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- Kalman filters provide a way to incorporate model information into sequential bayesian estimation.

General Approach for Kalman Filtering Problems

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2. Formulate the Measurement Model with matrices $\boldsymbol{\Phi}$, $\boldsymbol{V}$, $\boldsymbol{R}$.
3. Start with an estimate of 0 and noninformative prior $\hat{\boldsymbol{\beta}}(0|0) = \mathbf{0}$, and $P(0|0) = (\text{big number})\boldsymbol{I}$.
4. For K iterations do:
    4.1 Predict step $\hat{\boldsymbol{\beta}}(n|n-1)$, $P(n|n-1)$.
    4.2 Update step $\hat{\boldsymbol{\beta}}(n|n)$, $P(n|n)$.

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- Here $\epsilon_x[n]$ and $epsilon_y[n]$ are independent realizations of noise with variance σ_2.
- Given a starting position and velocity updates, it's easy to compute the position of the vehicle every Δt seconds:

$$r_x[n] = r_x[n-1] + v_x[n-1]\Delta t$$
$$r_y[n] = r_y[n-1] + v_y[n-1]\Delta t$$

Finding the State Equation from the Physical Model

- Write this as a state equation:

$$\boldsymbol{\beta}_n = \begin{bmatrix} r_x[n] \\ r_y[n] \\ v_x[n] \\ v_y[n] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\boldsymbol{H}} \underbrace{\begin{bmatrix} r_x[n-1] \\ r_y[n-1] \\ v_x[n-1] \\ v_y[n-1] \end{bmatrix}}_{\boldsymbol{\beta}_{n-1}} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\boldsymbol{J}} \underbrace{\begin{bmatrix} \eta_x[n] \\ \eta_y[n] \end{bmatrix}}_{\boldsymbol{\eta_k}}$$

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- Since η_x and η_y are independent, we can define the state covariance matrix as:

$$\boldsymbol{Q} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma^2 & 0 \\ 0 & 0 & 0 & \sigma^2 \end{bmatrix}$$

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- Next, lets look at measurements.

Visualizing the Problem

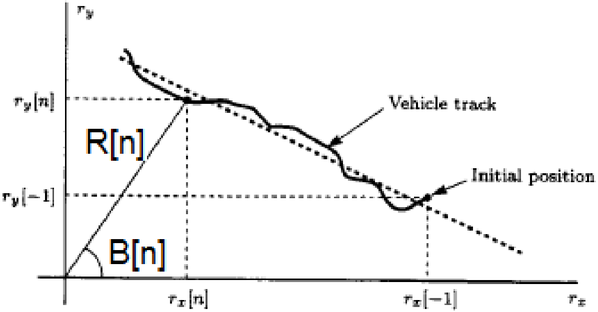

Figure 13.21 Typical track of vehicle moving in given direction at constant speed

Figure 1: Typical model for tracking a vehicle from a stationary point, such as an Air Traffic Control Tower

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- Here $\epsilon_R \sim N(0, \sigma_R^2)$ and $\epsilon_B \sim N(0, \sigma_B^2)$ are known independent gaussian random variables representing the sensor noise. Their covariance matrix is :

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- Note: Here we are not looking at any interference noise so $\boldsymbol{V} = \boldsymbol{0}$.

Linearize the Measurement Model

- We have a problem: The measurement model doesn't consist of linear functions!
- To address this issue, find the Jacobian of $\boldsymbol{C}$:

$$\boldsymbol{D_C} = \begin{bmatrix} \frac{\partial\sqrt{r_x[n]^2+r_y[n]^2}}{\partial r_x[n]} & \frac{\partial\sqrt{r_x[n]^2+r_y[n]^2}}{\partial r_y[n]} & \frac{\partial\sqrt{r_x[n]^2+r_y[n]^2}}{\partial v_x[n]} & \frac{\partial\sqrt{r_x[n]^2+r_y[n]^2}}{\partial v_y[n]} \\ \frac{\partial \arctan\frac{r_y[n]}{r_x[n]}}{\partial r_x[n]} & \frac{\partial \arctan\frac{r_y[n]}{r_x[n]}}{\partial r_y[n]} & \frac{\partial \arctan\frac{r_y[n]}{r_x[n]}}{\partial v_x[n]} & \frac{\partial \arctan\frac{r_y[n]}{r_x[n]}}{\partial v_y[n]} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{r_x[n]}{\sqrt{r_x[n]^2+r_y[n]^2}} & \frac{r_y[n]}{\sqrt{r_x[n]^2+r_y[n]^2}} & 0 & 0 \\ \frac{-r_y[n]}{r_x[n]^2+r_y[n]^2} & \frac{r_x[n]}{r_x[n]^2+r_y[n]^2} & 0 & 0 \end{bmatrix}$$

- In the 60s, engineers at NASA's Ames Research Center figured out that using $\boldsymbol{\Phi} = \boldsymbol{D_C}$ can work well for non-linear systems. This is called the **Extended Kalman Filter** and is critically important in control systems.
- Note: Here $\Phi = \Phi[n]$ changes every iteration.

Iterations: Steps 3 and 4

- Predict Step:

$$\hat{\boldsymbol{\beta}}(n|n-1) = H\hat{\boldsymbol{\beta}}(n-1|n-1)$$
$$P(n|n-1) = \boldsymbol{H}P(n-1|n-1)\boldsymbol{H}^T + \boldsymbol{JQJ}^T$$

- This is the mean and covariance of the new posterior distribution.
- Update Step:

$$\kappa[n] = P(n|n-1)\boldsymbol{D_C}[n]^T[R + \boldsymbol{D_C}[n]P(k|k-1)\boldsymbol{D_C}[n]^T]^{-1}$$
$$\hat{\boldsymbol{\beta}}(n|n) = \hat{\boldsymbol{\beta}}(n|n-1) + \kappa[n](\boldsymbol{y_n} - \boldsymbol{D_C}[n]\hat{\boldsymbol{\beta}}(n|n-1))$$
$$P(n|n) = P(n|n-1) - \kappa[n]\boldsymbol{D_C}[n]P(n|n-1)$$

- This changes the prior with a combination of new measurements (y_n) and prior information (contained in the gain $\kappa[n]$).

Results

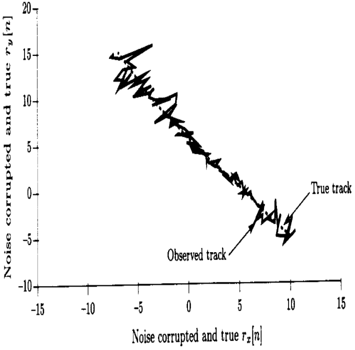

Figure 13.24 True and observed vehicle tracks

Figure 2: True Measurements

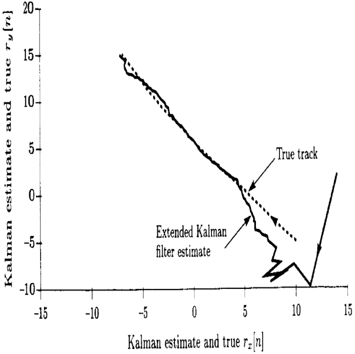

Figure 13.25 True and extended Kalman filter estimate

Figure 3: Extended kalman filter has high error at the start. but is quick to find the target

Comments

- Kalman Filters are used ubiquitously whenever there are sequential measurements.
- Usually the real distributions are approximated with Gaussians to use the formula.
- Originally was panned. Kalman originally published in Mechanical engineering. Then NASA found it and realized it was useful for navigating the Apollo spacecrafts.
- The main trick when applying KF is to figure out good state space and measurement matrices. This way the inverse computations don't blow up.