of this equation represents the area of the surface generated. Also $2\pi\bar{y}$ is the length of the path described by the center of gravity. This equation, therefore, expresses the result to be proved.

To prove the second theorem let the area be revolved about the x-axis. From the equation

$$\bar{y} = \frac{\int y\,dA}{A}$$

we get

$$2\pi \int y\,dA = 2\pi\bar{y}A.$$

Since $2\pi \int y\,dA$ is the volume generated, this equation is equivalent to theorem II.

Example 1. Find the area of the torus generated by revolving a circle of radius a about an axis in its plane at distance b (greater than a) from its center.

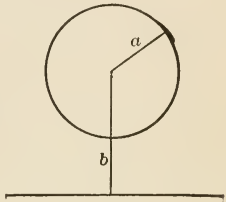

FIG. 39*a*.

Since the circumference of the circle is $2\pi a$ and the length of the path described by its center $2\pi b$, the area generated is

$$S = 2\pi a \cdot 2\pi b = 4\pi^2 ab.$$

Ex. 2. Find the center of gravity of the area of a semicircle by using Pappus's theorems.

When a semicircle of radius a is revolved about its diameter, the volume of the sphere generated is $\frac{4}{3}\pi a^3$. If $\bar{y}$ is the distance of the center of gravity of the semicircle from this diameter, by the second theorem of Pappus,

$$\tfrac{4}{3}\pi a^3 = 2\pi\bar{y}A = 2\pi\bar{y}\cdot\tfrac{1}{2}\pi a^2,$$

whence

$$\bar{y} = \frac{\frac{4}{3}\pi a^3}{\pi^2 a^2} = \frac{4a}{3\pi}.$$

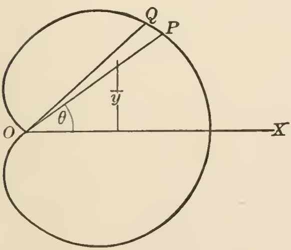

FIG. 39*b*.

Ex. 3. Find the volume generated by revolving the cardioid $r = a(1 + \cos\theta)$ about the initial line.

The area of the triangle OPQ is approximately

$$\tfrac{1}{2}r^2\,\Delta\theta,$$

and its center of gravity is $\frac{2}{3}$ of the distance from the vertex to the base. Hence

$$\bar{y} = \tfrac{2}{3}r\sin\theta.$$

By the second theorem of Pappus, the volume generated by OPQ is then approximately

$$2\pi y\,\Delta A = \tfrac{2}{3}\pi r^3 \sin\theta\,\Delta\theta.$$

The entire volume is therefore

$$v = \int_0^{\pi} \tfrac{2}{3}\pi r^3 \sin\theta\,d\theta = \tfrac{2}{3}\pi a^3 \int_0^{\pi} (1+\cos\theta)^3 \sin\theta\,d\theta$$

$$= -\frac{2}{3}\pi a^3 \frac{(1+\cos\theta)^4}{4}\Bigg|_0^{\pi} = \frac{8}{3}\pi a^3.$$

EXERCISES

1. By using Pappus's theorems find the lateral area and the volume of a right circular cone.

2. Find the volume of the torus generated by revolving a circle of radius a about an axis in its plane at distance b (greater than a) from its center.

3. A groove with cross-section an equilateral triangle of side $\frac{1}{2}$ inch is cut around a cylindrical shaft 6 inches in diameter. Find the volume of material cut away.

✓**4.** A steel band is placed around a cylindrical boiler 48 inches in diameter. A cross-section of the band is a semi-ellipse, its axes being 6 and $\sqrt{6}$ inches, respectively, the greater being parallel to the axis of the boiler. What is the volume of the band?

✓ **5.** The length of an arch of the cycloid

$$x = a(\phi - \sin\phi), \qquad y = a(1 - \cos\phi)$$

is $8a$, and the area generated by revolving it about the x-axis is $\frac{64}{3}\pi a^2$. Find the area generated by revolving the arch about the tangent at its highest point.

6. By the method of Ex. 3, page 81, find the volume generated by revolving the lemniscate $r^2 = 2a^2 \cos 2\theta$ about the x-axis.

7. Obtain a formula for the volume generated by revolving the polar element of area about the line $x = -a$. Apply this formula to obtain the volume generated by revolving about $x = -a$ the sector of the circle $r = a$ bounded by the radii $\theta = -\alpha$, $\theta = +\alpha$.

8. A variable circle revolves about an axis in its plane. If the distance from the center of the circle to the axis is $2a$ and its radius is $a \sin\theta$, where θ is the angle of rotation, find the volume of the horn-shaped solid that is generated.

9. Can the area of the surface in Ex. 8 be found in a similar way?

10. The vertex of a right circular cone is on the surface of a right circular cylinder and its axis cuts the axis of the cylinder at right angles. Find the volume common to the cylinder and cone (use sections determined by planes through the vertex of the cone and the generators of the cylinder).

40. Moment of Inertia. — The moment of inertia of a particle about an axis is the product of its mass and the square of its distance from the axis.

To find the moment of inertia of a continuous mass, we divide it into parts such that the points of each differ only infinitesimally in distance from the axis. Let Δm be such a part and R the distance of one of its points from the axis. Except for infinitesimals of higher order, the moment of inertia of Δm about the axis is $R^2 \Delta m$. The moment of inertia of the entire mass is therefore

$$I = \lim_{\Delta m \doteq 0} \sum R^2 \Delta m = \int R^2 \, dm. \tag{40}$$

By the moment of inertia of a length, area, or volume, we mean the value obtained by using the differential of length, area, or volume in place of dm in equation (40).

Example 1. Find the moment of inertia of a right circular cone of constant density about its axis.

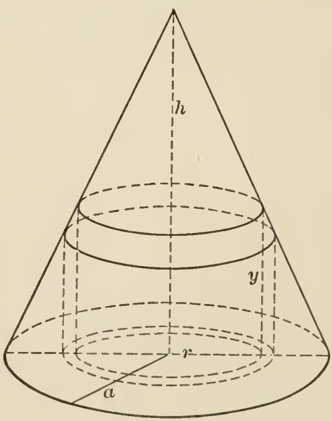

FIG. 40*a*.

Let ρ be the density, h the altitude, and a the radius of the base of the cone. Divide it into hollow cylindrical slices by means of cylindrical surfaces having the same axis as the cone. By similar triangles the altitude y of the cylindrical surface of radius r is

$$y = \frac{h}{a}(a - r).$$

Neglecting infinitesimals of higher order, the volume between the cylinders of radii r and $r + \Delta r$ is then

$$\Delta v = 2\pi r y\,\Delta r = \frac{2\pi h}{a}\,r\,(a - r)\,dr.$$

The moment of inertia is therefore

$$I = \int r^2\,dm = \int r^2 \rho\,dv = \frac{2\pi h\rho}{a}\int_0^a r^3 (a - r)\,dr = \frac{\pi\rho h a^4}{10}.$$

The mass of the cone is

$$M = \rho v = \tfrac{1}{3}\pi\rho a^2 h.$$

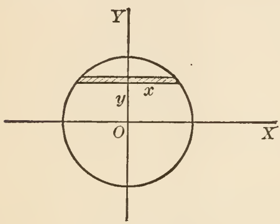

FIG. 40b.

Hence

$$I = \tfrac{3}{10} M a^2.$$

Ex. 2. Find the moment of inertia of the area of a circle about a diameter of the circle.

Let the radius be a and let the x-axis be the diameter about which the moment of inertia is taken. Divide the area into strips by lines parallel to the x-axis. Neglecting infinitesimals of higher order, the area of such a strip is $2x\,\Delta y$ and its moment of inertia $2xy^2\,\Delta y$. The moment of inertia of the entire area is therefore

$$I = \int 2xy^2\,dy = 2\int_{-a}^{a}\sqrt{a^2 - y^2}\,y^2\,dy = \frac{\pi a^4}{4}.$$

EXERCISES

1. Find the moment of inertia of the area of a rectangle about one of its edges.

2. Find the moment of inertia of a triangle about its base.

3. Find the moment of inertia of a triangle about an axis through its vertex parallel to its base.

4. Find the moment of inertia about the y-axis of the area bounded by the parabola $y^2 = 4ax$ and the line $x = a$.

5. Find the moment of inertia of the area in Ex. 4 about the line $x = a$.

6. Find the moment of inertia of the area of a circle about the axis perpendicular to its plane at the center. (Divide the area into rings with centers at the center of the circle.)

7. Find the moment of inertia of a cylinder of mass M and radius a about its axis.

8. Find the moment of inertia of a sphere of mass M and radius a about a diameter.

9. An ellipsoid is generated by revolving the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

about the x-axis. Find its moment of inertia about the x-axis.

10. Find the moment of inertia of a hemispherical shell of constant density about the diameter perpendicular to its plane face.

11. Prove that the moment of inertia about any axis is equal to the moment of inertia about a parallel axis through the center of gravity plus the product of the mass and the square of the distance between the two axes.

12. Use the answer to Ex. 6, and the theorem of Ex. 11 to determine the moment of inertia of a circular area about an axis, perpendicular to its plane at a point of the circumference.

41. Work Done by a Force. — Let a force be applied to a body at a fixed point. When the body moves work is done by the force. If the force is constant, the work is defined as the product of the force and the distance the point of application moves in the direction of the force. That is,

$$W = Fs, \tag{41a}$$

where W is the work, F the force, and s the distance moved in the direction of the force.

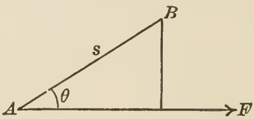

FIG. 41*a*.

If the direction of motion does not coincide with that of the force, the work done is the product of the force and the projection of the displacement on the force. Thus when the body moves from A to B (Fig. 41a) the work done by the force F is

$$W = Fs \cos\theta. \tag{41b}$$

If the force is variable, we divide the path into parts Δs. In moving the distance Δs, the force is nearly constant and so the work done is approximately $F \cos\theta\, \Delta s$. As the intervals Δs are taken shorter and shorter, this approximation becomes more and more accurate. The exact work is then the limit

$$W = \lim_{\Delta s \doteq 0} \sum F \cos\theta \, \Delta s = \int F \cos\theta \, ds. \tag{41c}$$

To determine the value of W, we express $F \cos\theta$ and ds in terms of a single variable. The limits of integration are the values of this variable at the two ends of the path. If the displacement is in the direction of the force, $\theta = 0$, $\cos\theta = 1$ and

$$W = \int F \, ds. \tag{41d}$$

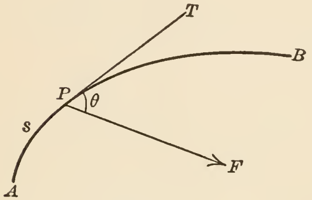

FIG. 41*b*.

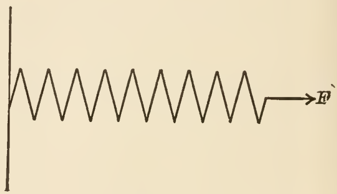

FIG. 41*c*.

Example 1. The amount a helical spring is stretched is proportional to the force applied. If a force of 100 lbs. is required to stretch the spring 1 inch, find the work done in stretching it 4 inches.

Let s be the number of inches the spring is stretched. The force is then

$$F = ks,$$

k being constant. When $s = 1$, $F = 100$ lbs. Hence $k = 100$ and

$$F = 100\, s.$$

The work done in stretching the spring 4 inches is

$$\int_0^4 F\, ds = \int_0^4 100\, s\, ds = 800 \text{ inch pounds} = 66\tfrac{2}{3} \text{ foot pounds.}$$

Ex. 2. A gas is confined in a cylinder with a movable piston. Assuming Boyle's law $pv = k$, find the work done by the pressure of the gas in pushing out the piston (Fig. 41*d*).

Let v be the volume of gas in the cylinder and p the pressure per unit area of the piston. If A is the area of the piston, pA is the total pressure of the gas upon it. If s is the distance the piston moves, the work done is

$$W = \int_{s_1}^{s_2} pA\, ds.$$

But $A\, ds = dv$. Hence

$$W = \int_{v_1}^{v_2} p\, dv = \int_{v_1}^{v_2} \frac{k}{v}\, dv = k \ln \frac{v_2}{v_1}$$

is the work done when the volume expands from v_1 to v_2.

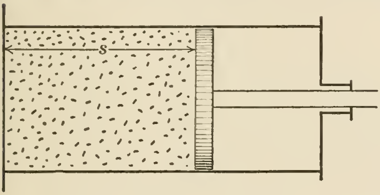

FIG. 41*d*.

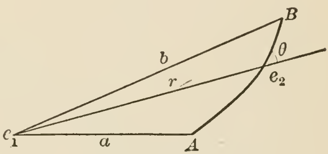

FIG. 41*e*.

Ex. 3. The force with which an electric charge e_1 repels a charge e_2 at distance r is

$$\frac{ke_1e_2}{r^2},$$

where k is constant. Find the work done by this force when the charge e_2 moves from $r = a$ to $r = b$, e_1 remaining fixed.

Let the charge e_2 move from A to B along any path AB (Fig. 41*e*). The work done by the force of repulsion is

$$W = \int F \cos\theta\, ds = \int F\, dr = \int_a^b \frac{ke_1e_2}{r^2}\, dr$$

$$= ke_1e_2\left(\frac{1}{a} - \frac{1}{b}\right).$$

The work depends only on the end points A and B and not on the path connecting them.

EXERCISES

1. According to Hooke's law the force required to stretch a bar from the length a to the length $a + x$ is

$$\frac{kx}{a},$$

where K is constant. Find the work done in stretching the bar from the length a to the length b.

2. Supposing the force of gravity to vary inversely as the square of the distance from the earth's center, find the work done by gravity on a meteor of weight w lbs., when it comes from an indefinitely great distance to the earth's surface.

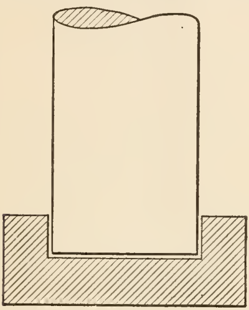

FIG. 41f.

3. If a gas expands without change of temperature, according to van der Waal's equation,

$$p = \frac{c}{v - b} - \frac{a}{v^2},$$

a, b, c being constant. Find the work done when the gas expands from the volume v_1 to the volume v_2.

4. The work in foot pounds required to move a body from one altitude to another is equal to the product of its weight in pounds and the height in feet that it is raised. Find the work required to pump the water out of a cylindrical cistern of diameter 4 ft. and depth 8 ft.

5. A vertical shaft is supported by a flat step bearing (Fig. 41f). The frictional force between a small part of the shaft and the bearing is μp, where p is the pressure between the two and μ is a constant. If the pressure per unit area is the same at all points of the supporting surface, and the weight of the shaft and its load is P, find the work of the frictional forces during each revolution of the shaft.

6. When an electric current flows a distance x through a homogeneous conductor of cross-section A, the resistance is

$$\frac{kx}{A},$$

where K is a constant depending on the material. Find the resistance when the current flows from the inner to the outer surface of a hollow cylinder, the two radii being a and b.

7. Find the resistance when the current flows from the inner to the outer surface of a hollow sphere.

8. Find the resistance when the current flows from one base of a truncated cone to the other.

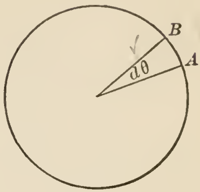

FIG. 41g.

9. When an electric current i flows an infinitesimal distance AB (Fig. 41g) it produces at any point O a magnetic force (perpendicular to the paper) equal to

$$\frac{i\,d\theta}{r},$$

where r is the distance between AB and O. Find the force at the center of a circle due to a current i flowing around it.

10. Find the magnetic force at the distance c from an infinite straight line along which a current i is flowing.

CHAPTER VII

APPROXIMATE METHODS

42. The Prismoidal Formula. — Let y_1, y_3, be two ordinates of a curve at distance h apart, and let y_2 be the ordinate midway between them. The area bounded by the x-axis, the curve, and the two ordinates is given approximately by the formula

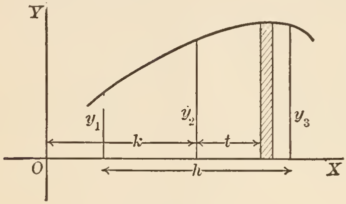

FIG. 42a.

$$A = \tfrac{1}{6} h (y_1 + 4 y_2 + y_3). \quad (42a)$$

This is called the prismoidal formula because of its similarity to the formula for the volume of a prismoid.

If the equation of the curve is

$$y = a + bx + cx^2 + dx^3, \quad (42b)$$

where a, b, c, d, are constants (some of which may be zero), the prismoidal formula gives the exact area. To prove this let k be the abscissa of the middle ordinate and t the distance of any other ordinate from it (Fig. 42a). Then

$$x = k + t.$$

If we substitute this value for x, (42b) takes the form

$$y = A + Bt + Ct^2 + Dt^3,$$

where A, B, C, D are constants. The ordinates y_1, y_2, y_3 are obtained by substituting $t = -\frac{h}{2}, 0, \frac{h}{2}$. Hence

$$y_1 + 4 y_2 + y_3 = 6 A + \tfrac{1}{2} Ch^2.$$

Also the area is

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} y\,dt = Ah + C\,\frac{h^3}{12}.$$

This is equivalent to

$$\frac{h}{6}\left(6\,A + \frac{1}{2}\,Ch^2\right) = \frac{h}{6}\,(y_1 + 4\,y_2 + y_3),$$

which was to be proved.

If the equation of the curve does not have the form (42b), it may be approximately equivalent to one of that type and so the prismoidal formula may give an approximate value for the area.

While we have illustrated the prismoidal formula by the area under a curve, it may be used equally well to determine a length or volume or any other quantity represented by a definite integral,

$$\int_a^b f\,(x)\;dx.$$

Since such an integral represents the area under the curve $y = f\,(x)$, its value can be found by replacing h in (42a) by $b - a$ and y_1, y_2, y_3 by $f\,(a)$, $f\left(\frac{a+b}{2}\right)$, $f\,(b)$ respectively.

Example 1. Find the area bounded by the x-axis, the curve $y = e^{-x^2}$, and the ordinates $x = 0$, $x = 2$.

The integral

$$\int e^{-x^2}\,dx$$

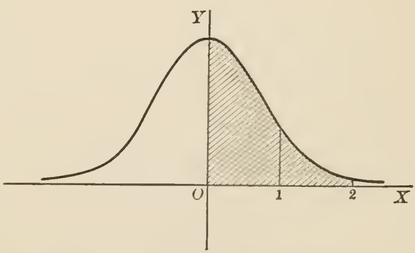

FIG. 42*b*.

cannot be expressed in terms of elementary functions. Therefore we cannot obtain the area by the methods that we have previously used. The ordinates y_1, y_2, y_3, in this case are

$$y_1 = 1, \qquad y_2 = e^{-1}, \qquad y_3 = e^{-2}.$$

The prismoidal formula, therefore, gives

$$A = \frac{2}{6}\left(1 + \frac{4}{e} + \frac{1}{e^2}\right) = 0.869.$$

The answer correct to 3 decimals (obtained from a table) is 0.882.

Ex. 2. Find the length of the parabola $y^2 = 4\,x$ from $x = 1$ to $x = 5$.

The length is given by the formula

$$s = \int_1^5 \sqrt{\frac{x+1}{x}}\,dx.$$

By integration we find $s = 4.726$. To apply the prismoidal formula, let

$$y = \sqrt{\frac{x+1}{x}}.$$

Then $h = 4$,

$$y_1 = \sqrt{2}, \qquad y_2 = \sqrt{\tfrac{4}{3}}, \qquad y_3 = \sqrt{\tfrac{6}{5}},$$

and

$$s = \tfrac{4}{6}\left(\sqrt{2} + 4\sqrt{\tfrac{4}{3}} + \sqrt{\tfrac{6}{5}}\right) = 4.752.$$

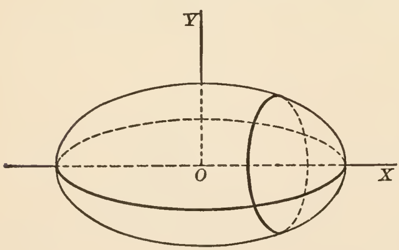

FIG. 42c.

Ex. 3. Find the volume of the spheroid generated by revolving the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

about the x-axis.

The section of the spheroid perpendicular to OX has the area

$$A = \pi y^2 = \pi b^2\left(1 - \frac{x^2}{a^2}\right).$$

Its volume is

$$V = \int_{-a}^{a} A\,dx.$$

Since A is a polynomial of the second degree in x (a special case of a third degree polynomial), the prismoidal formula gives the exact volume. The three cross-sections corresponding to $x = -a$, $x = 0$, $x = a$, are

$$A_1 = 0, \qquad A_2 = \pi b^2, \qquad A_3 = 0.$$

Hence

$$V = \frac{2a}{6}[A_1 + 4A_2 + A_3] = \frac{4}{3}\pi ab^2.$$

43. Simpson's Rule. — Divide the area between a curve and the x-axis into any even number of parts by means of equidistant ordinates $y_1, y_2, y_3, \ldots, y_n$. (An odd number of ordinates will be needed.) Simpson's rule for determining approximately the area between y_1 and y_n is

$$A = h\left(\frac{y_1 + 4y_2 + 2y_3 + 4y_4 + 2y_5 + \cdots + y_n}{1 + 4 + 2 + 4 + 2 + \cdots + 1}\right), \qquad (43)$$

h being the distance between the ordinates y_1 and y_n. In the numerator the end coefficients are 1. The others are alternately 4 and 2. The denominator is the sum of the coefficients in the numerator.

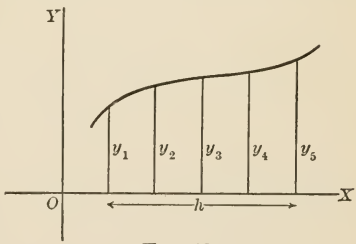

FIG. 43.

This formula is obtained by applying the prismoidal formula to the strips taken two at a time and adding the results. Thus if the area is divided into four strips by the ordinates y_1, y_2, y_3, y_4, y_5, the part between y_1 and y_3 has a base equal to $\frac{h}{2}$. Its area as given by the prismoidal formula is

$$\frac{1}{6}\frac{h}{2}(y_1 + 4y_2 + y_3).$$

Similarly the area between y_3 and y_5 is

$$\frac{1}{6}\frac{h}{2}(y_3 + 4\,y_4 + y_5).$$

The sum of the two is

$$A = h\left(\frac{y_1 + 4\,y_2 + 2\,y_3 + 4\,y_4 + y_5}{12}\right).$$

By using a sufficiently large number of ordinates in Simpson's formula, the result can be made as accurate as desired.

Example. Find ln 5 by Simpson's rule. Since

$$\text{in } 5 = \int_1^5 \frac{dx}{x},$$

we take $y = \dfrac{1}{x}$ in Simpson's formula. Dividing the interval into 4 parts we get

$$\ln 5 = 4\left(\frac{1 + 4\cdot\frac{1}{2} + 2\cdot\frac{1}{3} + 4\cdot\frac{1}{4} + \frac{1}{5}}{12}\right) = 1.622.$$

If we divide the interval into 8 parts, we get

$$\ln 5 = \tfrac{4}{24}\left(1 + \tfrac{8}{3} + \tfrac{2}{2} + \tfrac{8}{5} + \tfrac{2}{3} + \tfrac{8}{7} + \tfrac{2}{4} + \tfrac{8}{9} + \tfrac{1}{5}\right) = 1.6108.$$

The value correct to 4 decimals is

$$\ln 5 = 1.6094.$$

44. Integration in Series. — In calculating integrals it is sometimes convenient to expand a function in infinite series and then integrate the series. This is particularly the case when the integral contains constants for which numerical values are not assigned. For the process to be valid all series used should converge.

Example. Find the length of a quadrant of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Let a be greater than b. Introduce a parameter ϕ by the equation

$$x = a \sin \phi.$$

Substituting this value in the equation of the ellipse, we find

$$y = b \cos \phi.$$

Using these values of x and y we get

$$s = \int \sqrt{dx^2 + dy^2} = \int_0^{\frac{\pi}{2}} \sqrt{a^2 - (a^2 - b^2) \sin^2 \phi}\, d\phi.$$

This is an elliptic integral. It cannot be represented by an expression containing only a finite number of elementary functions. We therefore express it as an infinite series. By the binomial theorem

$$\sqrt{a^2 - (a^2 - b^2)\sin^2 \phi}$$
$$= a\left[1 - \frac{1}{2}\,\frac{a^2 - b^2}{a^2}\sin^2 \phi - \frac{1}{2.4}\left(\frac{a^2 - b^2}{a^2}\right)^2 \sin^4 \phi \ldots\right].$$

Since

$$\int_0^{\frac{\pi}{2}} \sin^2 \phi\, d\phi = \frac{\pi}{4}, \qquad \int_0^{\frac{\pi}{2}} \sin^4 \phi\, d\phi = \frac{3\pi}{16},$$

we find by integrating term by term

$$s = a\left[\frac{\pi}{2} - \frac{\pi}{8}\,\frac{a^2 - b^2}{a^2} - \frac{3\pi}{128}\left(\frac{a^2 - b^2}{a^2}\right)^2 \cdots\right]$$
$$= \frac{\pi a}{2}\left[1 - \frac{a^2 - b^2}{4a^2} - \frac{3}{64}\left(\frac{a^2 - b^2}{a^2}\right)^2 \cdots\right].$$

If a and b are nearly equal, the value of s can be calculated very rapidly from the series.

EXERCISES

1. Show that the prismoidal formula gives the correct volume in each of the following cases: (*a*) sphere, (*b*) cone, (*c*) cylinder, (*d*) pyramid, (*e*) segment of a sphere, (*f*) truncated cone or pyramid.

2. Find the error when the value of the integral $\int_1^5 x^4\, dx$ is found by the prismoidal formula.

In each of the following cases compare the value given by the prismoidal formula with the exact value determined by integration.

3. Area bounded by $y = \sqrt{x}$, $y = 0$, $x = 1$, $x = 3$.

4. Arc of the curve $y = x^3$ between $x = -2$, $x = +2$.

5. Volume generated by revolving about OX one arch of the sine curve $y = \sin x$.

6. Area of the surface of a hemisphere.

Compute each of the following by Simpson's rule using 4 intervals:

7. $\dfrac{\pi}{4} = \displaystyle\int_0^1 \frac{dx}{1 + x^2}$.

8. $\displaystyle\int_1^9 \frac{dx}{\sqrt{1 + x^3}}$.

9. Length of the curve $y = \ln x$ from $x = 1$ to $x = 5$.

10. Surface of the spheroid generated by rotating the ellipse $x^2 + 4y^2 = 4$ about the x-axis.

11. Volume of the solid generated by revolving about the x-axis the area bounded by $y = 0$, $y = \dfrac{1}{1 + x^2}$, $x = -2$, $x = 2$.

12. Find the value of

$$\int_0^1 \cos(x^2)\, dx,$$

by expanding in series.

13. Express

$$\int_0^a \frac{\sin(\lambda x)\, dx}{x}$$

as a series in powers of λ.

14. Find the length of a quadrant of the ellipse $x^2 + 2y^2 = 2$.

CHAPTER VIII

DOUBLE INTEGRATION

45. Double Integrals. — The notation

$$\int_a^b \int_c^d f(x, y)\, dx\, dy$$

is used to represent the result of integrating first with respect to y (leaving x constant) between the limits c, d and then with respect to x between the limits a, b.

As here defined the first integration is with respect to the variable whose differential stands last and its limits are attached to the last integral sign. Some writers integrate in a different order. In reading an article it is therefore necessary to know what convention the author uses.

Example. Find the value of the double integral

$$\int_0^1 \int_{-x}^x (x^2 + y^2)\, dx\, dy.$$

We integrate first with respect to y between the limits $-x$, x, then with respect to x between the limits 0, 1. The result is

$$\int_0^1 \int_{-x}^x (x^2+y^2)\, dx\, dy = \int_0^1 dx\, (x^2 y + \tfrac{1}{3} y^3)_{-x}^x = \int_0^1 \tfrac{8}{3} x^3\, dx = \tfrac{2}{3}.$$

46. Area as a Double Integral. — Divide the area between two curves $y = f(x)$, $y = F(x)$ into strips of width Δx. Let P be the point (x, y) and Q the point $(x + \Delta x, y + \Delta y)$. The area of the rectangle PQ is $\Delta x\, \Delta y$. The area of the rectangle RS (Fig. 46a) is

$$\Delta x \sum_{f(x)}^{F(x)} \Delta y = \Delta x \int_{f(x)}^{F(x)} dy.$$

The area bounded by the ordinates $x = a$, $x = b$ is then

$$A = \lim_{\Delta x \doteq 0} \sum_a^b \Delta x \int_{f(x)}^{F(x)} dy = \int_a^b \int_{f(x)}^{F(x)} dx\, dy.$$

If it is simpler to cut the area into strips parallel to the x-axis, the area is

$$A = \int \int dy\, dx,$$

the limits in the first integration being the values of x at the ends of a variable strip; those in the second integration, the values of y giving the limiting strips.

Example. Find the area bounded by the parabola $y^2 = 4\,ax + 4\,a^2$ and the straight line $y = 2\,a - x$ (Fig. 46b).

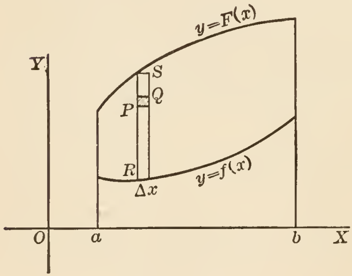

FIG. 46a.

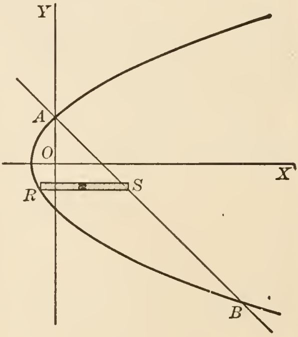

FIG. 46b.

Solving simultaneously, we find that the parabola and the line intersect at A (0, 2 a) and B (8 a, $-$ 6 a). Draw the strips parallel to the x-axis. The area is

$$A = \int_{-6a}^{2a} \int_{\frac{y^2-4a^2}{4a}}^{2a-y} dy\, dx = \int_{-6a}^{2a} \left(2\,a - y - \frac{y^2 - 4\,a^2}{4\,a}\right) dy = \frac{64}{3} a^2.$$

The limits in the first integration are the values of x at R and S, the ends of the variable strip. The limits in the second integration are the values of y at B and A, corresponding to the outside strips.

47. Volume by Double Integration. — To find the volume under a surface $z = f(x, y)$ and over a given region in the xy-plane.

The volume of the prism PQ standing on the base $\Delta x\,\Delta y$ (Fig. 47a) is

$$z\,\Delta x\,\Delta y.$$

The volume of the plate RT is then

$$\lim_{\Delta y \doteq 0} \sum_R^S z\,\Delta x\,\Delta y = \Delta x \int_{f(x)}^{F(x)} z\,dy,$$

$f(x)$, $F(x)$ being the values of y at R, S. The entire volume is the limit of the sum of such plates

$$\lim_{\Delta x \doteq 0} \sum_a^b \Delta x \int_{f(x)}^{F(x)} z\,dy = \int_a^b \int_{f(x)}^{F(x)} z\,dx\,dy,$$

a, b being the values of x corresponding to the outside plates.

Example. Find the volume bounded by the surface $az = a^2 - x^2 - 4y^2$ and the xy-plane.

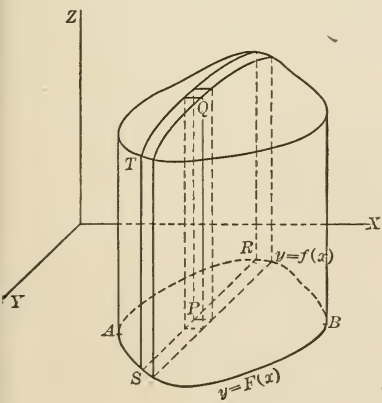

FIG. 47a.

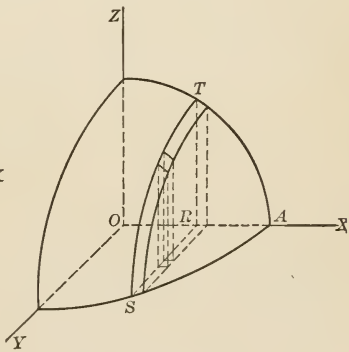

FIG. 47b.

Fig. 47b shows one-fourth of the required volume. At R, $y = 0$. At S, $z = 0$ and so

$$y = \tfrac{1}{2}\sqrt{a^2 - x^2}.$$

The limiting values of x at O and A are 0 and a. Therefore

$$v = 4\int_0^a \int_0^{\frac{1}{2}\sqrt{a^2-x^2}} z\,dx\,dy = 4\int_0^a \int_0^{\frac{1}{2}\sqrt{a^2-x^2}} \frac{1}{a}(a^2 - x^2 - 4\,y^2)\,dx\,dy$$

$$= \frac{4}{3\,a}\int_0^a (a^2 - x^2)^{\frac{3}{2}}\,dx = \frac{\pi a^3}{4}.$$

48. The Double Integral as the Limit of a Double Summation. — Divide a plane area by lines parallel to the coördinate axes into rectangles with sides Δx and Δy. Let (x, y) be any point within one of these rectangles. Form the product

$$f(x, y)\,\Delta x\,\Delta y.$$

This product is equal to the volume of the prism standing on the rectangle as base and reaching the surface $z = f(x, y)$ at some point over the base. Take the sum of such products

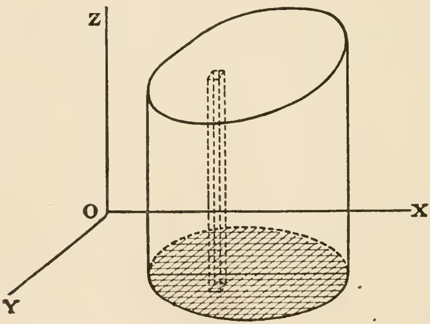

FIG. 48a.

for all the rectangles that lie entirely within the area. We represent this sum by the notation

$$\sum\sum f(x, y)\,\Delta x\,\Delta y.$$

When Δx and Δy are taken smaller and smaller, this sum approaches as limit the double integral

$$\int\int f(x, y)\,dx\,dy,$$

with the limits determined by the given area; for it approaches the volume over the area and that volume is equal to the double integral.

Whenever then a quantity is a limit of a sum of the form

$$\sum \sum f(x, y) \, \Delta x \, \Delta y$$

its value can be found by double integration. Furthermore, in the formation of this sum, infinitesimals of higher order than $\Delta x \, \Delta y$ can be neglected without changing the limit. For, if $\epsilon \, \Delta x \, \Delta y$ is such an infinitesimal, the sum of the errors thus made is

$$\sum \sum \epsilon \, \Delta x \, \Delta y.$$

When Δx and Δy approach zero, ϵ approaches zero. The sum of the errors approaches zero, since it is represented by a volume whose thickness approaches zero.

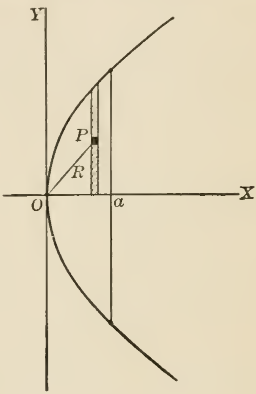

FIG. 48*b*.

Example 1. An area is bounded by the parabola $y^2 = 4\,ax$ and the line $x = a$. Find its moment of inertia about the axis perpendicular to its plane at the origin.

Divide the area into rectangles $\Delta x \, \Delta y$. The distance of any point $P\ (x, y)$ from the axis perpendicular to the plane at O is $R = OP = \sqrt{x^2 + y^2}$. If then (x, y) is a point within one of the rectangles, the moment of inertia of that rectangle is

$$R^2 \, \Delta x \, \Delta y = (x^2 + y^2) \, \Delta x \, \Delta y,$$

approximately. That the result is approximate and not exact is due to the fact that different points in the rectangle differ slightly in distance from the axis. This difference is, however, infinitesimal and, since R^2 is multiplied by $\Delta x\, \Delta y$, the resulting error is of higher order than $\Delta x\, \Delta y$. Hence in the limit

$$I = \int_0^a \int_{-2\sqrt{ax}}^{2\sqrt{ax}} (x^2 + y^2)\, dx\, dy = \frac{344}{105} a^4.$$

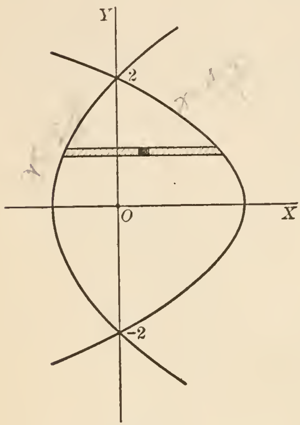

FIG. 48c.

Ex. 2. Find the center of gravity of the area bounded by the parabolas $y^2 = 4x + 4$, $y^2 = -2x + 4$.

By symmetry the center of gravity is seen to be on the x-axis. Its abscissa is

$$\bar{x} = \frac{\int x\, dA}{A}.$$

If we wish to use double integration we have merely to replace dA by $dx\, dy$ or $dy\, dx$. From the figure it is seen that the first integration should be with respect to x. Hence

$$\bar{x} = \frac{\int_{-2}^{2} \int_{\frac{1}{4}(y^2-4)}^{\frac{1}{2}(4-y^2)} x\, dy\, dx}{\int_{-2}^{2} \int_{\frac{1}{4}(y^2-4)}^{\frac{1}{2}(4-y^2)} dy\, dx} = \frac{\frac{16}{5}}{8} = \frac{2}{5}.$$

EXERCISES

Find the values of the following double integrals:

1. $\int_3^4 \int_1^2 \frac{dx\, dy}{(x+y)^2}$.
2. $\int_0^{2\pi} \int_{a\sin\theta}^{a} r\, d\theta\, dr$.
3. $\int_1^2 \int_x^{x\sqrt{3}} xy\, dx\, dy$.
4. $\int_0^{2\pi} \int_0^{\infty} e^{-kr^2} r\, d\theta\, dr$.
5. $\int_0^3 \int_y^2 (x^2 + y^2)\, dy\, dx$.
6. $\int_0^a \int_0^{\sqrt{a^2-y^2}} dy\, dx$.

7. Find the area bounded by the parabola $y^2 = 2x$ and the line $x = y$.

8. Find the area bounded by the parabola $y^2 = 4ax$, the line $x + y = 3a$, and the x-axis.

9. Find the area enclosed by the ellipse

$$(y - x)^2 + x^2 = 1.$$

10. Find the volume under the paraboloid $z = 4 - x^2 - y^2$ and over the square bounded by the lines $x = \pm 1$, $y = \pm 1$ in the xy-plane.

11. Find the volume bounded by the xy-plane, the cylinder $x^2 + y^2 = 1$, and the plane $x + y + z = 3$.

12. Find the volume in the first octant bounded by the cylinder $(x - 1)^2 + (y - 1)^2 = 1$ and the paraboloid $xy = z$.

13. Find the moment of inertia of the triangle bounded by the coördinate axes and the line $x + y = 1$ about the line perpendicular to its plane at the origin.

14. Find the moment of inertia of a square of side a about the axis perpendicular to its plane at one corner.

15. Find the moment of inertia of the triangle bounded by the lines $x + y = 2$, $x = 2$, $y = 2$ about the x-axis.

16. Find the moment of inertia of the area bounded by the parabola $y^2 = ax$ and the line $x = a$ about the line $y = -a$.

17. Find the moment of inertia of the area bounded by the hyperbola $xy = 4$ and the line $x + y = 5$ about the line $y = x$.

18. Find the moment of inertia of a cube about an edge.

19. A wedge is cut from a cylinder by a plane passing through a diameter of the base and inclined 45° to the base. Find its moment of inertia about the axis of the cylinder.

20. Find the center of gravity of the triangle formed by the lines $x = y$, $x + y = 4$, $x - 2y = 4$.

21. Find the center of gravity of the area bounded by the parabola $y^2 = 4ax + 4a^2$ and the line $y = 2a - x$.

49. Double Integration. Polar Coördinates. — Pass through the origin a series of lines making with each other equal angles $\Delta\theta$. Construct a series of circles with centers at the origin and radii differing by Δr. The lines and circles divide the plane into curved quadrilaterals (Fig. 49a).

Let r, θ be the coördinates of P, $r + \Delta r$, $\theta + \Delta\theta$ those of Q. Since PR is the arc of a circle of radius r and subtends the angle $\Delta\theta$ at the center, $PR = r\,\Delta\theta$. Also $RQ = \Delta r$.

When Δr and $\Delta\theta$ are very small PRQ will be approximately a rectangle with area

$$PR \cdot RQ = r\,\Delta\theta\,\Delta r.$$

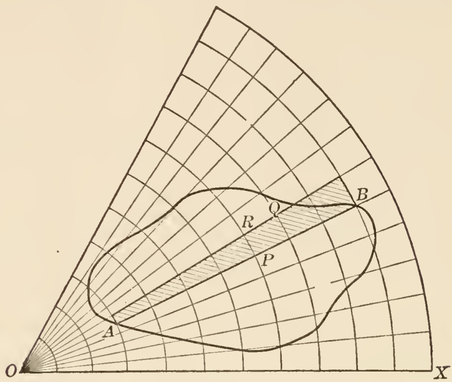

FIG. 49a.

It is very easy to show that the error is an infinitesimal of higher order than $\Delta\theta\,\Delta r$. (See Ex. 5, page 107.) Hence the sum

$$\sum\sum r\,\Delta\theta\,\Delta r,$$

taken for all the rectangles within a curve, gives in the limit the area of the curve in the form

$$A = \int\int r\,d\theta\,dr. \tag{49a}$$

The limits in the first integration are the values of r at the ends A, B of the strip across the area. The limits in the second integration are the values of θ giving the outside strips.

If it is more convenient the first integration may be with respect to θ. The area is then

$$A = \int\int r\,dr\,d\theta.$$

The first limits are the values of θ at the ends of a strip between two concentric circles (Fig. 49*b*). The second limits are the extreme values of r.

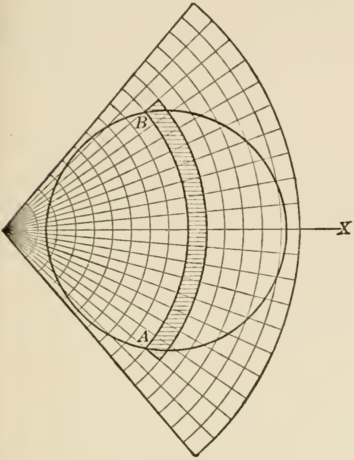

FIG. 49*b*.

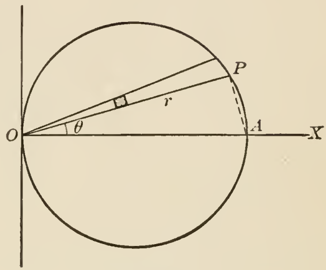

FIG. 49*c*.

The element of area in polar coördinates is

$$dA = r\,d\theta\,dr. \tag{49b}$$

We can use this in place of dA in finding moments of inertia, volumes, centers of gravity, or any other quantities expressed by integrals of the form

$$\int f(r, \theta)\,dA.$$

Example 1. Change the double integral

$$\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} (x^2 + y^2)\,dx\,dy$$

to polar coördinates.

The integral is taken over the area of the semicircle $y = \sqrt{2ax - x^2}$ (Fig. 49*c*). In polar coördinates the equation of this circle is $r = 2a\cos\theta$. The element of area

$dx\,dy$ can be replaced by $r\,d\theta\,dr$.* Also $x^2 + y^2 = r^2$. Hence

$$\int_0^{2a}\int_0^{\sqrt{2ax-x^2}}(x^2+y^2)\,dx\,dy = \int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta} r^2\cdot r\,d\theta\,dr.$$

The limits for r are the ends of the sector OP. The limits for θ give the extreme sectors $\theta = 0$, $\theta = \frac{\pi}{2}$.

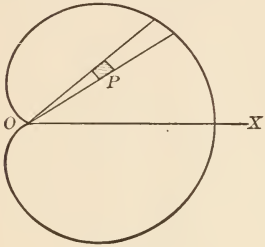

FIG. 49d.

Ex. 2. Find the moment of inertia of the area of the cardioid $r = a\ (1 + \cos\theta)$ about the axis perpendicular to its plane at the origin.

The distance from any point P (r, θ) (Fig. 49d) to the axis of rotation is

$$OP = r.$$

Hence the moment of inertia is

$$I = 2\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2\cdot r\,d\theta\,dr = \frac{a^4}{2}\int_0^{\pi}(1+\cos\theta)^4\,d\theta = \frac{35}{16}\pi a^4.$$

Ex. 3. Find the center of gravity of the cardioid in the preceding problem.

The ordinate of the center of gravity is evidently zero. Its abscissa is

$$x = \frac{\int x\,dA}{\int dA} = \frac{2\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\cos\theta\cdot r\,d\theta\,dr}{2\int\int r\,d\theta\,dr} = \frac{5}{6}a.$$

Ex. 4. Find the volume common to a sphere of radius $2a$ and a cylinder of radius a, the center of the sphere being on the surface of the cylinder.

* This does not mean that

$$dx\,dy = r\,d\theta\,dr,$$

but merely that the sum of all the rectangular elements in the circle is equal to the sum of all the polar elements.

Fig. 49e shows one-fourth of the required volume. Take a system of polar coördinates in the xy-plane. On the element of area $r\,d\theta\,dr$ stands a prism of height

$$z = \sqrt{4a^2 - r^2}.$$

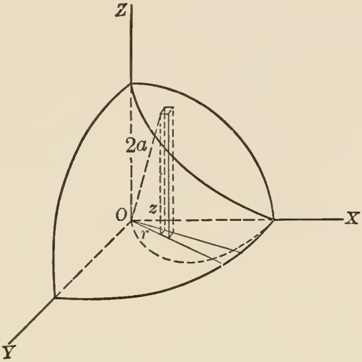

FIG. 49e.

The volume of the prism is $z \cdot r\,d\theta\,dr$ and the entire volume is

$$v = 4\int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta}\sqrt{4a^2 - r^2}\cdot r\,d\theta\,dr = 4\int_0^{\frac{\pi}{2}} \frac{(4a^2-r^2)^{\frac{3}{2}}}{-3}\Bigg|_0^{2a\cos\theta} d\theta$$

$$= \frac{32a^3}{3}\int_0^{\frac{\pi}{2}}(1 - \sin^3\theta)\,d\theta = \frac{16}{9}a^3(3\pi - 4).$$

EXERCISES

Find the values of the following integrals by changing to polar coördinates:

1. $\displaystyle\int_0^a\int_0^{\sqrt{a^2-y^2}}(x^2 + y^2)\,dy\,dx.$

2. $\displaystyle\int_0^{2a}\int_0^{\sqrt{2ax-x^2}}dx\,dy.$

3. $\displaystyle\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy.$

4. $\displaystyle\int_0^a\int_0^{\sqrt{a^2-x^2}}\sqrt{a^2 - x^2 - y^2}\,dx\,dy.$

5. Find the area bounded by two circles of radii a, $a + \Delta a$ and two lines through the origin, making with the initial line the angles α, $\alpha + \Delta\alpha$, respectively. Show that when Δa and $\Delta\alpha$ approach zero, the result differs from

$$a\,\Delta\alpha\,\Delta a$$

by an infinitesimal of higher order than $\Delta\alpha\,\Delta a$.

6. The central angle of a circular sector is 2α. Find the moment of inertia of its area about the bisector of the angle.

7. An area is bounded by the circle $r = a\sqrt{2}$ and the straight line $r = a \sec\left(\theta - \frac{\pi}{4}\right)$. Find its moment of inertia about the axis perpendicular to its plane at the origin.

8. Find the center of gravity of the area in Ex. 6.

9. The center of a circle of radius $2a$ lies on a circle of radius a. Find the moment of inertia of the area between them about the common tangent.

10. Find the moment of inertia of the area of the lemniscate $r^2 = 2a^2 \cos 2\theta$ about the axis perpendicular to its plane at the origin.

11. Find the moment of inertia of the area of the circle $r = 2a$ outside the parabola, $r = a \sec^2 \frac{\theta}{2}$ about the axis perpendicular to its plane at the origin.

12. Find the moment of inertia about the y-axis of the area within the circle $(x - a)^2 + (y - a)^2 = 2a^2$.

13. The density of a square lamina is proportional to the distance from one corner. Find its moment of inertia about an edge passing through that corner.

14. Find the moment of inertia of a cylinder about a generator.

15. Find the moment of inertia of a cone about its axis.

16. Find the volume under the spherical surface $x^2 + y^2 + z^2 = a^2$ and over the lemniscate $r^2 = a^2 \cos 2\theta$ in the xy-plane.

17. Find the volume bounded by the xy-plane, the paraboloid $az = x^2 + y^2$ and the cylinder $x^2 + y^2 = 2ax$.

18. Find the moment of inertia of a sphere of density ρ about a diameter.

19. Find the volume generated by revolving one loop of the curve $r = a \cos 2\theta$ about the initial line.

50. Area of a Surface. — Let an area A in one plane be projected upon another plane. The area of the projection is

$$A' = A \cos \phi,$$

when ϕ is the angle between the planes.

To show this divide A into rectangles by two sets of lines respectively parallel and perpendicular to the intersection MN of the two planes. Let a and b be the sides of one of these rectangles, a being parallel to MN. The projection of this rectangle will be a rectangle with sides

$$a' = a, \qquad b' = b \cos \phi,$$

and area

$$a'b' = ab \cos \phi.$$

The sum of the projections of all the rectangles is

$$\sum a'b' = \sum ab \cos \phi.$$

As the rectangles are taken smaller and smaller this approaches as limit

$$A' = A \cos \phi,$$

which was to be proved.

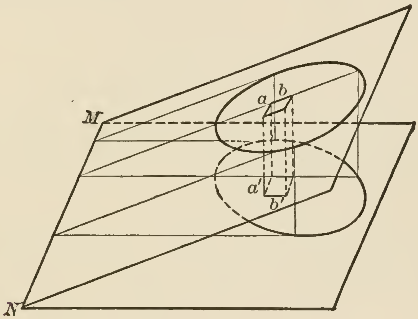

FIG. 50a.

To find the area of a curved surface, resolve it into elements whose projections on a coördinate plane are equal to the differential of area dA in that plane. The element of surface can be considered as lying approximately in a tangent plane. Its area is, therefore, approximately

$$\frac{dA}{\cos \phi},$$

where ϕ is the angle between the tangent plane and the coördinate plane on which the area is projected. The area of the surface is the limit

$$S = \int \frac{dA}{\cos \phi}.$$

The angle between two planes is equal to that between the perpendiculars to the planes. Therefore ϕ is equal to

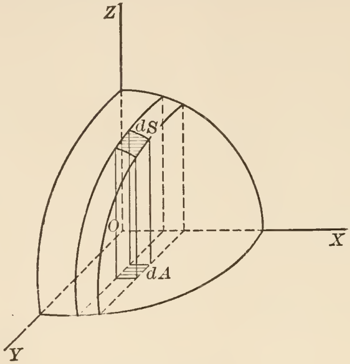

FIG. 50b.

the angle between the normal to the surface and the coordinate axis perpendicular to the plane on which we project.

If the equation of the surface is

$$F(x, y, z) = 0,$$

the cosine of the angle between its normal and the z-axis is (Differential Calculus, Art. 101)

$$\cos \gamma = \frac{\dfrac{\partial F}{\partial z}}{\sqrt{\left(\dfrac{\partial F}{\partial x}\right)^2 + \left(\dfrac{\partial F}{\partial y}\right)^2 + \left(\dfrac{\partial F}{\partial z}\right)^2}}.$$

The cosines of the angles between the normal and the x-axis or y-axis are obtained by replacing $\dfrac{\partial F}{\partial z}$ by $\dfrac{\partial F}{\partial x}$ or $\dfrac{\partial F}{\partial y}$. In finding areas the algebraic sign is assumed to be positive.

Example 1. Find the area of the sphere $x^2 + y^2 + z^2 = a^2$ within the cylinder $x^2 + y^2 = ax$.

Project on the xy-plane. The angle ϕ is then the angle γ between the normal to the sphere and the z-axis. Its cosine is

$$\cos \gamma = \frac{z}{\sqrt{x^2 + y^2 + z^2}} = \frac{z}{a}.$$

Using polar coördinates in the xy-plane,

$$z = \sqrt{a^2 - x^2 - y^2} = \sqrt{a^2 - r^2}.$$

Hence the area of the surface is

$$S = \int \frac{dA}{\cos \gamma} = 4 \int_0^{\frac{\pi}{2}} \int_0^{a \cos \theta} \frac{ar\, d\theta\, dr}{\sqrt{a^2 - r^2}} = 2\, a^2 (\pi - 2).$$

Ex. 2. Find the area of the surface of the cone $y^2 + z^2 = x^2$ in the first octant bounded by the plane $y + z = a$.

Project on the yz-plane. Then $\phi = \alpha$ and

$$\cos \alpha = \frac{x}{\sqrt{x^2 + y^2 + z^2}} = \frac{x}{\sqrt{2\, x^2}} = \frac{1}{\sqrt{2}}.$$

The area on the cone is therefore

$$S = \int_0^a \int_0^{a-y} \sqrt{2}\, dy\, dz = \frac{a^2\sqrt{2}}{2}.$$

EXERCISES

1. Find the area of the triangle cut from the plane

$$x + 2\, y + 3\, z = 6$$

by the coördinate planes.

2. Find the area of the surface of the cylinder $x^2 + y^2 = a^2$ between the planes $z = 0$, $z = mx$.

3. Find the area of the surface of the cone $x^2 + y^2 = z^2$ cut out by the cylinder $x^2 + y^2 = 2\, ax$.

4. Find the area of the plane $x + y + z = 2\, a$ in the first octant bounded by the cylinder $x^2 + y^2 = a^2$.

5. Find the area of the surface $z^2 = 2\, xy$ above the xy-plane bounded by the planes $y = 1$, $x = 2$.

6. Find the area of the surface of the cylinder $x^2 + y^2 = 2\, ax$ between the xy-plane and the cone $x^2 + y^2 = z^2$.

7. Find the area of the surface of the paraboloid $y^2 + z^2 = 2\, ax$, intercepted by the parabolic cylinder $y^2 = ax$ and the plane $x = a$.

8. Find the area intercepted on the cylinder in Ex. 4.

9. A square hole of side a is cut through a sphere of radius a. If the axis of the hole is a diameter of the sphere, find the area of the surface cut out.

CHAPTER IX

TRIPLE INTEGRATION

51. Triple Integrals. — The notation

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x, y, z)\, dx\, dy\, dz$$

is used to represent the result of integrating first with respect to z (leaving x and y constant) between the limits z_1 and z_2, then with respect to y (leaving x constant) between the limits y_1 and y_2, and finally with respect to x between the limits x_1 and x_2.

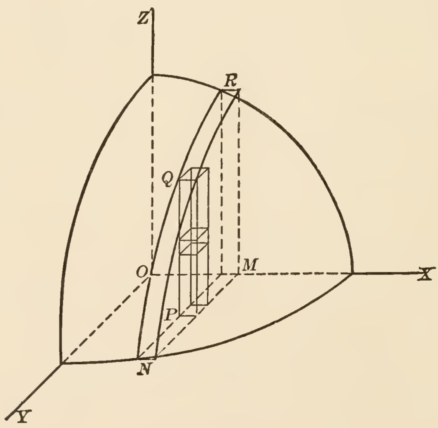

FIG. 52a.

52. Rectangular Coördinates. — Divide a solid into rectangular parallelepipeds of volume $\Delta x\, \Delta y\, \Delta z$ by planes parallel to the coördinate planes. To find the volume of the solid, first take the sum of the parallelepipeds in a vertical column PQ. The result is

$$\sum \Delta x\,\Delta y\,\Delta z = \Delta x\,\Delta y \int_{z_1}^{z_2} dz,$$

z_1 and z_2 being the values of z at the ends of the column. Then sum these columns along a base MN and so obtain the volume of the plate MNR. The result is

$$\lim_{\Delta y \doteq 0} \sum \Delta x\,\Delta y \int_{z_1}^{z_2} dz = \Delta x \int_{y_1}^{y_2} \int_{z_1}^{z_2} dy\,dz,$$

y_1 and y_2 being the limiting values of y in the plate. Finally, take the sum of these plates. The result is the triple integral

$$v = \lim_{\Delta x \doteq 0} \sum \Delta x \int_{y_1}^{y_2} \int_{z_1}^{z_2} dy\,dz = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} dx\,dy\,dz,$$

x_1, x_2 being the limiting values of x.

It may be more convenient to begin by integrating with respect to x or y. In any case the limits can be obtained from the consideration that the first integration is a summation of parallelepipeds to form a prism, the second a summation of prisms to form a plate, and the third integration a summation of plates.

Let (x, y, z) be any point of the parallelepiped $\Delta x\,\Delta y\,\Delta z$. Multiply $\Delta x\,\Delta y\,\Delta z$ by $f(x, y, z)$ and form the sum

$$\sum \sum \sum f(x, y, z)\,\Delta x\,\Delta y\,\Delta z$$

taken for all parallelepipeds in the solid. When Δx, Δy, and Δz approach zero, this sum approaches the triple integral

$$\int\int\int f(x, y, z)\,dx\,dy\,dz$$

as limit. It can be shown that terms of higher order than $\Delta x\,\Delta y\,\Delta z$ can be neglected in the sum without changing the limit.

The differential of volume in rectangular coördinates is

$$dv = dx\,dy\,dz.$$

This can be used in the formulas for moment of inertia, center of gravity, etc., those quantities being then determined by triple integration.

Example 1. Find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Fig. 52*a* shows one-eighth of the required volume. Therefore

$$v = 8\int\int\int dx\,dy\,dz.$$

The limits in the first integration are the values $z = 0$ at P and $z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$ at Q. The limits in the second integration are the values of y at M and N. At M, $y = 0$ and at N, $z = 0$, whence $y = b\sqrt{1 - \frac{x^2}{a^2}}$. Finally, the limits for x are 0 and a. Therefore

$$v = 8\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dx\,dy\,dz = \tfrac{4}{3}\pi abc.$$

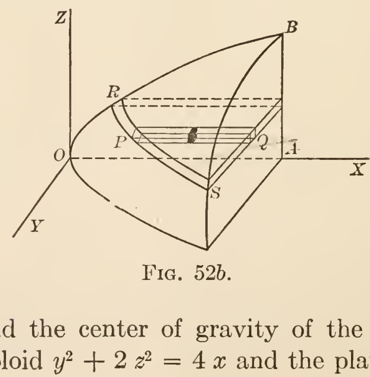

FIG. 52*b*.

Ex. 2. Find the center of gravity of the solid bounded by the paraboloid $y^2 + 2z^2 = 4x$ and the plane $x = 2$.

By symmetry $\bar{y}$ and $\bar{z}$ are zero. The x-coördinate is

$$\bar{x} = \frac{\int x\,dv}{\int dv} = \frac{4\int_0 \int_0^{\sqrt{8-2z^2}} \int_{\frac{1}{4}(y^2+2z^2)}^{2} x\,dz\,dy\,dx}{4\int\int\int dz\,dy\,dx} = \frac{4}{3}.$$

The limits for x are the values $x = \frac{1}{4}(y^2 + 2z^2)$ at P and $x = 2$ at Q. At S, $x = 2$ and $y = \sqrt{4x - 2z^2} = \sqrt{8 - 2z^2}$. The limits for y are, therefore, $y = 0$ at R and $y = \sqrt{8 - 2z^2}$ at S. The limits for z are $z = 0$ at A and $z = 2$ at B.

Ex. 3. Find the moment of inertia of a cube about an edge.

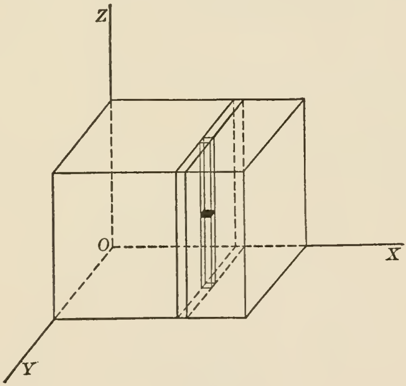

FIG. 52c.

Place the cube as shown in Fig. 52c and determine its moment of inertia about the z-axis. The distance of any point (x, y, z) from the z-axis is

$$R = \sqrt{x^2 + y^2}.$$

Hence the moment of inertia is

$$I = \int_0^a \int_0^a \int_0^a (x^2 + y^2)\,dx\,dy\,dz = \tfrac{2}{3}a^5,$$

where a is the edge of the cube.

EXERCISES

1. Find by triple integration the volume of the pyramid determined by the coördinate planes and the plane $x + y + z = 1$.

2. Find the moment of inertia of the pyramid in Ex. 1 about the x-axis.

3. A wedge is cut from a cylinder of radius a by a plane passing through a diameter of the base and inclined 45° to the base. Find its center of gravity.

4. Find the volume bounded by the paraboloid $\frac{y^2}{b^2} + \frac{z^2}{c^2} = 2\frac{x}{a}$ and the plane $x = a$.

5. Express the volume of the cone

$$(z - 1)^2 = x^2 + y^2$$

in the first octant as a triple integral in 6 ways by integrating with dx, dy, dz, arranged in all possible orders.

6. Find the volume bounded by the surfaces $y^2 = 4a^2 - 3ax$, $y^2 = ax$, $z = \pm h$.

7. Find the volume bounded by the cylinder $z^2 = 1 - x - y$ and the coördinate planes.

53. Cylindrical Coördinates. — Let M be the projection of P on the xy-plane. Let r, θ be the polar coördinates of M in the xy-plane. The cylindrical coördinates of P are r, θ, z.

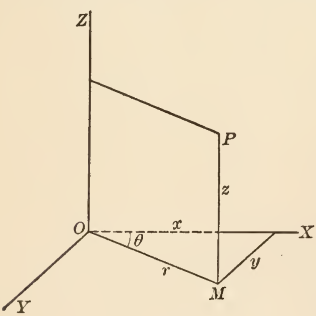

FIG. 53a.

From Fig. 53a it is evident that

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

By using these equations we can change any rectangular into a cylindrical equation.

The element of volume in cylindrical coördinates is the volume PQ, Fig. 53b, bounded by two cylindrical surfaces of radii r, $r + \Delta r$, two horizontal planes z, $z + \Delta z$, and two planes through the z-axis making angles θ, $\theta + \Delta\theta$ with OX. The base of PQ is equal to the polar element MN in the xy-plane. Its altitude PR is Δz. Hence

$$dv = r\,d\theta\,dr\,dz. \tag{53}$$

This value of dv can be used in the formulas for volume, center of gravity, moment of inertia, etc. In problems con-

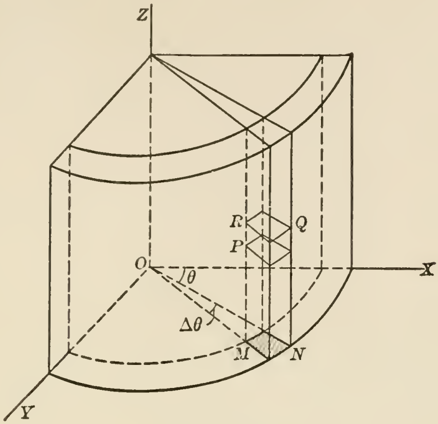

FIG. 53b.

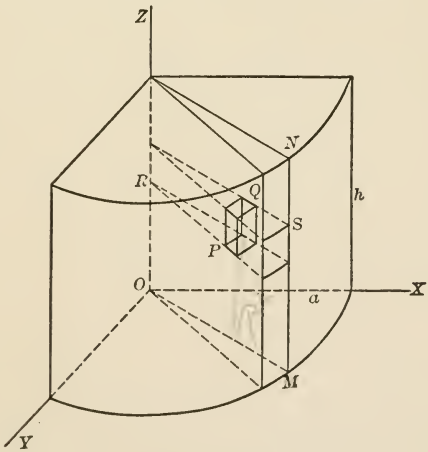

FIG. 53c.

nected with cylinders, cones, and spheres, the resulting integrations are usually much easier in cylindrical than in rectangular coördinates.

Example 1. Find the moment of inertia of a cylinder about a diameter of its base.

Let the moment of inertia be taken about the x-axis, Fig. 53c. The square of the distance from the element PQ to the x-axis is

$$R^2 = y^2 + z^2 = r^2 \sin^2\theta + z^2.$$

The moment of inertia is therefore

$$\int R^2\, dv = \int_0^{2\pi}\int_0^{h}\int_0^{a} (r^2 \sin^2\theta + z^2)\, r\, d\theta\, dz\, dr$$
$$= \frac{\pi a^2 h}{12}(3\,a^2 + 4\,h^2).$$

The first integration is a summation for elements in the wedge RS, the second a summation for wedges in the slice OMN, the third a summation for all such slices.

Ex. 2. Find the volume bounded by the xy-plane, the cylinder $x^2 + y^2 = ax$, and the sphere $x^2 + y^2 + z^2 = a^2$.

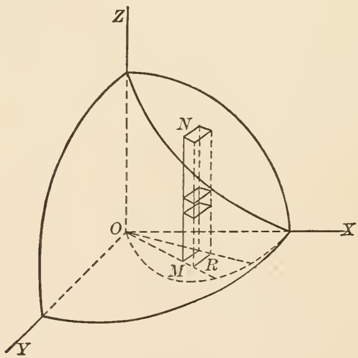

FIG. 53d.

In cylindrical coördinates, the equations of the cylinder and sphere are $r = a\cos\theta$ and $r^2 + z^2 = a^2$. The volume required is therefore

$$v = 2\int_0^{\frac{\pi}{2}}\int_0^{a\cos\theta}\int_0^{\sqrt{a^2-r^2}} r\, d\theta\, dr\, dz = \tfrac{1}{9}\, a^3\,(3\,\pi - 4).$$

54. Spherical Coördinates. — The spherical coördinates of the point P (Fig. 54a) are $r = OP$ and the two angles θ and ϕ. From the diagram it is easily seen that

$$x = r \sin \phi \cos \theta,$$
$$y = r \sin \phi \sin \theta,$$
$$z = r \cos \phi.$$

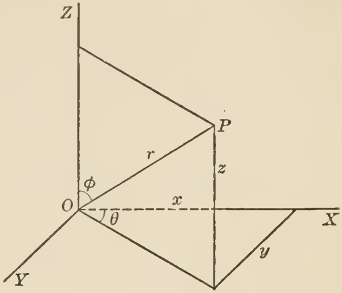

FIG. 54a.

The locus $r =$ const. is a sphere with center at O; $\theta =$ const. is the plane through OZ making the angle θ with OX; $\phi =$ const. is the cone generated by lines through O making the angle ϕ with OZ.

The element of volume is the volume $PQRS$ bounded

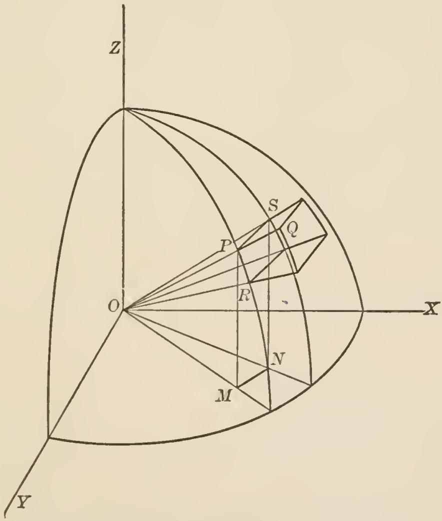

FIG. 54b.

by the spheres r, $r + \Delta r$, the planes θ, $\theta + \Delta\theta$, and the cones ϕ, $\phi + \Delta\phi$. When Δr, $\Delta\phi$, and $\Delta\theta$ are very small this is approximately a rectangular parallelepiped. Since $OP = r$ and $POR = \Delta\phi$,

$$PR = r\,\Delta\phi.$$

Also $OM = OP \sin\phi$ and the arc PS is approximately equal to its projection MN, whence

$$PS = MN = r \sin\phi\,\Delta\theta,$$

approximately. Consequently

$$\Delta v = PR \cdot PS \cdot PQ = r^2 \sin\phi\,\Delta\theta \cdot \Delta\phi \cdot \Delta r,$$

approximately. When the increments are taken smaller and smaller, the result becomes more and more accurate. Therefore

$$dv = r^2 \sin\phi\, d\theta\, d\phi\, dr. \tag{54}$$

Spherical coördinates work best in problems connected with spheres. They are also very useful in problems where the distance from a fixed point plays an important role.

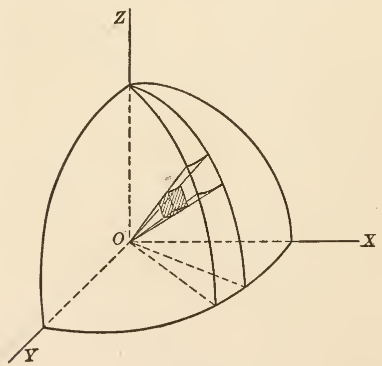

FIG. 54c.

Example. If the density of a solid hemisphere varies as the distance from the center, find its center of gravity.

Take the center of the sphere as origin and let the z-axis be perpendicular to the plane face of the hemisphere. By symmetry it is evident that $\bar{x}$ and $\bar{y}$ are zero. The density is $\rho = kr$, where k is constant. Also $z = r \cos\phi$. Hence

$$\bar{z} = \frac{\int z\,dm}{\int dm} = \frac{\int krz\,dv}{\int kr\,dv}$$

$$= \frac{\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^a r^4 \cos\phi \sin\phi \, d\theta\, d\phi\, dr}{\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^a r^3 \sin\phi \, d\theta\, d\phi\, dr} = \frac{2}{5}a.$$

EXERCISES

1. Find the volume bounded by the sphere $x^2 + y^2 + z^2 = 4$ and the paraboloid $x^2 + y^2 = 3z$.

2. A right cone is scooped out of a right cylinder of the same height and base. Find the distance of the center of gravity of the remainder from the vertex.

3. Find the volume bounded by the surface $z = e^{-(x^2+y^2)}$ and the xy-plane.

4. Find the moment of inertia of a cone about a diameter of its base.

5. Find the volume of the cylinder $x^2 + y^2 = 2ax$ intercepted between the paraboloid $x^2 + y^2 = 2az$ and the xy-plane.

6. Find the center of gravity of the volume common to a sphere of radius a and a cone of vertical angle 2α, the vertex of the cone being at the center of the sphere.

7. Find the center of gravity of the volume bounded by a spherical surface of radius a and two planes passing through its center and including an angle of 60°.

8. The vertex of a cone of vertical angle $\frac{\pi}{2}$ is on the surface of a sphere of radius a. If the axis of the cone is a diameter of the sphere, find the moment of inertia of the volume common to the cone and sphere about this axis.

55. Attraction. — Two particles of masses m_1, m_2, separated by a distance r, attract each other with a force

$$\frac{km_1m_2}{r^2},$$

where k is a constant depending on the units of mass, distance, and force used. A similar law expresses the attraction or repulsion between electric charges.

To find the attraction due to a continuous mass, resolve it into elements. Each of these attracts with a force given by the above law. Since the forces do not all act in the same direction we cannot obtain the total attraction by merely adding the magnitudes of the forces due to the several elements. The forces must be added geometrically. For this purpose we calculate the sum of the components along each coördinate axis. The force having these sums as components is the resultant attraction.

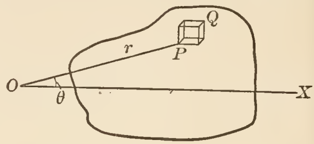

FIG. 55a.

If dm is the mass of an element at P, r its distance from O, and θ the angle between OX and OP, the attraction between this element and a unit particle at O is

$$k\frac{1 \cdot dm}{r^2} = \frac{k\,dm}{r^2}.$$

This force acts along OP. Its component along OX is

$$\frac{\cos\theta \cdot k\,dm}{r^2}.$$

The component along OX of the total attraction is then

$$X = \int \frac{k\cos\theta\,dm}{r^2}.$$

The calculation of this integral may involve single, double, or triple integration, depending on the form of the attracting mass.

Example 1. Find the attraction of a uniform wire of length $2\,l$, and mass M on a particle of unit mass at distance c along the perpendicular at the center of the wire.

Take the origin at the unit particle and the x-axis perpendicular to the wire. Since particles below OX attract downward just as much as those above OX attract upward, the vertical component of the total attraction is zero. The component along OX is, therefore, the total attraction. The mass of the length dy of the wire is

$$\frac{M\,dy}{2l}.$$

Hence

$$X = \frac{kM}{2l}\int \frac{\cos\theta\,dy}{r^2}.$$

For simplicity of integration it is better to use θ as variable. Then $y = c\tan\theta$, $dy = c\sec^2\theta\,d\theta$, and

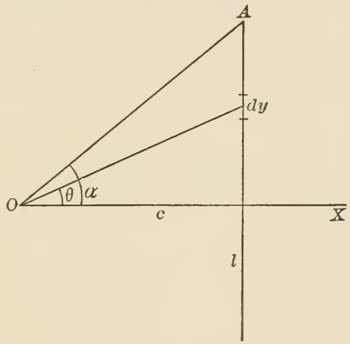

FIG. 55b.

$$X = \frac{kM}{2l}\int_{-\alpha}^{\alpha} \frac{\cos\theta\cdot c\sec^2\theta\,d\theta}{c^2\sec^2\theta} = \frac{kM}{cl}\sin\alpha,$$

where α is the angle XOA. In terms of l this is

$$X = \frac{kM}{c\sqrt{c^2+l^2}}.$$

Ex. 2. Find the attraction of a homogeneous cylinder of mass M upon a particle of unit mass on the axis at distance c from the end of the cylinder.

By symmetry it is clear that the total attraction will act along the axis of the cylinder. Take the origin at the attracting particle and let the y-axis be the axis of the cylinder.

Divide the cylinder into rings generated by rotating the elements $dx\,dy$ about the y-axis. The volume of such a ring is

$$2\pi x\,dx\,dy$$

and its mass is

$$dm = \frac{M}{\pi a^2 h}\cdot 2\pi x\,dx\,dy = \frac{2M}{a^2h}x\,dx\,dy.$$

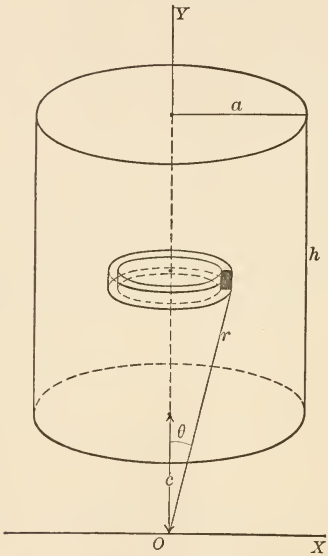

FIG. 55c.

Since all points of this ring are at the same distance from O and the joining lines make the same angle θ with OY, the vertical component of attraction is

$$k\int\frac{\cos\theta\,dm}{r^2} = k\int\frac{y\,dm}{r^3} = \frac{2Mk}{a^2h}\int_c^{c+h}\int_0^a\frac{xy\,dy\,dx}{(x^2+y^2)^{\frac{3}{2}}}$$

$$= \frac{2Mk}{a^2h}\left[h + \sqrt{a^2+c^2} - \sqrt{a^2+(c+h)^2}\right].$$

EXERCISES

1. Find the attraction of a uniform wire of mass M and length l on a particle of unit mass situated in the line of the wire at distance c from its end.

2. Find the attraction of a wire of mass M bent in the form of a semicircle of radius a on a unit particle at its center.

3. Find the attraction of a flat disk of mass M and radius a on a unit particle at the distance c in the perpendicular at the center of the disk.

4. Find the attraction of a homogeneous cone upon a unit particle situated at its vertex.

5. Show that, if a sphere is concentrated at its center, its attraction upon an outside particle will not be changed.

6. Find the attraction of a homogeneous cube upon a particle at one corner.

CHAPTER X

DIFFERENTIAL EQUATIONS

56. Definitions. — A differential equation is an equation containing differentials or derivatives. Thus

$$(x^2 + y^2)\,dx + 2\,xy\,dy = 0,$$

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} = 2$$

are differential equations.

A *solution* of a differential equation is an equation connecting the variables such that the derivatives or differentials calculated from that equation satisfy the differential equation. Thus $y = x^2 - 2\,x$ is a solution of the second equation above; for when $x^2 - 2\,x$ is substituted for y the equation is satisfied.

A differential equation containing only a single independent variable, and so containing only total derivatives, is called an *ordinary* differential equation. An equation containing partial derivatives is called a *partial* differential equation. We shall consider only ordinary differential equations in this book.

The *order* of a differential equation is the order of the highest derivative occurring in it.

57. Illustrations of Differential Equations. — Whenever an equation connecting derivatives or differentials is known, the equation connecting the variables can be determined by solving the differential equation. A number of simple cases were treated in Chapter I.

The fundamental problem of integral calculus is to find the function

$$y = \int f(x)\,dx,$$

when $f(x)$ is given. This is equivalent to solving the differential equation

$$dy = f(x)\,dx.$$

Often the slope of a curve is known as a function of x and y,

$$\frac{dy}{dx} = f(x, y).$$

The equation of the curve can be found by solving the differential equation.

In mechanical problems the velocity or acceleration of a particle may be known in terms of the distance s the particle has moved and the time t,

$$\frac{ds}{dt} = v, \qquad \frac{d^2s}{dt^2} = a.$$

The position s can be determined as a function of the time by solving the differential equation.

In physical or chemical problems the rates of change of the variables may be known as functions of the variables and the time. The values of those variables at any time can be found by solving the differential equations.

Example. Find the curve in which the cable of a suspension bridge hangs.

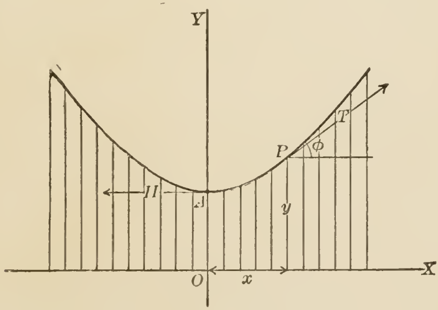

FIG. 57

Let the bridge be the x-axis and let the y-axis pass through the center of the cable. The portion of the cable AP is in equilibrium under three forces, a horizontal tension H at A, a tension PT in the direction of the cable at P, and the weight of the portion of the bridge between A and P. The weight of the cable, being very small in comparison with that of the bridge, is neglected.

The weight of the part of the bridge between A and P is proportional to x. Let it be Kx. Since the vertical components of force must be in equilibrium

$$T \sin \phi = Kx.$$

Similarly, from the equilibrium of horizontal components, we have

$$T \cos \phi = H.$$

Dividing the former equation by this, we get

$$\tan \phi = \frac{K}{H} x.$$

But $\tan \phi = \frac{dy}{dx}$. Hence

$$\frac{dy}{dx} = \frac{K}{H} x.$$

The solution of this equation is

$$y = \frac{K}{2H} x^2 + c.$$

The curve is therefore a parabola.

58. Constants of Integration. Particular and General Solutions. — To solve the equation

$$\frac{dy}{dx} = f(x),$$

we integrate once and so obtain an equation with one arbitrary constant,

$$y = \int f(x)\, dx + c.$$

To solve the equation

$$\frac{d^2y}{dx^2} = f(x)$$

we integrate twice. The result

$$y = \int \int f(x)\, dx^2 + c_1 x + c_2$$

contains two arbitrary constants. Similarly, the integral of the equation

$$\frac{d^n y}{dx^n} = f(x)$$

contains n arbitrary constants.

These illustrations belong to a special type. The rule indicated is, however, general. *The complete, or general, solution of a differential equation of the nth order in two variables contains n arbitrary constants.* If particular values are assigned to any or all of these constants, the result is still a solution. Such a solution is called a *particular* solution.

In most problems leading to differential equations the result desired is a particular solution. To find this we usually find the general solution and then determine the constants from some extra information contained in the statement of the problem.

Example 1. Show that

$$x^2 + y^2 - 2\,cx = 0$$

is the general solution of the differential equation

$$y^2 - x^2 - 2\,xy\frac{dy}{dx} = 0.$$

Differentiating $x^2 + y^2 - 2\,cx = 0$, we get

$$2\,x + 2\,y\frac{dy}{dx} - 2\,c = 0,$$

whence

$$\frac{dy}{dx} = \frac{c - x}{y}.$$

Substituting this value in the differential equation, it becomes

$$y^2 - x^2 - 2\,xy\frac{dy}{dx} = y^2 - x^2 - 2\,x\,(c - x) = y^2 + x^2 - 2\,cx = 0.$$

Hence $x^2 + y^2 - 2\,cx = 0$ is a solution. Since it contains one constant and the differential equation is one of the first order, it is the general solution.

Ex. 2. Find the differential equation of which $y = c_1e^x + c_2e^{2x}$ is the general solution.

Since the given equation contains two constants, the differential equation is one of the second order. We therefore differentiate twice and so obtain

$$\frac{dy}{dx} = c_1e^x + 2\,c_2e^{2x},$$

$$\frac{d^2y}{dx^2} = c_1e^x + 4\,c_2e^{2x}.$$

Eliminating c_1, we get

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = 2\,c_2e^{2x},$$

$$\frac{dy}{dx} - y = c_2e^{2x}.$$

Hence

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = 2\left(\frac{dy}{dx} - y\right)$$

or

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2\,y = 0.$$

This is an equation of the second order having $y = c_1e^x + c_2e^{2x}$ as solution. It is the differential equation required.

EXERCISES

In each of the following exercises, show that the equation given is a solution of the differential equation and state whether it is the general or a particular solution.

1. $y = ce^x + e^{-x}$, $\qquad \dfrac{d^2y}{dx^2} = y.$

2. $x^2 - y^2 = cx,$ $\qquad (x^2 + y^2)\,dx - 2\,xy\,dy = 0.$

3. $y = ce^x \sin x,$ $\qquad \dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2\,y = 0.$

4. $y = c_1 + c_2 \sin (x + c_3),$ $\qquad \dfrac{d^3y}{dx^3} + \dfrac{dy}{dx} = 0.$

Find the differential equation of which each of the following equations is the general solution:

5. $y = c_1x + \dfrac{c_2}{x}.$

6. $y = cxe^x.$

7. $y = c_1 \sin x + c_2 \cos x.$

8. $x^2y = c_1 + c_2 \ln x + c_3x^3.$

9. $x^2 + c_1xy + c_2y^2 = 0.$

59. Differential Equations of the First Order in Two Variables. — By solving for $\dfrac{dy}{dx}$ an equation of the first order in two variables x and y can be reduced to the form

$$\frac{dy}{dx} = f(x, y).$$

To solve this equation is equivalent to finding the curves with slope equal to $f(x, y)$. The solution contains one

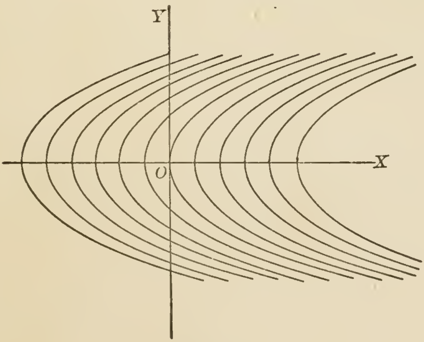

FIG. 59.

arbitrary constant. There is consequently an infinite number of such curves, usually one through each point of the plane.

We cannot always solve even this simple type of equation. In the following articles some cases will be discussed which frequently occur and for which general methods of solution are known.

60. Variables Separable. — A differential equation of the form

$$M\,dx + N\,dy = 0$$

is called separable if each of the coefficients M and N contains only one of the variables or is the product of a function of x and a function of y. By division the x's and dx can be brought together in the first term, the y's and dy in the second. The two terms can then be integrated separately and the sum of the integrals equated to a constant.

Example 1. $(1 + x^2)\,dy - xy\,dx = 0$.

Dividing by $(1 + x^2)\,y$, this becomes

$$\frac{dy}{y} = \frac{x\,dx}{1 + x^2},$$

whence

$$\ln y = \tfrac{1}{2}\ln(1 + x^2) + c.$$

If $c = \ln k$, this is equivalent to

$$\ln y = \ln\sqrt{1 + x^2} + \ln k = \ln k\sqrt{1 + x^2},$$

and so

$$y = k\sqrt{1 + x^2},$$

where k is an arbitrary constant.

Ex. 2. Find the curve in which the area bounded by the curve, coördinate axes, and a variable ordinate is proportional to the arc forming part of the boundary

Let A be the area and s the length of arc. Then

$$A = ks.$$

Differentiating with respect to x,

$$\frac{dA}{dx} = k\frac{ds}{dx},$$

or

$$y = k\sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

Solving for $\frac{dy}{dx}$,

$$\frac{dy}{dx} = \frac{\sqrt{y^2 - k^2}}{k},$$

whence

$$\frac{dy}{\sqrt{y^2 - k^2}} = \frac{dx}{k}.$$

The solution of this is

$$\ln\left(y + \sqrt{y^2 - k^2}\right) = \frac{x}{k} + c.$$

Therefore

$$y + \sqrt{y^2 - k^2} = e^{\frac{x}{k} + c} = e^c e^{\frac{x}{k}} = c_1 e^{\frac{x}{k}},$$

where c_1 is a new constant. Transposing y and squaring, we get

$$y^2 - k^2 = \left(c_1 e^{\frac{x}{k}}\right)^2 - 2\, c_1 e^{\frac{x}{k}} y + y^2.$$

Hence, finally,

$$y = \frac{c_1}{2} e^{\frac{x}{k}} + \frac{k^2}{2\, c_1} e^{-\frac{x}{k}}.$$

61. Exact Differential Equations. — An equation

$$du = 0,$$

obtained by equating to zero the total differential of a function u of x and y, is called an *exact* differential equation. The solution of such an equation is

$$u = c.$$

The condition that $M\,dx + N\,dy$ be an exact differential is (Diff. Cal., Art. 100)

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{61}$$

This equation, therefore, expresses the condition that

$$M\,dx + N\,dy = 0$$

be an exact differential equation.

An exact equation can often be solved by inspection. To find u it is merely necessary to obtain a function whose total differential is $M\,dx + N\,dy$.

If this cannot be found by inspection, it can be determined from the fact that

$$du = M\,dx + N\,dy$$

and so

$$\frac{\partial u}{\partial x} = M.$$

By integrating with y constant, we therefore get

$$u = \int M\,dx + f(y).$$

Since y is constant in the integration, the constant of integration may be a function of y. This function can be found by equating the total differential of u to $M\,dx + N\,dy$. Since $df(y)$ gives terms containing y only, *$f(y)$ can usually be found by integrating the terms in $N\,dy$ that do not contain x.* In exceptional cases this may not give the correct result. The answer should, therefore, be tested by differentiation.

Example 1. $(2x - y)\,dx + (4y - x)\,dy = 0.$

The equation is equivalent to

$$2x\,dx + 4y\,dy - (y\,dx + x\,dy) = d(x^2 + 2y^2 - xy) = 0.$$

It is therefore exact and its solution is

$$x^2 + 2y^2 - xy = c.$$

Ex. 2. $(\ln y - 2x)\,dx + \left(\frac{x}{y} - 2y\right)dy = 0.$

In this case

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(\ln y - 2x) = \frac{1}{y},$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{y} - 2y\right) = \frac{1}{y}.$$

These derivatives being equal, the equation is exact. Its solution is

$$x \ln y - x^2 - y^2 = c.$$

The part $x \ln y - x^2$ is obtained by integrating $(\ln y - 2x)\,dx$ with y constant. The term $-y^2$ is the integral of $-2y\,dy$, which is the only term in $\left(\frac{x}{y} - 2y\right) dy$ that does not contain x.

62. Integrating Factors. — If an equation of the form $M\,dx + N\,dy = 0$ is not exact it can be made exact by multiplying by a proper factor. Such a multiplier is called an *integrating factor*.

For example, the equation

$$x\,dy - y\,dx = 0$$

is not exact. But if it is multiplied by $\frac{1}{x^2}$, it takes the form

$$\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right) = 0$$

which is exact. It also becomes exact when multiplied by $\frac{1}{y^2}$ or $\frac{1}{xy}$. The functions $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{xy}$ are all integrating factors of $x\,dy - y\,dx = 0$.

While an equation of the form $M\,dx + N\,dy = 0$ always has integrating factors, there is no general method of finding them.

Example 1. $y(1 + xy)\,dx - x\,dy = 0$.

This equation can be written

$$y\,dx - x\,dy + xy^2\,dx = 0.$$

Dividing by y^2,

$$\frac{y\,dx - x\,dy}{y^2} + x\,dx = 0.$$

Both terms of this equation are exact differentials. The solution is

$$\frac{x}{y} + \frac{1}{2}x^2 = c.$$

Ex. 2. $(y^2 + 2xy)\,dx + (2x^2 + 3xy)\,dy = 0.$

This is equivalent to

$$y^2\,dx + 3xy\,dy + 2xy\,dx + 2x^2\,dy = 0.$$

Multiplying by y, it becomes

$$y^3\,dx + 3xy^2\,dy + 2xy^2\,dx + 2x^2y\,dy = d\,(xy^3 + x^2y^2) = 0.$$

Hence

$$xy^3 + x^2y^2 = c.$$

63. Linear Equations. — A differential equation of the form

$$\frac{dy}{dx} + Py = Q, \qquad \text{(63a)}$$

where P and Q are functions of x or constants, is called *linear*. The linear equation is one of the first degree in one of the variables (y in this case) and its derivative. Any functions of the other variable can occur.

If the linear equation is written in the form (63a),

$$e^{\int P\,dx}$$

is an integrating factor; for when multiplied by this factor the equation becomes

$$e^{\int P\,dx}\frac{dy}{dx} + ye^{\int P\,dx}P = e^{\int P\,dx}Q.$$

The left side is the derivative of

$$ye^{\int P\,dx}.$$

Hence

$$ye^{\int P\,dx} = \int e^{\int P\,dx}Q\,dx + c \qquad \text{(63b)}$$

is the solution.

Example 1. $\dfrac{dy}{dx} + \dfrac{2}{x}y = x^3.$

In this case

$$\int P\,dx = \int \frac{2}{x}\,dx = 2\ln x = \ln x^2.$$

Hence

$$e^{\int P\,dx} = e^{\ln x^2} = x^2.$$

The integrating factor is, therefore, x^2. Multiplying by x^2 and changing to differentials, the equation becomes

$$x^2\,dy + 2\,xy\,dx = x^5\,dx.$$

The integral is

$$x^2y = \tfrac{1}{6}\,x^6 + c.$$

Ex. 2. $(1 + y^2)\,dx - (xy + y + y^3)\,dy = 0.$

This is an equation of the first degree in x and dx. Dividing by $(1 + y^2)\,dy$, it becomes

$$\frac{dx}{dy} - \frac{y}{1 + y^2}\,x = y.$$

P is here a function of y and

$$\int P\,dy = \int \frac{-y\,dy}{1 + y^2} = -\frac{1}{2}\ln(1 + y^2) = \ln\frac{1}{\sqrt{1 + y^2}},$$

$$e^{\int P\,dy} = \frac{1}{\sqrt{1 + y^2}}.$$

Multiplying by the integrating factor, the equation becomes

$$\frac{dx}{\sqrt{1 + y^2}} - \frac{xy\,dy}{(1 + y^2)^{\frac{3}{2}}} = \frac{y\,dy}{\sqrt{1 + y^2}},$$

whence

$$\frac{x}{\sqrt{1 + y^2}} = \sqrt{1 + y^2} + c$$

and

$$x = 1 + y^2 + c\sqrt{1 + y^2}.$$

64. Equations Reducible to Linear Form. — An equation of the form

$$\frac{dy}{dx} + Py = Qy^n, \tag{64}$$

where P and Q are functions of x, can be made linear by a change of variable. Dividing by y^n, it becomes

$$y^{-n}\frac{dy}{dx} + Py^{-n+1} = Q.$$

If we take

$$y^{1-n} = u$$

as a new variable, the equation takes the form

$$\frac{1}{1-n}\frac{du}{dx} + Pu = Q,$$

which is linear.

Example. $\dfrac{dy}{dx} + \dfrac{2}{x}y = \dfrac{y^3}{x^3}.$

Division by y^3 gives

$$y^{-3}\frac{dy}{dx} + \frac{2}{x}y^{-2} = \frac{1}{x^3}.$$

Let

$$u = y^{-2}.$$

Then

$$\frac{du}{dx} = -2y^{-3}\frac{dy}{dx},$$

whence

$$y^{-3}\frac{dy}{dx} = -\frac{1}{2}\frac{du}{dx}.$$

Substituting these values, we get

$$-\frac{1}{2}\frac{du}{dx} + \frac{2}{x}u = \frac{1}{x^3},$$

and so

$$\frac{du}{dx} - \frac{4}{x}u = -\frac{2}{x^3}.$$

This is a linear equation with solution

$$u = \frac{1}{3x^2} + cx^4,$$

or, since $u = y^{-2}$,

$$\frac{1}{y^2} = \frac{1}{3x^2} + cx^4.$$

65. Homogeneous Equations. — A function $f(x, y)$ is said to be a homogeneous function of the nth degree if

$$f(tx, ty) = t^n f(x, y).$$

Thus $\sqrt{x^2 + y^2}$ is a homogeneous function of the first degree; for

$$\sqrt{x^2t^2 + y^2t^2} = t\sqrt{x^2 + y^2}.$$

It is easily seen that a polynomial whose terms are all of the nth degree is a homogeneous function of the nth degree.

The differential equation

$$M\,dx + N\,dy = 0$$

is called homogeneous if M and N are homogeneous functions of the same degree. To solve a homogeneous equation substitute

$$y = vx.$$

The new equation will be separable.

Example 1. $x\dfrac{dy}{dx} - y = \sqrt{x^2 + y^2}.$

This is a homogeneous equation of the first degree. Substituting $y = vx$, it becomes

$$x\left(v + x\frac{dv}{dx}\right) - vx = \sqrt{x^2 + v^2x^2}$$

whence

$$x\,\frac{dv}{dx} = \sqrt{1 + v^2}.$$

This is a separable equation with solution

$$x = c\left(v + \sqrt{1 + v^2}\right).$$

Replacing v by $\dfrac{y}{x}$, transposing, squaring, etc., the equation becomes

$$x^2 - 2\,cy = c^2.$$

Ex. 2. $y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0.$

Solving for $\frac{dy}{dx}$, we get

$$\frac{dy}{dx} = \frac{-x \pm \sqrt{x^2 + y^2}}{y},$$

or

$$y\,dy + x\,dx = \pm\sqrt{x^2 + y^2}\,dx.$$

This is a homogeneous equation of the first degree. It is much easier, however, to divide by $\sqrt{x^2 + y^2}$ and integrate at once. The result is

$$\frac{x\,dx + y\,dy}{\sqrt{x^2 + y^2}} = \pm\,dx,$$

whence

$$\sqrt{x^2 + y^2} = c \pm x$$

and

$$y^2 = c^2 \pm 2\,cx.$$

Since c may be either positive or negative, the answer can be written

$$y^2 = c^2 + 2\,cx.$$

66. Change of Variable. — We have solved the homogeneous equation by taking as new variable

$$v = \frac{y}{x}.$$

It may be possible to reduce any equation to a simpler form by taking some function u of x and y as a new variable or by taking two functions u and v as new variables. Such functions are often suggested by the equation. In other cases they may be indicated by the problem in the solution of which the equation occurs.

Example. $(x - y)^2\frac{dy}{dx} = a^2.$

Let $x - y = u$. Then

$$1 - \frac{dy}{dx} = \frac{du}{dx}$$

and the differential equation becomes

$$u^2 \left(1 - \frac{du}{dx}\right) = a^2,$$

whence

$$u^2 - a^2 = u^2 \frac{du}{dx}.$$

The variables are separable. The solution is

$$x = u + \frac{a}{2} \ln \frac{u - a}{u + a} + c$$

$$= x - y + \frac{a}{2} \ln \frac{x - y - a}{x - y + a} + c,$$

or

$$y = \frac{a}{2} \ln \frac{x - y - a}{x - y + a} + c.$$

EXERCISES

Solve the following differential equations:

1. $x^3\, dy - y^3\, dx = 0.$
2. $\tan x \sin^2 y\, dx + \cos^2 x \cot y\, dy = 0.$
3. $(xy^2 + x)\, dx + (y - x^2 y)\, dy = 0.$
4. $(xy^2 + x)\, dx + (x^2 y - y)\, dy = 0.$
5. $(3x^2 + 2xy - y^2)\, dx + (x^2 - 2xy - 3y^2)\, dy = 0.$
6. $x\frac{dy}{dx} - y = y^3.$
7. $x\,dx + y\,dy = a(x^2 + y^2)\, dy.$
8. $x\frac{dy}{dx} + y = y^2.$
9. $\frac{dy}{dx} - ay = e^{bx}.$
10. $x^2 \frac{dy}{dx} - 2xy = 3.$
11. $x^2 \frac{dy}{dx} - 2xy = 3y.$
12. $(2xy^2 - y)\, dx + x\, dy = 0.$
13. $(1 - x^2)\frac{dy}{dx} + 2xy = (1 - x^2)^2.$
14. $\tan x \frac{dy}{dx} - y = a.$
15. $x\frac{dy}{dx} - 3y + x^4 y^2 = 0.$
16. $\frac{dy}{dx} + y = xy^3.$

17. $(x^2 - 1)^{\frac{3}{2}}\,dy + \left(x^3 + 3\,xy\sqrt{x^2 - 1}\right)dx = 0.$

18. $x\,dx + (x + y)\,dy = 0.$

19. $(x^2 + y^2)\,dx - 2\,xy\,dy = 0.$

20. $y\,dx + (x + y)\,dy = 0.$

21. $(x^3 - 3\,x^2y)\,dx + (y^3 - x^3)\,dy = 0.$

22. $ye^y\,dx = (y^3 + 2\,xe^y)\,dy.$

23. $\left(xye^{\frac{x}{y}} + y^2\right)dx - x^2e^{\frac{x}{y}}\,dy = 0.$

24. $(x + y - 1)\,dx + (2\,x + 2\,y - 3)\,dy = 0.$

25. $3\,y^2\dfrac{dy}{dx} - y^3 = x.$

26. $e^y\left(\dfrac{dy}{dx} + 1\right) = e^x.$

27. $x\left(\dfrac{dy}{dx}\right)^2 - 2\,y\dfrac{dy}{dx} - x = 0.$

28. $\left(\dfrac{dy}{dx}\right)^2 - (x + y)\dfrac{dy}{dx} + xy = 0.$

29. $y^2\left(\dfrac{dy}{dx}\right)^2 + 2\,xy\dfrac{dy}{dx} - y^2 = 0.$

30. The differential equation for the charge q of a condenser having a capacity C connected in series with a circuit of resistance R is

$$R\frac{dq}{dt} + \frac{q}{C} = E,$$

where E is the electromotive force. Find v as a function of t if E is constant and $q = 0$ when $t = 0$.

31. The differential equation for the current induced by an electromotive force $E \sin \alpha t$ in a circuit having the inductance L and resistance R is

$$L\frac{di}{dt} + Ri = E \sin \alpha t.$$

Solve for i and determine the constants so that $i = I$ when $t = 0$.

Let PT be the tangent and PN the normal to a plane curve at $P\,(x, y)$ (Fig. 66a). Determine the curve or curves in each of the following cases:

32. The subtangent $TM = 3$ and the curve passes through (2, 2).

33. The subnormal $MN = a$ and the curve passes through (0, 0).

34. The intercept OT of the tangent on the x-axis is one-half the abscissa OM.

35. The length PT of the tangent is a constant a.

36. The length PN of the normal is a constant a.

37. The perpendicular from M to PT is a constant a.

Using polar coördinates (Fig. 66b), find the curve or curves in each of the following cases:

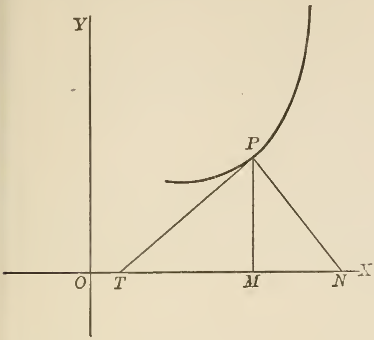

FIG. 66a.

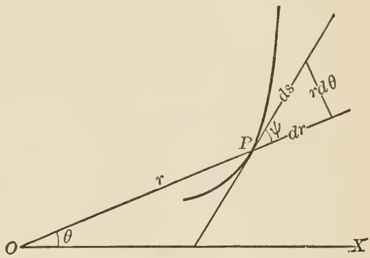

FIG. 66b.

38. The curve passes through (1, 0) and makes with OP a constant angle $\psi = \frac{\pi}{4}$.

39. The angles ψ and θ are equal.

40. The distance from O to the tangent is a constant a.

41. The projection of OP on the tangent at P is a constant a.

42. Find the curve passing through the origin in which the area bounded by the curve, x-axis, a fixed, and a variable ordinate is proportional to that ordinate.

43. Find the curve in which the length of arc is proportional to the angle between the tangents at its end.

44. Find the curve in which the length of arc is proportional to the difference of the abscissas at its ends.

45. Find the curve in which the length of any arc is proportional to the angle it subtends at a fixed point.

46. Find the curve in which the length of arc is proportional to the difference of the distances of its ends from a fixed point.

47. Oxygen flows through one tube into a liter flask filled with air while the mixture of oxygen and air escapes through another. If the action is so slow that the mixture in the flask may be considered uniform, what percentage of oxygen will the flask contain after 10 liters of gas have passed through? (Assume that air contains 21 per cent by volume of oxygen.)

67. Certain Equations of the Second Order. — There are two forms of the second order differential equation that occur in mechanical problems so frequently that they deserve special attention. These are

$$(1) \qquad \frac{d^2y}{dx^2} = f\left(x, \frac{dy}{dx}\right),$$

$$(2) \qquad \frac{d^2y}{dx^2} = f\left(y, \frac{dy}{dx}\right).$$

The peculiarity of these equations is that one of the variables (y in the first, x in the second) does not appear directly in the equation. They are both reduced to equations of the first order by the substitution

$$\frac{dy}{dx} = p.$$

This substitution reduces the first equation to the form

$$\frac{dp}{dx} = f(x, p).$$

This is a first order equation whose solution has the form

$$p = F(x, c_1),$$

or, since $p = \dfrac{dy}{dx}$,

$$\frac{dy}{dx} = F(x, c_1).$$

This is again an equation of the first order. Its solution is the result required.

In case of an equation of the second type, write the second derivative in the form

$$\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy} \cdot \frac{dy}{dx} = p\frac{dp}{dy}.$$

The differential equation then becomes

$$p\frac{dp}{dy} = f(y, p).$$

Solve this for p and proceed as before.

Example 1. $(1+x^2)\dfrac{d^2y}{dx^2}+1+\left(\dfrac{dy}{dx}\right)^2=0.$

Substituting p for $\dfrac{dy}{dx}$, we get

$$(1+x^2)\frac{dp}{dx}+1+p^2=0.$$

This is a separable equation with solution

$$p=\frac{c_1-x}{1+c_1x},$$

whence

$$dy=\frac{c_1-x}{1+c_1x}\,dx.$$

The integral of this is

$$y=-\frac{x}{c_1}+\frac{c_1^2+1}{c_1^2}\ln(1+c_1x)+c_2.$$

By a change of constants this becomes

$$y=cx+(1+c^2)\ln(c-x)+c'.$$

Ex. 2. $y\dfrac{d^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)^2=1.$

Substituting

$$\frac{dy}{dx}=p,\qquad \frac{d^2y}{dx^2}=p\frac{dp}{dy},$$

we get

$$yp\frac{dp}{dy}+p^2=1.$$

The solution of this is

$$y^2p^2=y^2+c_1.$$

Replacing p by $\dfrac{dy}{dx}$ and solving again, we get

$$y^2+c_1=(x+c_2)^2.$$

Ex. 3. Under the action of gravitation the acceleration of a falling body is $\frac{k}{r^2}$, where k is constant and r the distance from the center of the earth. Find the time required for the body to fall to the earth from a distance equal to that of the moon.

Let r_1 be the radius of the earth (about 4000 miles), r_2 the distance from the center of the earth to the moon (about 240,000 miles) and g the acceleration of gravity at the surface of the earth (about 32 feet per second). At the surface of the earth $r = r_1$ and

$$a = \frac{k}{{r_1}^2} = -g.$$

The negative sign is used because the acceleration is toward the origin ($r = 0$). Hence $k = -gr_1^2$ and the general value of the acceleration is

$$a = \frac{v\,dv}{dr} = -\frac{gr_1^2}{r^2},$$

where v is the velocity. The solution of this equation is

$$v^2 = \frac{2\,gr_1^2}{r} + C.$$

When $r = r_2$, $v = 0$. Consequently,

$$C = -2\,g\frac{r_1^2}{r_2^{\cancel{2}}}$$

and

$$v = \frac{dr}{dt} = -\sqrt{2\,gr_1^2\left(\frac{1}{r} - \frac{1}{r_2}\right)}.$$

The time of falling is therefore

$$t = \int_{r_1}^{r_2}\sqrt{\frac{rr_2}{2\,gr_1^2\,(r_2 - r)}}\,dr = 116 \text{ hours.}$$

This result is obtained by using the numerical values of r_1 and r_2 and reducing g to miles per hour.

68. Linear Differential Equations with Constant Coefficients. — A differential equation of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_n y = f(x), \quad (68a)$$

where $a_1, a_2, \ldots a_n$ are constants, is called a linear differential equation with constant coefficients. For practical applications this is the most important type of differential equation.

In discussing these equations we shall find it convenient to represent the operation $\frac{d}{dx}$ by D. Then

$$\frac{dy}{dx} = Dy, \qquad \frac{d^2 y}{dx^2} = D^2 y, \text{ etc.}$$

Equation (68a) can be written

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n) y = f(x). \quad (68b)$$

This signifies that if the operation

$$D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n \quad (68c)$$

is performed on y, the result will be $f(x)$. The operation consists in differentiating y, n times, $n - 1$ times, $n - 2$ times, etc., multiplying the results by 1, a_1, a_2, etc., and adding.

With the differential equation is associated an algebraic equation

$$r^n + a_1 r^{n-1} + a_2 r^{n-2} + \cdots + a_n = 0.$$

If the roots of this *auxiliary* equation are $r_1, r_2, \ldots r_n$, the polynomial (68c) can be factored in the form

$$(D - r_1)(D - r_2) \cdots (D - r_n). \quad (68d)$$

If we operate on y with $D - a$, we get

$$(D - a) y = \frac{dy}{dx} - ay.$$

If we operate on this with $D - b$, we get

$$(D - b) \cdot (D - a)\, y = (D - b)\left(\frac{dy}{dx} - ay\right)$$
$$= \frac{d^2y}{dx^2} - (a + b)\frac{dy}{dx} + ab.$$

The same result is obtained by operating on y with

$$(D - a)(D - b) = D^2 - (a + b)\, D + ab.$$

Similarly, if we operate in succession with the factors of (68d), we get the same result that we should get by operating directly with the product (68c).

69. Equation with Right Hand Member Zero. — To solve the equation

$$(D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_n)\, y = 0 \qquad (69a)$$

factor the symbolic operator and so reduce the equation to the form

$$(D - r_1)(D - r_2) \cdots (D - r_n)\, y = 0.$$

The value $y = c_1e^{r_1x}$ is a solution; for

$$(D - r_1)\, c_1e^{r_1x} = c_1r_1e^{r_1x} - r_1c_1e^{r_1x} = 0$$

and the equation can be written

$$(D-r_2) \cdots (D-r_n) \cdot (D-r_1)\, y = (D-r_2) \cdots (D-r_n) \cdot 0 = 0.$$

Similarly, $y = c_2e^{r_2x}$, $y = c_3e^{r_3x}$, etc., are solutions. Finally

$$y = c_1e^{r_1x} + c_2e^{r_2x} + \cdots + c_ne^{r_nx} \qquad (69b)$$

is a solution; for the result of operating on y is the sum of the results of operating on $c_1e^{r_1x}$, $c_2e^{r_2x}$, etc., each of which is zero.

If the roots $r_1, r_2, \ldots, r_n$ are all different, (69b) contains n constants and so is the complete solution of (69a). If, however, two roots r_1 and r_2 are equal

$$c_1e^{r_1x} + c_2e^{r_2x} = (c_1 + c_2)\, e^{r_1x}$$

contains only one abitrary constant $c_1 + c_2$ and (69b) contains only $n - 1$ arbitrary constants. In this case, however, xe^{r_1x} is also a solution; for

$$(D - r_1)\, xe^{r_1x} = r_1xe^{r_1x} + e^{r_1x} - r_1xe^{r_1x} = e^{r_1x}$$

and so

$$(D - r_1)^2\, xe^{r_1x} = (D - r_1)\, e^{r_1x} = 0.$$

If then two roots r_1 and r_2 are equal, the part of the solution corresponding to these roots is

$$(c_1 + c_2x)e^{r_1x}.$$

More generally, if m roots $r_1, r_2, \ldots r_m$ are equal, the part of the solution corresponding to them is

$$(c_1 + c_2x + c_3x^2 + \cdots + c_mx^{m-1})e^{r_1x}. \qquad (69c)$$

If the coefficients $a_1, a_2, \ldots a_n$, are real, imaginary roots occur in pairs

$$r_1 = \alpha + \beta\sqrt{-1}, \qquad r_2 = \alpha - \beta\sqrt{-1}.$$

The terms $c_1e^{r_1x}$, $c_2e^{r_2x}$ are imaginary but they can be replaced by two other terms that are real. Using these values of r_1 and r_2, we have

$$(D - r_1)\,(D - r_2) = (D - \alpha)^2 + \beta^2.$$

By performing the differentiations it can easily be verified that

$$[(D - \alpha)^2 + \beta^2] \cdot e^{\alpha x} \sin \beta x = 0,$$
$$[(D - \alpha)^2 + \beta^2] \cdot e^{\alpha x} \cos \beta x = 0.$$

Therefore

$$e^{\alpha x}\,[c_1 \sin \beta x + c_2 \cos \beta x] \qquad (69d)$$

is a solution. This function, in which α and β are real, can, therefore, be used as the part of the solution corresponding to two imaginary roots $r = \alpha \pm \beta\sqrt{-1}$.

To solve the differential equation

$$(D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_n)\, y = 0,$$

let $r_1, r_2, \ldots, r_n$ *be the roots of the auxiliary equation*

$$r^n + a_1 r^{n-1} + a_2 r^{n-2} + \cdots + a_n = 0.$$

If these roots are all real and different, the solution of the equation is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_n e^{r_n x}.$$

If m *of the roots* $r_1, r_2, \ldots, r_m$ *are equal, the corresponding part of the solution is*

$$(c_1 + c_2 x + c_3 x^2 + \cdots + c_m x^{m-1})\, e^{r_1 x}.$$

The part of the solution corresponding to two imaginary roots $r = \alpha \pm \beta\sqrt{-1}$ *is*

$$e^{\alpha x}\,[c_1 \sin \beta x + c_2 \cos \beta x].$$

Example 1. $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 2\,y = 0.$

This is equivalent to

$$(D^2 - D - 2)\,y = 0.$$

The roots of the auxiliary equation

$$r^2 - r - 2 = 0$$

are -1 and 2. Hence the solution is

$$y = c_1 e^{-x} + c_2 e^{2x}.$$

Ex. 2. $\dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 3\,y = 0.$

The roots of the auxiliary equation

$$r^3 + r^2 - 5\,r + 3 = 0$$

are $1, 1, -3$. The part of the solution corresponding to the two roots equal to 1 is

$$(c_1 + c_2 x)\, e^x.$$

Hence

$$y = (c_1 + c_2 x)\, e^x + c_3 e^{-3x}.$$

Ex. 3. $(D^2 + 2D + 2)y = 0.$

The roots of the auxiliary equation are

$$-1 \pm \sqrt{-1}.$$

Therefore $\alpha = -1$, $\beta = 1$ in (69d) and

$$y = e^{-x}[c_1 \sin x + c_2 \cos x].$$

70. Equation with Right Hand Member a Function of x.— Let $y = u$ be the *general solution* of the equation

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n)y = 0$$

and let $y = v$ be *any solution* of the equation

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots a_n)y = f(x). \qquad (70)$$

Then

$$y = u + v$$

is a solution of (70); for the operation

$$D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n$$

when performed on u gives zero and when performed on v gives $f(x)$. Furthermore, $u + v$ contains n arbitrary constants. Hence it is the general solution of (70).

The part u is called the *complementary function*, v the *particular integral*. To solve an equation of the form (70), first solve the equation with right hand member zero and then add to the result any solution of (70).

A particular integral can often be found by inspection. If not, the general form of the integral can usually be determined by the following rules:

1. If $f(x) = ax^n + a_1 x^{n-1} + \cdots + a_n$, assume

$$y = Ax^n + A_1 x^{n-1} + \cdots + A_n.$$

But, if 0 occurs m times as a root in the auxiliary equation, assume

$$y = x^m[Ax^n + A_1 x^{n-1} + \cdots + A_m].$$

2. If $f(x) = ce^{ax}$, assume

$$y = Ae^{ax}.$$

But, if a occurs m times as a root of the auxiliary equation, assume

$$y = Ax^m e^{ax}.$$

3. If $f(x) = a \cos \beta x + b \sin \beta x$, assume

$$y = A \cos \beta x + B \sin \beta x.$$

But, if $\cos \beta x$ and $\sin \beta x$ occur in the complementary function, assume

$$y = x [A \cos \beta x + B \sin \beta x].$$

4. If $f(x) = ae^{\alpha x} \cos \beta x + be^{\alpha x} \sin \beta x$, assume

$$y = Ae^{\alpha x} \cos \beta x + \beta e^{\alpha x} \sin \beta x.$$

But, if $e^{\alpha x} \cos \beta x$ and $e^{\alpha x} \sin \beta x$ occur in the complementary function, assume

$$y = xe^{\alpha x} [A \cos \beta x + B \sin \beta x].$$

If $f(x)$ contains terms of different types, take for y the sum of the corresponding expressions. Substitute the value of y in the differential equation and determine the constants so that the equation is satisfied.

Example 1. $\dfrac{d^2y}{dx^2} + 4y = 2x + 3.$

A particular solution is evidently

$$y = \tfrac{1}{4}(2x + 3).$$

Hence the complete solution is

$$y = c_1 \cos 2x + c_2 \sin 2x + \tfrac{1}{4}(2x + 3).$$

Ex. 2. $(D^2 + 3D + 2)y = 2 + e^x.$

Substituting $y = A + Be^x$, we get

$$2A + 6Be^x = 2 + e^x.$$

Hence

$$2A = 2, \qquad 6B = 1$$

and

$$y = 1 + \tfrac{1}{6}e^x + c_1 e^{-x} + c_2 e^{-2x}.$$

Ex. 3. $\dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} = x^2.$

The roots of the auxiliary equation are 0, 0, -1. Since 0 is twice a root, we assume

$$y = x^2 (Ax^2 + Bx + C) = Ax^4 + Bx^3 + Cx^2.$$

Substituting this value,

$$12\,Ax^2 + (24\,A + 6\,B)\,x + 6\,B + 2\,C = x^2.$$

Consequently,

$$12\,A = 1, \qquad 24\,A + 6\,B = 0, \qquad 6\,B + 2\,C = 0,$$

whence

$$A = \tfrac{1}{12}, \qquad B = -\tfrac{1}{3}, \qquad C = 1.$$

The solution is

$$y = \tfrac{1}{12}\,x^4 - \tfrac{1}{3}\,x^3 + x^2 + c_1 + c_2x + c_3e^{-x}.$$

71. Simultaneous Equations. — We consider only linear equations with constant coefficients containing one independent variable and as many dependent variables as equations. All but one of the dependent variables can be eliminated by a process analogous to that used in solving linear algebraic equations. The one remaining dependent variable is the solution of a linear equation. Its value can be found and the other functions can then be determined by substituting this value in the previous equations.

Example.

$$\frac{dx}{dt} + 2\,x - 3\,y = t,$$

$$\frac{dy}{dt} - 3\,x + 2\,y = e^{2t}.$$

Using D for $\dfrac{d}{dt}$, these equations can be written

$$(D + 2)\,x - 3\,y = t,$$
$$(D + 2)\,y - 3\,x = e^{2t}.$$

To eliminate y, multiply the first equation by $D + 2$ and the second by 3. The result is

$$(D + 2)^2\,x - 3\,(D + 2)\,y = 1 + 2\,t,$$
$$3\,(D + 2)\,y - 9\,x = 3\,e^{2t}.$$

Adding, we get

$$[(D+2)^2 - 9]\,x = 1 + 2\,t + 3\,e^{2\,t}.$$

The solution of this equation is

$$x = -\tfrac{2}{5}\,t - \tfrac{13}{25} + \tfrac{3}{7}\,e^{2\,t} + c_1e^t + c_2e^{-5\,t}.$$

Substituting this value in the first equation, we find

$$y = \tfrac{1}{3}\,(D+2)\,x - \tfrac{1}{3}\,t = -\tfrac{3}{5}\,t - \tfrac{12}{25} + \tfrac{4}{7}\,e^{2\,t} + c_1e^t - c_2e^{-5\,t}.$$

EXERCISES

Solve the following equations:

1. $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + x = 0.$

2. $(x+1)\dfrac{d^2y}{dx^2} - (x+2)\dfrac{dy}{dx} + x + 2 = 0.$

3. $\dfrac{d^2y}{dx^2} = a^2y.$

4. $\dfrac{d^2y}{dx^2} = -a^2y.$

5. $\dfrac{d^2s}{dt^2} = -\dfrac{k}{s^2}.$

6. $\dfrac{d^2s}{dt^2} + a^2\left(\dfrac{ds}{dt}\right)^2 = b^2.$

7. $x\dfrac{d^2y}{dx^2} = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}.$

8. $y\dfrac{d^2y}{dx^2} = 1 + \left(\dfrac{dy}{dx}\right)^2.$

9. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} = 0.$

10. $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} - 6\,y = 0.$

11. $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9\,y = 0.$

12. $\dfrac{d^2y}{dx^2} + y = 0.$

13. $\dfrac{d^3y}{dx^3} - 2\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} = 0.$

14. $\dfrac{d^4y}{dx^4} = y.$

15. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 3\,y = 0.$

16. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = 0.$

17. $\dfrac{d^4y}{dx^4} - 3\dfrac{d^2y}{dx^2} + 2\,y = 0.$

18. $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - y = 0.$

19. $\dfrac{d^2y}{dx^2} + y = x + 3.$

20. $\dfrac{d^2y}{dx^2} - 4\,y = e^x.$

21. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 6\,y = x^2.$

22. $\dfrac{dy}{dx} - y = \sin x.$

23. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} = 2\,x - 3.$

24. $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 5\,y = x + e^{3x}.$

25. $\dfrac{d^2y}{dx^2} - a^2y = e^{ax}.$

26. $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + y = \cos 2\,x.$

27. $\dfrac{d^3y}{dx^3} - y = x^3 - x^2.$

28. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 3\,y = e^{2x}\sin x.$

29. $\dfrac{d^2y}{dx^2} - 9y = e^{3x}\cos x.$

30. $\dfrac{d^4y}{dx^4} + \dfrac{d^3y}{dx^3} = \cos 4x.$

31. $\dfrac{d^2y}{dx^2} + 4y = \cos 2x.$

32. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = e^x + e^{-x}.$

33. $\dfrac{dy}{dt} + x = e^t, \quad \dfrac{dx}{dt} - y = e^{-t}.$

34. $\dfrac{dx}{dt} = x - 2y + 1, \quad \dfrac{dy}{dt} = x - y + 2.$

35. $4\dfrac{dx}{dt} - \dfrac{dy}{dt} + 3x = \sin t, \quad \dfrac{dx}{dt} + y = \cos t.$

36. $\dfrac{d^2y}{dt^2} = x, \quad \dfrac{d^2x}{dt^2} = y.$

37. Solve the equation

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 1$$

and determine the constants so that $y = 0$ and $\dfrac{dy}{dx} = 1$ when $x = 0$.

38. Solve $\dfrac{d^2y}{dx^2} = 3\sqrt{y}$ under the hypothesis that $y = 1$ and $\dfrac{dy}{dx} = 2$ when $x = 0$.

39. When a body sinks slowly in a liquid, its acceleration and velocity approximately satisfy the equation

$$a = g - kv,$$

g and k being constants. Find the distance passed over as a function of the time if the body starts from rest.

40. The acceleration and velocity of a body falling in the air approximately satisfy the equation $a = g - kv^2$, g and k being constants. Find the distance traversed as a function of the time if the body falls from rest.

41. A weight supported by a spiral spring is lifted a distance b and let fall. Its acceleration is given by the equation $a = -k^2s$, k being constant and s the displacement from the position of equilibrium. Find s in terms of the time t.

42. Find the velocity with which a meteor strikes the earth, assuming that it starts from rest at an indefinitely great distance and moves toward the earth with an acceleration inversely proportional to the square of its distance from the center.

43. A body falling in a hole through the center of the earth would have an acceleration toward the center proportional to its distance from the center. If the body starts from rest at the surface, find the time required to fall through.

44. A chain 5 feet long starts with one foot of its length hanging over the edge of a smooth table. The acceleration of the chain will be proportional to the amount over the edge. Find the time required to slide off.

45. A chain hangs over a smooth peg, 8 feet of its length being on one side and 10 on the other. Its acceleration will be proportional to the difference in length of the two sides. Find the time required to slide off.

SUPPLEMENTARY EXERCISES

CHAPTER II

1. $\int \frac{x\,dx}{a + bx^2}$.

2. $\int (a + bx)^2\,dx$.

3. $\int \frac{a + bx}{p + qx}\,dx$.

4. $\int x\sqrt{2 - 3x^2}\,dx$.

5. $\int \frac{(x + a)\,dx}{\sqrt{x^2 + 2ax + b}}$.

6. $\int \sqrt{\frac{x - 1}{x}}\,\frac{dx}{x^2}$.

7. $\int (x - 1)(x^2 - 2x)^{\frac{3}{2}}\,dx$.

8. $\int \frac{dx}{\sin ax}$.

9. $\int \frac{\sin ax}{\cos^2 ax}\,dx$.

10. $\int \frac{\cos 2x}{1 - \sin 2x}\,dx$.

11. $\int \frac{\sec^2 x \tan x\,dx}{a + b\sec^2 x}$.

12. $\int \frac{dx}{\sec x}$.

13. $\int \frac{\cot x\,dx}{1 - \sin x}$.

14. $\int \frac{\sin ax}{\cos ax}\,dx$.

15. $\int \frac{dx}{\cos x - \sin x}$.

16. $\int \frac{dx}{\sec x - \tan x}$.

17. $\int \frac{dx}{\sin^2 ax \cos^2 ax}$.

18. $\int \frac{\cos x\,dx}{\cos x + \sin x}$.

19. $\int \frac{dx}{\sqrt{\tan^2 x + 2}}$.

20. $\int \frac{dx}{x\sqrt{x^n - 1}}$.

21. $\int \frac{\sqrt{x^2 - 1}}{x}\,dx$.

22. $\int \frac{dx}{\sin^2 x - \cos^2 x}$.

23. $\int \frac{\cot x\,dx}{\sqrt{1 + \sin^2 x}}$.

24. $\int \frac{dx}{(2x - 1)\sqrt{4x^2 - 4x}}$.

25. $\int xe^{ax^2}\,dx$.

26. $\int \frac{e^{ax}\,dx}{b + ce^{ax}}$.

27. $\int \sec^2 x e^{\tan x}\,dx$.

28. $\int a^{b+cx}\,dx$.

29. $\int \frac{dx}{e^x + 1}$.

30. $\int \frac{dx}{e^x - e^{-x}}$.

31. $\int \tan x \ln \cos x\,dx$.

32. $\int \frac{dx}{a^2x^2 + 2abx + b^2}$.

33. $\int \frac{x\,dx}{\sqrt{x^2 - 2x + 3}}$.

34. $\int \frac{(2x + 3)\,dx}{(x - 1)\sqrt{x^2 - 2x}}$.

35. $\int \frac{dx}{x\sqrt{2x+3}}$.

36. $\int \frac{dx}{x^2\sqrt{ax^2+b}}$.

37. $\int (a^2-x^2)^{\frac{3}{2}}\,dx$.

38. $\int \frac{dx}{x\sqrt{ax^2+bx}}$.

39. $\int \sqrt{3-2x-x^2\,dx}$.

40. $\int \left(a^{\frac{2}{3}}-x^{\frac{2}{3}}\right)^{\frac{3}{2}}dx$.

41. $\int \cos^6 x\sin^2 x\,dx$.

42. $\int (1+\cos x)^{\frac{3}{2}}\,dx$.

43. $\int \tan 2x\sec^5 2x\,dx$.

44. $\int \cot^4 x\,dx$.

45. $\int \frac{dx}{\tan x+\cot x}$.

46. $\int (\sec x+\tan x)^2\,dx$.

47. $\int \frac{\tan x-1}{\tan x+1}\,dx$.

48. $\int \frac{\cos^3 x\,dx}{\sin^3 x}$.

49. $\int \sin 2x\cos^2 x\,dx$.

50. $\int \sqrt{1+\cos^2 x}\sin 2x\,dx$.

51. $\int \frac{dx}{x\sqrt{a^2x+b^2}}$.

52. $\int \frac{dx}{x^2\sqrt{x-2}}$.

53. $\int \frac{dx}{(x-1)\sqrt{x+2}}$.

54. $\int x(ax+b)^{\frac{3}{2}}\,dx$.

55. $\int \frac{px+q}{\sqrt{ax+b}}\,dx$.

56. $\int x^3\sqrt{a^2-x^2}\,dx$.

57. $\int \frac{x^2\,dx}{\sqrt{a^2-x^2}}$.

58. $\int \frac{x^2\,dx}{(x^2-1)^2}$.

59. $\int \frac{dx}{(a^2-x^2)^{\frac{3}{2}}}$.

60. $\int \frac{x^5\,dx}{\sqrt{a^3-x^3}}$.

61. $\int e^{x+\ln x}\,dx$.

62. $\int e^{ax}\sin bx\,dx$.

63. $\int \frac{x^2}{e^x}\,dx$.

64. $\int x\ln(cx+b)\,dx$.

65. $\int \frac{\ln(ax+b)}{x^2}\,dx$

66. $\int x\cot^{-1}x\,dx$.

67. $\int \frac{x\,dx}{(x^2-1)^2(x^2+1)}$.

68. $\int \frac{x^2\,dx}{(x^3+1)(x^3-2)}$.

69. $\int \frac{x\,dx}{x^4+1}$.

70. $\int \frac{x^5}{(x-1)^5}\,dx$.

71. $\int x^2\cos\frac{1}{2}x\,dx$.

72. $\int \frac{2x^2+3x}{(x-1)(x-2)(x+3)}\,dx$.

73. $\int \frac{(3x-5)\,dx}{x(x+3)^2}$.

74. $\int \frac{x\,dx}{x^3-8}$.

75. $\int \frac{dx}{(1-x^3)^{\frac{4}{3}}}$.

76. $\int \frac{x^3\,dx}{(1-2x^2)^{\frac{3}{2}}}$.

77. $\int \sec^4 x \tan^{\frac{1}{3}} x \, dx.$

78. $\int \sin 3x \cos 4x \, dx.$

79. $\int \sin^2 x \cos 2x \, dx.$

80. $\int \sin x \sin 5x \, dx.$

81. $\int \cos 2x \cos 3x \, dx.$

82. $\int (\cot x + \csc x)^2 \, dx.$

83. $\int \frac{(3x - 1)\, dx}{\sqrt{2 + 3x - x^2}}.$

84. $\int \frac{x^2 - 3x + 2}{\sqrt{x^2 - 4x + 3}} \, dx.$

85. $\int \sqrt{\frac{a - x}{a + x}} \, dx.$

86. $\int (\sin x - \cos x)^3 \, dx.$

87. $\int \frac{x^5 \, dx}{(x^2 - a^2)^2}.$

88. $\int \frac{\log (x + \sqrt{x^2 - 1})}{\sqrt{x^2 - 1}} \, dx.$

89. $\int \sec^7 x \, dx.$

90. $\int (x^2 + a^2)^{\frac{5}{2}} \, dx.$

CHAPTER IV

91. Find the area bounded by the x-axis and the parabola $y = x^2 - 4x + 5$.

92. Find the area bounded by the curves $y = x^3$, $y^2 = x$.

93. Find the area bounded by the parabola $y^2 = 2x$ and the witch $x = \frac{1}{y^2 + 1}$.

94. Find the area within a loop of the curve $y^2 = x^2 - x^4$.

95. Find the area of one of the sectors bounded by the hyperbola $x^2 - y^2 = 3$ and the lines $x = \pm 2y$.

96. Find the area bounded by the parabolas $y^2 = 2ax + a^2$, $y^2 + 2ax = 0$.

97. Find the area within the loop of the curve $x = \frac{3am}{1 + m^3}$, $y = \frac{3am^2}{1 + m^3}$.

98. Find the area bounded by the parabola $x = a \cos 2\phi$, $y = a \sin \phi$ and the line $x = -a$.

99. Find the area inclosed by the curve $x = a \cos^3 \phi$, $y = b \sin^3 \phi$.

100. Find the area bounded by the curve $x = a \sin \theta$, $y = a \cos^3 \theta$.

101. Find the area of one loop of the curve $r = a \cos n\theta$.

102. Find the area of a loop of the curve $r = a(1 - 2 \cos \theta)$.

103. Find the area between the curves $r = a(\cos \theta + 2)$, $r = a$.

104. Find the total area inclosed by the curve $r = a \sin \frac{1}{2} \theta$.

105. Find the area of the part of one loop of the curve $r^2 = a^2 \sin 3\theta$ outside the curve $r^2 = a \sin \theta$.

106. By changing to polar coördinates find the area within one loop of the curve $(x^2 + y^2)^2 = a^2xy$.

107. By changing to polar coördinates find the area of one of the regions between the circle $x^2 + y^2 = 2a^2$ and hyperbola $x^2 - y^2 = a^2$.

108. Find the area of one of the regions bounded by $\theta = \sin r$ and the line $\theta = 1$.

109. Find the volume generated by revolving an ellipse about the tangent at one of its vertices.

110. Find the volume generated by revolving about the y-axis the area bounded by the curve $y^2 = x^3$ and the line $x = 4$.

111. Find the volume generated by rotating about the y-axis the area between the x-axis and one arch of the cycloid $x = a(\phi - \sin\phi)$, $y = a(1 - \cos\phi)$.

112. Find the volume generated by rotating the area of the preceding problem about the tangent at the highest point of the cycloid.

113. Find the volume generated by revolving about the x-axis the part of the ellipse $x^2 - xy + y^2 = 1$ in the first quadrant.

114. Find the volume generated by revolving about $\theta = \frac{\pi}{2}$ the area enclosed by the curve $r^2 = a^2 \sin\theta$.

115. The ends of an ellipse move along the parabolas $z^2 = ax$, $y^2 = ax$ and its plane is perpendicular to the x-axis. Find the volume swept out between $x = 0$ and $x = c$.

116. The ends of a helical spring lie in parallel planes at distance h apart and the area of a cross section of the spring perpendicular to its axis is A. Find the volume of the spring.

117. The axes of two right circular cylinders of equal radius intersect at an angle α. Find the common volume.

118. A rectangle moves from a fixed point, one side varying as the distance from the point, and the other as the square of this distance. At the distance of 10 feet the rectangle becomes a square of side 4 ft. What is the volume then generated?

119. A cylindrical bucket filled with oil is tipped until half the bottom is exposed; if the radius is 4 inches and the altitude 12 inches find the amount of oil poured out.

120. Two equal ellipses with semi-axes 5 and 6 inches have the same major axis and lie in perpendicular planes. A square moves with its center in the common axis and its diagonals chords of the ellipses. Find the volume generated.

121. Find the volume bounded by the paraboloid $12z = 3x^2 + y^2$ and the plane $z = 4$.

CHAPTER V

122. Find the length of the arc of the curve

$$y = \tfrac{1}{2} x \sqrt{x^2 - 1} - \tfrac{1}{2} \ln \left(x + \sqrt{x^2 - 1}\right) \text{ between } x = 1 \text{ and } x = 3.$$

123. Find the arc of the curve $9\,y^2 = (2\,x - 1)^3$ cut off by the line $x = 5$.

124. Find the perimeter of the loop of the curve

$$9\,x^2 = (2\,y - 1)\,(y - 2)^2.$$

125. Find the length of the curve $x = t^2 + t$, $y = t^2 - t$ below the x-axis.

126. Find the length of an arch of the curve

$$x = a\sqrt{3}\,(2\,\phi - \sin 2\,\phi), \quad y = \frac{a}{3}\,(1 - \cos 3\,\phi).$$

127. Find the length of one quadrant of the curve

$$x = a \cos^3 \phi, \qquad y = b \sin^3 \phi.$$

128. Find the circumference of the circle

$$r = 2 \sin\theta + 3 \cos\theta.$$

129. Find the perimeter of one loop of the curve

$$r = a \sin^5 \left(\frac{\theta}{5}\right).$$

130. Find the area of the surface generated by revolving the arc of the curve $9\,y^2 = (2\,x - 1)^3$ between $x = 0$ and $x = 2$ about the y-axis.

131. Find the area of the surface generated by revolving one arch of the cycloid $x = a\,(\phi - \sin\phi)$, $y = a\,(1 - \cos\phi)$ about the tangent at its highest point.

132. Find the area of the surface generated by rotating the curve $r^2 = a^2 \sin 2\,\theta$ about the x-axis.

133. Find the area generated by revolving the loop of the curve $9\,x^2 = (2\,y - 1)\,(y - 2)^2$ about the x-axis.

134. Find the volume generated by revolving the area within the curve $y^2 = x^2\,(1 - x^2)$ about the y-axis.

135. The vertical angle of a cone is 90°, its vertex is on a sphere of radius a, and its axis is tangent to the sphere. Find the area of the cone within the sphere.

136. A cylinder with radius b intersects and is tangent to a sphere of radius a, greater than b. Find the area of the surface of the cylinder within the sphere.

137. A plane passes through the center of the base of a right circular cone and is parallel to an element of the cone. Find the areas of the two parts into which it cuts the lateral surface.

CHAPTER VI

138. Find the pressure on a square of side 4 feet if one diagonal is vertical and has its upper end in the surface.

139. Find the pressure on a segment of a parabola of base $2b$ and altitude h, if the vertex is at the surface and the axis of the parabola is vertical.

140. Find the pressure on the parabolic segment of the preceding problem if the vertex is submerged and the base of the segment is in the surface.

141. Find the pressure on the ends of a cylindrical tank 4 feet in diameter, if the axis is horizontal and the tank is filled with water under a pressure of 10 lbs. per square inch at the top of the tank.

142. A barrel 3 ft. in diameter is filled with equal parts of water and oil. If the axis is horizontal and the weight of oil half that of water, find the pressure on one end.

143. Find the moment of the pressure in Ex. 138 about the other diagonal of the square.

144. Weights of 1, 2, and 3 pounds are placed at the points (0, 0), (2, 1), (4, − 3). Find their center of gravity.

145. A trapezoid is formed by connecting one vertex of a rectangle to the middle point of the opposite side. Find its center of gravity.

146. Find the center of gravity of a sector of a circle with radius a and central angle 2α.

147. Find the center of gravity of the area within a loop of the curve $y^2 = x^2 - x^4$.

148. Find the center of gravity of the area bounded by the curve $y^2 = \dfrac{x^3}{2a - x}$ and its asymptote $x = 2a$.

149. Find the center of gravity of the area within one loop of the curve $r^2 = a^2 \sin\theta$.

150. Find the center of gravity of the area of the curve $x = a\sin^3\phi$, $y = b\sin^3\phi$ above the x-axis.

151. Find the center of gravity of the arc of the curve $9y^2 = (2x - 1)^3$ cut off by the line $x = 5$.

152. Find the center of gravity of the arc that forms the loop of the curve

$$9y^2 = (2x - 1)(x - 2)^2.$$

153. Find the center of gravity of the arc of the curve $x = t^2 + t$, $y = t^2 - t$ below the x-axis.

154. Show that the center of gravity of a pyramid of constant density is on the line joining the vertex to the center of gravity of the base, $\frac{3}{4}$ of the way from the vertex to the base.

155. Find the center of gravity of the surface of a right circular cone.

156. Show that the distance from the base to the center of gravity of the surface of an oblique cone is $\frac{1}{3}$ of the altitude. Is it on the line joining the vertex to the center of the base?

157. Find the center of gravity of the solid generated by rotating about the line $x = 4$, the area above the x-axis bounded by the parabola $y^2 = 4x$ and the line $x = 4$.

158. The arc of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ above the x-axis is rotated about the y-axis. Find the center of gravity of the volume and that of the area generated.

159. Assuming that the specific gravity of sea water at depth h in miles is

$$\rho = e^{.0075\,h},$$

find the center of gravity of a section of the water with vertical sides five miles deep.

160. By using Pappus's theorems, find the center of gravity of the arc of a semicircle.

161. The ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is rotated about a tangent inclined 45° to its axis. Find the volume generated.

162. The volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is $\frac{4}{3}\pi abc$. Use this to find the center of gravity of a quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

163. Find the volume generated by revolving one loop of the curve $r = a \sin \theta$ about the initial line.

164. A semicircle of radius a rotates about its bounding diameter while the diameter slides along the line in which it lies. Find the volume generated in one revolution.

165. The plane of a moving square is perpendicular to that of a fixed circle. One corner of the square is kept fixed at a point of the circle while the opposite corner moves around the circle. Find the volume generated.

166. Find the moment of inertia about the x-axis of the area bounded by the x-axis and the curve $y = 4 - x^2$.

167. Show that the moment of inertia of a plane area about an axis perpendicular to its plane at the origin is equal to the sum of its moments of inertia about the coordinate axes. Use this to find the moment of

inertia of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the axis perpendicular to its plane at its center.

168. Find the moment of inertia of the surface of a right circular cone about its axis.

169. The area bounded by the x-axis and the parabola $y^2 = 4\,ax - x^2$ is revolved about the x-axis. Find the moment of inertia about the x-axis of the volume thus generated.

170. From a right circular cylinder a right cone with the same base and altitude is cut. Find the moment of inertia of the remaining volume about the axis of the cylinder.

171. A torus is generated by rotating a circle of radius a about an axis in its plane at distance b, greater than a, from the center. Find the moment of inertia of the volume of the torus about its axis.

172. Find the moment of inertia of the area of the torus about its axis.

173. The kinetic energy of a moving mass is

$$\int \tfrac{1}{2}\, v^2\, dm,$$

where v is the velocity of the element of mass dm. Show that the kinetic energy of a homogeneous cylinder of mass M and radius a rotating with angular velocity ω about its axis is $\frac{1}{2}\, M\omega^2 a^2$.

174. Show that the kinetic energy of a uniform sphere of mass M and radius a rotating with angular velocity ω about a diameter is $\frac{1}{5}\, M\omega^2 a^2$.

175. When a gas expands without receiving or giving out heat, its pressure and volume are connected by the equation

$$pv^\gamma = k$$

where γ and k are constant. Find the work done in expanding from the volume v_1 to the volume v_2.

176. The work done by an electric current of i amperes and E volts is iE joules per second. If

$$E = E_0 \cos \omega t, \qquad i = I_0 \cos(\omega t + \alpha),$$

where E_0, I_0, ω are constants, find the work done in one cycle.

177. When water is pumped from one vessel into another at a higher level, show that the work in foot pounds required is equal to the product of the total weight of water in pounds and the distance in feet its center of gravity is raised.

CHAPTER VII

178. Find the volume of an ellipsoid by using the prismoidal formula.

179. A wedge is cut from a right circular cylinder by a plane which passes through the center of the base and makes with the base an angle α. Find the volume of the wedge by the prismoidal formula.

180. Find approximately the volume of a barrel 30 inches long if its diameter at the ends is 20 inches and at the middle 24 inches.

181. The width of an irregular piece of land was measured at intervals of 10 yards, the measurements being 52, 56, 67, 49, 45, 53, and 62 yards. Find its area approximately by using Simpson's rule.

Find the values of the following integrals approximately by Simpson's rule:

182. $\int_0^4 \sqrt{x^3+1}\,dx.$

183. $\int_1^{10} \frac{1}{x^2} \ln x\,dx.$

184. $\int_0^{\frac{2}{3}\pi} \sqrt{\sin x}\,dx.$

185. $\int_{-4}^4 \frac{dx}{1+x^4}.$

186. Find approximately the length of an arch of the curve $y = \sin x$.

187. Find approximately the area bounded by the x-axis, the curve $y = \frac{\sin x}{x}$, and the ordinates $x = 0$, $x = \pi$.

CHAPTER VIII

Express the following quantities as double integrals and determine the limits:

188. Area bounded by the parabola $y = x^2 - 2x + 3$ and the line $y = 2x$.

189. Area bounded by the circle $x^2 + y^2 = 2a^2$ and the curve

$$y^2 = \frac{x^3}{2a - x}.$$

190. Moment of inertia about the x-axis of the area within the circles

$$x^2 + y^2 = 5, \qquad x^2 + y^2 - 2x - 4y = 0.$$

191. Moment of inertia of the area within the loop of the curve $y^2 = x^2 - x^4$ about the axis perpendicular to its plane at the origin.

192. Volume bounded by the xy-plane the paraboloid $z = x^2 + y^2$ and the cylinder $x^2 + y^2 = 4$.

193. Volume bounded by the xy-plane the paraboloid $z = x^2 + y^2$ and the plane $z = 2x + 2y$.

194. Center of gravity of the solid bounded by the xz-plane, the cylinder $x^2 + z^2 = a^2$, and the plane $x + y + z = 4a$.

195. Volume generated by rotating about the x-axis one of the areas bounded by the circle $x^2 + y^2 = 5a^2$ and the parabola $y^2 = 4ax$.

In each of the following cases determine the region over which the integral is taken, interchange dx and dy, determine the new limits, and so find the value of the integral:

196. $\int_0^1 \int_0^x \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$.

197. $\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} (x+y)\,dy\,dx$.

198. $\int_0^1 \int_{\sqrt{y}}^1 \frac{1}{x} e^{-\frac{y}{x}}\,dy\,dx$.

199. $\int_0^1 \int_0^2 \sqrt{x^2+xy}\,dy\,dx$.

Express the following quantities as double integrals using polar coördinates:

200. Area within the cardioid $r = a(1+\cos\theta)$ and outside the circle $r = \frac{3}{2}a$.

201. Center of gravity of the area within the circle $r = a$ and outside the circle $r = 2a\sin\theta$.

202. Moment of inertia of the area cut from the parabola

$$r = \frac{2a}{1-\cos\theta}$$

by the line $y = x$, about the x-axis.

203. Volume within the cylinder $r = 2a\sin\theta$ and the sphere

$$x^2+y^2+z^2 = 4a^2.$$

204. Moment of inertia of a sphere about a tangent line.

205. Volume bounded by the paraboloid $z = x^2+y^2$ and the plane $z = 2x+2y$.

206. Find the area cut from the cone $x^2+y^2 = z^2$ by the plane $x = 2z-3$.

207. Find the area cut from the plane by the cone in Ex. 206.

208. Find the area of the surface $z^2+(x+y)^2 = a^2$ in the first octant.

209. Determine the area of the surface $z^2 = 2x$ cut out by the planes $y = 0$, $y = x$, $x = 1$.

CHAPTER IX

Express the following quantities as triple integrals:

210. Volume of an octant of a sphere of radius a.

211. Moment of inertia of the volume in the first octant bounded by the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1$ about the x-axis.

212. Center of gravity of the region in the first octant bounded by the paraboloid $z = xy$ and the cylinder $x^2+y^2 = a^2$.

213. Moment of inertia about the z-axis of the volume bounded by the paraboloid $z = x^2+y^2$ and the plane $z = 2x+3$.

214. Volume bounded by the cone $x^2 = y^2+2z^2$ and the plane $3x+y = 6$.

Express the following quantities as triple integrals in rectangular, cylindrical. and spherical coördinates, and evaluate one of the integrals:

215. Moment of inertia of a right circular cylinder about a line tangent to its base.

216. Moment of inertia of a segment cut from a sphere by a plane, about a diameter parallel to that plane.

217. Center of gravity of a right circular cone whose density varies as the distance from the center of the base.

218. Volume bounded by the xy-plane, the cylinder $x^2 + y^2 = 2\,ax$ and the cone $z^2 = x^2 + y^2$.

219. Find the attraction of a uniform wire of length l and mass M on a particle of unit mass at distance c from the wire in the perpendicular at one end.

220. Find the attraction of a right circular cylinder on a particle at the middle of its base.

221. Show that the attraction of a homogeneous shell bounded by two concentric spherical surfaces on a particle in the enclosed space is zero.

CHAPTER X

Solve the following differential equations:

222. $y\,dx + (x - xy)\,dy = 0.$

223. $\sin x \sin y\,dx + \cos x \cos y\,dy = 0.$

224. $(2\,xy - y^2 + 6\,x^2)\,dx + (3\,y^2 + x^2 - 2\,xy)\,dy = 0.$

225. $x\dfrac{dy}{dx} + y = x^3y.$

226. $x\dfrac{dy}{dx} + y = \cot x.$

227. $x\,dy - \left(y + e^{\frac{1}{x}}\right)dx = 0.$

228. $(1 + x^2)\,dy + (xy + x)\,dx = 0.$

229. $x\,dx + y\,dy = x\,dy - y\,dx.$

230. $(\sin x + y)\,dy + (y\cos x - x^2)\,dx = 0.$

231. $y\,(e^x + 2)\,dx + (e^x + 2\,x)\,dy = 0.$

232. $(xy^2 - x)\,dx + (y + xy)\,dy = 0.$

233. $(1 + x^2)\dfrac{dy}{dx} + xy = 2\,y.$

234. $x\,dy - y\,dx = \sqrt{x^2 + y^2}\,dx.$

235. $(x - y)\,dx + x\,dy = 0.$

236. $x\,dy - y\,dx = x\sqrt{x^2 + y^2}\,dx.$

237. $e^{x+y}\,dy + (1 + e^y)\,dx = 0.$

238. $(2\,x + 3\,y - 1)\,dx + (4\,x + 6\,y - 5)\,dy = 0.$

239. $(3\,y^2 + 3\,xy + x^2)\,dx = (x^2 + 2\,xy)\,dy.$

240. $(1 + x^2)\,dy + (xy - x^2)\,dx = 0.$

241. $(x^2y + y^4)\,dx - (x^3 + 2\,xy^3)\,dy = 0.$

242. $(y + 1)\left(\dfrac{dy}{dx}\right)^2 = x^4 - x^2.$

243. $2\dfrac{dy}{dx} + y + xy^3 = 0.$

244. $y\,dx = (y^3 - x)\,dy.$

245. $y\dfrac{dy}{dx} + y^2 \cot x = \cos x.$

246. $(x^2 - y^2)(dx + dy) = (x^2 + y^2)(dy - dx).$

247. $x^2\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 1.$

248. $\dfrac{d^2s}{dt^2} = \dfrac{a^2}{s^3}.$

249. $y\dfrac{d^2y}{dx^2} + 2\,y\dfrac{dy}{dx} + \left(\dfrac{dy}{dx}\right)^2 = 0.$

250. $(1 + x)\dfrac{d^2y}{dx^2} = \dfrac{dy}{dx}.$

251. $\dfrac{d^2y}{dx^2} = k\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}.$

252. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - 3\,y = x^2.$

253. $\dfrac{d^3y}{dx^3} - \dfrac{dy}{dx} = e^{2x}.$

254. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4\,y = \cos x.$

255. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 8\,y = 3\,x - 4.$

256. $\dfrac{d^3y}{dx^3} + 2\dfrac{dy}{dx} = x + 1.$

257. $\dfrac{d^3y}{dx^3} + 2\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 2\,y = e^x + 3.$

258. $\dfrac{d^2y}{dx^2} + a^2y = \sin ax.$

259. $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 2\,y = e^x \sin 2\,x.$

260. $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 2\,y = e^{-x} \sin 2\,x.$

261. $\dfrac{d^2y}{dx^2} + 9\,y = 2\cos 3\,x - 3\cos 2\,x.$

262. $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 5\,y = (e^x + 1)^2.$

263. $\dfrac{d^2y}{dx^2} - y = xe^{2x}$.

264. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 2y = x\cos x$.

265. $\dfrac{dy}{dt} + 2x = \sin t,\ \dfrac{dx}{dt} - 2y = \cos t$.

266. $\dfrac{dx}{dt} - \dfrac{dy}{dt} + x - y = t^2 - 3t - 3,\ x - y - \dfrac{dx}{dt} - \dfrac{dy}{dt} = t^2 - t - 3$.

267. According to Newton's Law, the rate at which a substance cools in air is proportional to the difference of the temperature of the substance and the temperature of air. If the temperature of air is 20° C. and the substance cools from 100° to 60° in 20 minutes, when will its temperature become 30°?

268. A particle moves in a straight line from a distance a towards a point with an acceleration which at distance r from the point is $k\,r^{-\frac{5}{3}}$. If the particle starts from rest, how long will be the time before it reaches the point?

269. A substance is undergoing transformation into another at a rate proportional to the amount of the substance remaining untransformed. If that amount is 34.2 when $t = 1$ hour and 11.6 when $t = 3$ hours, determine the amount at the start, $t = 0$, and find how many hours will elapse before only one per cent will remain.

270. Determine the shape of a reflector so that all the rays of light coming from a fixed point will be reflected in the same direction.

271. Find the curve in which a chain hangs when its ends are supported at two points and it is allowed to hang under its own weight. (See the example solved in Art. 57.)

272. By Hooke's Law the amount an elastic string of natural length l stretches under a force F is klF, k being constant. If the string is held vertical and allowed to elongate under its own weight w, show that the elongation is $\frac{1}{2}\,kwl$.

273. Assuming that the resistance of the air produces a negative acceleration equal to k times the square of the velocity, show that a projectile fired upward with a velocity v_1 will return to its starting point with the velocity

$$v_2 = \sqrt{\frac{gv_1^2}{g + kv_1^2}},$$

g being the acceleration of gravity.

274. Assuming that the density of sea water under a pressure of p pounds per square inch is

$$\rho = 1 + 0.000003\,p,$$

show that the surface of an ocean 5 miles deep is about 465 feet lower than it would be if water were incompressible. (A cubic foot of sea water weighs about 64 pounds.)

275. Show that when a liquid rotating with constant velocity is in equilibrium, its surface is a paraboloid of revolution.

276. Find the path described by a particle moving in a plane, if its acceleration is directed toward a fixed point and is proportional to the distance from the point.

ANSWERS TO EXERCISES

Page 5

1. $\frac{1}{5} x^5 - \frac{3}{4} x^4 + \frac{5}{3} x^3 + C.$
2. $\dfrac{1}{3} x^3 + \dfrac{1}{x} + C.$
3. $\frac{2}{3} x^{\frac{3}{2}} + 2 x^{\frac{1}{2}} + C.$
4. $\frac{1}{3} \sqrt{2x}\,(2x - 3) + C.$
5. $x^{\frac{1}{2}} (\frac{2}{7} x^3 + \frac{4}{5} x^2 + \frac{2}{3} x) + C.$
6. $a^{\frac{3}{2}} x - 2 a x^{\frac{3}{2}} + \frac{3}{2} a^{\frac{1}{2}} x^2 - \frac{2}{5} x^{\frac{5}{2}} + C.$
7. $\frac{1}{4} x^4 + \frac{1}{3} (a + b) x^3 + \frac{1}{2} abx^2 + C.$
8. $2x + 3 \ln x + C.$
9. $\frac{1}{2} y^2 + 4y + 4 \ln y + C.$
10. $x^{\frac{1}{3}} (\frac{3}{13} x^4 - \frac{3}{7} x^2 - 6) + C.$
11. $\ln (x + 1) + C.$
12. $-\dfrac{1}{x + 1} + C.$
13. $\sqrt{2x + 1} + C.$
14. $\frac{1}{2} \ln (x^2 + 2) + C.$
15. $\sqrt{x^2 - 1} + C.$
16. $-\dfrac{1}{4b(a + bx^2)^2} + C.$
17. $-\frac{1}{3} (a^2 - x^2)^{\frac{3}{2}} + C.$
18. $\frac{1}{3} \ln (a^3 + x^3) + C.$
19. $\frac{2}{9} (x^3 - 1)^{\frac{3}{2}} + C.$
20 $\ln (x^2 + ax + b) + C.$
21. $2\sqrt{x^2 + ax + b} + C.$
22. $-\dfrac{1}{5a} \ln (1 - at^5) + C.$
23. $-\frac{1}{5} (a^2 - t^2)^{\frac{5}{2}} + C.$
24. $x + 3 \ln (x - 2) + C.$
25. $\dfrac{1}{2} \ln (2x^2 + 1) - \dfrac{1}{4(2x^2 + 1)} + C.$
26. $\dfrac{1}{9} \left(1 - \dfrac{1}{x}\right)^9 + C.$
27. $-\dfrac{1}{n(n-1)(x^n + a)^{n-1}} + C.$
28. $\dfrac{\sqrt{2}\,(\sqrt{2x} - \sqrt{2a})^{11}}{11} + C.$
29. $\frac{1}{3} x^3 - \frac{4}{3} \ln (x^3 + 2) + C.$
30. $\frac{1}{8} x^8 - \frac{2}{5} x^5 + \frac{1}{2} x^2 + C.$

Pages 12, 13

1. 138.
2. $\frac{1}{2} gt^2 + 30t.$
3. $h = -\frac{1}{2} gt^2 + 100t + 60.$ It reaches the highest point when $t = 3.1$ sec., $h = 215.3$ ft.
4. $\frac{1}{3000}$ sec.

5. $x = t^2 - t + 1$, $y = t - \frac{1}{2}t^2 + 2$. These are parametric equations of the path. The rectangular equation is $x^2 + 4xy + 4y^2 - 12x - 22y + 31 = 0$.

6. About 53 miles.

7. $x = \dfrac{k}{4}\sqrt{3}\,t^2$, $y = \dfrac{k}{4}t^2 + V_0 t$.

8. $y = 2x - \frac{1}{2}x^2 - \frac{3}{2}$.

9. $y = e^x$.

10. $(-\frac{5}{2}, -\frac{17}{4})$.

11. $6y = x^3 - 3x^2 + 3x + 13$.

12. $-12\frac{1}{2}$.

14. About 4 per cent.

15. $x = x_0 e^{kt}$, where x is the number at time t, x_0 the number at time $t = 0$, and k is constant.

17. 17 minutes.

18. 11.4 minutes.

19. 11.6 years.

Pages 18, 19

1. $-(\frac{1}{2}\cos 2x + \frac{1}{3}\sin 2x) + C$.

2. $\dfrac{5}{2}\sin\left(\dfrac{2x-3}{5}\right) + C$.

3. $-\dfrac{1}{n}\cos(nt + \alpha) + C$.

4. $3\tan\frac{1}{3}\theta + C$.

5. $-4\csc\dfrac{\theta}{4} + C$.

6. $\frac{1}{2}\sin^2\theta + C$.

7. $\tan x + C$.

8. $-\frac{1}{2}\cot 2x + C$.

9. $-\csc x + C$.

10. $\frac{1}{4}\sec^4 x + C$.

11. $2\left(\csc\dfrac{\theta}{2} - \cot\dfrac{\theta}{2}\right) + C$.

12. $\frac{1}{2}\sin(x^2 - 1) + C$.

13. $\frac{1}{3}(\tan 3x + \sec 3x) + C$.

14. $\tan x + x - 2\ln(\sec x + \tan x) + C$.

15. $\ln(1 + \sin x) + C$.

16. $\theta + \cos^2\theta + C$.

17. $\sin x + \ln(\csc x - \cot x) + C$.

18. $\frac{1}{3}\sin^3 x + C$.

19. $\frac{1}{4}\tan^4 x + C$.

20. $\frac{1}{2}\tan^2 x + C$.

21. $-\frac{1}{6}\cos^6 x + C$.

22. $\frac{1}{2}\ln(1 + 2\tan x) + C$.

23. $-\frac{1}{2}\ln(1 - \sin 2x) + C$.

24. $\dfrac{1}{a}\ln(1 + \tan ax) + C$.

25. $\dfrac{1}{\sqrt{2}}\sin^{-1}\dfrac{x\sqrt{2}}{\sqrt{3}} + C$.

26. $\frac{1}{\sqrt{3}} \tan^{-1} \frac{x\sqrt{3}}{2} + C.$

27. $\frac{1}{2} \sec^{-1} \frac{x\sqrt{3}}{2} + C.$

28. $\frac{1}{6} \tan^{-1} 2y + C.$

29. $\frac{1}{\sqrt{7}} \ln \left(x\sqrt{7} + \sqrt{7x^2 + 1}\right) + C.$

30. $\frac{1}{3} \sec^{-1} \frac{ax}{3} + C.$

31. $\frac{1}{4\sqrt{3}} \ln \frac{2x + \sqrt{3}}{2x - \sqrt{3}} + C.$

32. $\frac{1}{2}$ $\ln \left(2x + \sqrt{4x^2 - 3}\right) + C.$

33. $-3\sqrt{4 - x^2} - 2 \sin^{-1} \frac{x}{2} + C.$

34. $2\sqrt{x^2 + 4} + 3 \ln \left(x + \sqrt{x^2 + 4}\right) + C.$

35. $\frac{1}{8} \ln (4x^2 - 5) + \frac{1}{\sqrt{5}} \ln \frac{2x - \sqrt{5}}{2x + \sqrt{5}} + C.$

36. $\frac{5}{3}\sqrt{3x^2 - 9} - \frac{2}{\sqrt{3}} \ln \left(x + \sqrt{x^2 - 3}\right) + C.$

37. $\sin^{-1}\left(\frac{\sin x}{\sqrt{2}}\right) + C.$

38. $-\sqrt{2 - \sin^2 x} + C.$

39. $\tan^{-1}(\sin x) + C.$

40. $\sec^{-1}(\tan x) + C.$

41. $\ln \left(\sec x + \sqrt{\sec^2 x + 1}\right).$

42. $2\sqrt{1 - \cos\theta} + C.$

43. $\frac{1}{4} \ln \frac{2 + \ln x}{2 - \ln x} + C.$

44. $-\frac{1}{2}\sqrt{\cos^2 x - \sin^2 x} + C.$

45. $\frac{1}{2} \sin^{-1} \frac{x^2}{a^2} + C.$

46. $-\frac{1}{k^2} e^{-k^2 x} + C.$

47. $\frac{1}{2a}(e^{2ax} - e^{-2ax}) + 2x + C.$

48. $\frac{1}{3} \ln (1 + e^{3x}) + C.$

49. $\ln (e^x + e^{-x}) + C.$

50. $e^{-\frac{1}{x}} + C.$

51. $\tan^{-1}(e^x) + C.$

52. $\ln \frac{1 - e^x}{1 + e^x} + C.$

53. $\frac{1}{a} \sin^{-1}(e^{ax}) + C.$

54. $\tan^{-1}(e^x) + C.$

Page 20

1. $\frac{1}{2} \tan^{-1} \frac{x + 3}{2} + C.$

2. $\frac{1}{2} \sin^{-1} \frac{2x - 1}{\sqrt{3}} + C.$

3. $\frac{1}{\sqrt{3}} \ln \left(3x + 2 + \sqrt{9x^2 + 12x + 6}\right) + C.$

4. $\frac{1}{\sqrt{5}} \sin^{-1} \frac{(2x - 1)\sqrt{5}}{3} + C.$

5. $\frac{1}{\sqrt{3}} \sec^{-1} \frac{(x - 3)\sqrt{6}}{3} + C.$

6. $\frac{1}{b-a}\ln\frac{x+a}{x+b}+C.$

7. $\frac{1}{4}\ln(4x^2-4x-2)+\frac{\sqrt{3}}{2}\ln\frac{2x-1-\sqrt{3}}{2x-1+\sqrt{3}}+C.$

8. $\frac{2}{3}\sqrt{3x^2-6x+1}+\frac{1}{\sqrt{3}}\ln\left[3(x-1)+\sqrt{9x^2-18x+3}\right]+C.$

9. $\frac{1}{6}\ln(3x^2+2x+2)+\frac{1}{3\sqrt{5}}\tan^{-1}\frac{3x+1}{\sqrt{5}}+C.$

10. $\frac{1}{\sqrt{2}}\sec^{-1}\frac{2x+1}{\sqrt{2}}+\frac{1}{2}\ln\left(2x+1+\sqrt{4x^2+4x-1}\right)+C.$

11. $-\frac{3}{\sqrt{x^2-2x+3}}+C.$

12. $\sqrt{x^2-x-2}+\frac{3}{2}\ln\left(2x-1+\sqrt{4x^2-4x-8}\right)+C.$

13. $\frac{1}{\sqrt{17}}\ln\left(\frac{4e^x+3-\sqrt{17}}{4e^x+3+\sqrt{17}}\right)+C.$

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1. $-\cos x+\frac{1}{3}\cos^3 x+C.$
2. $\sin x-\frac{2}{3}\sin^3 x+\frac{1}{5}\sin^5 x+C.$
3. $\sin x-\frac{2}{3}\cos^3 x+\frac{2}{3}\sin^3 x-\cos x+C.$
4. $-\frac{1}{3}\cos^3 x+\frac{1}{5}\cos^5 x+C.$
5. $\frac{2}{5}\sin^5\frac{1}{2}x-\frac{4}{7}\sin^7\frac{1}{2}x+\frac{2}{9}\sin^9\frac{1}{2}x+C.$
6. $\frac{1}{18}\sin^6 3\theta-\frac{1}{24}\sin^8 3\theta+C.$
7. $-\frac{2}{3}\cos^3\theta+\cos\theta+C.$
8. $\sin x+\frac{1}{2}\sin^2 x+C.$
9. $\cos x+\ln(\csc x-\cot x)+C.$
10. $\cos^2\theta-\frac{1}{4}\cos^4\theta-\ln\cos\theta+C.$
11. $\tan x+\frac{1}{3}\tan^3 x+C.$
12. $-(\cot y+\frac{4}{3}\cot^3 y+\frac{6}{5}\cot^5 y+\frac{4}{7}\cot^7 y+\frac{1}{9}\cot^9 y)+C.$
13. $\tan x-x+C.$
14. $2\tan\theta-\sec\theta-\theta+C.$
15. $\frac{2}{3}\sec^3\frac{1}{2}x+C.$
16. $\frac{1}{14}\sec^7 2x-\frac{1}{5}\sec^5 2x+\frac{1}{6}\sec^3 2x+C.$
17. $-\frac{1}{2}\csc^2 x-\ln\sin x+C.$
18. $\frac{1}{6}\sec^6 x-\frac{3}{4}\sec^4 x+\frac{3}{2}\sec^2 x+\ln\cos x+C.$
19. $-\frac{1}{5}\cot^5 x-\frac{1}{3}\cot^3 x+C.$
20. $\frac{1}{2}\tan^2 x+\ln\tan x+C.$
21. $\frac{x}{2}-\frac{1}{4a}\sin(2ax)+C.$
22. $\frac{x}{2}+\frac{1}{4a}\sin(2ax)+C.$

23. $\frac{1}{16}x - \frac{1}{64}\sin 4x - \frac{1}{48}\sin^3 2x + C.$
24. $\frac{1}{16}x - \frac{1}{32}\sin 2x + \frac{1}{24}\sin^3 x + C.$
25. $\frac{5}{16}x - \frac{1}{4}\sin 2x + \frac{3}{64}\sin 4x + \frac{1}{48}\sin^3 2x + C.$
26. $\tan x + \sec x + C.$
27. $\tan \frac{1}{2}x + C.$
28. $2\left(\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right) + C.$
29. $\frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\ln\left(x + \sqrt{x^2 - a^2}\right) + C.$
30. $\frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\ln\left(x + \sqrt{x^2 + a^2}\right) + C.$
31. $\frac{x}{2}\sqrt{x^2 + a^2} - \frac{a^2}{2}\ln\left(x + \sqrt{x^2 + a^2}\right) + C.$
32. $-\frac{x}{a^2\sqrt{x^2 - a^2}} + C.$
33. $\frac{1}{a}\ln\frac{x}{a + \sqrt{a^2 - x^2}} + C.$
34. $-\frac{\sqrt{2ax - x^2}}{ax} + C.$
35. $\frac{1}{\sqrt{a^2 - x^2}} + C.$
36. $\frac{1}{5}(x^2 + a^2)^{\frac{5}{2}} - \frac{a^2}{3}(x^2 + a^2)^{\frac{3}{2}} + C.$
37. $-\frac{\sqrt{x^2 + a^2}}{a^2x} + C.$
38. $\frac{x-2}{2}\sqrt{x^2 - 4x + 5} + \frac{1}{2}\ln\left(x - 2 + \sqrt{x^2 - 4x + 5}\right) + C.$
39. $\frac{11 - 4x}{32}\sqrt{2 - 2x - 4x^2} + \frac{19}{64}\sin^{-1}\frac{4x + 1}{3} + C.$

Page 30

1. $\frac{x^2}{2} + 4x - 2\ln(x - 1) + 12\ln(x - 2) + C.$
2. $3\ln x - \ln(x + 1) + C.$
3. $\ln\frac{(x - 1)(x + 1)}{x} + C.$
4. $\frac{x}{4} + \ln x - \frac{7}{16}\ln(2x - 1) - \frac{9}{16}\ln(2x + 1) + C.$
5. $\frac{3}{4}\ln(x + 3) - \frac{1}{8}\ln(x + 1) - \frac{5}{8}\ln(x + 5) + C.$
6. $\frac{1}{2}\ln(2x - 1) - 3\ln(2x - 3) + \frac{5}{2}\ln(2x - 5) + C.$
7. $x + \frac{1}{x} + \ln\frac{(x - 1)^2}{x} + C.$

8. $\frac{1}{4}\ln(x+1)+\frac{3}{4}\ln(x-1)-\frac{1}{2(x-1)}+C.$

9. $\frac{1}{4}\left(\frac{2x}{1-x^2}+\ln\frac{x+1}{x-1}\right)+C.$

10. $x-8\ln(x+1)-\frac{72x^2+96x+40}{3(x+1)^3}+C.$

11. $\frac{x^2+2x+3}{x^3}+\ln\frac{x-1}{x}+C.$

12. $\frac{1}{2}\ln\frac{x-2}{x+2}-\frac{1}{2}\frac{x}{x^2-4}+C.$

13. $\frac{1}{2(4-x^2)}+C.$

14. $x+\frac{1}{4}\ln\frac{x-1}{x+1}-\frac{1}{2}\tan^{-1}x+C.$

15. $\frac{1}{3}\ln\frac{x+1}{\sqrt{x^2-x+1}}+\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}+C.$

16. $\frac{1}{3}\ln(x^3+1)+C.$

17. $\frac{1}{6(x+1)}+\frac{1}{4}\ln\frac{x-1}{x+1}-\frac{2\sqrt{3}}{9}\tan^{-1}\frac{2x-1}{\sqrt{3}}+C.$

18. $\frac{1}{2(1-x^2)}+\ln\frac{x-1}{x+1}+C.$

19. $-\frac{8}{x^3-8}+\frac{1}{6}\ln\frac{x-2}{\sqrt{x^2+2x+4}}+\frac{1}{2\sqrt{3}}\tan^{-1}\frac{x+1}{\sqrt{3}}+C.$

20. $3(x+1)^{\frac{1}{3}}+\ln\left[(x+1)^{\frac{1}{3}}-1\right]-\sqrt{3}\tan^{-1}\frac{2(x+1)^{\frac{1}{3}}+1}{\sqrt{3}}+C.$

21. $-\frac{1}{5}x^{\frac{5}{12}}+\frac{1}{4}x^{\frac{1}{3}}-\frac{1}{3}x^{\frac{1}{4}}-x^{\frac{1}{12}}+\frac{1}{2}\ln\frac{1+x^{\frac{1}{12}}}{1-x^{\frac{1}{12}}}+C.$

22. $\frac{2}{5a^2}(ax+b)^{\frac{5}{2}}-\frac{2b}{3a^2}(ax+b)^{\frac{3}{2}}+C.$

23. $2\sqrt{x+2}-\ln(x+3)-2\tan^{-1}\sqrt{x+2}+C.$

24. $4x^{\frac{1}{4}}+2\ln\left(x^{\frac{1}{4}}-1\right)+\ln\left(x^{\frac{1}{2}}+1\right)-2\tan^{-1}x^{\frac{1}{4}}+C.$

25. $\frac{1}{3}(x+1)^{\frac{3}{2}}+\frac{1}{3}(x-1)^{\frac{3}{2}}+C.$

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1. $\frac{1}{4}\cos 2x+\frac{x}{2}\sin 2x+C.$

2. $\frac{x^2}{2}\ln x-\frac{x^2}{4}+C.$

3. $x\sin^{-1}x+\sqrt{1-x^2}+C.$

4. $\frac{x^2+1}{2}\tan^{-1}x-\frac{1}{2}x+C.$

5. $x\ln\left(x+\sqrt{a^2+x^2}\right)-\sqrt{a^2+x^2}+C.$

6. $2\sqrt{x-1}\ln x - 4\sqrt{x-1} + 4\tan^{-1}\sqrt{x-1} + C.$

7. $\ln x \ln(\ln x) - \ln x + C.$

8. $\dfrac{1}{3}x^3\sec^{-1}x - \dfrac{x}{6}\sqrt{x^2-1} - \dfrac{1}{6}\ln\left(x+\sqrt{x^2-1}\right) + C.$

9. $x - (1+e^{-x})\ln(1+e^x) + C.$

10. $(x^2 - 2x + 2)e^x + C.$

11. $-(x^3 + 3x^2 + 6x + 6)e^{-x} + C.$

12. $\dfrac{x-1}{2}\sin 2x - \dfrac{2x^2 - 4x + 1}{4}\cos 2x + C.$

13. $\dfrac{x}{2}\sqrt{x^2-a^2} - \dfrac{a^2}{2}\ln\left(x+\sqrt{x^2-a^2}\right) + C.$

14. $\dfrac{x}{2}\sqrt{a^2+x^2} + \dfrac{a^2}{2}\ln\left(x+\sqrt{a^2+x^2}\right) + C.$

15. $\dfrac{e^{2x}}{13}(2\sin 3x - 3\cos 3x) + C.$

16. $\dfrac{e^x}{2}(\cos x + \sin x) + C.$

17. $-\dfrac{e^{-x}}{5}(\sin 2x + 2\cos 2x) + C.$

18. $\frac{1}{2}(\sec\theta\tan\theta + \ln(\sec\theta + \tan\theta)] + C.$

19. $\frac{1}{2}\cos x - \frac{1}{10}\cos 5x + C.$

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1. $\frac{5}{3}$.
2. 2.829.
3. -0.630.

Pages 45, 46

1. $1 - \frac{1}{3}\sqrt{3}$.
2. $\dfrac{\pi}{3}$.
3. -20.
4. 2.
5. $\frac{2}{3}$.
6. 1.807.
7. 0.2877.
8. 0.
9. $\frac{3}{2}a$.
10. 2.
11. ∞.
12. $\dfrac{\pi}{2}$.
13. $\dfrac{1}{2k^2}$.
14. 0.5493.
15. $\dfrac{\pi}{4} + \dfrac{1}{2}$.
16. 1.786.
17. 0.4055.
18. 0.2877.
19. $\dfrac{a^2}{2}(1 - \ln 2)$.

Pages 49, 50

1. 11.
2. $\frac{4}{9}\sqrt{3}\,(4-\sqrt{2})$.
3. $\frac{9}{4}\sqrt[3]{3}+6$.
4. $\frac{4}{3}$.
5. 9.248.
6. πab.
7. $\frac{1}{6}a^2$.
8. $5\frac{1}{3}$.
9. $\frac{8}{15}$.
11. $\frac{16}{3}a^2$.
12. $\frac{9}{2}$.
13. $2\pi+\frac{4}{3},\ 6\pi-\frac{4}{3}$.
14. $4\sqrt{3}+\frac{16}{3}\pi$.
15. $5\left(\frac{\pi}{4}-\tan^{-1}\frac{1}{2}\right)+\frac{2}{3}$
17. $3\pi a^2$.
18. $\frac{3}{2}\pi a^2$.
19. $\pi(b^2+2ab)$.
20. 2$\pi\sqrt{3}$.
21. $3\pi a^2$.
22. $\frac{3}{4}\pi ab$.

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? 2. $\frac{\pi}{4}a^2$. $\pi/2$
3. $\frac{2}{3}a^2\sqrt{3}$.
? 4. $\frac{a^2}{4}(e^{4\pi}-1)$.
5. $\frac{a^2}{2}$.
6. $\frac{9}{2}\pi$.
7. $\frac{3}{2}\pi a^2$.
8. $\frac{8}{3}a^2$.
9. $2a^2\left(1+\frac{4}{3}\sqrt{2}\right)$.
10. $16\pi^3a^2$.
11. $a^2\left(\frac{\pi}{3}+\frac{1}{2}\sqrt{3}\right)$.
13. $\frac{a^2}{4}(\pi-1)$.
14. $(10\pi+9\sqrt{3})\frac{a^2}{32}$.
15. πa^2.

Pages 55, 56

3. $\frac{16}{15}\pi$.
4. $\frac{\pi a^2}{6}$.
5. $\frac{\pi a^3}{4}\left(e^2+4-\frac{1}{e^2}\right)$.
6. $\frac{1}{2}\pi^2$.
7. $\frac{4}{3}\pi a^3(1-\cos^4\alpha)$, where a is the radius of the sphere and 2α the vertical angle of the cone.
8. $\frac{16}{5}\pi a^3$.
9. $\frac{32}{15}\pi a^3$.
10. $\frac{32}{3}\pi a^3$.
11. $5\pi^2a^3$.
12. $\frac{32}{105}\pi a^3$.
13. $\frac{8}{3}\pi a^3$.
14. $\frac{13}{4}\pi^2a^3$.
15. $8\pi\sqrt{3}$.
16. $\frac{\pi a^3}{16}\sqrt{2}$.

Page 59

2. $\frac{2}{3}a^3\tan\alpha$.
3. $\frac{8}{3}a^3$.
4. $\frac{4}{3}\pi ab^2$.
5. $\frac{1}{2}\pi a^2h$.
6. Ah.
8. $\frac{4}{3}a^3\left(1+\frac{\pi}{2}\right)$
9. $\frac{4}{3}a^2h$.
10. $\frac{4}{3\sqrt{2}}\pi a^3$.

Page 63

2. $\frac{8}{27}(10\sqrt{10}-1)$.
3. $\ln(2+\sqrt{3})$.
4. $\frac{3}{4}+\frac{1}{2}\ln 2$.
5. 2.003.
6. $6a$.
7. $a\left(1-\frac{1}{e}\right)$.
8. $8a$.
9. $2\pi^2 a$.

Page 64

3. $\frac{\sqrt{a^2+1}}{a}(e-1)$.
4. $\frac{4a}{\sqrt{3}}$.
5. $2a[\sqrt{2}+\ln(1+\sqrt{2})]$.
6. $\frac{16}{3}a$.
7. $8a$.
8. $\frac{3}{2}\pi a$.

Pages 66, 67

3. $2\pi b\left(b+\frac{a^2}{\sqrt{a^2-b^2}}\sin^{-1}\frac{\sqrt{a^2-b^2}}{a}\right)$.
4. $\frac{12}{5}\pi a^2$.
5. $\frac{a^2}{2}\left(e^2-\frac{1}{e^2}+4\right)$.
6. $\frac{64}{3}\pi a^2$.
7. $\frac{32}{5}\pi a^2$.
8. $\frac{1}{2}\pi a^2\sqrt{2}(4-\pi)$.
9. $8\pi[\sqrt{2}+\ln(1+\sqrt{2})]$.
10. $4\pi a^2$.

Page 69

1. $\frac{4}{3}\pi^3 a^2$.
2. $2a^2$.
3. $16a^2$.
4. $\frac{1}{4}\pi a^2(a+2b\sqrt{3})$.
5. $\frac{4}{3}\pi a^3(2\sqrt{2}-1)$.
6. $\frac{2}{3}\pi a^3(1-\cos\alpha)$.
7. $2ah(\pi-2)$.
8. $\frac{1}{2}\pi a\sqrt{h^2+4a^2}$.
9. $\frac{1}{9}a^2h(9\pi-16)$.

Pages 71, 72

1. 45,000 lbs.
2. 33,750 lbs.
3. $\frac{1}{3}wbh^2$.
4. $\frac{1}{6}wbh^2$.
5. $\frac{2}{3}wab^2$, where a is the semi-axis in the surface and b the vertical semi-axis.
6. $300{,}000\,w$.
7. $40\pi w$.

Pages 78–80

1. $\frac{1}{2}pa^2b$, where p is the pressure per unit area, a the width, and b the height of the door.
2. $\frac{1}{12}wa^3b$.
3. $\frac{1}{12}wbh^2(4c+3h)$.
4. The intersection of the medians.

5. $(\frac{3}{5}a, 0)$.

6. $\bar{x} = \frac{4a}{3\pi}$, $\bar{y} = \frac{4b}{3\pi}$.

7. $\left(\frac{a}{5}, \frac{a}{5}\right)$.

8. $\left(0, \frac{256a}{315\pi}\right)$.

9. $\bar{y} = \frac{\pi}{8}$.

10. $(\frac{9}{20}a, \frac{9}{20}a)$.

11. $\left(\frac{5}{6}a, \frac{16a}{9\pi}\right)$.

12. $\bar{y} = \frac{5}{6}a$.

13. $\bar{x} = \frac{1}{8}\pi\sqrt{2}\,a$.

14. At distance $\frac{2a}{\pi}$ from the bounding diameter.

15. $\bar{y} = \frac{a(e^4 + 4e^2 - 1)}{4e(e^2 - 1)}$.

16. $(\frac{2}{5}a, \frac{2}{5}a)$.

17. (0.399, 1.520).

18. $\bar{y} = \frac{4}{3}a$.

19. On the axis $\frac{1}{4}$ of the distance from the base to the vertex.

20. At distance $\frac{3}{8}a$ from the plane face of the hemisphere, where a is the radius.

21. $(\frac{8}{3}, 0)$.

22. Its distance from the plane face is $\frac{8}{15}$ of the radius.

23. On the axis at distance $\frac{3}{8}a(1 + \cos\alpha)$ from the vertex, a being the radius and α the angle of the sector.

24. $(\frac{4}{5}a, 0)$.

25. The distance of the center of gravity from the base of the cylinder is $\frac{3}{32}\pi a \tan\alpha$.

26. At the middle of the radius perpendicular to the plane face.

28. $\bar{x} = \frac{6\sqrt{3} + 1}{15\sqrt{3} - 5}$.

Pages 82, 83

2. $2\pi^2a^2b$.

3. $\frac{\pi}{32}(12\sqrt{3} - 1)$.

4. $\pi(36\pi + \sqrt{6})\sqrt{6}$.

5. $\frac{32}{3}\pi a^2$.

6. $\frac{1}{3}\pi a^3[3\ln(1+\sqrt{2}) - \sqrt{2}]$.

7. $\frac{2}{3}\pi a^3(3\alpha + 2\sin\alpha)$.

8. π^2a^3.

10. $\frac{2}{3}\pi a^3(4\sin\alpha - \sin^3\alpha)\tan\alpha$, where a is the radius of the cylinder and 2α the vertical angle of the cone.

Pages 84, 85

1. $\frac{1}{3}a^3b$, where b is the edge about which the rectangle is revolved.
2. $\frac{1}{12}bh^3$, where b is the base and h the altitude.
3. $\frac{1}{4}bh^3$, where b is the base and h the altitude.
4. $\frac{8}{7}a^4$.
5. $\frac{64}{105}a^4$.
6. $\frac{1}{2}\pi a^4$.

7. $\frac{1}{2} Ma^2h$, where h is the altitude.
8. $\frac{2}{5} Ma^2$.
9. $\dfrac{8\pi\rho ab^4}{15}$, where ρ is the density.
10. $\frac{2}{3} Ma^2$, where M is the mass and a the radius.
12. $\frac{3}{2} \pi a^4$

Pages 88, 89

1. $\dfrac{k(b-a)^2}{2a}$.
2. aw ft. lbs., where a is the radius of the earth in feet.
3. $c \ln \dfrac{v_2 - b}{v_1 - b} + \dfrac{a}{v_2} - \dfrac{a}{v_1}$.
4. 25,133 ft. lbs.
5. $\frac{4}{3} \pi\mu Pa$, where a is the radius of the shaft.
6. $\dfrac{k}{2\pi h} \ln \dfrac{b}{a}$, where h is the altitude of the cylinder.
7. $\dfrac{k}{4\pi}\left(\dfrac{1}{a} - \dfrac{1}{b}\right)$, where a and b are the inner and outer radii.
8. $\dfrac{kh}{\pi ab}$, where a and b are the radii of the ends and h the altitude
9. $\dfrac{2\pi i}{r}$.
10. $\dfrac{2i}{c}$.

Pages 95, 96

2. 8.5.
7. 0.785392.
8. 1.26.
9. 4.38.
10. 21.48.
11. 4.27.
12. 0.9045.
13. $a\lambda - \dfrac{a^3\lambda^3}{3\lfloor 3} + \dfrac{a^5\lambda^5}{5\lfloor 5} + \cdots$.
14. 1.91.

Pages 102, 103

1. $\ln \frac{25}{24}$.
2. $\frac{1}{2}\pi a^2$.
3. $\frac{15}{4}$.
4. $\dfrac{\pi}{k}$.
5. -1.
6. $\dfrac{\pi a^2}{4}$.
7. $\frac{2}{3}$.
8. $\frac{10}{3} a^2$.
9. π.
10. $13\frac{1}{3}$.
11. 3π.
12. π.
13. $\frac{1}{6}$.
14. $\frac{2}{3} a^4$.
15. 4.
16. $\frac{8}{5} a^4$.
17. $16 \ln 2 - \frac{75}{8}$.
18. $\frac{2}{3} a^5$.
19. $\frac{2}{5} a^5$.
20. $(\frac{2}{3}, -\frac{2}{3})$.
21. $(\frac{12}{5} a, -2a)$.

Pages 107, 108

1. $\dfrac{\pi a^4}{8}$.
2. $\dfrac{\pi a^2}{2}$.
3. $\dfrac{\pi}{4}$.
4. $\dfrac{\pi a^3}{6}$.
6. $\frac{1}{8} a^4 (2\alpha - \sin 2\alpha)$.
7. $\frac{1}{12} a^4 (6\pi - 8)$.
8. On the bisector at the distance $\dfrac{2a \sin \alpha}{3\alpha}$ from the center.
9. $\frac{75}{4} \pi a^4$.
10. $\frac{1}{2} \pi a^4$.
11. $a^4 (4\pi - \frac{96}{35})$.
12. $3\pi a^4$.
15. $\frac{3}{10} Ma^2$, M being the mass and a the radius of the base.
16. $\frac{1}{9} a^3 (3\pi - 4)$.
17. $\frac{3}{2} \pi a^3$.
18. $\frac{8}{15} \pi \rho a^5$.
19. $(8\sqrt{2} - 9) \dfrac{2\pi a^3}{105}$.

Page 111

1. $3\sqrt{14}$.
2. There are two areas between the planes each equal to $2ma^2$.
3. Two areas are determined each equal to $\pi a^2 \sqrt{2}$.
4. $\frac{1}{4} \pi a^2 \sqrt{3}$.
5. 4.
6. $8a^2$.
7. $\frac{1}{12} \pi a^2 (3\sqrt{3} - 1)$.
8. $a^2 (\pi - 2)$.
9. $8a^2 \tan^{-1} \frac{1}{5} \sqrt{2}$.

Page 116

1. $\frac{1}{6}$.
2. $\frac{1}{30}$.
3. Its distance from the base is $\frac{3}{32} \pi a$.
4. πabc.
6. $\frac{32}{9} a^2 h$.
7. $\frac{4}{15}$.

Page 121

1. $\frac{19}{6} \pi$.
2. $\frac{3}{8} h$, where h is the altitude.
3. π.
4. $\dfrac{\pi a^2 h}{60} (2h^2 + 3a^2)$.
5. $\frac{3}{2} \pi a^3$.
6. On the axis of the cone at the distance $\frac{3}{8} a (1 + \cos \alpha)$ from the vertex.
7. If the two planes are $\theta = \pm \dfrac{\pi}{6}$, the spherical coördinates of the center of gravity are $r = \dfrac{9}{16} a$, $\theta = 0$, $\phi = \dfrac{\pi}{2}$.
8. $\frac{11}{30} \pi a^5$.

Page 125

1. $\dfrac{kM}{c\,(c+l)}$.

2. $\dfrac{2\,kM}{\pi a^2}$.

3. $\dfrac{2\,kMc}{a^2}\left[\dfrac{1}{c}-\dfrac{1}{\sqrt{c^2+a^2}}\right]$.

4. $2\,\pi k\rho h\,(1-\cos\alpha)$, where ρ is the density, h the altitude, and $2\,\alpha$ the vertical angle of the cone.

6. The components along the edge through the corner are each equal to

$$\frac{2\,Mk}{a^2}\left[\frac{\pi}{12}+\ln\frac{2+\sqrt{2}}{1+\sqrt{3}}\right].$$

Pages 130, 131

5. $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-y=0$.

6. $x\,dy-y\,(x+1)\,dx=0$.

7. $\dfrac{d^2y}{dx^2}+y=0$.

8. $x^3\dfrac{d^3y}{dx^3}+6\,x^2\dfrac{d^2y}{dx^2}+4\,x\dfrac{dy}{dx}-4\,y=0$.

9. $y\,dx=x\,dy$.

Pages 141–143

1. $x^2-y^2=cx^2y^2$.
2. $\tan^2 x-\cot^2 y=c$.
3. $y^2+1=c\,(x^2-1)$.
4. $x^2y^2+x^2-y^2=c$.
5. $x^3+x^2y-xy^2-y^3=c$.
6. $y^2=cx^2\,(y^2+1)$.
7. $x^2+y^2=ce^{2ay}$.
8. $xy=c\,(y-1)$.
9. $y=ce^{ax}+\dfrac{1}{b-a}\,e^{bx}$.
10. $y=cx^2-\dfrac{1}{x}$.
11. $y=cx^2e^{-\frac{3}{x}}$.
12. $x^2y=x+cy$.
13. $y=(1-x^2)\,(x+c)$.
14. $y=c\sin x-a$.
15. $7\,x^3=y\,(x^7+c)$.
16. $\dfrac{1}{y^2}=x+\dfrac{1}{2}+ce^{2x}$.
17. $x^4+4\,y\,(x^2-1)^{\frac{3}{2}}=c$.
18. $\ln\,(x^2+xy+y^2)+\dfrac{2}{\sqrt{3}}\tan^{-1}\dfrac{x+2\,y}{x\sqrt{3}}=c$.
19. $x^2-y^2=cx$.
20. $y^2+2\,xy=c$.
21. $x^4-4\,x^3y+y^4=c$.
22. $\dfrac{x}{y^2}=c-e^{-y}$.
23. $e^{\frac{x}{y}}+\ln x=c$.
24. $x+2y+\ln\,(x+y-2)=c$.
25. $y^3=ce^x-x-1$.
26. $e^y=\frac{1}{2}\,e^x+ce^{-x}$.
27. $y=\dfrac{c}{2}-\dfrac{x^2}{2\,c}$.
28. $y=\frac{1}{2}\,x^2+c$, or $y=ce^x$.
29. $y^2=2\,cx+c^2$.
30. $q=Ec\left(1-e^{-\frac{t}{Rc}}\right)$.
31. $i=Ie^{-\frac{R}{L}t}+\dfrac{E}{R^2+\alpha^2L^2}\left[R\sin\alpha t-L\alpha\left(\cos\alpha t-e^{-\frac{R}{L}t}\right)\right]$.

32. $y^3 = 8\,e^{x-2}$.
33. $y^2 = 2\,ax$.
34. $y = cx^2$.
35. $x = a \ln \dfrac{y}{a + \sqrt{a^2 - y^2}} + \sqrt{a^2 - y^2} + c$.
36. $y^2 + (x - c)^2 = a^2$.
37. $y = \dfrac{c}{2}\, e^{\frac{x}{a}} + \dfrac{a^2}{2\,c}\, e^{-\frac{x}{a}}$.
38. $r = e^{\theta}$.
39. $r = c \sin \theta$.
40. $r = a \sec (\theta + c)$.
41. $a\theta = \sqrt{r^2 - a^2} - a \sec^{-1} \dfrac{r}{a} + c$.
42. $y = e^{\frac{x}{k}}$.
43. A circle.
44. A straight line.
45. A circle with the fixed point on its circumference or at its center.
46. The logarithmic spirals $r = ce^{k\theta}$.
47. 0.999964.

Pages 154–156

1. $y = c_1 \ln x - \frac{1}{4}\, x^2 + c_2$.
2. $y = x + c_1 x e^x + c_2$.
3. $y = c_1 e^{ax} + c_2 e^{-ax}$.
4. $y = c_1 \sin ax + c_2 \cos ax$.
5. $t = \displaystyle\int \sqrt{\frac{s}{2\,k + c_1 s}}\, ds + c_2$.
6. $s = \dfrac{1}{a^2} \ln (c_1 e^{abt} - e^{-abt}) + c_2$.
7. $y = \dfrac{c_1}{4}\, x^2 - \dfrac{1}{2\,c_1} \ln x + c_2$.
8. $y = \dfrac{1}{2\,c_1} \left[e^{c_1 x + c_2} + e^{-(c_1 x + c_2)}\right]$.
9. $y = c_1 + c_2 e^{4\,x}$.
10. $y = c_1 e^{6\,x} + c_2 e^{-x}$.
11. $y = (c_1 + c_2 x)\, e^{3\,x}$.
12. $y = c_1 \cos x + c_2 \sin x$.
13. $y = c_1 + c_2 e^{-x} + c_3 e^{3\,x}$.
14. $y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$.
15. $y = e^x\,[c_1 \cos (x\sqrt{2}) + c_2 \sin (x \sqrt{2})]$.
16. $y = e^{-\frac{1}{2}x} \left[c_1 \cos \dfrac{x\sqrt{3}}{2} + c_2 \sin \dfrac{x\sqrt{3}}{2}\right]$.

17. $y = c_1e^x + c_2e^{-x} + c_3e^{x\sqrt{2}} + c_4e^{-x\sqrt{2}}$.
18. $y = (c_1 + c_2x + c_3x^2)\,e^x$.
19. $y = x + 3 + c_1\cos x + c_2\sin x$.
20. $y = c_1e^{2x} + c_2e^{-2x} - \frac{1}{3}e^x$.
21. $y = c_1e^{2x} + c_2e^{-3x} - \frac{1}{6}x^2 - \frac{1}{18}x - \frac{7}{108}$.
22. $y = ce^x - \frac{1}{2}(\sin x + \cos x)$.
23. $y = c_1 + c_2e^{2x} - \frac{1}{2}x^2 + x$.
24. $y = c_1e^{-x} + c_2e^{-5x} + \frac{1}{5}x - \frac{6}{25} + \frac{1}{32}e^{3x}$.
25. $y = c_1e^{ax} + c_2e^{-ax} + \dfrac{x}{2a}e^{ax}$.
26. $y = e^{\frac{1}{2}x}\left[c_1\cos\dfrac{x\sqrt{3}}{2} + c_2\sin\dfrac{x\sqrt{3}}{2}\right] - \dfrac{1}{13}(2\sin 2x + 3\cos 2x)$.
27. $y = c_1e^x + e^{-\frac{1}{2}x}\left[c_1\cos\dfrac{x\sqrt{3}}{2} + c_2\sin\dfrac{x\sqrt{3}}{2}\right] - x^3 + x^2 - 6$.
28. $y = c_1e^x + c_2e^{3x} - \frac{1}{2}e^{2x}\sin x$.
29. $y = c_1e^{3x} + c_2e^{-3x} + \frac{1}{37}e^{3x}(6\sin x - \cos x)$.
30. $y = c_1 + c_2x + c_3x^2 + c_4e^{-x} + \frac{1}{1088}(4\cos 4x - \sin 4x)$.
31. $y = c_1\cos 2x + (c_2 + \frac{1}{4}x)\sin 2x$.
32. $y = e^{-x}(c_1 + c_2x + \frac{1}{2}x^2) + \frac{1}{5}e^x$.
33. $x = c_1\cos t + c_2\sin t + \frac{1}{2}(e^t - e^{-t})$,
    $y = c_1\sin t - c_2\cos t + \frac{1}{2}(e^t - e^{-t})$.
34. $y = c_1\cos t + c_2\sin t - 1$,
    $x = (c_1 + c_2)\cos t + (c_2 - c_1)\sin t - 3$.
35. $x = c_1e^t + c_3e^{-3t}$,
    $y = c_1e^{-t} + 3c_2e^{-3t} + \cos t$.
36. $x = c_1e^t + c_2e^{-t} + c_3\cos t + c_4\sin t$,
    $y = c_1e^t + c_2e^{-t} - c_3\cos t - c_4\sin t$.
37. $y = x$.
38. $2y^{\frac{1}{2}} = x + 2$.
39. $s = \dfrac{g}{k}t + \dfrac{g}{k^2}(e^{-kt} - 1)$.
40. $s = \dfrac{1}{k}\ln\left(\dfrac{e^{t\sqrt{gk}} + e^{-t\sqrt{gk}}}{2}\right)$.
41. $s = b\cos(kt)$.
42. About 7 miles per second.
43. About $42\frac{1}{2}$ minutes.
44. $t = \sqrt{\dfrac{5}{g}}\ln(5 + \sqrt{24})$.
45. $t = \dfrac{3}{\sqrt{g}}\ln(9 + 4\sqrt{5})$.

TABLE OF INTEGRALS

1. $\int u^n\,du = \dfrac{u^{n+1}}{n+1}$, if n is not -1.

2. $\int \dfrac{du}{u} = \ln u.$

3. $\int \dfrac{du}{u^2+a^2} = \dfrac{1}{a}\tan^{-1}\dfrac{u}{a}.$

4. $\int \dfrac{du}{u^2-a^2} = \dfrac{1}{2\,a}\ln\dfrac{u-a}{u+a}.$

5. $\int e^u\,du = e^u.$

6. $\int a^u\,du = \dfrac{a^u}{\ln a}.$

Integrals of Trigonometric Functions

7. $\int \sin u\,du = -\cos u.$

8. $\int \sin^2 u\,du = \frac{1}{2}u - \frac{1}{4}\sin 2\,u = \frac{1}{2}(u - \sin u\cos u).$

9. $\int \sin^4 u\,du = \frac{3}{8}u - \frac{1}{4}\sin 2\,u + \frac{1}{32}\sin 4\,u.$

10. $\int \sin^6 u\,du = \frac{5}{16}u - \frac{1}{4}\sin 2\,u + \frac{1}{48}\sin^3 2\,u + \frac{3}{64}\sin 4\,u.$

11. $\int \cos u\,du = \sin u.$

12. $\int \cos^2 u\,du = \frac{1}{2}u + \frac{1}{4}\sin 2\,u = \frac{1}{2}(u + \sin u\cos u).$

13. $\int \cos^4 u\,du = \frac{3}{8}u + \frac{1}{4}\sin 2\,u + \frac{1}{32}\sin 4\,u.$

14. $\int \cos^6 u\,du = \frac{5}{16}u + \frac{1}{4}\sin 2\,u - \frac{1}{48}\sin^3 2\,u + \frac{3}{64}\sin 4\,u.$

15. $\int \tan u\,du = -\ln\cos u.$

16. $\int \cot u\,du = \ln\sin u.$

17. $\int \sec u \, du = \ln(\sec u + \tan u) = \ln \tan\left(\frac{u}{2} + \frac{\pi}{4}\right).$

18. $\int \sec^2 u \, du = \tan u.$

19. $\int \sec^3 u \, du = \frac{1}{2} \sec u \tan u + \frac{1}{2} \ln(\sec u + \tan u).$

20 $\int \csc u \, du = \ln(\csc u - \cot u) = \ln \tan \frac{u}{2}.$

21. $\int \csc^2 u \, du = -\cot u.$

22. $\int \csc^3 u \, du = -\frac{1}{2} \csc u \cot u + \frac{1}{2} \ln(\csc u - \cot u).$

INTEGRALS CONTAINING $\sqrt{a^2 - u^2}$

23. $\int \sqrt{a^2 - u^2} \, du = \frac{u}{2} \sqrt{a^2 - u^2} + \frac{a^2}{2} \sin^{-1} \frac{u}{a}.$

24. $\int u^2 \sqrt{a^2 - u^2} \, du = \frac{u}{8} (2u^2 - a^2) \sqrt{a^2 - u^2} + \frac{a^4}{8} \sin^{-1} \frac{u}{a}.$

25. $\int \frac{\sqrt{a^2 - u^2}}{u} du = \sqrt{a^2 - u^2} + a \ln \frac{a - \sqrt{a^2 - u^2}}{u}.$

26. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a}.$

27. $\int \frac{u^2 \, du}{\sqrt{a^2 - u^2}} = -\frac{u}{2} \sqrt{a^2 - u^2} + \frac{a^2}{2} \sin^{-1} \frac{u}{a}.$

28. $\int \frac{du}{u \sqrt{a^2 - u^2}} = \frac{1}{a} \ln \frac{a - \sqrt{a^2 - u^2}}{u}.$

29. $\int \frac{du}{u^2 \sqrt{a^2 - u^2}} = -\frac{\sqrt{a^2 - u^2}}{a^2 u}.$

30. $\int (a^2 - u^2)^{\frac{3}{2}} du = \frac{u}{8} (5a^2 - 2u^2) \sqrt{a^2 - u^2} + \frac{3a^4}{8} \sin^{-1} \frac{u}{a}.$

31. $\int \frac{du}{(a^2 - u^2)^{\frac{3}{2}}} = \frac{u}{a^2 \sqrt{a^2 - u^2}}.$

INTEGRALS CONTAINING $\sqrt{u^2 - a^2}$

32. $\int \sqrt{u^2 - a^2} \, du = \frac{u}{2} \sqrt{u^2 - a^2} - \frac{a^2}{2} \ln\left(u + \sqrt{u^2 - a^2}\right).$

33. $\int u^2 \sqrt{u^2 - a^2} \, du = \frac{u}{8} (2u^2 - a^2) \sqrt{u^2 - a^2} - \frac{a^4}{8} \ln\left(u + \sqrt{u^2 - a^2}\right).$

34. $\displaystyle\int \frac{\sqrt{u^2 - a^2}}{u}\, du = \sqrt{u^2 - a^2} - a \sec^{-1} \frac{u}{a}.$

35. $\displaystyle\int \frac{du}{\sqrt{u^2 - a^2}}\, du = \ln \left(u + \sqrt{u^2 - a^2}\right).$

36. $\displaystyle\int \frac{u^2\, du}{\sqrt{u^2 - a^2}} = \frac{u}{2} \sqrt{u^2 - a^2} + \frac{a^2}{2} \ln \left(u + \sqrt{u^2 - a^2}\right).$

37. $\displaystyle\int \frac{du}{u \sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{u}{a}.$

38. $\displaystyle\int \frac{du}{u^2 \sqrt{u^2 - a^2}} = \frac{\sqrt{u^2 - a^2}}{a^2 u}.$

39. $\displaystyle\int (u^2 - a^2)^{\frac{3}{2}}\, du = \frac{u}{8} (2\, u^2 - 5\, a^2) \sqrt{u^2 - a^2} + \frac{3\, a^4}{8} \ln \left(u + \sqrt{u^2 - a^2}\right).$

40. $\displaystyle\int \frac{du}{(u^2 - a^2)^{\frac{3}{2}}} = - \frac{u}{\sqrt{u^2 - a^2}}.$

INTEGRALS CONTAINING $\sqrt{u^2 + a^2}$

41. $\displaystyle\int \sqrt{u^2 + a^2}\, du = \frac{u}{2} \sqrt{u^2 + a^2} + \frac{a^2}{2} \ln \left(u + \sqrt{u^2 + a^2}\right).$

42. $\displaystyle\int u^2 \sqrt{u^2 + a^2}\, du = \frac{u}{8} (2\, u^2 + a^2) \sqrt{u^2 + a^2} - \frac{a^4}{8} \ln \left(u + \sqrt{u^2 + a^2}\right).$

43. $\displaystyle\int \frac{\sqrt{u^2 + a^2}}{u}\, du = \sqrt{u^2 + a^2} + a \ln \frac{\sqrt{u^2 + a^2} - a}{u}.$

44. $\displaystyle\int \frac{du}{\sqrt{u^2 + a^2}} = \ln \left(u + \sqrt{u^2 + a^2}\right).$

45. $\displaystyle\int \frac{u^2\, du}{\sqrt{u^2 + a^2}} = \frac{u}{2} \sqrt{u^2 + a^2} - \frac{a^2}{2} \ln \left(u + \sqrt{u^2 + a^2}\right).$

46. $\displaystyle\int \frac{du}{u \sqrt{u^2 + a^2}} = \frac{1}{a} \ln \frac{\sqrt{u^2 + a^2} - a}{u}.$

47. $\displaystyle\int \frac{du}{u^2 \sqrt{u^2 + a^2}} = - \frac{\sqrt{u^2 + a^2}}{a^2 u}.$

48. $\displaystyle\int (u^2 + a^2)^{\frac{3}{2}}\, du = \frac{u}{8} (2\, u^2 + 5\, a^2) \sqrt{u^2 + a^2} + \frac{3\, a^4}{8} \ln \left(u + \sqrt{u^2 + a^2}\right).$

49. $\displaystyle\int \frac{du}{(u^2 + a^2)^{\frac{3}{2}}} = \frac{u}{a^2 \sqrt{u^2 + a^2}}.$

OTHER INTEGRALS

50. $$\int \sqrt{\frac{px+q}{ax+b}}\,dx$$
$$= \frac{1}{a}\sqrt{(ax+b)(px+q)}$$
$$- \frac{bp-aq}{a\sqrt{ap}} \ln\left(\sqrt{p(ax+b)} + \sqrt{a(px+q)}\right)$$
$$= \frac{1}{a}\sqrt{(ax+b)(px+q)} - \frac{bp-aq}{a\sqrt{-ap}} \tan^{-1} \frac{\sqrt{-ap(ax+b)}}{a\sqrt{px+q}}.$$

51. $$\int e^{ax} \sin bx\,dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2+b^2}.$$

52. $$\int e^{ax} \cos bx\,dx = \frac{e^{ax}(b \sin bx + a \cos bx)}{a^2+b^2}.$$

0–509

N	0	1	2	3	4	5	6	7	8	9
0	......	0.0000	0.6931	1.0986	1.3863	1.6094	1.7918	1.9459	2.0794	2.1972
1	2.3026	2.3979	2.4849	2.5649	2.6391	2.7081	2.7726	2.8332	2.8904	2.9444
2	9957	3.0445	3.0910	3.1355	3.1781	3.2189	3.2581	3.2958	3.3322	3.3673
3	3.4012	4340	4657	4965	5264	5553	5835	6109	6376	6636
4	6889	7136	7377	7612	7842	8067	8286	8501	8712	8918
5	9120	9318	9512	9703	9890	4.0073	4.0254	4.0431	4.0604	4.0775
6	4.0943	4.1109	4.1271	4.1431	4.1589	1744	1897	2047	2195	2341
7	2485	2627	2767	2905	3041	3175	3307	3438	3567	3694
8	3820	3944	4067	4188	4308	4427	4543	4659	4773	4886
9	4998	5109	5218	5326	5433	5539	5643	5747	5850	5951
10	6052	6151	6250	6347	6444	6540	6634	6728	6821	6913
11	7005	7095	7185	7274	7362	7449	7536	7622	7707	7791
12	7875	7958	8040	8122	8203	8283	8363	8442	8520	8598
13	8675	8752	8828	8903	8978	9053	9127	9200	9273	9345
14	9416	9488	9558	9628	9698	9767	9836	9904	9972	5.0039
15	5.0106	5.0173	5.0239	5.0304	5.0370	5.0434	5.0499	5.0562	5.0626	0689
16	0752	0814	0876	0938	0999	1059	1120	1180	1240	1299
17	1358	1417	1475	1533	1591	1648	1705	1761	1818	1874
18	1930	1985	2040	2095	2149	2204	2257	2311	2364	2417
19	2470	2523	2575	2627	2679	2730	2781	2832	2883	2933
20	2983	3033	3083	3132	3181	3230	3279	3327	3375	3423
21	3471	3519	3566	3613	3660	3706	3753	3799	3845	3891
22	3936	3982	4027	4072	4116	4161	4205	4250	4293	4337
23	4381	4424	4467	4510	4553	4596	4638	4681	4723	4765
24	4806	4848	4889	4931	4972	5013	5053	5094	5134	5175
25	5215	5255	5294	5334	5373	5413	5452	5491	5530	5568
26	5607	5645	5683	5722	5759	5797	5835	5872	5910	5947
27	5984	6021	6058	6095	6131	6168	6204	6240	6276	6312
28	6348	6384	6419	6454	6490	6525	6560	6595	6630	6664
29	6699	6733	6768	6802	6836	6870	6904	6937	6971	7004
30	7038	7071	7104	7137	7170	7203	7236	7268	7301	7333
31	7366	7398	7430	7462	7494	7526	7557	7589	7621	7652
32	7683	7714	7746	7777	7807	7838	7869	7900	7930	7961
33	7991	8021	8051	8081	8111	8141	8171	8201	8230	8260
34	8289	8319	8348	8377	8406	8435	8464	8493	8522	8551
35	8579	8608	8636	8665	8693	8721	8749	8777	8805	8833
36	8861	8889	8916	8944	8972	8999	9026	9054	9081	9108
37	9135	9162	9189	9216	9243	9269	9296	9322	9349	9375
38	9402	9428	9454	9480	9506	9532	9558	9584	9610	9636
39	9661	9687	9713	9738	9764	9789	9814	9839	9865	9890
40	9915	9940	9965	9989	6.0014	6.0039	6.0064	6.0088	6.0113	6.0137
41	6.0162	6.0186	6.0210	6.0234	0259	0283	0307	0331	0355	0379
42	0403	0426	0450	0474	0497	0521	0544	0568	0591	0615
43	0638	0661	0684	0707	0730	0753	0776	0799	0822	0845
44	0868	0890	0913	0936	0958	0981	1003	1026	1048	1070
45	1092	1115	1137	1159	1181	1203	1225	1247	1269	1291
46	1312	1334	1356	1377	1399	1420	1442	1463	1485	1506
47	1527	1549	1570	1591	1612	1633	1654	1675	1696	1717
48	1738	1759	1779	1800	1821	1841	1862	1883	1903	1924
49	1944	1964	1985	2005	2025	2046	2066	2086	2106	2126
50	2146	2166	2186	2206	2226	2246	2265	2285	2305	2324
N	0	1	2	3	4	5	6	7	8	9

e = 2.71828

log e =

N	0	1	2	3	4	5	6	7	8	9
50	6.2146	6.2166	6.2186	6.2206	6.2226	6.2246	6.2265	6.2285	6.2305	6.2324
51	2344	2364	2383	2403	2422	2442	2461	2480	2500	2519
52	2538	2558	2577	2596	2615	2634	2653	2672	2691	2710
53	2729	2748	2766	2785	2804	2823	2841	2860	2879	2897
54	2916	2934	2953	2971	2989	3008	3026	3044	3063	3081
55	3099	3117	3135	3154	3172	3190	3208	3226	3244	3261
56	3279	3297	3315	3333	3351	3368	3386	3404	3421	3439
57	3456	3474	3491	3509	3526	3544	3561	3578	3596	3613
58	3630	3648	3665	3682	3699	3716	3733	3750	3767	3784
59	3801	3818	3835	3852	3869	3886	3902	3919	3936	3953
60	3969	3986	4003	4019	4036	4052	4069	4085	4102	4118
61	4135	4151	4167	4184	4200	4216	4232	4249	4265	4281
62	4297	4313	4329	4345	4362	4378	4394	4409	4425	4441
63	4457	4473	4489	4505	4520	4536	4552	4568	4583	4599
64	4615	4630	4646	4661	4677	4693	4708	4723	4739	4754
65	4770	4785	4800	4816	4831	4846	4862	4877	4892	4907
66	4922	4938	4953	4968	4983	4998	5013	5028	5043	5058
67	5073	5088	5103	5117	5132	5147	5162	5177	5191	5206
68	5221	5236	5250	5265	5280	5294	5309	5323	5338	5352
69	5367	5381	5396	5410	5425	5439	5453	5468	5482	5497
70	5511	5525	5539	5554	5568	5582	5596	5610	5624	5639
71	5653	5667	5681	5695	5709	5723	5737	5751	5765	5779
72	5793	5806	5820	5834	5848	5862	5876	5889	5903	5917
73	5930	5944	5958	5971	5985	5999	6012	6026	6039	6053
74	6067	6080	6093	6107	6120	6134	6147	6161	6174	6187
75	6201	6214	6227	6241	6254	6267	6280	6294	6307	6320
76	6333	6346	6359	6373	6386	6399	6412	6425	6438	6451
77	6464	6477	6490	6503	6516	6529	6542	6554	6567	6580
78	6593	6606	6619	6631	6644	6657	6670	6682	6695	6708
79	6720	6733	6746	6758	6771	6783	6796	6809	6821	6834
80	6846	6859	6871	6884	6896	6908	6921	6933	6946	6958
81	6970	6983	6995	7007	7020	7032	7044	7056	7069	7081
82	7093	7105	7117	7130	7142	7154	7166	7178	7190	7202
83	7214	7226	7238	7250	7262	7274	7286	7298	7310	7322
84	7334	7346	7358	7370	7382	7393	7405	7417	7429	7441
85	7452	7464	7476	7488	7499	7511	7523	7534	7546	7558
86	7569	7581	7593	7604	7616	7627	7639	7650	7662	7673
87	7685	7696	7708	7719	7731	7742	7754	7765	7776	7788
88	7799	7811	7822	7833	7845	7856	7867	7878	7890	7901
89	7912	7923	7935	7946	7957	7968	7979	7991	8002	8013
90	8024	8035	8046	8057	8068	8079	8090	8101	8112	8123
91	8134	8145	8156	8167	8178	8189	8200	8211	8222	8233
92	8244	8255	8265	8276	8287	8298	8309	8320	8330	8341
93	8352	8363	8373	8384	8395	8405	8416	8427	8437	8448
94	8459	8469	8480	8491	8501	8512	8522	8533	8544	8554
95	8565	8575	8586	8596	8607	8617	8628	8638	8648	8659
96	8669	8680	8690	8701	8711	8721	8732	8742	8752	8763
97	8773	8783	8794	8804	8814	8824	8835	8845	8855	8865
98	8876	8886	8896	8906	8916	8926	8937	8947	8957	8967
99	8977	8987	8997	9007	9017	9027	9037	9048	9058	9068
100	9078	9088	9098	9108	9117	9127	9137	9147	9157	9167
N	0	1	2	3	4	5	6	7	8	9

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The numbers refer to the pages.

Differentials

$dc = 0$

$d(u+v) = du + dv$

$d(uv) = v\,du + u\,dv.$

$d(cu) = c\,du$

$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$

$du^n = n u^{n-1} du$

$d \sin u = \cos u \, du.$

$d \cos u = -\sin u \, du.$

$d \tan u = \sec^2 u \, du$

$d \cot u = -\csc^2 u \, du$

$d \sec u = \sec u \tan u \, du.$

$d \csc u = -\csc u \cot u \, du.$

$d \sin^{-1} u = \frac{du}{\sqrt{1-u^2}}$

$d \cos^{-1} u = -\frac{du}{\sqrt{1-u^2}}$

$d \tan^{-1} u = \frac{du}{1+u^2}$

$d \cot^{-1} u = -\frac{du}{1+u^2}$

$d \sec^{-1} u = \frac{du}{u\sqrt{u^2-1}}$

$d \csc^{-1} u = -\frac{du}{u\sqrt{u^2-1}}$

$de^u = e^u du$

$da^u = a^u \log_e a \, du.$

$d \log_e u = \frac{du}{u}$

$d \log_a u = \frac{\log_a e \, du}{u}$

$\sin 45^\circ = \cos 45^\circ = \frac{1}{2}\sqrt{2}$

$\tan 45^\circ = \cot 45^\circ = 1$

$\sec 45^\circ = \csc 45^\circ = \sqrt{2}$

$\sin 30^\circ = \cos 60^\circ = \frac{1}{2}.$

$\cos 30^\circ = \sin 60^\circ = \frac{1}{2}\sqrt{3}$

$\tan 30^\circ = \cot 60^\circ = \frac{1}{3}\sqrt{3}$

$\sec 30^\circ = \csc 60^\circ = \frac{2}{3}\sqrt{3}$

$\csc 30^\circ = \sec 60^\circ = 2$

$\cot 30^\circ = \tan 60^\circ = \sqrt{3}$

$\sin^2 A + \cos^2 A = 1$

$\tan A = \frac{\sin A}{\cos A}$

$1 + \tan^2 A = \sec^2 A$

$1 + \cot^2 A = \csc^2 A.$