Frequentist Detection of Composite Hypotheses

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Contents

READING: [Hero 2015, §8], [Johnson 2013, §5].

Preliminaries

IN composite testing of two hypotheses, we have $\mathrm{sp}_\Theta(0)$ and $\mathrm{sp}_\Theta(1)$ that form a *partition* of the parameter space sp_Θ:

$$\mathrm{sp}_\Theta(0) \cup \mathrm{sp}_\Theta(1) = \mathrm{sp}_\Theta, \qquad \mathrm{sp}_\Theta(0) \cap \mathrm{sp}_\Theta(1) = \emptyset$$

and that we wish to identify *which* of the two hypotheses is true:

$$\mathbb{H}_0 : \Theta \in \mathrm{sp}_\Theta(0) \qquad \text{versus}$$
$$\mathbb{H}_1 : \Theta \in \mathrm{sp}_\Theta(1).$$

null hypothesis

alternative hypothesis

Reminder: Level-α test and power of a test

Definition 1. A test ϕ is said to be of (false-alarm) level $\alpha \in [0, 1]$ if its probabilities of false alarm are upper-bounded by α:

$$\max_{\theta \in \mathrm{sp}_\Theta(0)} \mathrm{E}[\phi(X) \,|\, \theta] \leq \alpha. \tag{1}$$

Definition 2. The *power function* of a test ϕ is

$$\beta(\theta) = \mathrm{E}[\phi(\boldsymbol{X}) \,|\, \theta]$$

for $\theta \in \mathrm{sp}_\Theta(1)$.

Here, we adopt the classical Neyman-Pearson approach: given an upper bound α on the false-alarm probability, maximize the detection probability.

We consider all tests $\phi(\boldsymbol{X})$ that satisfy (1), where

$$\max_{\theta \in \mathrm{sp}_\Theta(0)} \mathrm{E}\big[\phi(\boldsymbol{X}) \,|\, \theta\big] \tag{2}$$

is the *size of the test* $\phi(\boldsymbol{X})$. Therefore, the condition (1) states that we focus on tests whose size is upper-bounded by α.

UMP Tests

Definition 3. Among all tests $\phi(\boldsymbol{X})$ whose size is upper-bounded by α, we say that $\phi_{\mathrm{UMP}}(\boldsymbol{X})$ is a *uniformly most powerful (UMP)* if it satisfies

$$\mathrm{E}[\phi_{\mathrm{UMP}}(\boldsymbol{X}) \,|\, \theta] \geq \mathrm{E}[\phi(\boldsymbol{X}) \,|\, \theta]$$

for all $\theta \in \mathrm{sp}_\Theta(1)$.

This is a strong statement and few hypothesis-testing problems have UMP tests.

☞ Neyman-Pearson tests for simple hypotheses are UMP.

> Hence, to find an UMP test for composite hypotheses, we need to first write a likelihood ratio for the simple hypothesis test with
>
> $$\mathrm{sp}_\Theta(0) = \{\theta_0\}, \qquad \mathrm{sp}_\Theta(1) = \{\theta_1\}, \qquad \mathrm{sp}_\Theta = \{\theta_0, \theta_1\}$$
>
> and then transform it in such a way that unknown quantities (e.g., θ_0 and θ_1) disappear from the test statistic.

1) If such a transformation can be found, there is hope that a UMP test exists.

2) However, we still need to find out how to set a decision threshold such that the upper bound (1) is satisfied.

Positive DC Level in AWGN

CONSIDER the following composite hypothesis-testing problem:

$$\mathbb{H}_0: \qquad \Theta = 0$$

i.e., $\mathrm{sp}_\Theta(0) = \{0\}$

versus

$$\mathbb{H}_1: \qquad \Theta > 0$$

i.e., $\mathrm{sp}_\Theta(1) = (0, +\infty)$

where the measurements $\boldsymbol{X} = (X[n])_{n=0}^{N-1}$ are conditionally independent, identically distributed (i.i.d.) given $\Theta = \theta$, modeled as

$$\{X[n] \mid \Theta = \theta\} = \theta + W[n]$$

and $(W[n])_{n=0}^{N-1}$ a zero-mean additive white Gaussian noise (AWGN) with known variance σ^2, i.e.,

$$W[n] \sim \mathcal{N}(0, \sigma^2)$$

which implies

$$f_{\boldsymbol{X}\,|\,\Theta}(\boldsymbol{x} \mid \theta) = \frac{1}{\sqrt{(2\pi\sigma^2)^N}} \exp\Big[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-\theta)^2\Big]. \tag{3}$$

A sufficient statistic for θ is

$$\overline{x} = \frac{1}{N}\sum_{n=1}^{N} x[n].$$

Now, find the probability density function (pdf) of $\overline{x}$ given $\Theta = \theta$:

$$f_{\overline{X}|\Theta}(\overline{x} \mid \theta) = \mathcal{N}\Big(\overline{x}\,\Big|\,\theta, \frac{\sigma^2}{N}\Big). \tag{4}$$

We follow 1 and start by writing the classical Neyman-Pearson test for the simple hypotheses with $\mathrm{sp}_\Theta^{\mathrm{simple}}(0) = \{0\}$ and $\mathrm{sp}_\Theta^{\mathrm{simple}}(1) = \{\theta_1\}$, $\theta_1 \in (0, +\infty)$:

$$\frac{f_{\overline{X}\,|\,\Theta}(\overline{x} \mid \theta_1)}{f_{\overline{X}\,|\,\Theta}(\overline{x} \mid 0)} = \frac{\left(2\pi\frac{\sigma^2}{N}\right)^{-1/2} \exp\left[-\frac{1}{2\sigma^2/N}(\overline{x}-\theta_1)^2\right]}{\left(2\pi\frac{\sigma^2}{N}\right)^{-1/2} \exp\left(-\frac{1}{2\sigma^2/N}\overline{x}^2\right)} \underset{\mathbb{H}_0}{\overset{\mathbb{H}_1}{\gtrless}} \lambda. \tag{5}$$

Taking log and rearranging leads to

$$\theta_1 \overline{x} \underset{\mathbb{H}_0}{\overset{\mathbb{H}_1}{\gtrless}} \eta.$$

Since we know that $\theta_1 > 0$, we can divide both sides of the above expression by θ_1, yielding

$$\overline{x} \underset{\mathbb{H}_0}{\overset{\mathbb{H}_1}{\gtrless}} \tau.$$

Hence, we have transformed our likelihood ratio in such a way that θ_1 disappears from the test statistic, i.e., we have accomplished 1.

Now, on to 2. How to determine the threshold τ such that the upper bound (1) is satisfied? We know:

$$f_{\overline{X}\,|\,\Theta}(\overline{x}\,|\,0) = \mathcal{N}\left(\overline{x}\,\middle|\,0, \frac{\sigma^2}{N}\right)$$

and, therefore,

$$\begin{aligned} \mathrm{E}\big[\phi(\boldsymbol{X})\,|\,0\big] &= \Pr_{\boldsymbol{X}\,|\,\Theta}\left\{\overline{X} > \tau\,\middle|\,0\right\} \\ &= \Pr_{\boldsymbol{X}|\Theta}\left\{\frac{\overline{X}-0}{\sqrt{\sigma^2/N}} > \frac{\tau}{\sqrt{\sigma^2/N}}\,\middle|\,0\right\} \\ &= Q\left(\frac{\tau}{\sqrt{\sigma^2/N}}\right). \end{aligned} \tag{6}$$

$\frac{\overline{X}-0}{\sqrt{\sigma^2/N}}$ is standard normal random variable

Since $\mathbb{H}_0$ is a simple hypothesis,

$$\begin{aligned} \max_{\theta \in \mathrm{sp}_\Theta(0)} \mathrm{E}\big[\phi(\boldsymbol{X})\,|\,\theta\big] &= \mathrm{E}\big[\phi(\boldsymbol{X})\,|\,0\big] \\ &= Q\left(\frac{\tau}{\sqrt{\sigma^2/N}}\right) \end{aligned}$$

The most powerful (MP) test is achieved if the upper bound α in (1) is reached by equality:

$$\tau = \sqrt{\frac{\sigma^2}{N}}\,Q^{-1}(\alpha). \tag{7}$$

Hence, we have accomplished 2 because this τ yields exactly size α for our test $\phi(\boldsymbol{X})$.

To determine the receiver operating characteristic (ROC) of the above test, substitute (7) into the power function:

$$\begin{aligned} \Pr_{\boldsymbol{X}\,|\,\Theta}\left\{\overline{X} > \tau\,|\,\theta\right\} &= \Pr_{\boldsymbol{X}\,|\,\Theta}\left\{\frac{\overline{X}-\theta}{\sqrt{\sigma^2/N}} > \frac{\tau-\theta}{\sqrt{\sigma^2/N}}\,\middle|\,\theta\right\} \\ &= Q\left(\frac{\tau-\theta}{\sqrt{\sigma^2/N}}\right) \\ &= Q\left(Q^{-1}(\alpha) - \frac{\theta}{\sqrt{\sigma^2/N}}\right). \end{aligned} \tag{8}$$

$\frac{\overline{X}-\theta}{\sqrt{\sigma^2/N}}$ is standard normal random variable

Positive versus negative DC level

CONSIDER the following composite hypothesis-testing problem:

$$\mathbb{H}_0\colon \Theta \le 0 \qquad \text{versus}$$
$$\mathbb{H}_1\colon \Theta > 0$$

i.e., $\mathrm{sp}_\Theta(0) = (-\infty, 0]$

i.e., $\mathrm{sp}_\Theta(1) = (0, +\infty)$

where the measurements $\boldsymbol{X} = \big(X[n]\big)_{n=0}^{N-1}$ follow (3). Write the classical Neyman-Pearson test for the simple hypotheses with $\mathrm{sp}_\Theta^{\text{simple}}(0) = \{\theta_0\}$ and $\mathrm{sp}_\Theta^{\text{simple}}(1) = \{\theta_1\}$, where $\theta_0 \in (-\infty, 0]$ and $\theta_1 \in (0, +\infty)$:

$$\frac{f_{\overline{X}\,|\,\Theta}(\overline{x}\,|\,\theta_1)}{f_{\overline{X}\,|\,\Theta}(\overline{x}\,|\,0)} = \frac{\left(2\pi\dfrac{\sigma^2}{N}\right)^{-1/2} \exp\left[-\dfrac{1}{2\sigma^2/N}(\overline{x}-\theta_1)^2\right]}{\left(2\pi\dfrac{\sigma^2}{N}\right)^{-1/2} \exp\left(-\dfrac{1}{2\sigma^2/N}\overline{x}^2\right)} \underset{\mathbb{H}_0}{\overset{\mathbb{H}_1}{\gtrless}} \lambda \tag{9}$$

where

$$\theta_0 < \theta_1.$$

Taking log and rearranging leads to

$$(\theta_1 - \theta_0)\overline{x} \underset{\mathbb{H}_0}{\overset{\mathbb{H}_1}{\gtrless}} \eta$$

and, since $\theta_0 < \theta_1$, to

$$\phi(\boldsymbol{x}): \quad \overline{x} \underset{\mathbb{H}_0}{\overset{\mathbb{H}_1}{\gtrless}} \tau.$$

Hence, we have transformed our likelihood ratio in such a way that θ_0 and θ_1 disappear from the test statistic, i.e., we accomplished 1.

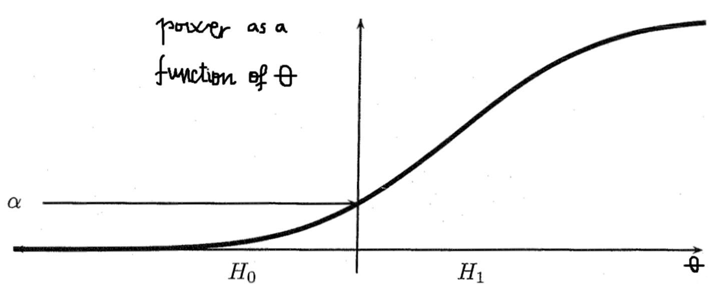

FIGURE 10.1. The power function . The size of the test is the largest probability of rejecting H_0 when H_0 is true. This occurs at $\theta = 0$, hence the size is power(0). We choose the critical value c so that power$(0) = \alpha$.

Figure 1: Power as a function of θ.

The power function of this test is

$$\Pr_{\boldsymbol{X}\,|\,\Theta}\left\{\overline{X} > \tau \,|\, \theta\right\} = \Pr_{\boldsymbol{X}\,|\,\Theta}\left\{\frac{\overline{X}-\theta}{\sigma/\sqrt{N}} > \frac{\tau-\theta}{\sigma/\sqrt{N}} \,\middle|\, \theta\right\}$$

$$= Q\left(\frac{\tau - \theta}{\sigma/\sqrt{N}}\right)$$

which is an increasing function of θ, see Fig. 1. Compute the test size (2):

$$\begin{aligned}\max_{\theta \in \mathrm{sp}_\Theta(0)} \mathrm{E}\big[\phi(\boldsymbol{X}) \,|\, \theta\big] &= \max_{\theta \in \mathrm{sp}_\Theta(0)} \mathrm{Pr}_{X \,|\, \Theta}\left\{\overline{X} > \tau \,|\, \theta\right\} \\ &= \max_{\theta \in (-\infty, 0]} Q\left(\frac{\tau - \theta}{\sigma/\sqrt{N}}\right) \\ &= Q\left(\frac{\tau}{\sigma/\sqrt{N}}\right).\end{aligned}$$

The MP test is achieved if the upper bound α in (1) is reached by equality:

$$\tau = \frac{\sigma}{\sqrt{N}} Q^{-1}(\alpha).$$

Hence, we have accomplished 2 because this τ yields exactly size α for our test $\phi(\boldsymbol{X})$.

Completely Unknown DC Level in AWGN

CONSIDER now the composite hypothesis-testing problem:

$\mathbb{H}_0\colon \Theta = 0$ versus — i.e., $\mathrm{sp}_\Theta(0) = \{0\}$

$\mathbb{H}_1\colon \Theta \neq 0$ — i.e., $\mathrm{sp}_\Theta(1) = (-\infty, +\infty) \backslash \{0\}$

where the measurements $\boldsymbol{X} = \big(X[n]\big)_{n=0}^{N-1}$ follow (3). We start by writing the classical Neyman-Pearson test for the simple hypotheses with $\mathrm{sp}_\Theta(0) = \{0\}$ and $\mathrm{sp}_\Theta(1) = \{\theta_1 \neq 0\}$:

$$\theta_1\, \overline{x} > \eta.$$

We cannot accomplish 1 because θ_1 cannot be removed from the test statistic; therefore, UMP test does not exist for the above problem.

Monotone Likelihood-Ratio Criterion

CONSIDER a scalar parameter θ. We say that the likelihood function $f_{\boldsymbol{X} \,|\, \Theta}(\boldsymbol{x} \,|\, \boldsymbol{\theta})$ belongs to the monotone likelihood-ratio (MLR) family if the pdfs (or probability mass functions (pmfs)) from this family

i) satisfy the identifiability condition for θ (i.e., these pdfs are distinct for different values of θ) and

ii) there is a scalar statistic $T(\boldsymbol{x})$ such that, for $\theta_0 < \theta_1$, the likelihood ratio

$$\Lambda(\boldsymbol{x}\,;\,\theta_0,\theta_1) = \frac{f_{\boldsymbol{X}\,|\,\Theta}(\boldsymbol{x}\,|\,\theta_1)}{f_{\boldsymbol{X}\,|\,\Theta}(\boldsymbol{x}\,|\,\theta_0)}$$

is a monotonically increasing function of $T(\boldsymbol{x})$.

If $f_{\boldsymbol{X}\,|\,\Theta}(\boldsymbol{x}\,|\,\boldsymbol{\theta})$ belongs to the MLR family, then use the following test:

$$\phi_\lambda(\boldsymbol{x}) = \begin{cases} 1, & \text{for } T(\boldsymbol{x}) \geq \lambda, \\ 0, & \text{for } T(\boldsymbol{x}) < \lambda \end{cases}$$

and set

$$\alpha = \mathrm{E}\big[\phi(\boldsymbol{X})\,|\,\theta_0\big] = \Pr_{\boldsymbol{X}|\Theta}\big\{T(\boldsymbol{X}) \geq \lambda\,|\,\theta_0\big\} \tag{10}$$

i.e., use this condition to find the threshold λ.

This test has the following properties:

- With α given by (10), $\phi_\lambda(\boldsymbol{x})$ is UMP test of size α for testing

$$\begin{aligned} &\mathbb{H}_0 : \Theta \leq \theta_0 \quad \text{versus} \\ &\mathbb{H}_1 : \Theta > \theta_0. \end{aligned}$$

- For each λ, the power function

$$\mathrm{E}\big[\phi(\boldsymbol{X})\,|\,\theta\big] = \Pr_{\boldsymbol{X}|\Theta}\big\{T(\boldsymbol{X}) \geq \lambda\,|\,\theta\big\} \tag{11}$$

is a monotonically increasing function of θ.

☞ NOTE: Consider the one-parameter exponential family

$$f_{\boldsymbol{X}|\Theta}(x\,|\,\theta) = h(\boldsymbol{x})\exp[\eta(\theta)T(\boldsymbol{x}) - B(\theta)]. \tag{12}$$

Then, if $\eta(\theta)$ is a monotonically increasing function of θ, the class of pdfs (pmfs) (12) satisfies the MLR conditions.

✻ EXAMPLE: Exponential random variables. Consider conditionally i.i.d. measurements $\boldsymbol{X} = \big(X[n]\big)_{n=0}^{N-1}$ given the parameter $\theta > 0$, following the exponential pdf:

$$\begin{aligned} f_{X|\Theta}(x[n]\,|\,\theta) &= \mathrm{Expon}\Big(x[n]\,\Big|\,\frac{1}{\theta}\Big) \\ &= \frac{1}{\theta}\exp\big(-\theta^{-1}x[n]\big)\,\mathbb{1}_{(0,+\infty)}(x[n]). \end{aligned}$$

The likelihood function of θ for all observations $\boldsymbol{x}$ is

$$f_{\boldsymbol{X}\,|\,\Theta}(\boldsymbol{x}\,|\,\theta) = \frac{1}{\theta^N}\,\exp\big[-\theta^{-1}T(\boldsymbol{x})\big]\,\mathbb{1}_{(0,+\infty)}\Big(\min_{n=0,\dots,N-1} x[n]\Big)$$

where

$$T(\boldsymbol{x}) = \sum_{n=0}^{N-1} x[n].$$

Since $f_{X|\Theta}(x \mid \theta)$ belongs to the one-parameter exponential family (12) and $\eta(\theta) = -\theta^{-1}$ is a monotonically increasing function of θ, $f_{X \mid \Theta}(x \mid \theta)$ satisfies the MLR conditions. Therefore,

$$\phi_\lambda(x) = \begin{cases} 1, & T(x) \geq \lambda, \\ 0, & T(x) < \lambda \end{cases}.$$

is UMP for testing

$$\begin{aligned} \mathbb{H}_0 &: \Theta \leq \theta_0 \quad \text{versus} \\ \mathbb{H}_1 &: \Theta > \theta_0. \end{aligned}$$

The sum of i.i.d. exponential random variables follows the Erlang pdf[1]:

[1] Erlang pdf a special case of the gamma pdf

$$\begin{aligned} f_{T|\Theta}(T \mid \theta) &= \frac{1}{\theta^N} \frac{T^{N-1}}{(N-1)!} \exp\big(-T/\theta\big)\, \mathbb{1}_{(0,+\infty)}(T) \\ &= \text{Gamma}(T \mid N, \theta^{-1}). \end{aligned}$$

Hence, the size of the test can be written as

$$\begin{aligned} \alpha &= \Pr_{X|\Theta}\big(T(X) \geq \lambda \mid \theta_0\big) \\ &= \frac{1}{\theta_0^N} \int_\lambda^{+\infty} \frac{t^{N-1}}{(N-1)!} \exp\Big(-\frac{t}{\theta_0}\Big)\, \mathrm{d}t \\ &= \Big[1 + \frac{\lambda}{\theta_0} + \cdots + \frac{1}{(N-1)!}\Big(\frac{\lambda}{\theta_0}\Big)^{N-1}\Big] \exp\Big(-\frac{\lambda}{\theta_0}\Big) \end{aligned}$$

where the integral was evaluated using integration by parts. For $N = 1$, we have

$$\lambda = \theta_0 \ln\Big(\frac{1}{\alpha}\Big).$$

Generalized Likelihood Ratio Test

In binary composite hypothesis testing, we have $\text{sp}_\Theta(0)$ and $\text{sp}_\Theta(1)$ that form a partition of the parameter space sp_Θ:

$$\text{sp}_\Theta(0) \cup \text{sp}_\Theta(1) = \text{sp}_\Theta, \ \text{sp}_\Theta(0) \cap \text{sp}_\Theta(1) = \emptyset$$

and wish to identify which of the two hypotheses is true:

$$\begin{aligned} \mathbb{H}_0 &: \Theta \in \text{sp}_\Theta(0) \quad \text{null hypothesis} \quad \text{versus} \\ \mathbb{H}_1 &: \Theta \in \text{sp}_\Theta(1) \quad \text{alternative hypothesis}. \end{aligned}$$

In generalized likelihood-ratio (GLR) tests, we replace the unknown parameters by their maximum-likelihood (ML) estimates under the two hypotheses. Hence, accept $\mathbb{H}_1$ if

$$\Lambda_{\text{GLR}}(x) = \frac{\max_{\theta \in \text{sp}_\Theta(1)} f_{X \mid \Theta}(x \mid \theta)}{\max_{\theta \in \text{sp}_\Theta(0)} f_{X \mid \Theta}(x \mid \theta)} > \tau.$$

This test has no UMP optimality properties, but often works well in practice.

Completely unknown DC level in AWGN

CONSIDER the composite hypothesis-testing problem:

$$\begin{aligned} \mathbb{H}_0 : \quad & \Theta = 0 \\ \text{versus} & \\ \mathbb{H}_1 : \quad & \Theta \neq 0 \end{aligned}$$

i.e., $\mathrm{sp}_\Theta(0) = \{0\}$

i.e., $\mathrm{sp}_\Theta(1) = (-\infty, +\infty) \backslash \{0\}$

where the measurements $\boldsymbol{X} = \big(X[n]\big)_{n=0}^{N-1}$ follow (3). Our GLR test accepts $\mathbb{H}_1$ if

$$\Lambda_{\mathrm{GLR}}(\boldsymbol{x}) = \frac{\max_{\theta \in \mathrm{sp}_\Theta(1)} f_{\overline{X}|\Theta}(\overline{x} \,|\, \theta)}{f_{\overline{X}|\Theta}(\overline{x} \,|\, 0)} > \tau.$$

Now,

$$\overline{x} = \arg \max_{\theta \in \mathrm{sp}_\Theta(1)} f_{\overline{X}|\Theta}(\overline{x} \,|\, \theta)$$

and

$$\begin{aligned} f_{\overline{X}|\Theta}(\overline{x} \,|\, 0) &= \mathcal{N}\Big(\overline{x} \,\Big|\, 0, \frac{\sigma^2}{N}\Big) \\ &= \frac{1}{\sqrt{2\pi \frac{\sigma^2}{N}}} \exp\Big(-0.5 \frac{\overline{x}^2}{\sigma^2/N}\Big) \\ f_{\overline{X}|\Theta}(\overline{x} \,|\, \overline{x}) &= \mathcal{N}\Big(\overline{x} \,|\, 0, \frac{\sigma^2}{N}\Big) = \frac{1}{\sqrt{2\pi\sigma^2/N}} \end{aligned}$$

yielding

$$\ln \Lambda_{\mathrm{GLR}}(\boldsymbol{x}) = \frac{N\overline{x}^2}{2\sigma^2}.$$

Therefore, we accept $\mathbb{H}_1$ if

$$(\overline{x})^2 > \gamma$$

i.e.,

$$|\overline{x}| > \eta.$$

We compare this detector with the (not realizable, also called *clairvoyant*) UMP detector that assumes the knowledge of the sign of θ under $\mathbb{H}_1$. Assuming that the sign of θ under $\mathbb{H}_1$ is *known*, we can construct the UMP detector, whose region of convergence (ROC) curve is given by

$$P_{\mathrm{D}} = Q(Q^{-1}(P_{\mathrm{FA}}) - d)$$

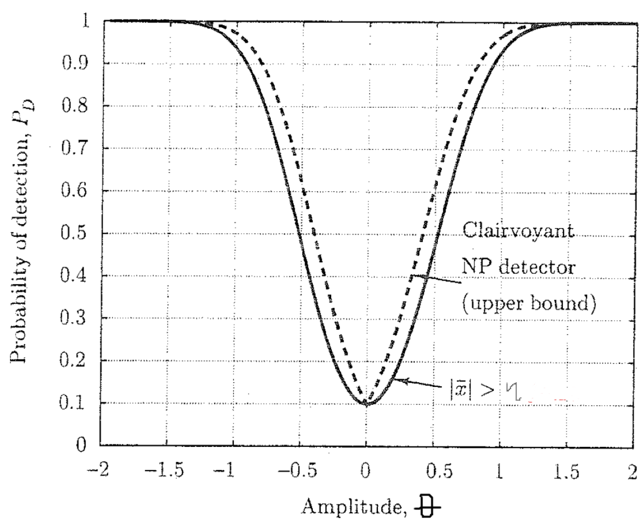

Figure 2: Detection probabilities of GLR and clairvoyant detectors as functions of the DC level θ.

where $d = \sqrt{N\theta^2/\sigma^2}$ and θ is the value of the parameter under $\mathbb{H}_1$; see (8) for the case where $\theta > 0$ under $\mathbb{H}_1$. All other detectors have P_{D} below this upper bound.

✻ Performance analysis. Decide $\mathbb{H}_1$ if $|\bar{x}| > \eta$. To make sure that the GLR test is implementable, we must be able to specify a threshold η so that the false-alarm probability is upper-bounded by a given size α. This is possible in our example:

$$\begin{aligned}
\underbrace{\mathrm{E}_{\boldsymbol{X}\,|\,\Theta}[\phi(\boldsymbol{X})\,|\,0]}_{P_{\mathrm{FA}}=\alpha} &= \Pr_{\bar{X}\,|\,\Theta}\{|\bar{X}| > \eta\,|\,0\} \\
&= 2\Pr_{\bar{X}\,|\,\Theta}\{\bar{X} > \eta\,|\,0\} \\
&= 2\Pr_{\bar{X}\,|\,\Theta}\left\{\frac{\bar{X}}{\sqrt{\sigma^2/N}} > \frac{\eta}{\sqrt{\sigma^2/N}}\,\middle|\,0\right\} \\
&= 2Q\Big(\frac{\eta}{\sqrt{\sigma^2/N}}\Big)
\end{aligned}$$

and

$$\begin{aligned}
\underbrace{\mathrm{E}_{\boldsymbol{X}\,|\,\Theta}\big[\phi(\boldsymbol{X})\,|\,\theta\big]}_{P_{\mathrm{D}}} &= \Pr_{\bar{X}\,|\,\Theta}(|\bar{X}| > \eta\,|\,\theta) \\
&= \Pr_{\bar{X}\,|\,\Theta}(\bar{X} > \eta\,|\,\theta) + \Pr_{\bar{X}\,|\,\Theta}(\bar{X} < -\eta\,|\,\theta) \\
&= Q\Big(\frac{\eta-\theta}{\sqrt{\sigma^2/N}}\Big) + Q\Big(\frac{\eta+\theta}{\sqrt{\sigma^2/N}}\Big) \\
&= Q\bigg(Q^{-1}\Big(\frac{\alpha}{2}\Big) - \frac{\theta}{\sqrt{\sigma^2/N}}\bigg) + Q\bigg(Q^{-1}\Big(\frac{\alpha}{2}\Big) + \frac{\theta}{\sqrt{\sigma^2/N}}\bigg).
\end{aligned}$$

use symmetry

In this case, the GLR test is only slightly inferior to the clairvoyant detector, see Fig. 2.

DC level in WGN with DC level a and noise variance σ^2 both unknown

RECALL that σ^2 is called a *nuisance parameter* because we are interested only in θ. Here, the GLR test for

$$\mathbb{H}_0\colon \Theta = 0 \quad \text{versus}$$
$$\mathbb{H}_1\colon \Theta \neq 0$$

i.e., $\mathrm{sp}_\Theta(0) = \{0\}$

i.e. $\mathrm{sp}_\Theta(1) = (-\infty, +\infty)\backslash\{0\}$

where the measurements $\boldsymbol{X} = \big(X[n]\big)_{n=0}^{N-1}$ follow (3). Here,

$$\max_{\theta,\sigma^2} f_{\boldsymbol{X}\,|\,\Theta,\sigma^2}(\boldsymbol{x}\,|\,\theta,\sigma^2) = \frac{1}{[2\pi\widehat{\sigma}_1^2(\boldsymbol{x})]^{N/2}}\mathrm{e}^{-N/2}$$
$$\max_{\sigma^2} f_{\boldsymbol{X}\,|\,\Theta,\sigma^2}(\boldsymbol{x}\,|\,0,\sigma^2) = \frac{1}{[2\pi\widehat{\sigma}_0^2(\boldsymbol{x})]^{N/2}}\mathrm{e}^{-N/2}$$

where

$$\widehat{\sigma}_0^2(\boldsymbol{x}) = \frac{1}{N}\sum_{n=1}^{N} x^2[n]$$
$$\widehat{\sigma}_1^2(\boldsymbol{x}) = \frac{1}{N}\sum_{n=1}^{N} \big(x[n] - \overline{x}\big)^2.$$

Hence,

$$\Lambda_{\mathrm{GLR}}(\boldsymbol{x}) = \left(\frac{\widehat{\sigma}_0^2(\boldsymbol{x})}{\widehat{\sigma}_1^2(\boldsymbol{x})}\right)^{N/2}$$

i.e., GLR test fits data with the “best” DC-level signal $\widehat{\theta}_{\mathrm{ML}} = \overline{x}$, finds the residual variance estimate $\widehat{\sigma}_1^2$, and compares this estimate with the variance estimate $\widehat{\sigma}_0^2$ under the null case (i.e., for $\theta = 0$). When sufficiently strong signal is present, $\widehat{\sigma}_1^2 \ll \widehat{\sigma}_0^2$ and $\Lambda_{\mathrm{GLR}}(\boldsymbol{x}) \gg 1$.

Note that

$$\begin{aligned}
\widehat{\sigma}_1^2(\boldsymbol{x}) &= \frac{1}{N}\sum_{n=1}^{N}(\overline{x} - x[n])^2 \\
&= \frac{1}{N}\sum_{n=1}^{N}(x^2[n] - 2\overline{x}x[n] + \overline{x}^2) \\
&= \left(\frac{1}{N}\sum_{n=1}^{N} x^2[n]\right) - 2\overline{x}^2 + \overline{x}^2 \\
&= \widehat{\sigma}_0^2(\boldsymbol{x}) - \overline{x}^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
2\ln\Lambda_{\mathrm{GLR}}(\boldsymbol{x}) &= N\ln\left[\frac{\widehat{\sigma}_0^2(\boldsymbol{x})}{\widehat{\sigma}_0^2(\boldsymbol{x}) - \overline{x}^2}\right] \\
&= N\ln\left[\frac{1}{1 - \overline{x}^2/\widehat{\sigma}_0^2(\boldsymbol{x})}\right].
\end{aligned}$$

Note that

$$0 \le \frac{\overline{x}^2}{\widehat{\sigma}_0^2(\boldsymbol{x})} \le 1$$

and $\ln\left[1/(1-z)\right]$ is monotonically increasing on $z \in (0, 1)$. Therefore, an equivalent test can be constructed as follows:

$$T(\boldsymbol{x}) = \frac{\overline{x}^2}{\widehat{\sigma}_0^2(\boldsymbol{x})} > \tau.$$

The pdf of $T(\boldsymbol{X})$ given $\Theta = 0$ *does not* depend on σ^2 and, therefore, a threshold for GLR test can be determined to satisfy the upper bound on the false-alarm probability.

Acronyms

AWGN additive white Gaussian noise. 3

GLR generalized likelihood-ratio. 8–12

i.i.d. independent, identically distributed. 3, 7, 8

ML maximum-likelihood. 8

MLR monotone likelihood-ratio. 6, 7

MP most powerful. 4, 6

pdf probability density function. 3, 6–8, 12

pmf probability mass function. 6, 7

ROC receiver operating characteristic. 4, 9

UMP uniformly most powerful. 2, 6, 8, 9

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