Sequential Bayesian Updates

February 25, 2019

Bayesian Search Theory

- In the 1960's the U.S. Navy faced a couple of problems.
- They had misplaced the USS *Scorpion* in 1968 and a B-52 bomber crashed in 1966 (along with it's payload, a hydrogen bomb).
- How were these wrecks eventually found?
- Actually, just using the idea of sequential bayesian estimation from the lecture slides!
- Most figures in this example are taken from this Metron presentation.

Scorpion Itinerary

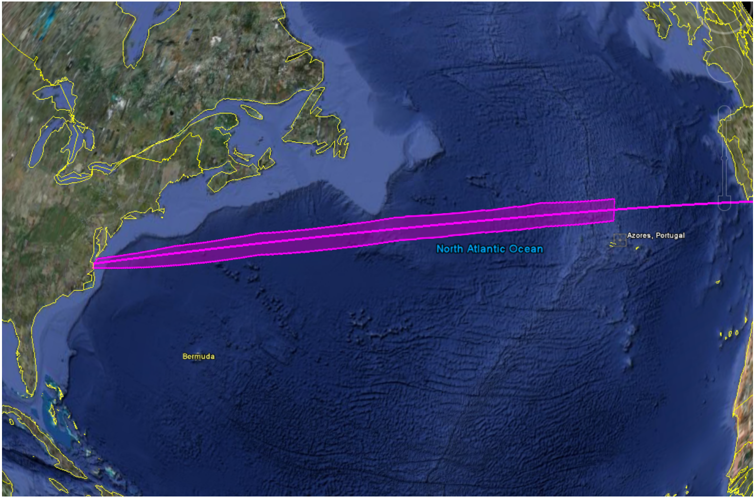


Figure 1: Itinerary and uncertainty region for the USS *Scorpion*.

Mathematical Model for the search

- Divide up the search area into N cells, denoted by $i = 1, 2, ...N$.
- Let $\theta_i \in \{0, 1\}$ denote whether the target is in cell i.
- Define $p_i = \mathrm{p}(\theta_i = 1)$ as the **prior distribution** based on knowledge of the target's location.
- Define $x_i^j \in \{0, 1\}$ as the result of the j^{th} search of cell i.
- Define $q = \mathrm{p}(x_i^j = 1 | \theta_i = 1)$ as the probability of detecting the target in any cell. - i.e. have the same probability of successfully searching every cell.

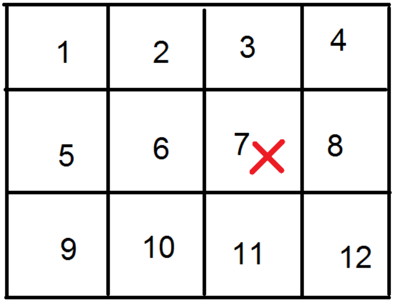
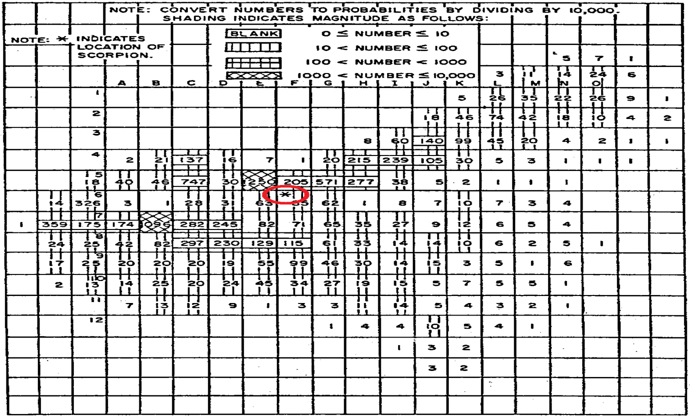


Figure 2: Example of a search grid with the target in one cell

The Original *Scorpion* Prior


NOTE: CONVERT NUMBERS TO PROBABILITIES BY DIVIDING BY 10,000.
SHADING INDICATES MAGNITUDE AS FOLLOWS:

BLANK: 0 ≤ NUMBER ≤ 10
10 < NUMBER ≤ 100
100 < NUMBER < 1000
1000 < NUMBER ≤ 10,000

NOTE: * INDICATES LOCATION OF SCORPION.

			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O		
																5	7	1	
														3	11	14	24	6	
		1											5	26	35	22	26	9	1
		2										18	46	74	42	18	10	4	2
		3								8	60	140	99	45	20	4	2	1	1
		4	2	21	137	16	7	1	20	215	239	105	30	5	3	1	1	1	
		5 18	40	46	747	30	1250	205	571	277	38	5	2	1	1	1			
	14	6 326	3	1	28	31	63	* 55	62	1	8	7	10	7	3	4			
1	359	7 175	174	1096	282	245	82	71	65	35	27	9	12	6	5	4			
	24	8 25	42	82	297	230	129	115	61	33	14	14	10	6	2	5	1		
	17	9 25	20	20	20	19	55	99	46	30	14	15	3	5	1	6			
	2	10 13	14	25	20	24	45	34	27	19	15	5	7	5	5	1			
		11	7	13	12	9	1	3	3	11	14	5	4	3	2	1			
		12							1	4	4	10	5	4	1				
											1	3	2						
												3	2						

FIGURE 2. Overall A Priori distribution for *Scorpion* search

Figure 3: Original Prior Distribution for the *Scorpion* search. The sub was found 200 yards from the original highest probability cell!

- Another key assumption: if the target isn't there we won't find it (no false alarms), i.e.:
$$\mathrm{p}(x_i = 0|\theta_i = 0) = 1$$
- With this assumption the procedure for an Bayesian Search is:
  - Set up a prior $\mathrm{p}(\theta_i)$ for every cell.
  - Search a single cell i.
  - If the target is found, stop.
  - If not, update the probability of the target being located in cell i, $\mathrm{p}(\theta_i = 1|\text{Search did not detect})$.
  - Update the probability in all the other cells based on the search in cell i, $\mathrm{p}(\theta_j = 1|\text{Search did not detect in in cell } i)$.
  - Set the updated probabilities as the new prior.
  - Search the next cell, and repeat.

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- $$\frac{(1-q)p_i}{p_i(1-q) + (1-p_i)(1)} = p_i \frac{1-q}{1-p_i q}$$

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- This changes every other cell as well.

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- But $\mathrm{p}(\text{Search did not detect in cell } i|\theta_j = 1) = 1$, so the update for the other cells is:

$$\mathrm{p}(\theta_j = 1|x_i = 0) = \frac{p_j}{p_i(1-q) + (1-p_i)}$$

Searching the next cell

- Create a new prior over every cell:

$$p_j = \mathrm{p}(\theta_j = 1 | x_i = 0)$$

- Then apply the same formulas as before.
- The key ingredient for a successful search is an accurate prior.
- For shipwrecks priors are usually a combination of a normal distribution centered on the itinerary/last known points and the output of many fluid flow simulations to approximate drift.

Visual Example - Air France Flight 447

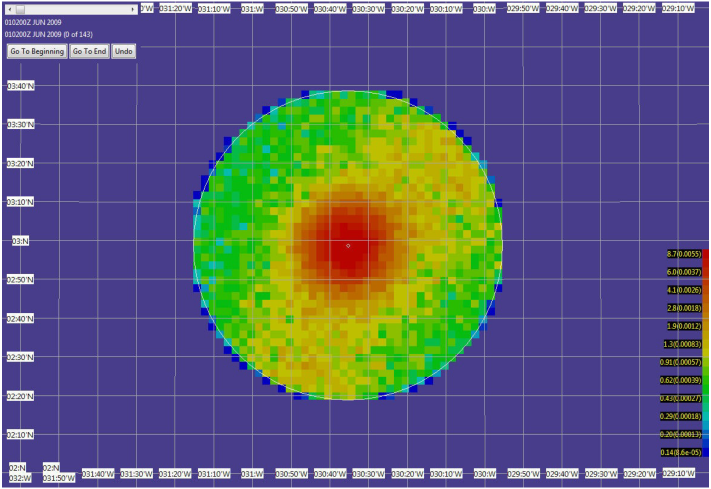

Figure 4: The prior distribution for the Air France Crash generated by Metron

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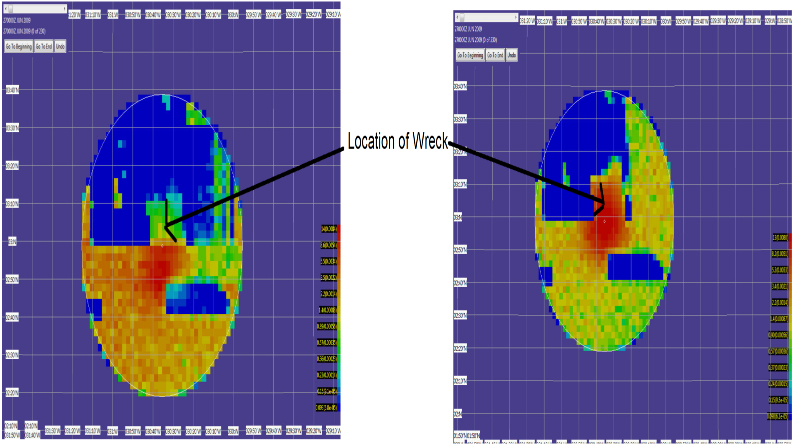

Figure 5: Sequential Posterior Distributions generated after searching for a while. Note that it becomes "better" to search some locations over again as probabilities are updated.

Comments

- How do you choose where to search? How much time do you spend in each cell?
- If you don't assume a continuous path, this is actually a convex optimization problem, so it has a global solution to maximize the probability of finding the target.
- Most searches incorporate several different teams and search methods, with different probabilities of missing the target.
- In the Malaysian Air case, there is not much good prior information available, but searchers did use Bayesian approaches.