ESE 524 - Homework 7

Problems
Assigned date: 04/16/19
Due date: 04/25/19
Total Points: 100 + 20 (extra credits)

1) **Poisson distribution - Composite Hypothesis Test**
Suppose $X_1, X_2, \ldots, X_N$ are i.i.d. random samples, and they follow Poisson(λ). Suppose we have the following hypothesis test:

$$\mathcal{H}_0 : \lambda = \lambda_0$$
$$\mathcal{H}_1 : \lambda \neq \lambda_0$$

where λ_0 is a known value.

a) (**15 pts**) Give the generalized likelihood ratio test (GLRT), Wald test, and Rao test.
b) (**10 pts**) Using the central limit theorem (CLT) and the Rao test you have in a), show that under $\mathcal{H}_0$, the test statistic for Rao test follows χ_1^2 distribution asymptotically.

Hint: For the Rao test, the restricted estimate of λ (under $\mathcal{H}_0$) is λ_0 in this case.

2) **Binomial Distribution - Composite Hypothesis Test**
The number of successes in n trials is to be used to test the null hypothesis that the parameter θ of a binomial population equals $\frac{1}{2}$ against the alternative that it doesn't equal $\frac{1}{2}$.

a) (**10 pts**) Find an expression for the likelihood ratio statistic.

b) (**5 pts**) Use the result of part (a) to show that the critical region of the likelihood ratio test can be written as

$$x \cdot \ln(x) + (n - x) \cdot \ln(n - x) \geq K,$$

where x is the observed number of successes, and K is a constant that depends on the size of the critical region.

c) (**10 pts**) Study the minimum and the symmetry of the function $f(x) = x \cdot \ln(x) + (n - x) \cdot \ln(n - x)$, and show that the critical region of this likelihood ratio test can also be written as:

$$|x - \frac{n}{2}| \geq K',$$

where K' is also a constant.

3) **Stock Market Analysis - Classical Linear Model**
(**25 pts**) It is desired to detect trends in stock market data. To do so we assume that the data are modeled as

$$x[n] = A + Bn + w[n], \quad n = 0, 1, \ldots, N - 1$$

where $w[n]$ is white Gaussian noise with variance σ^2. The average stock price A is unknown but is of no interest to us. More importantly, we wish to test whether $B = 0$ or $B \neq 0$, i.e., that a trend is present. Find the GLRT statistic for this problem.

Hint: Refer to p. 21 of l7.pdf. To get full points, derive the expression of $T(\boldsymbol{x})$ as

$$T(\boldsymbol{x}) = \frac{(N \sum nx[n] - \sum n \sum x[n])^2}{N\sigma^2[N \sum n^2 - (\sum n)^2]}.$$

4) **Uniformly Most Powerful Test**
Let x have a density given by:

$$p(x;\theta) = \frac{1+\theta x}{2}$$

for $-1 \leq x \leq 1$.

a) (**5 pts**) If the hypotheses are:

$$\mathcal{H}_0 : \theta = \theta_0$$
$$\mathcal{H}_1 : \theta = \theta_1,$$

where $\theta_0 \in [-1, 0]$ and $\theta_1 \in [0, 1]$ are known, find the likelihood ratio test and the threshold corresponding to the level α.

b) (**5 pts**) Is there a uniformly most powerful test for the hypotheses:

$$\mathcal{H}_0 : \theta = 0$$
$$\mathcal{H}_1 : \theta > 0.$$

If yes, find the UMP test.

c) (**15 pts**) Find the generalized likelihood ratio test for the hypotheses:

$$\mathcal{H}_0 : \theta \leq 0$$
$$\mathcal{H}_1 : \theta > 0,$$

and find the threshold for a level α.

5) **Extra Credit**:
(**20 pts**) Come up with an example and solution illustrating one or more concepts from class so far. This example should be something you believe would be good to present in class to help other students understand a concept from the lectures. MATLAB (or other software) simulations are encouraged. Problems can be inspired by or explore applications from literature, but should not just copy the results of a paper. We will also accept larger ”projects” which may take more time to complete but will be worth more points.