Radar Detection

- Example 32 from "Statistical Methods for Signal Processing" by Hero
- Consider again an (unrealistic) air traffic control radar.
- Prior Knowledge - a plane is as likely to be landing as not.
- Prior Knowledge - we know what a "clean" airplane signal looks like.
- Assumption - it's equally bad to miss a plane as it is to falsely detect one.

Setting up Hypotheses

- In this example we can exploit the fact that we know what a clean signal looks like with a filter that "matches" the incoming signal with the known signal.
- Let $s(t)$ be the signal of a plane coming towards the runway.
- Let $h(t) = s(t - T)$ be the impulse response of the match filter. T is the period that the radar recieves signals, i.e. it repeats it's pulse every T seconds.
- Model the incoming signal by $x(t) = s(t) + w(t)$, where $w(t) \sim N(0, \frac{N_0}{2})$.
- $w(t)$ is "white noise", a continuous analog to $w[n]$ we've seen before.
- $\mathbb{E}(w(t)) = 0$ and $\mathbb{E}(w(t_1)w(t_2)) = \begin{cases} 0 \; t_1 \neq t_2 \\ \frac{N_0}{2} \; t_1 = t_2 \end{cases}$

Output of Matched Filter

- The output of the matched filter is given by convolution:

$$y(T) = \int_0^T h(T-\tau)x(\tau)d\tau = \int_0^T s(\tau)x(\tau)d\tau$$

- This formula comes from the fact that $h(T-\tau) = s((T-tau)-T) = s(\tau)$.
- Integration is a linear transformation, so $y(T) \sim N(\mu, \sigma^2)$.
- Lets figure out the mean and variance of our new random variable for each hypothesis case.

H_0: No plane

- Define the null hypothesis as "no plane": $H_0 : x(t) = 0 + w(t)$.
- Then

$$y(T) = \int_0^T s(\tau)x(\tau)d\tau = \int_0^T s(\tau)w(\tau)d\tau$$

- The Expected value of $Y(T)$ is:

$$\mathbb{E}[y(t)] = \mathbb{E}\left[\int_0^T s(\tau)w(\tau)d\tau\right] = \int_0^T s(\tau)\mathbb{E}[w(\tau)]d\tau = 0$$

- The Variance of $y(T)$ is:

$$\mathbb{E}[(y(T)-\mathbb{E}[y(T)])^2] = \mathbb{E}[y(T)^2] = \mathbb{E}\left[\int_0^T s(\tau_1)w(\tau_1)d\tau_1 \int_0^T s(\tau_2)w(\tau_2)d\tau_2\right] =$$

$$\mathbb{E}\left[\int_0^T \int_0^T s(\tau_1)w(\tau_1)s(\tau_2)w(\tau_2)d\tau_1 d\tau_2\right] = \int_0^T \int_0^T s(\tau_1)s(\tau_2)\mathbb{E}[w(\tau_1)w(\tau_2)]d\tau_1 d\tau_2$$

No Plane Continued

- $\mathbb{E}[w(t)w(\tau)] = \frac{N_0}{2}$ if $t = \tau$, so we only have to look at the case where $\tau_1 = \tau_2$.
- The variance integral then simplifies to:

$$\int_0^T s(\tau)^2 \frac{N_0}{2} d\tau = \frac{N_0}{2} \boldsymbol{S}$$

- Here $\boldsymbol{S} = \int_0^T s(\tau)^2 d\tau$ is the "energy" of the signal.
- Therefore, when there is no plane, the null hypothesis is

$$H_0 : \; f_0(y|H_0) \sim N(0, \frac{N_0}{2}\boldsymbol{S})$$

H_1: Plane coming in

- Define $H_1 : x(t) = s(t) + w(t)$ as the alternative hypothesis that a plane is coming in.
- $y(T) = \int_0^T s(\tau)(s(\tau) + w(\tau))d\tau = \int_0^T s(\tau)^2 d\tau + \int_0^T s(\tau)w(\tau)d\tau$
- The Expected value under H_1 is:

$$\mathbb{E}[y(T)] = \mathbb{E}\left[\int_0^T s(\tau)^2 d\tau\right] + \mathbb{E}\left[\int_0^T s(\tau)w(\tau)d\tau\right] =$$

$$\boldsymbol{S} + 0 = \boldsymbol{S}$$

- The variance under H_1 is:

$$\mathbb{E}[y(T)^2] - \mathbb{E}[y(T)]^2 =$$

$$\mathbb{E}\left[\left(\int_0^T s(\tau_1)^2 d\tau_1 + \int_0^T s(\tau_1)w(\tau_1)d\tau_1\right)\left(\int_0^T s(\tau_2)^2 d\tau_2 + \int_0^T s(\tau_2)w(\tau_2)d\tau_2\right)\right] -$$

- $$S^2 - \mathbb{E}\left[\int_0^T s(\tau_1)^2 d\tau_1 \int_0^T s(\tau_2)^2 d\tau_2\right] + \mathbb{E}\left[\int_0^T s(\tau_1)^2 d\tau_1 \int_0^T s(\tau_2)w(\tau_2) d\tau_2\right] +$$
$$\mathbb{E}\left[\int_0^T s(\tau_1)w(\tau_1) d\tau_1 \int_0^T s(\tau_2)^2 d\tau_2\right] + \mathbb{E}\left[\int_0^T s(\tau_1)w(\tau_1) d\tau_1 \int_0^T s(\tau_2)w(\tau_2) d\tau_2\right] =$$
$$\boldsymbol{S}^2 + 0 + 0 + \frac{N_0}{2}\boldsymbol{S} - \boldsymbol{S}^2 = \frac{N_0}{2}\boldsymbol{S}$$
- So under the alternate hypothesis:
$$H_1 : \ f_1(y|H_1) \sim N(\boldsymbol{S}, \frac{N_0}{2}\boldsymbol{S})$$
- Note: The variances are the same here, but the means are different, so to detect whether or not a plane is coming in we can use the simple hypothesis test.

Bayesian Hypothesis Test

- Set up a bayesian hypothesis test with $L(0|H_1) = L(1|H_0)$ and $P(H_0) = P(H_1) = .5$.
- then the likelihood ratio test becomes:

$$\frac{f_1(y|H_1)}{f_0(y|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P(H_0)L(1|H_0)}{P(H_1)L(0|H_1)}$$

- Substituting in formulas:

$$\frac{\frac{1}{\sqrt{2\pi\frac{N_0}{2}\boldsymbol{S}}}\exp(-\frac{2}{N_0\boldsymbol{S}}(y-\boldsymbol{S})^2)}{\frac{1}{\sqrt{2\pi\frac{N_0}{2}\boldsymbol{S}}}\exp(-\frac{2}{N_0\boldsymbol{S}}(y)^2)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

- Simlifiying:

$$\exp(\frac{2}{N_0\boldsymbol{S}}(y\boldsymbol{S} - 0.5\boldsymbol{S}^2)) \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

Solving for $y(T)$

- Solve for y:

$$\frac{2}{N_0 \boldsymbol{S}}(y\boldsymbol{S} - 0.5\boldsymbol{S}^2) \underset{H_0}{\overset{H_1}{\gtrless}} 0 \rightarrow y = y(T) \underset{H_0}{\overset{H_1}{\gtrless}} 0.5\boldsymbol{S}$$

- Probability of false detection:

$$P_F = P(y(T) > .5\boldsymbol{S}|H_0) = P(\frac{y(T)}{\sqrt{\frac{N_0\boldsymbol{S}}{2}}} > \frac{.5\boldsymbol{S}}{\sqrt{\frac{N_0\boldsymbol{S}}{2}}}|H_0) =$$

$$\int_{\sqrt{2}/2\sqrt{\boldsymbol{S}/N_0}}^{\infty} \frac{1}{\sqrt{2\pi}} exp(-x^2/2)dx = Q(\sqrt{2}/2\sqrt{\boldsymbol{S}/N_0})$$

- Probability of Missing:

$$P_F = P(y(T) < .5\boldsymbol{S}|H_1) = P(\frac{y(T) - \boldsymbol{S}}{\sqrt{\frac{N_0\boldsymbol{S}}{2}}} > \frac{.5\boldsymbol{S} - \boldsymbol{S}}{\sqrt{\frac{N_0\boldsymbol{S}}{2}}}|H_1) = Q(\sqrt{2}/2\sqrt{\boldsymbol{S}/N_0})$$

The last step is true by the symmetry of H_0, H_1

Comments

- Match filters are used widely in communications, radar, etc.
- When the signal is known, you can get extremely low error rates by designing the proper signal shape.
- If the signal is unknown, need a different model.

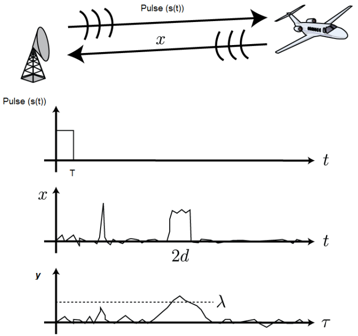

Figure 1: Visualization of the process, likelihood ratio critical point is drawn on y