ESE 524 - Homework 1

Assigned date: 01/22/19
Due Date: 02/05/19
Total Points: 100

(1) **Transformation of Random Variables**

a) (**10 pts**) Let $X \sim \text{Unif}[0,1]$ be a uniform random variable. Find the distribution of $Y = -\ln(X)$ where $\ln(\cdot)$ is the natural log.

b) (**10 pts**) Let X and Y be independent univariate $\mathcal{N}(0,1)$ random variables. Let R denote the length of the vector $[X, Y]'$, and let Θ denote the angle the vector makes with the x-axis. In other words, if X and Y are the Cartesian coordinates of a random point in the plane, then $R \geq 0$ and $-\pi < \Theta \leq \pi$ are the corresponding polar coordinates. Find the joint density of R and Θ.

(2) **Probability**

Let $X \sim \text{Unif}[0,1]$ be a uniformly-distributed random variable. Suppose we know $X + Y = 1$ in advance, then

a) **(5 pts)** Derive the distribution of Y.
b) **(10 pts)** Derive the distribution of $Z = \max\{X, Y\}$.
c) **(5 pts)** Compute the expectation of Z and $M = \min\{X, Y\}$.

(3) **Estimator performance**

Let $X_1, \dots, X_N$ be N independent and identically distributed (i.i.d.) samples drawn from $\mathcal{N}(\mu, \sigma^2)$, where the mean μ is known in advance, while the variance σ^2 is unknown. We have three estimators to estimate the variance,

$$\hat{\sigma}_1^2 = \frac{1}{N}\sum_{i=1}^{N}(X_i - \mu)^2,$$

$$\hat{\sigma}_2^2 = \frac{1}{N}\sum_{i=1}^{N}\left(X_i - \frac{1}{N}\sum_{j=1}^{N}X_j\right)^2,$$

$$\hat{\sigma}_3^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(X_i - \frac{1}{N}\sum_{j=1}^{N}X_j\right)^2.$$

a) **(15 pts)** Check if $\hat{\sigma}_1^2$, $\hat{\sigma}_2^2$, and $\hat{\sigma}_3^2$ are unbiased.
b) **(5 pts)** For any biased estimator above, check if it is asymptotically unbiased.

(4) **Sufficient Statistics**

Find a sufficient statistic for the following distributions.

a) **(10 pts) Normal distribution**

We consider a joint normal distribution for which the mean μ is unknown, but the variance σ^2 is known:

$$f(x_1, \dots, x_n|\mu) = (2\pi)^{-n/2}\sigma^{-n}\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2\right)$$

what about the case when neither σ nor μ is known?

b) **(5 pts) Uniform distribution**

Now suppose the X_is are uniformly distributed on $[0, \theta]$ where θ is unknown, with the joint density given as

$$f(x_1, \dots, x_n|\theta) = \theta^{-n}\mathbb{I}(x_i \leq \theta, \forall i)$$

where $\mathbb{I}(\cdot)$ is the indicator function.

c) **(5 pts) Gamma distribution**
Now suppose X_is have gamma distribution with β known and α unknown:

$$f(x_1, \dots, x_n|\alpha) = \frac{\beta^{n\alpha}}{\Gamma(\alpha)^n}(\prod_{i=1}^{n} x_i^{\alpha-1}) \exp(-\beta \sum_{i=1}^{n} x_i)$$

(5) **Matlab Problem: Exploring Bias**
Let $\mathbf{x} = [x[0], x[1], ..., x[N-1]]$ be i.i.d. samples from a Normal distribution with mean μ and variance σ^2. In an algebraic mishap, you are given estimators of the form

$$\hat{\mu} = \frac{1}{N-1} \sum_{n=0}^{N-1} x[n]$$

$$\hat{\sigma}^2 = \frac{1}{N} \sum_{n=0}^{N-1} (x[n] - \frac{1}{N} \sum_{n=0}^{n-1} x[n])^2$$

a) **(5 pts)** Compute the theoretical bias of $\hat{\mu}$. Note that you have already computed the bias of the variance estimator in a previous problem.

b) **(5 pts)** For a fixed variance, $\sigma^2 = 1$, vary μ from $0, 10, 20, 30, ..., 100$. Generate 1000 random samples $\mathbf{x}$ of length N=50 in MATLAB and compute the estimator of μ for each realization. Compute the average value of the estimator, and create a table comparing the true value of μ and the bias of $\hat{\mu}$.

c) **(5 pts)** For a fixed mean $\mu = 0$, vary σ^2 from $1, 5, 10, 15, ..., 50$. Again generate 1000 random samples of length N=50 and compute the estimator of σ^2 for each realization. Compute the average value and variacne of the estimator and create a table comparing the true value of σ^2, the average estimate of σ^2, and the estimator variance for each value of σ^2.

d) **(5 pts)** Fix $\mu = 10$ and $\sigma^2 = 5$. Generate 1000 random samples for each N from $10, 50, 100, 200, ..., 1000$, and compute the average and variance of both estimators. What happens to the estimator bias and variance as N increases? Does the variance approach the Cramer Rao bound?