ESE 524 - Homework 6

Problems
Assigned date: 04/09/19
Due date: 04/25/19
Total Points: 100 + 20 (extra credits)

1) **Hypothesis Testing**:
Consider the simple binary hypothesis-testing problem

$$f(x|H_0) = \begin{cases} c_0, & |x| \le 1 \\ 0, & \text{otherwise} \end{cases}$$

vs.

$$f(x|H_1) = \begin{cases} c_1(3 - |x|), & |x| \le 3 \\ 0, & \text{otherwise} \end{cases}$$

where c_0 and c_1 are normalizing constants.

a) **(5 pts)** Determine c_0 and c_1.

b) **(10 pts)** Assuming that the priors are equal, i.e., $\pi_0 = \pi_1 = 0.5$, and $0 - 1$ loss is used, find the Bayes' decision rule and specify the range of x for accepting H_0.

c) **(5 pts)** Based on the above Bayes' decision rule, find minimum Bayes risk.

2) **Uniformly Most Powerful Test**:
Consider the hypothesis test

$$H_0 : Y_1, \ldots, Y_n \sim \text{Binomial}(m, \theta_0), \quad \text{vs.}$$
$$H_1 : Y_1, \ldots, Y_n \sim \text{Binomial}(m, \theta), \quad \theta > \theta_0$$

where $\theta_0 \in (0, 1)$ and m are assumed to be known constants, and $Y_1, \ldots, Y_n$ are i.i.d.

a) **(10 pts)** For a fixed $\theta > \theta_0$, what is the general form of the Neyman-Pearson optimal test?

b) **(10 pts)** Does there exist a Uniformly Most Powerful (UMP) test for θ_0 vs $\theta > \theta_0$?

3) **Neyman-Pearson Detector**:

(**20 pts**) Let $\boldsymbol{x} = [x[1], ..., x[N]]^T$. A Neyman-Pearson detector is to be used to distinguish between two hypotheses based on the log-likelihood ratio. The two hypotheses $\mathcal{H}_0$ and $\mathcal{H}_1$ are:

$$\mathcal{H}_0 : p(x[n]|\lambda = \lambda_0) \sim \exp(-\lambda_0)\frac{\lambda_0^x}{x!}$$
$$\mathcal{H}_1 : p(x[n]|\lambda = \lambda_1) \sim \exp(-\lambda_1)\frac{\lambda_1^x}{x!}$$

Find the log-likelihood ratio. Compute the probability of false alarm for size α, i.e., $P_{FA} = \alpha$. If $\lambda_0 = 1$, $\lambda_1 = e$, and $\alpha = 0.1$, what is the threshold value for the likelihood ratio test? (Using MATLAB or any other software will help to compute the threshold value.)

4) **Binary Communication – Bayesian Detection**
A communication source generates binary digits 0 and 1 and transmits the respective signals $(s_t(i))_{t=0}^{N-1}$ for $i = 0, 1$. A sequence of independently identically distributed noise $(n_t(i))_{t=0}^{N-1}$ is added to produce the measurements $(x_t(i))_{t=0}^{N-1}$ with $x_t \sim \mathcal{N}(s_t(i), \sigma^2)$. Test $\mathcal{H}_0 : i = 0$ vs $\mathcal{H}_1 : i = 1$. Assume $\sum_{t=0}^{N-1} s_t^2(i) = E_s$ and $\sum_{t=0}^{N-1} s_t(0)s_t(1) = \rho E_s$, $-1 < \rho < 1$.

   a) **(10 pts)** What is the likelihood detector for the above test?

   b) **(5 pts)** What are the expressions of P_{FA} (false alarm) and P_{D} (detection) for $E_s/\sigma^2 = 10$ and $\rho = 1/2$?

   c) **(5 pts)** Where is the operating point on the ROC for the Bayes detector when $p(H_0) = 7/16$, $p(H_1) = 9/16$, $L(0|1) = L(1|0) = 1$. (Assume $L(0|0) = L(1|1) = 0$).

5) **Matlab Problem, Experimenting with ROC Curves**:
   a) **(5 pts)** Consider the Example "DC Level in AWGN" example in `24_bayesdetex.pdf` on Blackboard. Let $\theta_0 = 5$, $\theta_1 = 10$, and $\sigma_2 = 1$. Recreate Figure 1 from the notes.

   b) **(5 pts)** From a non-bayesian point of view, η' is not a fixed threshold, but is a parameter to choose. Varying η' (in equation (3)) from $-\infty$ to ∞, compute the probability of detection, P_D, the probability of false alarm, P_{FA}, and plot the ROC curve. Change θ_1 to 5.5 and plot the ROC curve again on the same plot. Which θ_1 is easier to distinguish from θ_0?

   In Bayesian Detection, η' is determined by the ratio $\dfrac{\pi_0 L(1|0)}{\pi_1 L(0|1)}$. Choose several combinations of π_0, π_1, $L(0|1)$, $L(1|0)$, and highlight their corresponding P_D's and P_{FA}'s on the ROC curves you created earlier.

   c) **(10 pts)**Repeat parts (a) and (b) for the "deciding between two rates for Poisson measurements" example. Set $\lambda_0 = 5$ with $\lambda_1 = 10$ for the first ROC and $\lambda_1 = 6$ for the second ROC curve.

6) **Extra Credit**:
(**20 pts**) Come up with an example and solution illustrating one or more concepts from class so far. This example should be something you believe would be good to present in class to help other students understand a concept from the lectures. MATLAB (or other software) simulations are encouraged. Problems can be inspired by or explore applications from literature, but should not just copy the results of a paper. We will also accept larger "projects" which may take more time to complete but will be worth more points.