Detecting Parametric Signals in Noise Having Exactly Known Pdf/Pmf

Reading:

- Ch. 5 in Kay-II.
- (Part of) Ch. III.B in Poor.

Detecting Parametric Signals in Noise w Having Exactly Known Pdf/Pmf $p_w(w)$ (Bayesian Decision-theoretic Approach)

Consider the simple binary signal-detection problem:

$$\begin{aligned} \mathcal{H}_0 : &\qquad \boldsymbol{x} = \boldsymbol{\mu}_0(\varphi) + \boldsymbol{w} \quad \text{versus} \\ \mathcal{H}_1 : &\qquad \boldsymbol{x} = \boldsymbol{\mu}_1(\varphi) + \boldsymbol{w} \end{aligned}$$

where

- $\boldsymbol{\mu}_0(\varphi)$ and $\boldsymbol{\mu}_1(\varphi)$ are *known* vector-valued functions of the nuisance parameter φ and
- the noise probability density or mass function (pdf/pmf) $p_{\boldsymbol{w}}(\boldsymbol{w})$ is *exactly known*.

Recall our discussion on handling nuisance parameters on pp. 17–18 of handout # 5. Since we have simple hypotheses, we need to specify the Bernoulli prior pmf for the two hypotheses, using prior probabilities

$$\pi_0, \quad \pi_1 = 1 - \pi_0 \quad \text{(the Bernoulli pmf)}.$$

Specializing the result in eq. (15) in handout # 5 (where we have assumed that the hypotheses and φ are independent *a* *priori*, see eq. (14) in handout # 5) to the above scenario, we obtain the following Bayes' decision rule:

$$\Lambda(\boldsymbol{x}) = \underbrace{\frac{\int p_{\boldsymbol{w}}\big(\boldsymbol{x} - \boldsymbol{\mu}_1(\varphi)\big) \overbrace{\pi(\varphi)}^{\text{prior pdf of } \varphi} d\varphi}{\int p_{\boldsymbol{w}}\big(\boldsymbol{x} - \boldsymbol{\mu}_0(\varphi)\big)\, \pi(\varphi)\, d\varphi}}_{\text{integrated likelihood ratio}} \overset{\mathcal{H}_1}{\gtrless} \frac{\pi_0\, \mathrm{L}(1\,|\,0)}{\pi_1\, \mathrm{L}(0\,|\,1)}. \tag{1}$$

Example. Detection of on-off keying signals with unknown phase in additive white Gaussian noise (AWGN): Choose AWGN with $\Sigma = \sigma^2 I$ and known noise variance σ^2, $\boldsymbol{\mu}_0(\varphi) = \mathbf{0}$, and

$$\begin{aligned} \boldsymbol{\mu}_1(\varphi) &= \boldsymbol{s}(\varphi) = \begin{bmatrix} s[0, \varphi] \\ s[1, \varphi] \\ \vdots \\ s[N-1, \varphi] \end{bmatrix} \\ &= \begin{bmatrix} a_0 \sin(0 \cdot \omega_c + \varphi) \\ a_1 \sin(1 \cdot \omega_c + \varphi) \\ \vdots \\ a_{N-1} \sin((N-1) \cdot \omega_c + \varphi) \end{bmatrix} \end{aligned}$$

where $a_1, a_2, \ldots, a_N$ is a known amplitude sequence, ω_c is known carrier frequency, and φ is an unknown phase angle, independent of the noise, following $\pi(\varphi) = \mathrm{uniform}(0, 2\pi)$.

Now, (1) reduces to

$$\Lambda(\boldsymbol{x}) = \frac{p(\boldsymbol{x} \,|\, \mathcal{H}_1)}{p(\boldsymbol{x} \,|\, \mathcal{H}_0)}$$

$$= \frac{\int_0^{2\pi} \frac{1}{2\pi} p(\boldsymbol{x} \,|\, \mathcal{H}_1, \varphi)\, d\varphi}{\int_0^{2\pi} \frac{1}{2\pi} \underbrace{p(\boldsymbol{x} \,|\, \mathcal{H}_0, \varphi)}_{\text{indep. of } \varphi}\, d\varphi}$$

$$= \frac{\int_0^{2\pi} \frac{1}{2\pi} \exp\{-\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} (x[n] - s[n,\varphi])^2\}\, d\varphi}{\exp\{-\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} (x[n])^2\}}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \exp\left\{ \frac{1}{\sigma^2} \Big[\Big(\sum_{n=0}^{N-1} x[n] s[n,\varphi] \Big) - \tfrac{1}{2} \sum_{n=0}^{N-1} (s[n,\varphi])^2 \Big] \right\} d\varphi.$$

We prefer to choose ω_{c} equal to an integer multiple of $2\pi/N$. For dense signal sampling (N large) and ω_{c} *not close* to 0 or π, we have [see eq. (III.B.67) in Poor]:

$$\sum_{n=0}^{N-1} (s[n,\varphi])^2 = \sum_{n=0}^{N-1} a_n^2 \sin^2(n \cdot \omega_{\mathrm{c}} + \varphi) \approx \tfrac{1}{2} \sum_{n=0}^{N-1} a_n^2 = \frac{N}{2} \overline{a^2}$$

where we have used the identity $\sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x$ and the following definition:

$$\overline{a^2} \triangleq \frac{1}{N} \cdot \sum_{n=0}^{N-1} a_n^2.$$

Thus,

$$\Lambda(\boldsymbol{x}) \approx \frac{1}{2\pi} \int_0^{2\pi} \exp\Big\{\frac{1}{\sigma^2}\Big[(\sum_{n=0}^{N-1} x[n]s[n,\varphi]) - \tfrac{1}{4}\,N\,\overline{a^2}\Big]\Big\}\,d\varphi$$

$$= \frac{\exp\Big(-\frac{1}{4}\,\frac{N\,\overline{a^2}}{\sigma^2}\Big)}{2\pi} \cdot \int_0^{2\pi} \exp\Big\{\frac{1}{\sigma^2}\,(\sum_{n=0}^{N-1} x[n]s[n,\varphi])\Big\}\,d\varphi$$

$$= \frac{\exp\Big(-\frac{1}{4}\,\frac{N\,\overline{a^2}}{\sigma^2}\Big)}{2\pi} \cdot$$

$$\int_0^{2\pi} \exp\Big\{\frac{1}{\sigma^2}\,\sum_{n=0}^{N-1} x[n]\,a_n \cos\big(n\,\omega_{\mathrm{c}} + \varphi - \tfrac{1}{2}\,\pi\big)\Big\}\,d\varphi$$

$$= \frac{\exp\Big(-\frac{1}{4}\,\frac{N\,\overline{a^2}}{\sigma^2}\Big)}{2\pi}$$

$$\cdot \int_0^{2\pi} \exp\Big(\frac{1}{\sigma^2}\,\mathrm{Re}\Big\{\sum_{n=0}^{N-1} x[n]\,a_n e^{j\,(n\,\omega_{\mathrm{c}}+\varphi-\frac{1}{2}\,\pi)}\Big\}\Big)\,d\varphi$$

$$= \frac{\exp\Big(-\frac{1}{4}\,\frac{N\,\overline{a^2}}{\sigma^2}\Big)}{2\pi}$$

$$\cdot \int_{\text{any } 2\pi \text{ interval}} \exp\Big(\frac{1}{\sigma^2}\,\mathrm{Re}\big\{z(\boldsymbol{x})\exp\big[j\,(\varphi - \tfrac{1}{2}\,\pi)\big]\big\}\Big)\,d\varphi \quad (2)$$

where

$$z(\boldsymbol{x}) = \sum_{n=0}^{N-1} x[n]\, a_n \exp(j\, n\, \omega_{\rm c}).$$

Clearly, (2) does not depend on $\angle z(\boldsymbol{x})$ and is, therefore, a function of $z(\boldsymbol{x})$ only through its magnitude $|z(\boldsymbol{x})|$. Furthermore, (2) is an increasing function of $|z(\boldsymbol{x})|$, implying that we can simplify our test to

$$\Big|\sum_{n=0}^{N-1} x[n]\, a_n \exp(j\, n\, \omega_{\rm c})\Big| \overset{\mathcal{H}_1}{\gtrless} \text{a threshold } \gamma \quad \Longleftrightarrow$$

$$\Big|\underbrace{\sum_{n=0}^{N-1} x[n]\, a_n \exp(-j\, n\, \omega_{\rm c})}_{\text{Fourier transform of } x[n]\, a_n}\Big| \overset{\mathcal{H}_1}{\gtrless} \text{a threshold} \quad \Longleftrightarrow$$

$$\Big\{\underbrace{\sum_{n=0}^{N-1} x[n]\, a_n \cos(n\, \omega_{\rm c})}_{\text{quadrature component}}\Big\}^2 + \Big\{\underbrace{\sum_{n=0}^{N-1} x[n]\, a_n \sin(n\, \omega_{\rm c})}_{\text{quadrature component}}\Big\}^2$$

$$\overset{\mathcal{H}_1}{\gtrless} \text{a threshold}$$

in both the Bayesian and Neyman-Pearson scenarios (as usual, only the choice of the threshold differs between the two scenarios). In this case, we can evaluate the integral (2) *exactly*, yielding

$$\Lambda(\boldsymbol{x}) \approx \exp\Big(-\tfrac{1}{4}\,\frac{N\,\overline{a^2}}{\sigma^2}\Big)\cdot I_0\Big(\frac{|z(\boldsymbol{x})|}{\sigma^2}\Big)$$

where $I_0(\cdot)$ denotes the zeroth-order modified Bessel function of the first kind, which can be defined as follows:

$$I_0(|z|) = \frac{1}{2\pi}\cdot\int_0^{2\pi} e^{\mathrm{Re}\{z e^{j\varphi}\}}\,d\varphi.$$

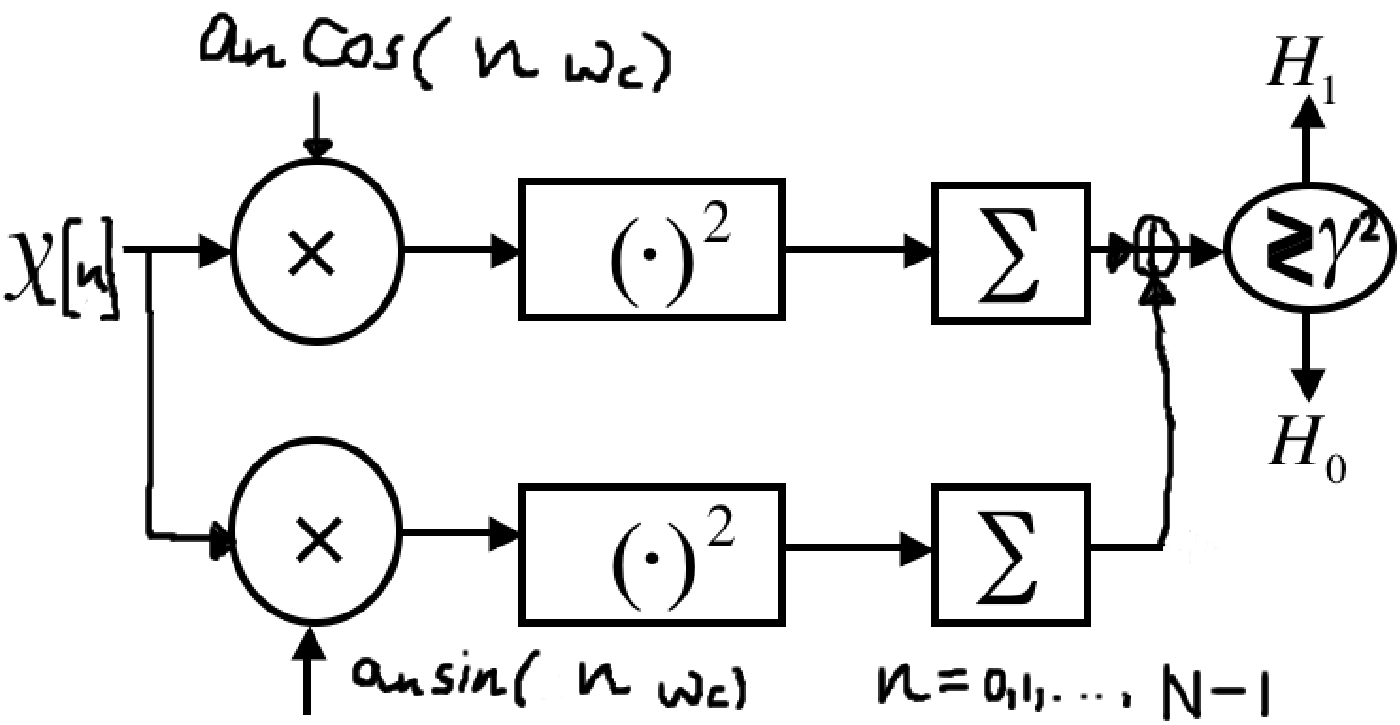
[Here, we can easily show that the right-hand side of the above expression is a function of $|z|$ only (i.e. independent of the phase $\angle z$):

$$\begin{aligned}
&\frac{1}{2\pi}\cdot\int_0^{2\pi} e^{\mathrm{Re}\{z e^{j\varphi}\}}\,d\varphi \\
&= \quad \frac{1}{2\pi}\cdot\int_0^{2\pi} e^{\mathrm{Re}\{|z|\,e^{j(\angle z+\varphi)}\}}\,d\varphi \\
&\overset{\theta=\angle z+\varphi}{=} \quad \underbrace{\frac{1}{2\pi}\cdot\int_{\text{any } 2\pi \text{ interval}} e^{\mathrm{Re}\{|z|\,e^{j\theta}\}}\,d\theta}_{\text{a function of } |z| \text{ only}} \\
&= \quad \frac{1}{2\pi}\cdot\int_{\text{any } 2\pi \text{ interval}} e^{|z|\ \mathrm{Re}\{e^{j\theta}\}}\,d\theta \\
&= \quad \int_0^{2\pi} e^{|z|\ \cos\theta}\,d\theta \ \stackrel{\triangle}{=}\ I_0(|z|).
\end{aligned}$$

] Therefore, our Bayes' decision rule simplifies to

$$|z(\boldsymbol{x})| \overset{\mathcal{H}_1}{\gtrless} \underbrace{\sigma^2 \cdot I_0^{-1}\Big(\exp\Big(\tfrac{1}{4}\,\frac{N\,\overline{a^2}}{\sigma^2}\Big) \cdot \frac{\pi_0\,\mathrm{L}(1\,|\,0)}{\pi_1\,\mathrm{L}(0\,|\,1)}\Big)}_{\gamma}$$

leading to the following receiver structure:



Note: Here, to implement the the maximum-likelihood detection rule:

$$|z(\boldsymbol{x})| \overset{\mathcal{H}_1}{\gtrless} \underbrace{\sigma^2 \cdot I_0^{-1}\Big(\exp\Big(\tfrac{1}{4}\,\frac{N\,\overline{a^2}}{\sigma^2}\Big)\Big)}_{\text{maximum-likelihood rule threshold}}$$

we need to know the AWGN noise variance σ^2.

Detecting A Stochastic Signal in AWGN (Neyman-Pearson Approach)

Some signals have unknown waveform (e.g. speech signals or NDE defect responses). We may need to use stochastic models to describe such signals. We start with a simple independent-signal model, described below.

Estimator-correlator. Consider the following hypothesis test:

$$\begin{aligned}
\mathcal{H}_0 : &\quad x[n] = w[n], \quad n = 1, 2, \ldots, N \\
\mathcal{H}_1 : &\quad x[n] = s[n] + w[n], \quad n = 1, 2, \ldots, N
\end{aligned}$$

where

- $s[n]$ are zero-mean independent Gaussian random variables with *known* variances $\sigma_{s,n}^2$, i.e. $s[n] \sim \mathcal{N}(0, \sigma_{s,n}^2)$,
- the noise $w[n]$ is AWGN with *known variance* σ^2, i.e. $w[n] \sim \mathcal{N}(0, \sigma^2)$,
- $s[n]$ and $w[n]$ are independent.

Here is an alternative formulation. Consider the following family of pdfs:

$$p(\boldsymbol{x} \mid C_s) = \mathcal{N}(\mathbf{0}, C_s + \sigma^2 I)$$
$$= \frac{1}{\sqrt{\prod_{n=1}^{N}[2\pi(\sigma^2 + c_{s,n})]}} \cdot \exp\Big[-\sum_{n=1}^{N} \frac{(x[n])^2}{2(\sigma^2 + c_{s,n})}\Big]$$
$$C_s = \operatorname{diag}\{c_{s,1}, c_{s,2}, \ldots, c_{s,N}\}$$

with

$$\boldsymbol{x} = \begin{bmatrix} x[1] \\ x[2] \\ \vdots \\ x[N] \end{bmatrix}$$

and the following (equivalent) hypotheses:

$\mathcal{H}_0 : \quad C_s = 0$ (signal absent) versus

$\mathcal{H}_1 : \quad C_s = \operatorname{diag}\{\sigma_{s,1}^2, \sigma_{s,2}^2, \ldots, \sigma_{s,N}^2\}$ (signal present).

Clearly, we have *integrated* $s[n]$, $n = 1, 2, \ldots, N$ *out* to obtain the marginal likelihood $p(\boldsymbol{x} \mid \Sigma_s)$ under $\mathcal{H}_1$.

Here, the only discrimination between the two hypotheses is in variance of the measurements (i.e. power of the received signal). The Neyman-Pearson detector computes the likelihood ratio:

$$\Lambda(\boldsymbol{x}) = \frac{p(\boldsymbol{x} \mid \operatorname{diag}\{\sigma_{s,1}^2, \sigma_{s,2}^2, \ldots, \sigma_{s,N}^2\})}{p(\boldsymbol{x} \mid 0)}.$$

Note that

$$\frac{1}{\sigma^2} - \frac{1}{\sigma^2 + \sigma_{s,n}^2} = \frac{1}{\sigma^2} \cdot \underbrace{\frac{\sigma_{s,n}^2}{\sigma^2 + \sigma_{s,n}^2}}_{\triangleq\, \kappa_n}$$

where we define

$$\kappa_n \;\triangleq\; \frac{\sigma_{s,n}^2}{\sigma^2 + \sigma_{s,n}^2}.$$

Let us compute the log likelihood ratio:

$$\begin{aligned}
\log \Lambda(\boldsymbol{x}) \quad &= \quad \underbrace{\text{const}}_{\text{not a function of } \boldsymbol{x}} \\
&\quad - \sum_{n=1}^{N} \frac{(x[n])^2}{2\,(\sigma^2 + \sigma_{s,n}^2)} + \sum_{n=1}^{N} \frac{(x[n])^2}{2\,\sigma^2} \\
&= \quad \underbrace{\text{const}}_{\text{not a function of } \boldsymbol{x}} \quad + \frac{2}{\sigma^2} \cdot \sum_{n=1}^{N} \kappa_n \cdot (x[n])^2
\end{aligned}$$

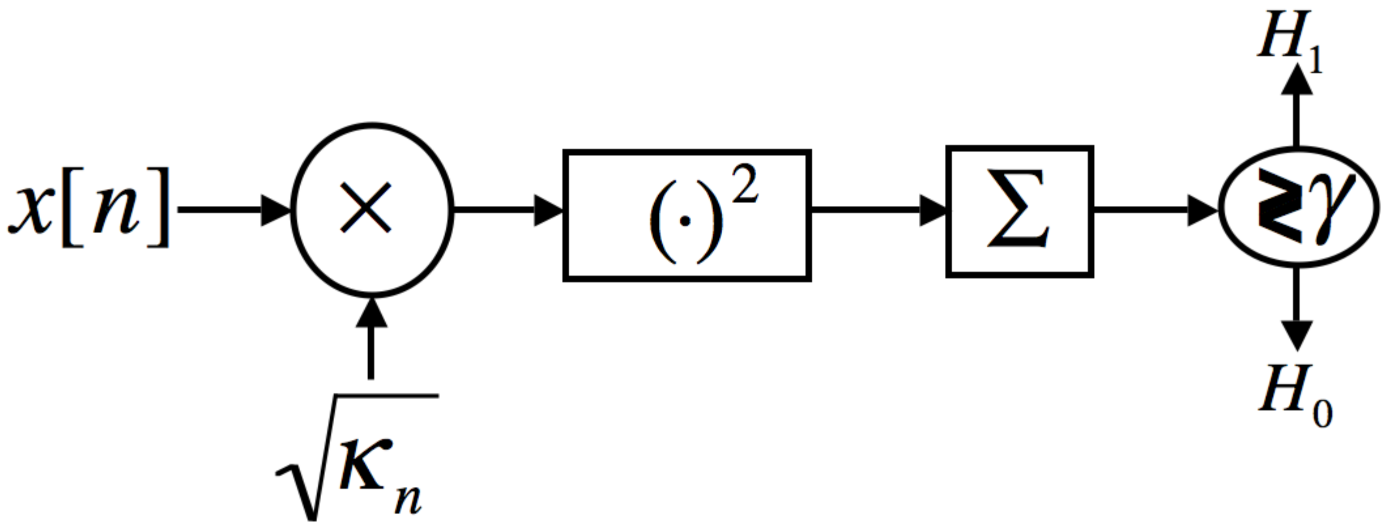
and, therefore, our test simplifies to (after ignoring the constant terms and scaling the log likelihood ratio by the positive constant $\sigma^2/2$):

$$T(\boldsymbol{x}) = \sum_{n=1}^{N} \kappa_n \cdot (x[n])^2 \overset{\mathcal{H}_1}{\gtrless} \gamma \quad \text{(a threshold).}$$

We first provide two interpretations of this detector and then generalize it to the case of correlated signal $s[n]$ having known covariance.

Filter-squarer interpretation:

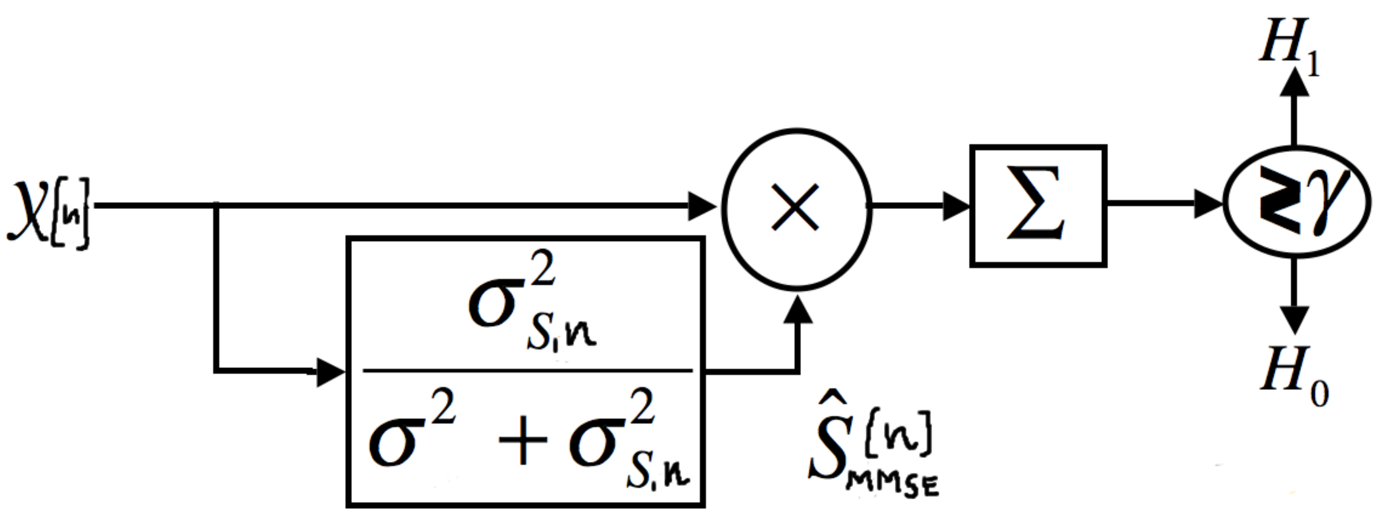
$$\sum_{n=1}^{N} \kappa_n \cdot (x[n])^2 = \sum_{n=1}^{N} (\sqrt{\kappa_n} \cdot x[n])^2.$$



Estimator-correlator interpretation:

$$\sum_{n=1}^{N} \kappa_n \cdot (x[n])^2 = \sum_{n=1}^{N} x[n] \cdot (\kappa_n \, x[n])$$

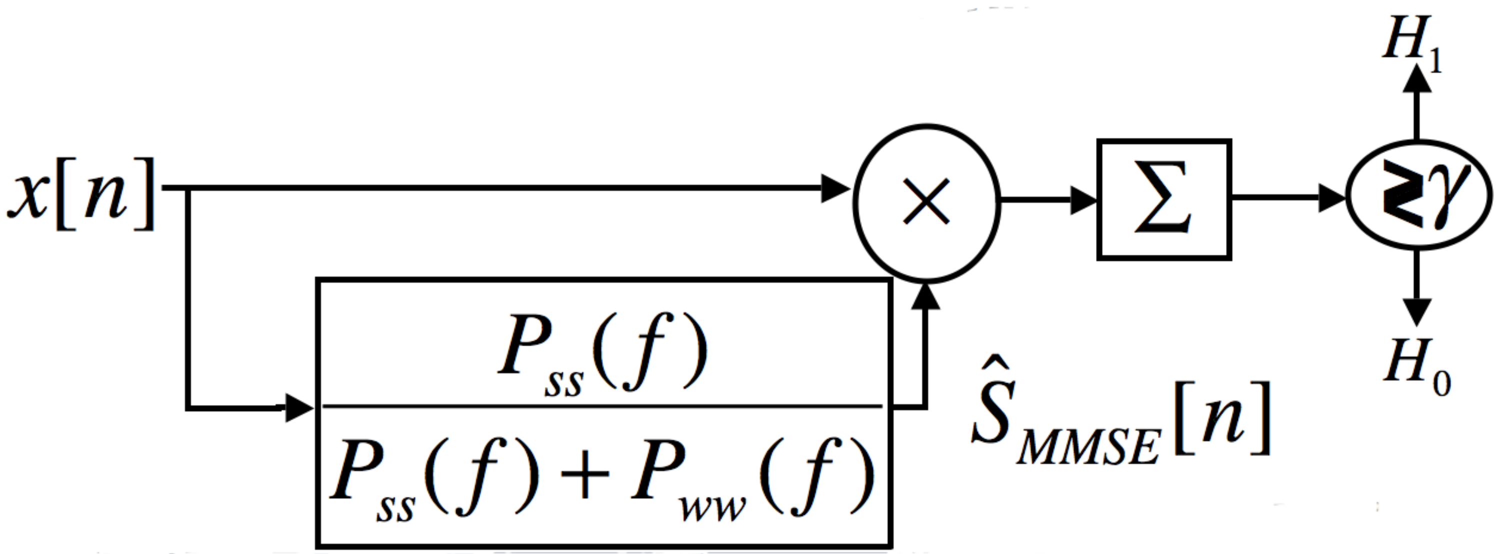
$$= \sum_{n=1}^{N} x[n] \cdot \underbrace{\frac{\sigma_{s,n}^2}{\sigma^2 + \sigma_{s,n}^2} x[n]}_{\mathrm{E}[s[n] \,|\, y[n]] = \widehat{s}_{\mathrm{MMSE}}[n]} .$$



(Asymptotic) Estimator-correlator: Wide-sense Stationary (WSS) signal $s[n]$ in WSS noise $w[n]$. Suppose that $s[n]$ and $w[n]$ are zero-mean WSS sequences with power spectral densities (PSDs)

$$P_{ss}(f) \quad \text{and} \quad P_{ww}(f), \quad f \in [-\tfrac{1}{2}, \tfrac{1}{2}].$$

Then, our estimator-correlator structure remains the same, with the estimator block modified accordingly:



which is an asymptotic estimator-correlator.

Estimator-correlator for Correlated Signal in the Form of a Linear Model

We extend the estimator-correlator to the case of correlated signal:

$$\mathcal{H}_0: \quad \boldsymbol{x} = \boldsymbol{w} \quad \text{versus}$$
$$\mathcal{H}_1: \quad \boldsymbol{x} = \underbrace{H\boldsymbol{\theta}}_{\text{signal } \boldsymbol{s}} + \boldsymbol{w}$$

where

- H is a known $N \times p$ matrix and $N \geq p$.
- the noise $\boldsymbol{w}$ is zero-mean Gaussian with known covariance matrix $\Sigma_{\boldsymbol{w}}$:
$$p(\boldsymbol{w}) = \mathcal{N}(\mathbf{0}, \Sigma_{\boldsymbol{w}}).$$
- $\boldsymbol{\theta}$ is *unknown*, with the following prior pdf:
$$\pi(\boldsymbol{\theta}) = \mathcal{N}(\mathbf{0}, \Sigma_{\boldsymbol{\theta}})$$
  where $\Sigma_{\boldsymbol{\theta}}$ is known.

We focus on the following general family of pdfs:

$$p(\boldsymbol{x} \,|\, C_s) = \mathcal{N}(\mathbf{0}, C_s + \Sigma_{\boldsymbol{w}})$$

where C_s is a positive-definite covariance matrix. Then, the above hypothesis-testing problem can be equivalently stated as

$$\begin{aligned} \mathcal{H}_0 : \quad & C_s = 0 \quad \text{(signal absent)} \quad \text{versus} \\ \mathcal{H}_1 : \quad & C_s = H\,\Sigma_{\boldsymbol{\theta}} H^T \quad \text{(signal present)}. \end{aligned}$$

The estimator-correlator test statistic is now

$$T(\boldsymbol{x}) \;=\; \tfrac{1}{2}\,\boldsymbol{x}^T\,\Sigma_{\boldsymbol{w}}^{-1}\,\boldsymbol{x} - \tfrac{1}{2}\,\boldsymbol{x}^T\,(H\,\Sigma_{\boldsymbol{\theta}} H^T + \Sigma_{\boldsymbol{w}})^{-1}\,\boldsymbol{x}$$

Recall the matrix inversion lemma:

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}$$

and use it as follows:

$$\begin{aligned} (\Sigma_{\boldsymbol{w}} + H\,\Sigma_{\boldsymbol{\theta}} H^T)^{-1} = {} & \Sigma_{\boldsymbol{w}}^{-1} \\ & -\Sigma_{\boldsymbol{w}}^{-1} H(\Sigma_{\boldsymbol{\theta}}^{-1} + H^T\,\Sigma_{\boldsymbol{w}}^{-1} H)^{-1} H^T\,\Sigma_{\boldsymbol{w}}^{-1} \end{aligned}$$

yielding

$$T(\boldsymbol{x}) = \tfrac{1}{2}\cdot \boldsymbol{x}^T\,\Sigma_{\boldsymbol{w}}^{-1} H \;\underbrace{(\Sigma_{\boldsymbol{\theta}}^{-1} + H^T\,\Sigma_{\boldsymbol{w}}^{-1} H)^{-1} H^T\,\Sigma_{\boldsymbol{w}}^{-1}\boldsymbol{x}}_{\mathrm{E}\,[\boldsymbol{\theta}\,|\,\boldsymbol{x}] \,\triangleq\, \widehat{\boldsymbol{\theta}}_{\mathrm{MMSE}},\ \text{see handout } \#\ 4}\;. \quad (3)$$

Example: Detecting a Sinusoid in a Rayleigh-fading Channel (Neyman-Pearson Approach)

Over a short time interval, the channel output is a constant-amplitude sinusoid with random amplitude and phase, i.e.

$$s[n] = A\cos(2\pi f_0 n + \varphi) = a\,\cos(2\pi f_0 n) + b\,\sin(2\pi f_0 n)$$

for $n = 0, 1, \ldots, N-1$. Let us choose independent, identically distributed (i.i.d.) Rayleigh fading:

$$\boldsymbol{\theta} = \begin{bmatrix} a \\ b \end{bmatrix} \sim \mathcal{N}(\mathbf{0}, \underbrace{\sigma_{\boldsymbol{\theta}}^2 I}_{\Sigma_{\boldsymbol{\theta}}})$$

implying

$$\begin{aligned} A &= \sqrt{a^2+b^2} \sim \text{ a Rayleigh random variable} \\ \varphi &\sim \text{uniform}(0, 2\pi). \end{aligned}$$

With these assumptions, $s[n]$ is a WSS Gaussian random process, since

$$
\begin{aligned}
&\mathrm{E}\left(s[n]\right) = 0 \\
&\mathrm{E}\left(s[n]s[n+k]\right) = \mathrm{E}\left\{[a\,\cos 2\pi f_0 n + b\,\sin 2\pi f_0 n]\right. \\
&\quad \left.\cdot [a\,\cos 2\pi f_0 (n+k) + b\,\sin 2\pi f_0 (n+k)]\right\} \\
&\quad = \sigma_{\boldsymbol{\theta}}^2 \cdot [\cos 2\pi f_0 n \cos 2\pi f_0 (n+k) \\
&\quad + \sin 2\pi f_0 n \sin 2\pi f_0 (n+k)] \\
&\quad = \sigma_{\boldsymbol{\theta}}^2 \cdot \left\{ \frac{\exp(j2\pi f_0 n) + \exp(-j2\pi f_0 n)}{2} \right. \\
&\quad \cdot \frac{\exp(j2\pi f_0 (n+k)) + \exp(-j2\pi f_0 (n+k))}{2} \\
&\quad + \frac{\exp(j2\pi f_0 n) - \exp(-j2\pi f_0 n)}{2j} \\
&\quad \left. \cdot \frac{\exp(j2\pi f_0 (n+k)) - \exp(-j2\pi f_0 (n+k))}{2j} \right\} \\
&\quad = \sigma_{\boldsymbol{\theta}}^2 \cos 2\pi f_0 k = r_{ss}[k]
\end{aligned}
$$

for $n = 0, 1, \ldots, N-1$. We now construct a linear model with

$$
H = \begin{bmatrix} 1 & 0 \\ \cos 2\pi f_0 & \sin 2\pi f_0 \\ \vdots & \vdots \\ \cos 2\pi f_0 (N-1) & \sin 2\pi f_0 (N-1) \end{bmatrix}
$$

and

$$\Sigma_{\boldsymbol{w}} = \sigma^2 I \quad \text{(AWGN)}.$$

Note that σ^2 and $\sigma_{\boldsymbol{\theta}}^2$ are assumed *known*. Now, (3) reduces to

$$\begin{aligned} T(\boldsymbol{x}) &= \tfrac{1}{2} \cdot \boldsymbol{x}^T \Sigma_{\boldsymbol{w}}^{-1} H (\Sigma_{\boldsymbol{\theta}}^{-1} + H^T \Sigma_{\boldsymbol{w}}^{-1} H)^{-1} H^T \Sigma_{\boldsymbol{w}}^{-1} \boldsymbol{x} \\ &= \frac{1}{2\,(\sigma^2)^2} \cdot \boldsymbol{x}^T H\, (\sigma_{\boldsymbol{\theta}}^{-2} I + \sigma^{-2} H^T H)^{-1} H^T \boldsymbol{x} \end{aligned}$$

For large N and f_0 not too close to 0 or $\frac{1}{2}$,

$$H^T H \approx (N/2)\, I$$

see p. 157 in Kay-II. Furthermore,

$$H^T \boldsymbol{x} = \left[\begin{array}{c} \sum_{n=0}^{N-1} x[n] \cos 2\pi f_0 n \\ \sum_{n=0}^{N-1} x[n] \sin 2\pi f_0 n \end{array} \right].$$

After scaling by the positive constant $2\,(\sigma^2)^2/(\sigma_{\boldsymbol{\theta}}^{-2} + \sigma^{-2} N/2)$, our test statistic $T(\boldsymbol{x})$ simplifies to

$$T'(\boldsymbol{x}) = \frac{1}{N} \Big| \sum_{n=0}^{N-1} x[n] \exp(-j 2\pi f_0 n) \Big|^2$$

which is nothing but the periodogram of $x[n]$, $n = 1, 2, \ldots, N$ evaluated at frequency $f = f_0$, also known as the *quadrature or noncoherent matched filter*.

Performance Analysis for the Neyman-Pearson Setup:

Define

$$\boldsymbol{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = H^T \boldsymbol{x}.$$

Under $\mathcal{H}_0$, we have only noise, implying that

$$\boldsymbol{\xi} \sim \mathcal{N}(\mathbf{0}, H^T \sigma^2 I H \approx \underbrace{\frac{N\sigma^2}{2}}_{\triangleq s_0^2} I).$$

Under $\mathcal{H}_1$, we have

$$\boldsymbol{\xi} \sim \mathcal{N}\Big(\mathbf{0}, H^T (H \underbrace{\Sigma_{\boldsymbol{\theta}}}_{\sigma_{\boldsymbol{\theta}}^2 I} H^T + \sigma^2 I) H\Big).$$

We approximate the covariance matrix of $\boldsymbol{\xi}$ under $\mathcal{H}_1$ as follows:

$$\begin{aligned} H^T (H \Sigma_{\boldsymbol{\theta}} H^T + \sigma^2 I) H &\approx \sigma_{\boldsymbol{\theta}}^2 (N/2)^2 I + \sigma^2 (N/2) I \\ &= \underbrace{\frac{N}{2} \left(\frac{N}{2} \sigma_{\boldsymbol{\theta}}^2 + \sigma^2 \right)}_{\triangleq s_1^2} I. \end{aligned}$$

Now, under $\mathcal{H}_0$,

$$T'(\boldsymbol{x}) = \frac{1}{N}(\xi_1^2 + \xi_2^2) = \frac{s_0^2}{N} \underbrace{\Big[\Big(\frac{\overbrace{\xi_1}^{\text{standard normal random variable}}}{s_0} \Big)^2 + \Big(\frac{\xi_2}{s_0}\Big)^2\Big]}_{\chi_2^2 \ \text{ under } \mathcal{H}_0}$$

implying that

$$\begin{aligned}
P_{\mathrm{FA}} &= P[T'(\boldsymbol{X}) > \gamma \,|\, C_s = 0] \\
&= \overbrace{P\Big[\underbrace{\frac{N\,T'(\boldsymbol{X})}{s_0^2}}_{\chi_2^2} > \frac{N\,\gamma}{s_0^2} \,\Big|\, C_s = 0\Big]}^{Q_{\chi_2^2}\left(\frac{N\,\gamma}{s_0^2}\right)} \\
&= \exp(-\tfrac{1}{2}\,N\,\gamma/s_0^2)
\end{aligned}$$

see eq. (2.10) in Kay-II. Similarly, under $\mathcal{H}_1$,

$$\begin{aligned}
P_{\mathrm{D}} &= P[T'(\boldsymbol{X}) > \gamma \,|\, C_s = \sigma_{\boldsymbol{\theta}}^2\, H H^T] \\
&= P\Big[\frac{NT'(\boldsymbol{X})}{s_1^2} > \frac{N\gamma}{s_1^2} \,\Big|\, C_s = \sigma_{\boldsymbol{\theta}}^2\, H H^T\Big] \\
&= \exp(-\tfrac{1}{2}\,N\,\gamma/s_1^2).
\end{aligned}$$

But

$$-\tfrac{1}{2}\,N\,\gamma = s_0^2 \log P_{\mathrm{FA}}$$

leading to

$$P_{\mathrm{D}} = \exp\left(\frac{s_0^2}{s_1^2}\log P_{\mathrm{FA}}\right) = {P_{\mathrm{FA}}}^{s_0^2/s_1^2}.$$

Recall the expressions for s_0^2 and s_1^2 and compute their ratio:

$$\frac{s_1^2}{s_0^2} = \frac{\frac{N}{2}\left(\frac{N}{2}\,\sigma_{\boldsymbol{\theta}}^2 + \sigma^2\right)}{\frac{N\sigma^2}{2}} = \frac{N}{2}\,\frac{\sigma_{\boldsymbol{\theta}}^2}{\sigma^2} + 1 = \frac{\overline{\eta}}{2} + 1$$

where

$$\overline{\eta} = \frac{N\sigma_{\boldsymbol{\theta}}^2}{\sigma^2} = \frac{N\mathrm{E}\,[\,\overbrace{A^2}^{a^2+b^2}\;/2]}{\sigma^2} \equiv \text{average SNR.}$$

Hence

$$P_{\mathrm{D}} = {P_{\mathrm{FA}}}^{\frac{1}{1+\overline{\eta}/2}}.$$

P_{D} increases slowly with the average signal-to-noise ratio (SNR) $\overline{\eta}$ (see Figure 5.7 in Kay-II) because Rayleigh fading causes amplitude to be small with high probability.

Coherent channel $\Rightarrow$ matched filter.

Noncoherent channel $\Rightarrow$ quadrature matched filter.

Compare Figures 5.7 and 4.5 in Kay-II.

Noncoherent FSK in a Rayleigh-fading Channel (Bayesian decision-theoretic detection for 0-1 loss)

$$\begin{aligned}
\mathcal{H}_0 \colon x[n] &= A\cos(2\pi f_0 n + \varphi) + w[n], \quad n = 0, 1, \ldots, N-1 \\
\mathcal{H}_1 \colon x[n] &= A\cos(2\pi f_1 n + \varphi) + w[n], \quad n = 0, 1, \ldots, N-1
\end{aligned}$$

where, as before, A and φ are random phase and amplitude and

$$\boldsymbol{w} = \begin{bmatrix} w[0] \\ w[1] \\ \vdots \\ w[N-1] \end{bmatrix} \sim \mathcal{N}(\mathbf{0}, \Sigma_{\boldsymbol{w}}).$$

We now focus on the following family of pdfs:

$$p(\boldsymbol{x} \mid H) = \mathcal{N}(\mathbf{0}, H\, \Sigma_{\boldsymbol{\theta}}\, H^T + \Sigma_{\boldsymbol{w}})$$

where $\Sigma_{\boldsymbol{\theta}}$ is known. We can rewrite the above detection problem using the linear model as follows:

$$\begin{aligned}
\mathcal{H}_0 : &\qquad H = H_0 \quad \text{versus} \\
\mathcal{H}_1 : &\qquad H = H_1
\end{aligned}$$

where $\boldsymbol{x} = \begin{bmatrix} x[1] \\ x[2] \\ \vdots \\ x[N] \end{bmatrix}$,

$$H_0 = \begin{bmatrix} 1 & 0 \\ \cos 2\pi f_0 & \sin 2\pi f_0 \\ \vdots & \vdots \\ \cos 2\pi f_0(N-1) & \sin 2\pi f_0(N-1) \end{bmatrix}$$

and

$$H_1 = \begin{bmatrix} 1 & 0 \\ \cos 2\pi f_1 & \sin 2\pi f_1 \\ \vdots & \vdots \\ \cos 2\pi f_1(N-1) & \sin 2\pi f_1(N-1) \end{bmatrix}.$$

For *a priori* equiprobable hypotheses

$$\pi(H = H_0) = \pi(H = H_1) = \tfrac{1}{2}$$

we have the maximum-likelihood test:

$$\frac{p(\boldsymbol{x} \,|\, H = H_1)}{p(\boldsymbol{x} \,|\, H = H_0)} \overset{\mathcal{H}_1}{\gtrless} 1$$

i.e.

$$\frac{\frac{1}{|H_1\Sigma_{\boldsymbol{\theta}}H_1^T+\Sigma_{\boldsymbol{w}}|^{1/2}}\,\exp\left[\frac{1}{2}\,\boldsymbol{x}^T\,\Sigma_{\boldsymbol{w}}^{-1}\,\boldsymbol{x}-\frac{1}{2}\,\boldsymbol{x}^T\,(H_1\Sigma_{\boldsymbol{\theta}}H_1^T+\Sigma_{\boldsymbol{w}})^{-1}\,\boldsymbol{x}\right]}{\frac{1}{|H_0\Sigma_{\boldsymbol{\theta}}H_0^T+\Sigma_{\boldsymbol{w}}|^{1/2}}\,\exp\left[\frac{1}{2}\,\boldsymbol{x}^T\,\Sigma_{\boldsymbol{w}}^{-1}\,\boldsymbol{x}-\frac{1}{2}\,\boldsymbol{x}^T\,(H_0\Sigma_{\boldsymbol{\theta}}H_0^T+\Sigma_{\boldsymbol{w}})^{-1}\,\boldsymbol{x}\right]}\ \overset{\mathcal{H}_1}{\gtrless}\ 1$$

which can be written as

$$\frac{\frac{1}{|H_1\Sigma_{\boldsymbol{\theta}}H_1^T+\Sigma_{\boldsymbol{w}}|^{1/2}}\,\exp\left[\frac{1}{2}\cdot\boldsymbol{x}^T\Sigma_{\boldsymbol{w}}^{-1}H_1(\Sigma_{\boldsymbol{\theta}}^{-1}+H_1^T\Sigma_{\boldsymbol{w}}^{-1}H_1)^{-1}H_1^T\Sigma_{\boldsymbol{w}}^{-1}\boldsymbol{x}\right]}{\frac{1}{|H_0\Sigma_{\boldsymbol{\theta}}H_0^T+\Sigma_{\boldsymbol{w}}|^{1/2}}\,\exp\left[\frac{1}{2}\cdot\boldsymbol{x}^T\Sigma_{\boldsymbol{w}}^{-1}H_0(\Sigma_{\boldsymbol{\theta}}^{-1}+H_0^T\Sigma_{\boldsymbol{w}}^{-1}H_0)^{-1}H_0^T\Sigma_{\boldsymbol{w}}^{-1}\boldsymbol{x}\right]}\ \overset{\mathcal{H}_1}{\gtrless}\ 1.$$

To further simplify the above test, we adopt additional assumptions. In particular, we assume i.i.d. Rayleigh fading:

$$\Sigma_{\boldsymbol{\theta}}=\sigma_{\boldsymbol{\theta}}^2\,I$$

and AWGN

$$\Sigma_{\boldsymbol{w}}=\sigma^2\,I\quad\text{(AWGN)}.$$

where σ^2 and $\sigma_{\boldsymbol{\theta}}^2$ are *known*. For large N and f_i, $i\in\{0,1\}$ not too close to 0 or $\frac{1}{2}$,

$$H_i^T H_i\approx(N/2)\,I$$

see p. 157 in Kay-II. Now, we apply this approximation and the identities $|PQ|=|P|\cdot|Q|$ and $|I+AB|=|I+BA|$ to further simplify the above determinant expressions:

$$\begin{aligned}
|H_i \Sigma_{\boldsymbol{\theta}} H_i^T + \Sigma_{\boldsymbol{w}}| &= |\sigma_{\boldsymbol{\theta}}^2 \, H_0 \, H_0^T + \sigma^2 \, I| \\
&= |\sigma^2 \, I| \cdot |I + \frac{\sigma_{\boldsymbol{\theta}}^2}{\sigma^2} \, H_0 \, H_0^T| \\
&\approx (\sigma^2)^N \cdot \left| I_2 + \frac{\sigma_s^2}{\sigma^2} \frac{N}{2} \, I_2 \right|
\end{aligned}$$

for $i \in \{0, 1\}$. Applying the above approximation and assumptions yields the simplified maximum-likelihood test:

$$\frac{\exp\left[\frac{1}{2\,\sigma^4} \cdot \boldsymbol{x}^T H_1 \, (\sigma_{\boldsymbol{\theta}}^2 \, I + \frac{N}{2\,\sigma^2} \, I)^{-1} \, H_1^T \boldsymbol{x}\right]}{\exp\left[\frac{1}{2\,\sigma^4} \cdot \boldsymbol{x}^T H_0 \, (\sigma_{\boldsymbol{\theta}}^2 \, I + \frac{N}{2\,\sigma^2} \, I)^{-1} \, H_0^T \boldsymbol{x}\right]} \overset{\mathcal{H}_1}{\gtrless} 1.$$

and, equivalently,

$$\boldsymbol{x}^T H_1 H_1^T \boldsymbol{x} - \boldsymbol{x}^T H_0 H_0^T \boldsymbol{x} \overset{\mathcal{H}_1}{\gtrless} 0$$

or

$$\frac{1}{N} \left| \sum_{n=0}^{N-1} x[n] \exp(-j 2\pi f_1 n) \right|^2 \overset{\mathcal{H}_1}{\gtrless} \frac{1}{N} \left| \sum_{n=0}^{N-1} x[n] \exp(-j 2\pi f_0 n) \right|^2.$$

For performance analysis of this detector (i.e. computing an approximate expression for the average error probability), see Example 5.6 in Kay-II.

Preliminaries

Let us define a matrix-variate circularly symmetric complex Gaussian pdf of an $p \times q$ random matrix Z with mean M (of size $p \times q$) and positive-definite covariance matrices S and Σ (of dimensions $p \times p$ and $q \times q$, respectively) as follows:

$$\begin{aligned}
\mathcal{N}_{p\times q}(Z \,|\, M, S, \Sigma) &= \frac{1}{\pi^{pq}\,|S|^q\,|\Sigma|^p} \\
&\quad \cdot \exp\big\{-\operatorname{tr}[\Sigma^{-1}(Z-M)^H S^{-1}(Z-M)]\big\} \\
&\propto \exp\big\{-\operatorname{tr}[\Sigma^{-1}(Z-M)^H S^{-1}(Z-M)]\big\} \\
&\propto \exp\big[-\operatorname{tr}(\Sigma^{-1}Z^H S^{-1}Z) + \operatorname{tr}(\Sigma^{-1}Z^H S^{-1}M) \\
&\quad + \operatorname{tr}(\Sigma^{-1}M^H S^{-1}Z)\big]
\end{aligned}$$

where "^{H}" denotes the Hermitian (conjugate) transpose.

Noncoherent Detection of Space-time Codes in a Rayleigh Fading Channel

Consider the multiple-input multiple-output (MIMO) flat-fading channel where the $n_{\mathrm{R}} \times 1$ vector signal received by the receiver array at time t is modeled as

$$\boldsymbol{x}(t) = H\boldsymbol{\phi}(t) + \boldsymbol{w}(t), \quad t = 1, \ldots, N$$

where H is the $n_{\mathrm{R}} \times n_{\mathrm{T}}$ channel-response matrix, $\boldsymbol{\phi}(t)$ is the $n_{\mathrm{T}} \times 1$ vector of symbols transmitted by n_{T} transmitter antennas and received by the receiver array at time t, and $\boldsymbol{w}(t)$ is additive noise. Note that we can write the above model as

$$\underbrace{[\boldsymbol{x}(1) \cdots \boldsymbol{x}(N)]}_{X} = H \underbrace{[\boldsymbol{\phi}(1) \cdots \boldsymbol{\phi}(N)]}_{\varPhi} + \underbrace{[\boldsymbol{w}(1) \cdots \boldsymbol{w}(N)]}_{\boldsymbol{W}} \quad (4)$$

Here, $\varPhi$ is a space-time code (multivariate "symbol") belonging to an M-ary constellation:

$$\varPhi \in \{\varPhi_0, \varPhi_1, \ldots, \varPhi_{M-1}\}.$$

We adopt *a priori* equiprobable hypotheses:[1]

$$\pi(\mathit{\Phi}) = \frac{1}{M} \cdot i_{\{\mathit{\Phi}_0, \mathit{\Phi}_1, \ldots, \mathit{\Phi}_{M-1}\}}(\mathit{\Phi})$$

and assume that $\boldsymbol{w}(t)$, $t = 1, 2, \ldots, N$ are zero-mean i.i.d. with known covariance

$$\mathrm{cov}(\boldsymbol{w}(t)) = \sigma^2 \, I_{n_\mathrm{R}}.$$

Now, the likelihood function for the measurement model (4) is

$$\begin{aligned} p(X \mid \mathit{\Phi}, H) &= \mathcal{N}_{n_\mathrm{R} \times N}(X \mid H \, \mathit{\Phi}, \sigma^2 \, I, I) \\ &= \frac{1}{\pi^{n_\mathrm{R} N} \, |\sigma^2 I|^N} \cdot \exp\Big\{ -\frac{1}{\sigma^2} \, \mathrm{tr}[(X - H \, \mathit{\Phi})^H (X - H \, \mathit{\Phi})] \Big\}. \end{aligned}$$

We assume that $\mathit{\Phi}$ and H are independent *a priori*, i.e.

$$\pi(\mathit{\Phi}, H) = \pi(\mathit{\Phi}) \, \pi(H)$$

where $\pi(\mathit{\Phi})$ is given in (5) and $\pi(H)$ is chosen according to

[1] Here, $i_A(x)$ denotes the indicator function:

$$i_A(x) = \begin{cases} 1, & x \in A, \\ 0, & \text{otherwise} \end{cases} .$$

the following separable Rayleigh-fading model:

$$\begin{aligned}\pi(H) &= \mathcal{N}_{n_\mathrm{R}\times n_\mathrm{T}}(H \mid 0, I, \overbrace{\Delta_h}^{\text{transmitter fading corr. matrix}}) \\ &= \frac{1}{\pi^{n_\mathrm{R} n_\mathrm{T}}\, |\Delta_h|^{n_\mathrm{R}}} \cdot \exp\big[-\operatorname{tr}(\Delta_h^{-1} H^H H)\big].\end{aligned}$$

Now

$$\begin{aligned}p(\Phi, H \mid X) &\propto p(X \mid \Phi, H)\, \pi(\Phi)\, \pi(H) \\ &\propto \exp\Big[\frac{1}{\sigma^2}\operatorname{tr}(\Phi^H H^H X) + \frac{1}{\sigma^2}\operatorname{tr}(X^H H \Phi) \\ &\qquad -\frac{1}{\sigma^2}\operatorname{tr}(\Phi^H H^H H \Phi)\Big] \cdot i_{\{\Phi_0, \Phi_1, \ldots, \Phi_{M-1}\}}(\Phi) \\ &\qquad \cdot \exp\big[-\operatorname{tr}(\Delta_h^{-1} H^H H)\big] \\ &\propto \exp\Big\{\frac{1}{\sigma^2}\operatorname{tr}(\Phi^H H^H X) + \frac{1}{\sigma^2}\operatorname{tr}(X^H H \Phi) \\ &\qquad -\operatorname{tr}\Big[\underbrace{\Big(\frac{1}{\sigma^2}\,\Phi\,\Phi^H + \Delta_h^{-1}\Big)}_{C_H(\Phi)^{-1}} H^H H\Big]\Big\} \cdot i_{\{\Phi_0, \Phi_1, \ldots, \Phi_{M-1}\}}(\Phi)\end{aligned}$$

implying that

$$
\begin{aligned}
& p(H \mid \Phi, X) \\
& \quad \propto \quad \exp\left\{\frac{1}{\sigma^2} \operatorname{tr}(\Phi^H H^H X) + \frac{1}{\sigma^2} \operatorname{tr}(X^H H \Phi) \right. \\
& \qquad\qquad \left. - \operatorname{tr}\left[\left(\frac{1}{\sigma^2} \Phi \, \Phi^H + \Delta_h^{-1}\right) H^H H\right]\right\} \\
& \quad \propto \quad \exp\Big[- \operatorname{tr}\left\{C_H(\Phi)^{-1} \, H^H \, H\right\} + \operatorname{tr}\left\{C_H(\Phi)^{-1} \, \widehat{H}(\Phi)^H \, H\right\} \\
& \qquad\qquad + \operatorname{tr}\left\{C_H(\Phi)^{-1} \, H^H \, \widehat{H}(\Phi)\right\}\Big] \\
& \quad = \quad \mathcal{N}_{n_{\mathrm{R}} \times n_{\mathrm{T}}}\Big(H \mid \underbrace{\frac{1}{\sigma^2} X \, \Phi^H \, (\frac{1}{\sigma^2} \Phi \, \Phi^H + \Delta_h^{-1})^{-1}}_{\triangleq \, \widehat{H}(\Phi)}, I, \\
& \qquad\qquad \underbrace{\left(\frac{1}{\sigma^2} \Phi \, \Phi^H + \Delta_h^{-1}\right)^{-1}}_{\triangleq \, C_H(\Phi)}\Big) \\
& \quad = \quad \frac{1}{\pi^{n_{\mathrm{R}} n_{\mathrm{T}}} \, |C_H(\Phi)|^{n_{\mathrm{R}}}} \\
& \qquad\qquad \cdot \exp\Big(- \operatorname{tr}\left\{C_H(\Phi)^{-1} \, [H - \widehat{H}(\Phi)]^H \, [H - \widehat{H}(\Phi)]\right\}\Big)
\end{aligned}
$$

where we have defined

$$\begin{aligned}
\widehat{H}(\Phi) &\triangleq \frac{1}{\sigma^2} X\, \Phi^H\, C_H(\Phi) \\
C_H(\Phi) &\triangleq \Big(\frac{1}{\sigma^2}\, \Phi\, \Phi^H + \Delta_h^{-1}\Big)^{-1}
\end{aligned}$$

Note the following useful facts:

$$\begin{aligned}
\mathrm{tr}[C_H(\Phi)^{-1} H^H \widehat{H}(\Phi)] &= \frac{1}{\sigma^2}\,\mathrm{tr}[C_H(\Phi)^{-1} H^H X\, \Phi^H\, C_H(\Phi)] \\
&= \frac{1}{\sigma^2}\,\mathrm{tr}(H^H X\, \Phi^H) \\
\mathrm{tr}[C_H(\Phi)^{-1} \widehat{H}(\Phi)^H H] &= \frac{1}{\sigma^2}\,\mathrm{tr}[C_H(\Phi)^{-1} C_H(\Phi)\, \Phi X^H H] \\
&= \frac{1}{\sigma^2}\,\mathrm{tr}(\Phi X^H H).
\end{aligned}$$

To obtain the marginal posterior pmf of Φ, we apply our "notorious" trick:

$$p(\Phi \mid X) = \underbrace{\frac{p(\Phi, H \mid X)}{p(H \mid \Phi, X)}}_{\propto}$$

keep track of the terms containing Φ and H

$$\exp\left[\frac{1}{\sigma^2}\,\mathrm{tr}(\Phi^H H^H X) + \frac{1}{\sigma^2}\,\mathrm{tr}(X^H H \Phi) - \mathrm{tr}(C_H(\Phi)^{-1} H^H H)\right]$$

$$\cdot i_{\{\Phi_0, \Phi_1, \ldots, \Phi_{M-1}\}}(\Phi) \cdot |C_H(\Phi)|^{n_{\mathrm{R}}}$$

$$\cdot \exp\left\{\mathrm{tr}\left[C_H(\Phi)^{-1}\,(H - \widehat{H}(\Phi))^H\,(H - \widehat{H}(\Phi))\right]\right\}$$

$$\begin{aligned}
\propto \quad & i_{\{\Phi_0, \Phi_1, \ldots, \Phi_{M-1}\}}(\Phi) \\
& \cdot |C_H(\Phi)|^{n_{\mathrm{R}}} \cdot \exp\{\mathrm{tr}[C_H(\Phi)^{-1} \widehat{H}(\Phi)^H \widehat{H}(\Phi)]\} \\
= \quad & i_{\{\Phi_0, \Phi_1, \ldots, \Phi_{M-1}\}}(\Phi) \cdot |C_H(\Phi)|^{n_{\mathrm{R}}} \\
& \cdot \exp\left\{\frac{1}{(\sigma^2)^2}\,\mathrm{tr}[C_H(\Phi)^{-1} C_H(\Phi)\,\Phi\, X^H\, X\, \Phi^H\, C_H(\Phi)]\right\} \\
= \quad & i_{\{\Phi_0, \Phi_1, \ldots, \Phi_{M-1}\}}(\Phi) \cdot |C_H(\Phi)|^{n_{\mathrm{R}}} \\
& \cdot \exp\left\{\frac{1}{(\sigma^2)^2}\,\mathrm{tr}[\Phi\, X^H\, X\, \Phi^H\, C_H(\Phi)]\right\}
\end{aligned}$$

and the maximum-likelihood test becomes:

$$\mathcal{X}_m^\star = \Big\{ \boldsymbol{x} : m \;\; = \;\; \arg \max_{l \in \{0,1,\ldots,M-1\}} \Big(|C_H(\Phi_l)|^{n_\mathrm{R}} \cdot \exp\Big(\frac{1}{(\sigma^2)^2} \operatorname{tr}[\Phi_l \, X^H \, X \, \Phi_l^H \, C_H(\Phi_l)] \Big) \Big) \Big\}.$$

Unitary space-time codes and i.i.d. fading: Suppose that $\Phi_m \Phi_m^H = I_{n_\mathrm{T}}$ for all m and the fading is i.i.d., i.e.

$$\Delta_h = \psi_h^2 \, I_{n_\mathrm{T}}.$$

Then, the above maximum-likelihood test greatly simplifies:

$$\mathcal{X}_m^\star = \Big\{ \boldsymbol{x} : m \;\; = \;\; \arg \max_{l \in \{0,1,\ldots,M-1\}} \operatorname{tr}[\Phi_l \, X^H \, X \, \Phi_l^H] \Big\}$$

which is the detector proposed in

B.M. Hochwald and T.L. Marzetta, *IEEE Trans. Inform. Theory*, vol. 46, pp. 543–564, March 2000.