Bayesian Detection Examples

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Contents

DC Level in AWGN: Simple Hypotheses

THE measurements $\boldsymbol{X} = \big(X[n]\big)_{n=0}^{N-1}$ are modeled as

$$(X[n])_{n=0}^{N-1} = \Theta + W[n]$$

where $(W[n])_{n=0}^{N-1}$ is zero-mean white Gaussian noise with known variance σ^2, which implies

$$f_{\boldsymbol{X}\,|\,\Theta}(\boldsymbol{x} \mid \theta) = \frac{1}{\sqrt{(2\pi\sigma^2)^N}} \exp\Big[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-\theta)^2\Big]$$

Our two hypotheses are

$$\begin{aligned} \mathbb{H}_0 : &\quad \Theta = \theta_0 \\ \text{versus} & \\ \mathbb{H}_1 : &\quad \Theta = \theta_1 \end{aligned}$$

where $\theta_1 > \theta_0$ without loss of generality. The Bayes' decision rule is

$$\begin{aligned} \underbrace{\Lambda(\boldsymbol{x})}_{\text{likelihood ratio}} &= \frac{f_{\boldsymbol{X}\,|\,\Theta}(\boldsymbol{x} \mid \theta_1)}{f_{\boldsymbol{X}\,|\,\Theta}(\boldsymbol{x} \mid \theta_0)} \\ &= \frac{(2\pi\sigma^2)^{-N/2}\exp\big[-\sum_{n=0}^{N-1}(x[n]-\theta_1)^2/(2\sigma^2)\big]}{(2\pi\sigma^2)^{-N/2}\exp\big[-\sum_{n=0}^{N-1}(x[n]-\theta_0)^2/(2\sigma^2)\big]} \\ &\underset{\mathbb{H}_0}{\overset{\mathbb{H}_1}{\gtrless}} \eta \triangleq \frac{\mathbb{L}(1\,|\,0)}{\mathbb{L}(0\,|\,1)}\,\frac{\Pr(\mathbb{H}_0)}{\Pr(\mathbb{H}_1)}. \end{aligned} \tag{1}$$

see (6) in handout Bayesdet

Now,

$$\begin{aligned}\ln \Lambda(\boldsymbol{x}) &= \frac{1}{2\sigma^2}\Big[\sum_{n=1}^{N}(x[n]-\theta_0)^2\Big] - \frac{1}{2\sigma^2}\Big[\sum_{n=1}^{N}(x[n]-\theta_1)^2\Big] \\ &= \frac{\theta_1-\theta_0}{\sigma^2}\Big(\sum_{n=1}^{N}x[n]\Big) - \frac{N(\theta_1^2-\theta_0^2)}{2\sigma^2} \end{aligned} \tag{2}$$

which reduces to

$$\frac{1}{N}\sum_{n=1}^{N}x[n] = \overline{x} \underset{\mathbb{H}_0}{\overset{\mathbb{H}_1}{\gtrless}} \eta' \tag{3}$$

where

$$\begin{aligned}\eta' &= \frac{\theta_0+\theta_1}{2} + \frac{\sigma^2}{N(\theta_1-\theta_0)}\ln\eta \\ &= \frac{\theta_0+\theta_1}{2} + \frac{\sigma^2}{N(\theta_1-\theta_0)}\ln\Big[\frac{\pi_0\mathbb{L}(1\,|\,0)}{\pi_1\mathbb{L}(0\,|\,1)}\Big]\end{aligned} \tag{4}$$

see Fig. 1.

⁂ Comments:

- The first term in (4) is the mid-point between the means θ_0 and θ_1 under the two hypotheses.
- The second term in (4) reflects the influence of the losses $\mathbb{L}(1\,|\,0)$ and $\mathbb{L}(0\,|\,1)$ and the prior probabilities π_0 and π_1 of the two hypotheses. This term varies as N^{-1} and therefore decreases as the number of observations grows.

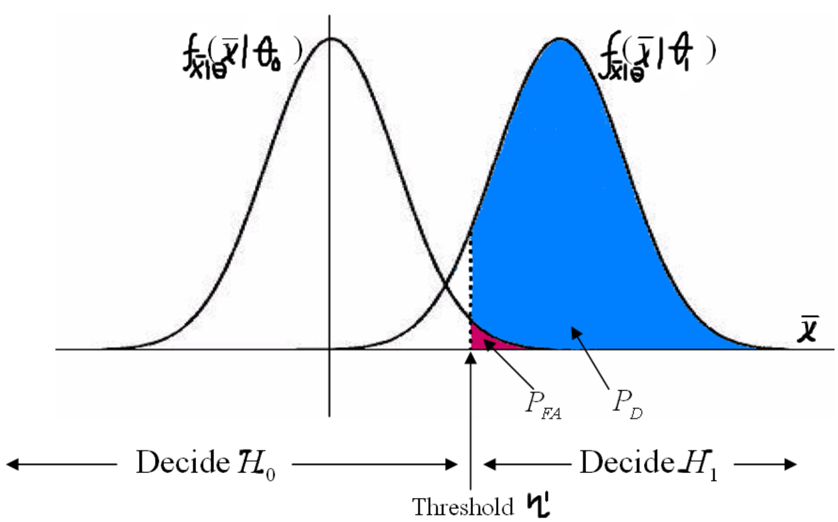

Figure 1: Probabilities of detection and false alarm for DC level in additive white Gaussian noise (AWGN).

Solution using sufficiency

First apply sufficiency. A sufficient statistic for θ is $\bar{x}$; now, find its probability density function (pdf) given θ:

$$f_{\bar{X}|\Theta}(\bar{x} \mid \theta) = \mathcal{N}\big(\bar{x} \,\big|\, \theta, \sigma^2/N\big).$$

The Bayes' rule is

see (16) in handout Bayesdet

$$\frac{f_{\bar{X}|\Theta}(\bar{x} \mid \theta_1)}{f_{\bar{X}|\Theta}(\bar{x} \mid \theta_0)} = \frac{(2\pi\sigma^2/N)^{-1/2}\exp\Big[-\dfrac{1}{2\sigma^2/N}(\bar{x}-\theta_1)^2\Big]}{(2\pi\sigma^2/N)^{-1/2}\exp\Big[-\dfrac{1}{2\sigma^2/N}(\bar{x}-\theta_0)^2\Big]} \underset{\mathbb{H}_0}{\overset{\mathbb{H}_1}{\gtrless}} \eta$$

which leads to (3).

ML rule

For 0–1 loss and equiprobable hypotheses

$$\pi_0 = \Pr(\Theta = \theta_0) = \pi_1 = \Pr(\Theta = \theta_1) = 0.5$$

the second term in (4) is zero; in this case, the Bayes' decision rule is known as the maximum-likelihood (ML) rule:

$$\bar{x} \underset{\mathbb{H}_0}{\overset{\mathbb{H}_1}{\gtrless}} \frac{\theta_0+\theta_1}{2}.$$

In this example, the ML rule *does not* require the knowledge of the noise variance σ^2 to declare its decision. However, the knowledge of σ^2 is *key* to assessing the detection performance.

We now derive the average error probability for the ML rule[1]. First, note that

$$\{\bar{X} \mid \Theta = \theta\} \sim \mathcal{N}\big(\theta, \sigma^2/N\big).$$

[1] For 0–1 loss, Bayes risk becomes average error probability.

- Probability of false alarm for ML rule. $\big\{\bar{X} \mid \Theta = \theta_0\big\} \sim \mathcal{N}\big(\theta_0, \sigma^2/N\big)$ and

$$\begin{aligned} P_{\mathrm{FA}} &= \Pr_{\boldsymbol{X}|\Theta}\Big(\bar{X} > \frac{\theta_0+\theta_1}{2} \,\Big|\, \theta_0\Big) \\ &= \Pr_{\boldsymbol{X}|\Theta}\Big(\frac{\bar{X}-\theta_0}{\sqrt{\sigma^2/N}} > \frac{(\theta_0+\theta_1)/2-\theta_0}{\sqrt{\sigma^2/N}} \,\Big|\, \theta_0\Big) \\ &= Q\big(0.5\sqrt{N(\theta_1-\theta_0)^2/\sigma^2}\big). \end{aligned}$$

$\dfrac{\bar{X}-\theta_0}{\sqrt{\sigma^2/N}}$ is a standard normal random variable for $\theta = \theta_0$

$Q(\cdot)$ is the complementary cumulative distribution function (cdf) of the standard normal distribution

- Probability of detection for ML rule. $\{\bar{X} \mid \Theta = \theta_1\} \sim \mathcal{N}(\theta_1, \sigma^2/N)$ and

$\dfrac{\bar{X}-\theta_1}{\sqrt{\sigma^2/N}}$ is a standard normal random variable for $\theta = \theta_1$

$$
\begin{aligned}
P_{\mathrm{D}} &= \Pr_{\boldsymbol{X}|\Theta}\Big(\overline{X} > \frac{\theta_0+\theta_1}{2} \,\Big|\, \theta_1\Big) \\
&= \Pr_{\boldsymbol{X}|\Theta}\Big(\frac{\overline{X}-\theta_1}{\sqrt{\sigma^2/N}} > \frac{(\theta_0+\theta_1)/2-\theta_1}{\sqrt{\sigma^2/N}} \,\Big|\, \theta_0\Big) \\
&= Q\Big(-0.5\sqrt{\frac{N(\theta_1-\theta_0)^2}{\sigma^2}}\Big) \\
&= \Phi\Big(0.5\sqrt{\frac{N(\theta_1-\theta_0)^2}{\sigma^2}}\Big)
\end{aligned}
$$

because

$$\Phi(-x) = Q(x).$$

Therefore,

$$
\begin{aligned}
P_{\mathrm{M}} &= 1 - P_{\mathrm{D}} \\
&= 1 - \Phi\Big(0.5\sqrt{\frac{N(\theta_1-\theta_0)^2}{\sigma^2}}\Big) \\
&= Q\Big(0.5\sqrt{\frac{N(\theta_1-\theta_0)^2}{\sigma^2}}\Big)
\end{aligned}
$$

Now, use (14) in handout `Bayesdet` to compute the minimum average error probability achieved by the ML test:

$$
\begin{aligned}
\text{av. error prob.} &= 0.5P_{\mathrm{FA}} + 0.5P_{\mathrm{M}} \\
&= 0.5Q\Big(0.5\sqrt{\frac{N(\theta_1-\theta_0)^2}{\sigma^2}}\Big) + 0.5Q\Big(0.5\sqrt{\frac{N(\theta_1-\theta_0)^2}{\sigma^2}}\Big) \\
&= Q\Big(0.5\sqrt{\frac{N(\theta_1-\theta_0)^2}{\sigma^2}}\Big) \\
&= Q\big(0.5\sqrt{d^2}\big)
\end{aligned}
$$

where

$$
\begin{aligned}
d^2 &= \frac{\big[\mathrm{E}(\overline{X}\,|\,\theta_1) - \mathrm{E}(\overline{X}\,|\,\theta_0)\big]^2}{\mathrm{var}(\overline{X}\,|\,\theta_0)} \\
&= \frac{(\theta_1-\theta_0)^2}{\sigma^2/N}
\end{aligned}
$$

is the deflection coefficient, see (3) in handout `introdet`.

Deciding between Two Rates for Poisson Measurements

THE measurements $\boldsymbol{X} = (X[n])_{n=0}^{N-1}$ are independent, identically distributed (i.i.d.) given $\Lambda = \lambda$, modeled as

$$
\begin{aligned}
&\mathbb{H}_0: \quad \{X[n]\,|\,\lambda_0\} \sim \text{Poisson}(\lambda_0) \\
&\text{versus} \\
&\mathbb{H}_1: \quad \{X[n]\,|\,\lambda_1\} \sim \text{Poisson}(\lambda_1)
\end{aligned}
$$

where λ_1 and λ_0 are known constants and

$$\lambda_1 > \lambda_0.$$

We know that

$$T = T(\boldsymbol{x}) = \sum_{n=0}^{N-1} x[n]$$

is a sufficient statistic for inference on λ and that

$$\{T(\boldsymbol{x}) \mid \lambda\} \sim \text{Poisson}(N\lambda).$$

i.e.,

$$f_{T(\boldsymbol{X})|\Lambda}(T(\boldsymbol{x}) \mid \lambda) = \frac{1}{T(\boldsymbol{x})!}\lambda^{T(\boldsymbol{x})}\mathrm{e}^{-N\lambda} \tag{5}$$

and the two hypotheses can be written as:

$$\begin{aligned} \mathbb{H}_0 : &\quad \Lambda = \lambda_0 \\ &\text{versus} \\ \mathbb{H}_1 : &\quad \Lambda = \lambda_1 \end{aligned}$$

The Bayes' decision rule is

see (16) in handout Bayesdet

$$\begin{aligned} \frac{f_{\boldsymbol{X}|\Lambda}(\boldsymbol{x} \mid \lambda_1)}{f_{\boldsymbol{X}|\Lambda}(\boldsymbol{x} \mid \lambda_0)} &= \frac{\lambda_1^{T(\boldsymbol{x})}\mathrm{e}^{-N\lambda_1}}{\lambda_0^{T(\boldsymbol{x})}\mathrm{e}^{-N\lambda_0}} \\ &= \Big(\frac{\lambda_1}{\lambda_0}\Big)^{T(\boldsymbol{x})}\mathrm{e}^{-N(\lambda_1-\lambda_0)} \\ &\underset{\mathbb{H}_0}{\overset{\mathbb{H}_1}{\gtrless}} \eta = \frac{\pi_0\mathbb{L}(1 \mid 0)}{\pi_1\mathbb{L}(0 \mid 1)}. \end{aligned}$$

Now,

$$\ln\Lambda(\boldsymbol{x}) = \ln\frac{\lambda_1}{\lambda_0}T(\boldsymbol{x}) - N(\lambda_1 - \lambda_0)$$

After simple manipulations, we reduce our test to

$\ln(\lambda_1/\lambda_0) > 0$, hence we can divide both sides with it without affecting the inequality

$$\overline{x} \underset{\mathbb{H}_0}{\overset{\mathbb{H}_1}{\gtrless}} \eta'$$

where $\overline{x} \triangleq \sum_{n=0}^{N-1} x[n]/N$ and

$$\begin{aligned} \eta' &= \frac{\lambda_1 - \lambda_0}{\ln(\lambda_1/\lambda_0)} + \frac{\ln\eta}{N\ln(\lambda_1/\lambda_0)} \\ &= \frac{\lambda_1 - \lambda_0}{\ln(\lambda_1/\lambda_0)} + \frac{1}{N\ln(\lambda_1/\lambda_0)}\ln\frac{\pi_0\mathbb{L}(1 \mid 0)}{\pi_1\mathbb{L}(0 \mid 1)}. \end{aligned} \tag{6}$$

✻ COMMENTS:

- The first term in (6) is located between the means λ_0 and λ_1 under the two hypotheses, which is easy to verify by using the inequality

$$1 - \frac{1}{x} \leq \ln x \leq x - 1.$$

- The second term reflects the influence of the losses $\mathbb{L}(1 \mid 0)$ and $\mathbb{L}(0 \mid 1)$ and prior probabilities π_0 and π_1 for the two hypotheses. This terms varies as N^{-1} and therefore decreases as the number of observations becomes very large. It vanishes in the case of 0-1 loss and equiprobable hypotheses.

✻ PROBABILITIES of false alarm and detection. Write our test as

$$T \underset{\mathbb{H}_0}{\overset{\mathbb{H}_1}{\gtrless}} N\eta'.$$

We have

$$P_{\text{FA}} = \sum_{m=\lceil N\eta' \rceil}^{+\infty} \frac{(N\lambda_0)^m}{m!} \mathrm{e}^{-\lambda_0}$$
$$P_{\text{D}} = \sum_{m=\lceil N\eta' \rceil}^{+\infty} \frac{(N\lambda_1)^m}{m!} \mathrm{e}^{-\lambda_1}$$

where $\lceil \gamma \rceil$ denotes the smallest integer larger than or equal to γ.

Noncoherent Detection: Simple Hypotheses with a Nuisance Parameter

CONSIDER detecting on-off keying signals with unknown phase in additive white Gaussian noise. The measurements $\boldsymbol{X} = (X[n])_{n=0}^{N-1}$ are conditionally i.i.d. given $\Theta = \theta$ and $U = u$, modeled as

$$X[n] = \Theta a_n \sin(n\omega + U) + W[n]$$

for $n = 0, 1, \ldots, N-1$, where $\Theta \in \mathrm{sp}_\Theta = \{0, 1\}$, $\mathrm{sp}_\Theta(0) = \{0\}$, and $\mathrm{sp}_\Theta(1) = \{1\}$,

- $W[n] \sim \mathcal{N}(0, \sigma^2)$ is zero-mean white Gaussian noise with known variance σ^2,
- $a_0, a_1, \ldots, a_{N-1}$ is a known amplitude sequence,
- ω is a known carrier frequency,
- U is an unknown phase angle (nuisance parameter), independent of the noise.

Then,

$$f_{\boldsymbol{X}\mid\Theta,U}(\boldsymbol{x} \mid \theta, u) = \frac{1}{\sqrt{(2\pi\sigma^2)^N}} \exp\left(-\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} \left\{x[n] - \theta a_n \sin(n\omega + u)\right\}^2\right) \tag{7}$$

and the two hypotheses can be written as:

$$\begin{aligned} \mathbb{H}_0 : &\qquad \Theta = 0 \\ \text{versus} & \\ \mathbb{H}_1 : &\qquad \Theta = 1. \end{aligned}$$

Assume that Θ and U are independent, i.e.,

$$f_{\Theta,U}(\theta, u) = p_\Theta(\theta) f_U(u)$$

where

$$\pi_0 = p_\Theta(0), \qquad \pi_1 = p_\Theta(1) = 1 - \pi_0$$

describe the prior probability mass function (pmf) of the binary random variable Θ and

$$f_U(u) = \mathrm{U}(u \,|\, 0, 2\pi).$$

uniform pdf

Now, (19) in handout `Bayesdet` becomes

$$\begin{aligned} \Lambda(\boldsymbol{x}) &= \frac{f_{\boldsymbol{X}|\Theta}(\boldsymbol{x} \,|\, 1)}{f_{\boldsymbol{X}|\Theta}(\boldsymbol{x} \,|\, 0)} \\ &= \frac{\int f_{\boldsymbol{X}|\Theta,U}(\boldsymbol{x}|1,u) f_U(u)\,\mathrm{d}u}{\int f_{\boldsymbol{X}|\Theta,U}(\boldsymbol{x}|0,u) f_U(u)\,\mathrm{d}u} \\ &= \frac{\int_0^{2\pi} \frac{1}{2\pi} \exp\Big\{-\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} (x[n] - a_n \sin(n\omega + u))^2\Big\}\,\mathrm{d}u}{\exp\Big(-\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} x^2[n]\Big)} \\ &= \frac{1}{2\pi} \int_0^{2\pi} \exp\Big\{\frac{1}{\sigma^2}\Big[\sum_{n=0}^{N-1} x[n] a_n \sin(n\omega + u) - 0.5 \sum_{n=0}^{N-1} a_n^2 \sin^2(n\omega + u)\Big]\Big\}\,\mathrm{d}u \\ &\underset{\mathbb{H}_0}{\overset{\mathbb{H}_1}{\gtrless}} \eta = \frac{\pi_0 \mathbb{L}(1 \,|\, 0)}{\pi_1 \mathbb{L}(0 \,|\, 1)}. \end{aligned}$$

For dense signal sampling (N large) and ω *not close* to 0 or π, we have

$$\sum_{n=0}^{N-1} a_n^2 \sin^2(n\omega + u) \approx 0.5 \sum_{n=0}^{N-1} a_n^2 = \frac{N}{2} \overline{a^2}$$

see [Poor 1994, eq. (III.B.67)])

where we have used the identity $\sin^2 x = 0.5 - 0.5\cos(2x)$ and

$$\overline{a^2} \triangleq \frac{1}{N} \sum_{n=0}^{N-1} a_n^2.$$

Thus,

$$
\begin{aligned}
\Lambda(\boldsymbol{x}) &\approx \frac{1}{2\pi}\int_0^{2\pi} \exp\Big\{\frac{1}{\sigma^2}\Big[-0.25N\overline{a^2} + \sum_{n=0}^{N-1} x[n]a_n \sin(n\omega+u)\Big]\Big\}\,\mathrm{d}u \\
&= \frac{\exp\big(-0.25N\overline{a^2}/\sigma^2\big)}{2\pi}\int_0^{2\pi} \exp\Big\{\frac{1}{\sigma^2}\sum_{n=0}^{N-1} x[n]a_n \sin(n\omega+u)\Big\}\,\mathrm{d}u \\
&= \frac{\exp\big(-0.25N\overline{a^2}/\sigma^2\big)}{2\pi}\int_0^{2\pi} \exp\left\{\frac{1}{\sigma^2}\sum_{n=0}^{N-1} x[n]a_n \frac{\mathrm{e}^{\mathrm{j}(n\omega+u)}-\mathrm{e}^{-\mathrm{j}(n\omega+u)}}{2\mathrm{j}}\right\}\mathrm{d}u \\
&= \frac{\exp\big(-0.25N\overline{a^2}/\sigma^2\big)}{2\pi}\int_0^{2\pi} \exp\Big\{\frac{1}{2\mathrm{j}}\underbrace{\frac{1}{\sigma^2}\sum_{n=0}^{N-1} x[n]a_n\mathrm{e}^{\mathrm{j}n\omega}}_{r(\boldsymbol{x})\mathrm{e}^{\mathrm{j}\varphi}}\,\mathrm{e}^{\mathrm{j}u} - \frac{1}{2\mathrm{j}}\underbrace{\frac{1}{\sigma^2}\sum_{n=0}^{N-1} x[n]a_n\mathrm{e}^{-\mathrm{j}n\omega}}_{r(\boldsymbol{x})\mathrm{e}^{-\mathrm{j}\varphi}}\,\mathrm{e}^{-\mathrm{j}u}\Big\}\,\mathrm{d}u \\
&= \frac{\exp\big(-0.25N\overline{a^2}/\sigma^2\big)}{2\pi}\int_0^{2\pi} \exp\left[r(\boldsymbol{x})\frac{\mathrm{e}^{\mathrm{j}(u+\varphi)}-\mathrm{e}^{-\mathrm{j}(u+\varphi)}}{2\mathrm{j}}\right]\mathrm{d}u \\
&= \frac{\exp\big(-0.25N\overline{a^2}/\sigma^2\big)}{2\pi}\int_0^{2\pi} \exp\left[r(\boldsymbol{x})\frac{\mathrm{e}^{\mathrm{j}\psi}-\mathrm{e}^{-\mathrm{j}\psi}}{2\mathrm{j}}\right]\mathrm{d}\psi \\
&= \frac{\exp\big(-0.25N\overline{a^2}/\sigma^2\big)}{2\pi}\int_0^{2\pi} \exp\left[r(\boldsymbol{x})\sin\psi\right]\mathrm{d}\psi \\
&= \exp\big(-0.25N\overline{a^2}/\sigma^2\big)\,I_0(r(\boldsymbol{x}))
\end{aligned}
$$

where

$$
r(\boldsymbol{x}) = \frac{1}{\sigma^2}\Big|\sum_{n=0}^{N-1} x[n]a_n\mathrm{e}^{\mathrm{j}n\omega}\Big|, \qquad \varphi = \angle\Big(\frac{1}{\sigma^2}\sum_{n=0}^{N-1} x[n]a_n\mathrm{e}^{\mathrm{j}n\omega}\Big)
$$

and

$$
I_0(r) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{r\sin\psi}\,\mathrm{d}\psi
$$

denotes the zeroth-order modified Bessel function of the first kind. Therefore, the Bayes' decision rule simplifies to

$$
r(\boldsymbol{x}) \underset{\mathbb{H}_0}{\overset{\mathbb{H}_1}{\gtrless}} \sigma^2 I_0^{-1}\Big(\exp\big(0.25N\overline{a^2}/\sigma^2\big)\eta\Big)
$$

which yields the receiver structure in Fig. 2 upon observing that

$$
r^2(\boldsymbol{x}) = \frac{1}{\sigma^2}\underbrace{\Big[\sum_{n=0}^{N-1} x[n]a_n\cos(n\omega)\Big]^2}_{\text{quadrature component}} + \frac{1}{\sigma^2}\underbrace{\Big[\sum_{n=0}^{N-1} x[n]a_n\sin(n\omega)\Big]^2}_{\text{quadrature component}}.
$$

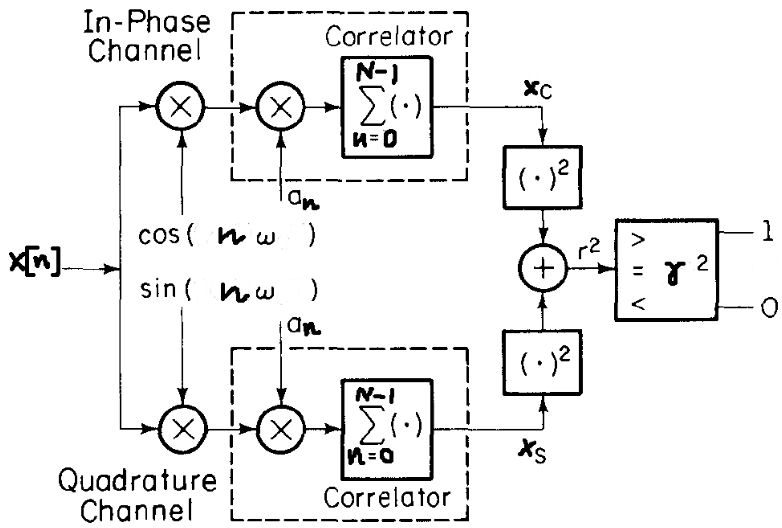

Figure 2: Noncoherent receiver structure.

Coherent Detection in Gaussian Noise: Simple Hypotheses

THE SPACE of the parameter $\boldsymbol{\mu}$ and its partitions are

$$\mathrm{sp}_{\boldsymbol{\mu}} = \{\boldsymbol{\mu}_0, \boldsymbol{\mu}_1\}, \qquad \mathrm{sp}_{\boldsymbol{\mu}}(0) = \{\boldsymbol{\mu}_0\}, \qquad \mathrm{sp}_{\boldsymbol{\mu}}(1) = \{\boldsymbol{\mu}_1\}.$$

The measurement vector $\boldsymbol{x}$ given $\boldsymbol{\mu}$ is modeled using

$$\begin{aligned} f_{\boldsymbol{X}|\boldsymbol{\mu}}(\boldsymbol{x}\,|\,\boldsymbol{\mu}) &= \mathcal{N}(\boldsymbol{x}\,|\,\boldsymbol{\mu}, C) \\ &= \frac{1}{\sqrt{\det(2\pi C)}} \exp\big[-0.5(\boldsymbol{x}-\boldsymbol{\mu})^T C^{-1} (\boldsymbol{x}-\boldsymbol{\mu})\big] \end{aligned} \tag{8}$$

where C is a known positive-definite covariance matrix. Our Bayes' decision rule is:

see (6) in handout `Bayesdet`

$$\begin{aligned} \frac{f_{\boldsymbol{X}|\boldsymbol{\mu}}(\boldsymbol{x}\,|\,\boldsymbol{\mu}_1)}{f_{\boldsymbol{X}|\boldsymbol{\mu}}(\boldsymbol{x}\,|\,\boldsymbol{\mu}_0)} &= \frac{\exp\big[-0.5(\boldsymbol{x}-\boldsymbol{\mu}_1)^T C^{-1} (\boldsymbol{x}-\boldsymbol{\mu}_1)\big]}{\exp\big[-0.5(\boldsymbol{x}-\boldsymbol{\mu}_0)^T C^{-1} (\boldsymbol{x}-\boldsymbol{\mu}_0)\big]} \\ &\mathop{\gtrless}_{\mathbb{H}_0}^{\mathbb{H}_1} \eta = \frac{\pi_0 \mathbb{L}(1\,|\,0)}{\pi_1 \mathbb{L}(0\,|\,1)}. \end{aligned}$$

Therefore,

$$-0.5(\boldsymbol{x}-\boldsymbol{\mu}_1)^T C^{-1} (\boldsymbol{x}-\boldsymbol{\mu}_1) + 0.5(\boldsymbol{x}-\boldsymbol{\mu}_0)^T C^{-1} (\boldsymbol{x}-\boldsymbol{\mu}_0) \mathop{\gtrless}_{\mathbb{H}_0}^{\mathbb{H}_1} \ln\eta$$

i.e.,

$$(\boldsymbol{\mu}_1-\boldsymbol{\mu}_0)^T C^{-1} \big[\boldsymbol{x} - 0.5(\boldsymbol{\mu}_0+\boldsymbol{\mu}_1)\big] \mathop{\gtrless}_{\mathbb{H}_0}^{\mathbb{H}_1} \ln\eta$$

and, finally,

$$T(\boldsymbol{x}) = \boldsymbol{s}^T C^{-1} \boldsymbol{x} \underset{\mathbb{H}_0}{\overset{\mathbb{H}_1}{\gtrless}} \ln \eta + 0.5(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)^T C^{-1}(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_0) \triangleq \gamma$$

where we have defined

$$\boldsymbol{s} \triangleq \boldsymbol{\mu}_1 - \boldsymbol{\mu}_0.$$

NOTE: The DC-level-in-AWGN example is a special case: Use

$$\begin{aligned} C &= I \\ \boldsymbol{\mu}_0 &= \theta_0 \mathbf{1}_{N\times 1} \\ \boldsymbol{\mu}_1 &= \theta_1 \mathbf{1}_{N\times 1} \\ \boldsymbol{s} &= (\theta_1 - \theta_0)\mathbf{1}_{N\times 1} \end{aligned}$$

where $\mathbf{1}_{N\times 1}$ is the $N \times 1$ vector of ones; then, our Bayes' test becomes

$$T(\boldsymbol{x}) = (\theta_1 - \theta_0)\mathbf{1}^T \boldsymbol{x} \underset{\mathbb{H}_0}{\overset{\mathbb{H}_1}{\gtrless}} \ln \eta + 0.5(\theta_1 - \theta_0)^2 N$$

which is the same as (3).

Probabilities of false alarm and detection/miss

GIVEN $\boldsymbol{\mu}$, $T(\boldsymbol{X})$ is a linear combination of Gaussian random variables, implying that it is also Gaussian, with mean and variance:

$$\begin{aligned} \mathrm{E}_{\boldsymbol{X}|\boldsymbol{\mu}}[T(\boldsymbol{x}) \,|\, \boldsymbol{\mu}] &= \boldsymbol{s}^T C^{-1} \boldsymbol{\mu} \\ \mathrm{var}_{\boldsymbol{X}|\boldsymbol{\mu}}[T(\boldsymbol{x}) \,|\, \boldsymbol{\mu}] &= \boldsymbol{s}^T C^{-1} \boldsymbol{s}. \end{aligned}$$

not a function of $\boldsymbol{\mu}$

* PROBABILITY of false alarm.

$$\begin{aligned} P_{\mathrm{FA}} &= \Pr_{\boldsymbol{X}|\boldsymbol{\mu}}\{T(\boldsymbol{x}) > \gamma \,|\, \boldsymbol{\mu}_0\} \\ &= \Pr\{\boldsymbol{X}|\boldsymbol{\mu}\} \frac{T(\boldsymbol{x}) - \boldsymbol{s}^T C^{-1}\boldsymbol{\mu}_0}{\sqrt{\boldsymbol{s}^T C^{-1} \boldsymbol{s}}} > \frac{\gamma - \boldsymbol{s}^T C^{-1}\boldsymbol{\mu}_0}{\sqrt{\boldsymbol{s}^T C^{-1} \boldsymbol{s}}} \\ &= Q\left(\frac{\gamma - \boldsymbol{s}^T C^{-1}\boldsymbol{\mu}_0}{\sqrt{\boldsymbol{s}^T C^{-1} \boldsymbol{s}}}\right) \end{aligned}$$

$\frac{T(\boldsymbol{x}) - \boldsymbol{s}^T C^{-1}\boldsymbol{\mu}_0}{\sqrt{\boldsymbol{s}^T C^{-1} \boldsymbol{s}}}$ is a standard normal random variable

* PROBABILITY of detection.

$$\begin{aligned} P_{\mathrm{D}} &= 1 - P_{\mathrm{M}} \\ &= \Pr_{\boldsymbol{X}|\boldsymbol{\mu}}\{T(\boldsymbol{x}) > \gamma | \boldsymbol{\mu}_1\} \\ &= \Pr_{\boldsymbol{X}|\boldsymbol{\mu}}\left\{\frac{T(\boldsymbol{x}) - \boldsymbol{s}^T C^{-1}\boldsymbol{\mu}_1}{\sqrt{\boldsymbol{s}^T C^{-1} \boldsymbol{s}}} > \frac{\gamma - \boldsymbol{s}^T C^{-1}\boldsymbol{\mu}_1}{\sqrt{\boldsymbol{s}^T C^{-1} \boldsymbol{s}}}\right\} \\ &= Q\left(\frac{\gamma - \boldsymbol{s}^T C^{-1}\boldsymbol{\mu}_1}{\sqrt{\boldsymbol{s}^T C^{-1} \boldsymbol{s}}}\right). \end{aligned}$$

$\frac{T(\boldsymbol{x}) - \boldsymbol{s}^T C^{-1}\boldsymbol{\mu}_1}{\sqrt{\boldsymbol{s}^T C^{-1} \boldsymbol{s}}}$ is a standard normal random variable

0–1 loss & equiprobable hypotheses

FOR 0-1 loss and practical case of equiprobable hypotheses

$$\pi_0 = \pi_1 = 0.5 \tag{9}$$

our Bayes' test simplifies to the ML test

$$T(\boldsymbol{x}) \underset{\mathbb{H}_0}{\overset{\mathbb{H}_1}{\gtrless}} 0.5(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)^T C^{-1}(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_0) = 0.5\boldsymbol{s}^T C^{-1}(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_0) = \gamma$$

and the probabilities of false alarm and detection/miss simplify as well:

$$P_{\text{FA}} = P_{\text{M}} = Q\Big(0.5\sqrt{\boldsymbol{s}^T C^{-1}\boldsymbol{s}}\Big)$$

In this case, the average error probability is

$$\begin{aligned} P_{\text{av}} &= 0.5P_{\text{FA}} + 0.5P_{\text{M}} \\ &= Q\big(0.5\sqrt{\boldsymbol{s}^T C^{-1}\boldsymbol{s}}\big) \\ &= Q\big(0.5\sqrt{d^2}\big) \end{aligned}$$

where

$$\begin{aligned} d^2 &= \frac{\big[\mathrm{E}_{\boldsymbol{X}\mid\boldsymbol{\mu}}\big(\boldsymbol{s}^T C^{-1}\boldsymbol{X} \mid \boldsymbol{\mu}_1\big) - \mathrm{E}_{\boldsymbol{X}\mid\boldsymbol{\mu}}\big(\boldsymbol{s}^T C^{-1}\boldsymbol{X} \mid \boldsymbol{\mu}_0\big)\big]^2}{\mathrm{var}_{\boldsymbol{X}\mid\boldsymbol{\mu}}\big(\boldsymbol{s}^T C^{-1}\boldsymbol{X} \mid \boldsymbol{\mu}_0\big)} \\ &= \boldsymbol{s}^T C^{-1}\boldsymbol{s} \\ &= (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)^T C^{-1}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0) \end{aligned}$$

is the deflection coefficient, see (3) in handout `introdet`.

Acronyms

AWGN additive white Gaussian noise. 2, 10

cdf cumulative distribution function. 3

i.i.d. independent, identically distributed. 4, 6

ML maximum-likelihood. 3, 4, 11

pdf probability density function. 3, 7

pmf probability mass function. 6, 7

References

Poor, H. Vincent (1994). *An Introduction to Signal Detection and Estimation*. 2nd ed. New York: Springer (cit. on p. 7).