In Class Example: Quick and Dirty Gradient Descent and Newton rhapson tutorial

April 16, 2019

Sinusoid Composite Hypothesis Examples

- Section 7.6 in Kay's detection theory
- $H_0 : \boldsymbol{x}[n] = \boldsymbol{w}[n]$
- $H_1 : \boldsymbol{x}[n] = Acos(2\pi f_0(n - n_0) + \phi) + \boldsymbol{w}[n]$
- We want to form the hypothesis test when:
  - A is unknown
  - A and ϕ are unknown
  - A, ϕ,and f_0 are unknown
  - All parameters unknown

Unknown Amplitude

- The generalized likelihood ratio test for the unknown amplitude is:

$$\frac{\exp(\frac{-1}{2\sigma^2}\Sigma_{n=0}^{N-1}(x[n]^2 - 2x[n]\hat{A}cos(2\pi f_0(n-n_0)+\phi) + \hat{A}^2\cos(2\pi f_0(n-n_0)+\phi)^2)}{\exp(\frac{-1}{2\sigma^2}\Sigma_{n=0}^{N-1}x[n]^2)}$$

- $\hat{A}$ is the maximum likelihood estimate of A based on x:

$$\hat{A} = \arg\max_{A}\Sigma_{n=0}^{N-1}(x[n]^2 - x[n]Acos(2\pi f_0(n-n_0)+\phi) + A^2\cos(2\pi f_0(n-n_0)+\phi)^2))$$

- Take the derivative with respect to A:

$$\Sigma_{n=0}^{N-1} - 2x[n]\cos(2\pi f_0(n-n_0)+\phi) + 2A\cos(2\pi f_0(n-n_0)+\phi) = 0$$

- This means that $\hat{A} = \frac{\Sigma_{n=0}^{N-1}x[n]\cos(2\pi f_0(n-n_0)+\phi)}{\Sigma_{n=0}^{N-1}\cos(2\pi f_0(n-n_0)+\phi)^2}$

Unknown Amlitude Cont.

- Using the fact that $\Sigma_{n=0}^{N-1} x[n]\cos(2\pi f_0(n-n_0)+\phi) = \hat{A}\Sigma_{n=0}^{N-1}\cos(2\pi f_0(n-n_0)+\phi)^2$ we can rewrite the likelihood ratio:

$$-\Sigma_{n=0}^{N-1} - 2\hat{A}^2\cos(2\pi f_0(n-n_0)+\phi)^2 + \hat{A}^2\cos(2\pi f_0(n-n_0)+\phi)^2 \underset{H_0}{\overset{H_1}{\gtrless}} \log\lambda'$$

$$\hat{A}^2 \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\lambda'}{\Sigma_{n=0}^{N-1}\cos(2\pi f_0(n-n_0)+\phi)^2}$$

- This means that our final test statistic is:

$$\left(\Sigma_{n=0}^{N-1} x[n]\cos(2\pi f_0(n-n_0)+\phi)\right)^2 \underset{H_0}{\gtrless} \lambda'$$

Performance of Only Amplitude Detector

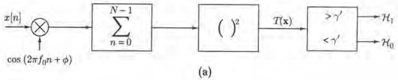

Figure 1: Block Diagram of Detector when Only amplitude is unknown

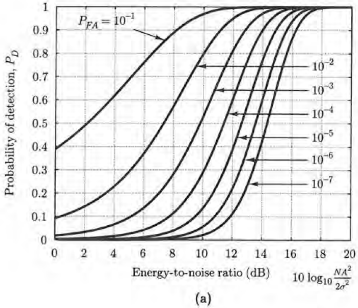

Figure 2: Performance of Detector when Only amplitude is unknown

Amplitude and Phase Unknown

- When Amplitude and Phase are unknown, we have to reduce to the case where either $A > 0$ or $A < 0$, otherwise shifting the phase by π means we have two functions that can produce the same signal.
- Assume $A > 0$, then find $\hat{A}$ and $\hat{\phi}$ from:

$$\arg\max_{A,\phi} \Sigma_{n=0}^{N-1}(x[n]^2 - x[n]Acos(2\pi f_0(n-n_0)+\phi) + A^2\cos(2\pi f_0(n-n_0)+\phi)^2))$$

- Now we have to set the gradient equal to $(0,0)^T$:

$$\nabla \log(p(x;A,\phi)) =$$

$$\begin{bmatrix} \Sigma_{n=0}^{N-1} - 2x[n]\cos(2\pi f_0(n-n_0)+\phi) + 2A\cos(2\pi f_0(n-n_0)+\phi) \\ \Sigma_{n=0}^{N-1} 2x[n]A\sin(2\pi f_0(n-n_0)+\phi) - A^2\sin(2\pi f_0(n-n_0)+\phi) \end{bmatrix}$$

- Using some trig identities we can approximate the solutions with :

$$\hat{A} = \sqrt{(\frac{2}{N}\Sigma_{n=0}^{N-1} x[n]\cos(2\pi f_0 n))^2 + (\frac{2}{N}\Sigma_{n=0}^{N-1} x[n]\sin(2\pi f_0 n))^2}$$

$$\hat{\phi} = \arctan(\frac{\frac{2}{N}\Sigma_{n=0}^{N-1} x[n]\cos(2\pi f_0 n)}{\frac{2}{N}\Sigma_{n=0}^{N-1} x[n]\sin(2\pi f_0 n)})$$

Amlitude and Phase Unknown

- Now the Likelihood ratio is:

$$-1/2\sigma^2\Sigma_{n=0}^{N-1} -2\hat{A}^2\cos(2\pi f_0(n-n_0)+\hat{\phi})^2+\hat{A}^2\cos(2\pi f_0(n-n_0)+\hat{\phi})^2 \underset{H_0}{\gtrless} \log\lambda$$

- Using the substitution $\hat{\alpha_1} = \hat{A}\cos(\hat{\phi})$ and $\hat{\alpha_2} = -\hat{A}sin(\hat{\phi})$ and some more trigonometry we can simplify this expression into:

$$\frac{N}{4\sigma^2}(\hat{\alpha}_1^2+\hat{\alpha}_2^2) \underset{H_0}{\gtrless} \log\lambda$$

- But it turns out that $\hat{\alpha}_1^2+\hat{\alpha}_2^2 = \frac{2^2}{N^2\sigma^2}((\Sigma_{n=0}^{N-1}x[n]\cos(2\pi f_0 n)^2+(\Sigma_{n=0}^{N-1}x[n]\sin(2\pi f_0 n)^2)$, so the test becomes:

$$\frac{1}{N\sigma^2}((\Sigma_{n=0}^{N-1}x[n]\cos(2\pi f_0 n)^2+(\Sigma_{n=0}^{N-1}x[n]\sin(2\pi f_0 n)^2) = \frac{I(f_0)}{\sigma^2} \underset{H_0}{\gtrless} \log\lambda$$

- This is either the sum of two correlators similar to the unknown amplitude case, or something called the periodogram, which estimates $|X(f)|^2$, where $X(f)$ is the fourier transform of $x[n]$.

Amplitude and Phase Unknown

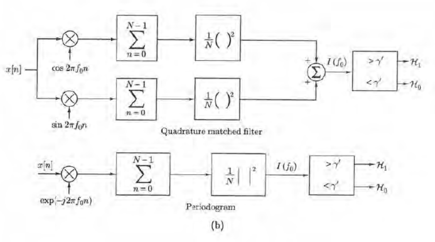

Figure 3: Block Diagram of Detector when Amplitude and Phase are unknown, this is sometimes called an incoherent detector or quadrature matched filter.

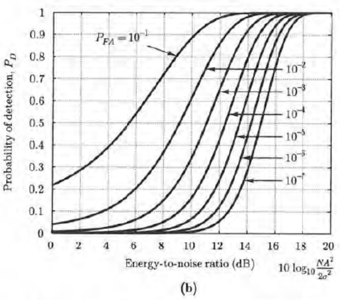

Figure 4: Performance of Detector when Amplitude and Phase are unknown

Performance Comparison

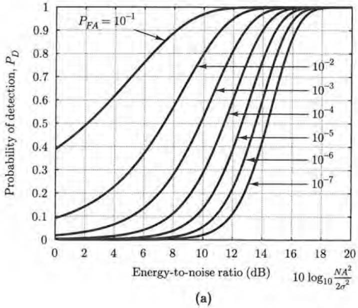

Figure 5: Performance of Detector when Only amplitude is unknown

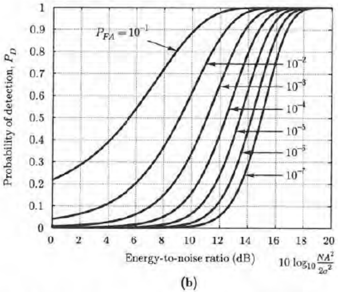

Figure 6: Performance of Detector when Amplitude and Phase are unknown - note that the PD is generally worse than the corresponding points in the first graph.

Amplitude, Phase, and Frequency Unknown

- When the Amplitude, Phase, and Frequency are unknown we need to find the maximum of:

$$\arg\max_{A,\phi} \Sigma_{n=0}^{N-1}(x[n]^2 - x[n]Acos(2\pi f_0(n-n_0)+\phi) + A^2\cos(2\pi f_0(n-n_0)+\phi)^2))$$

  across three variables.
- To do this, define $I(f_0) = \frac{2}{N}((\Sigma_{n=0}^{N-1}x[n]\cos(2\pi f_0 n)^2 + (\Sigma_{n=0}^{N-1}x[n]\sin(2\pi f_0 n)^2)$ as the periodogram of $x[n]$ as a function of f_0.
- To perform the maximization, note that if we know the best $\hat{f}_0$ we can compute $\hat{A}$ and $\hat{\phi}$ the same way as the previous case - this is an example of concentrated likelihood.
- Also, The likelihood under H_0 is not a function of any of the variables so:

$$\max_{A,f_0,\phi} p(x;A,f_0,\phi) = \frac{\max_{A,f_0,\phi} p(x;A,f_0,\phi)}{p(x;H_0)} = \max_{f_0}\frac{p(x;\hat{A},\hat{\phi},f_0)}{p(x;H_0)} =$$

$$\max_{f_0}\log\frac{p(x;\hat{A},\hat{\phi},f_0)}{p(x;H_0)} = \max_{f_0}\frac{I(f_0)}{\sigma_2}$$

Amplitude, Phase, and Frequency Unknown cont.

- So the likelihood ratio test becomes:

$$\max_{f_0} \frac{I(f_0)}{\sigma_2} \underset{H_0}{\gtrless} \log \lambda$$

- This is the same as finding the maximum of the FFT output in matlab and comparing it to your threshold.
- In general the performance gets worse as the frequency inreases.

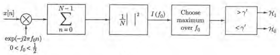

Figure 7: Block Diagram of Detector when Amplitude, Phase, and Frequency are unknown

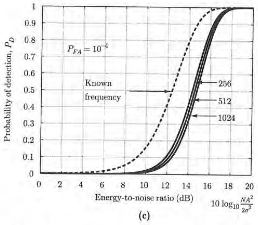

Figure 8: Performance of Detector when Amplitude, Phase, and Frequency are unknown

Unknown Amplitude, Phase, Frequency, and Arrival Time

- The parameter n_0 is the delay of the signal, also called the "Arrival Time". Assume we have a "long" set of data.
- Estimating the arrival time is finding the exact time window $[n_0, n_0 + N - 1]$ steps that the signal is active.
- First, we have to modify our $\hat{A}$ and $\hat{\phi}$ expressions from the previous cases
- Let $\hat{\alpha}_1 = \frac{2}{N}\Sigma_{n=n_0}^{n_0+N-1} x[n]\cos(2\pi\hat{f}_0(n-n_0))$ and $\hat{\alpha}_2 = \frac{2}{N}\Sigma_{n=n_0}^{n_0+N-1} x[n]\sin(2\pi\hat{f}_0(n-n_0))$
- Then, given a frequency and arrival time:

$$\hat{A} = \sqrt{\hat{\alpha}_1^2 + \hat{\alpha}_2^2}$$

$$\hat{\phi} = \arctan(\frac{-\hat{\alpha_1}}{\hat{\alpha_2}})$$

- Our likelihood ratio is still the periodogram, but for a window $[n_0, n_0 + N - 1]$

Unkown Amplitude, Phase, Frequency, and Arrival Time

- If we know the arrival time, we can use the same maximum as in the three paramter case.
- So, starting at $n_0 = 0$ we have to perform a frequency analysis and find the maximum frequency. Then set $n_0 = 1$ and find the same thing. Plot this for every n_0 and find the maximum frequency. This is called the short time periodogram (or short time FFT).
- This can be computed with the spectogram command in matlab, and is widely used.

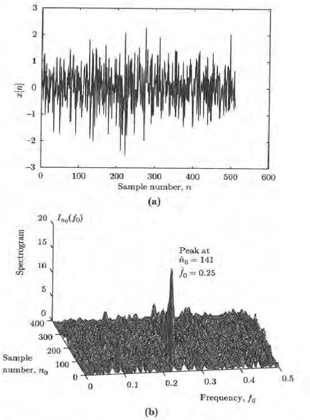

Figure 9: Example of signal and its short term FFT

Comments

- The generalized likelihood ratio test is a combination of MLE and the NP test for simple hypotheses.
- As more parameters are unknown, our detection performance generally goes down - same thing happened in estimation.
- As more parameters are unknown our MLE is more complicated.
- This example featured concentrated likelihood - i.e. we could express $\hat{A}$ and $\hat{\phi}$ in terms of $\hat{f}_0$ and $\hat{n}_0$.
- The Short Time Fourier Transform is commonly used in signal processing, and particularly in audio engineering.