ESE 524 - Homework 4

Assigned date: 03/05/19
Due Date: 03/19/19
Total Points: 100 + 20 Extra Credit

1) **Bayesian Estimation**
Suppose that we have two systems:

$$y_1 = \theta + n_1$$
$$y_2 = \theta + n_2$$

where $\theta \sim \mathcal{N}(0, \sigma_\theta^2)$, $n_1 \sim \mathcal{N}(0, \sigma_{n1}^2)$, and $n_2 \sim \mathcal{N}(0, \sigma_{n2}^2)$. In addition, θ, n_1, and n_2 are mutually independent.

(a) (**10 pts**) Suppose that we have observed y_1, can you infer the output of y_2, i.e., $p(y_2|y_1)$?
Hint: You can determine the joint distribution of y_1 and y_2 at first and then compute $p(y_2|y_1)$.

(b) (**10 pts**) Suppose we only observe y_1, what is $p(\theta|y_1)$? What is the Bayesian MMSE estimate of θ_1? If we further observe y_2, what is $p(\theta|y_1, y_2)$? What is the Bayesian MMSE estimate of θ_1 now? Compare the Bayesian MSE of these two estimators.

2) **Posterior Distribution**
Let X be a binomial distribution with parameters n and θ. Suppose that the prior distribution of the parameter θ has Beta probability density function given as:

$$\pi(\theta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha - 1}(1 - \theta)^{\beta - 1},$$

where $0 < \theta < 1$.

(a) (**10 pts**) Find the posterior density of θ, i.e., $p(\theta|X = x)$

(b) (**10 pts**) Find the Bayesian minimum mean-squared error (BMMSE) estimator of θ.

3) **Lecture l4.pdf, Pages 16-19**
Let $X_1, \ldots, X_n$ be independent and identically distributed as a Gaussian distribtuion given as

$$p(X_k|\theta) \sim \mathcal{N}(\theta, \sigma^2).$$

The parameter of interest θ is also follows a Gaussian distribution given as

$$\pi(\theta) \sim \mathcal{N}(m, \sigma_\theta^2).$$

(a) (**10 pts**) The conditional distribution of $\theta|x$, i.e., $p(\theta|x)$?

(b) (**5 pts**) What are the conditional mean and variance of $f(\theta|x)$?

(c) (**5 pts**) Compare the $\mathbb{E}(\theta|x)$ to $\hat{\theta}_{\mathrm{ML}}$, the maximum likelihood estimator for this problem.

4) **Jeffreys' Prior**
For a Gaussian distribution, we have

$$p(\boldsymbol{x}|\sigma^2) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{N}(x_i - \theta)^2\right]$$

where $\boldsymbol{x} = [x_1, x_2, \dots, x_N]^T$, θ is the mean, which is known. Suppose that we have known the Jeffreys' prior for σ^2, which is

$$\pi(\sigma^2) \propto \frac{1}{\sigma^2}.$$

(a) (**5 pts**) Regard σ as the parameter for the Gaussian distribution rather than σ^2, and find $\sqrt{\mathcal{I}(\sigma)}$, where $\mathcal{I}(\sigma)$ is the Fisher information for σ.

(b) (**5 pts**) Use $\phi = \log\sigma^2$ to re-parameterize the Gaussian distribution, and find $\sqrt{\mathcal{I}(\phi)}$, where $\mathcal{I}(\phi)$ is the Fisher information for ϕ.

(c) (**5 pts**) Use change-of-variables transformation to compute the prior $\pi(\sigma)$ based on $\pi(\sigma^2)$.

(d) (**5 pts**) Use change-of-variables transformation to compute the prior $\pi(\phi)$ based on $\pi(\sigma^2)$.

Hint: the change-of-variables transformation for pdfs is referred to that $p_Y(y) = \{p_X(x) \cdot |dx/dy|\}|_{x=h^{-1}(y)}$ if $Y = h(X)$. The above problems shows that we have two alternative ways to compute the Jeffreys' priors for transformed parameters.

5) **MATLAB Problem**
Assume you are interested in examining the proportion of defective products coming out of a production line. Denote $\theta = \frac{\text{\# of defective items}}{\text{\# of total items}}$.

(a) (**10 pts**) Let $x_i \sim \text{Bern}(\theta)$, $i = 0, 1, ..., N-1$ be the (i.i.d.) examination results for the first N products off the line. Assume there were n_f defective products in the sample. Compute two posterior distributions p$(\theta|x)$, one with a uniform prior, and one with Jeffreys' prior.
Hint: The uniform distribution is a special case of the Beta distribution with both parameters equal to 1.

(b) (**5 pts**) Let the "true" value of $\theta_{true} = 0.25$ and $N = 100$. Generate N random samples using the true value of theta. For each prior from the previous part, plot the likelihood, prior, and posterior as functions of θ. What is the value of the MLE estimator for this problem? How does it compare to the maximum of the posterior distribution?
Hint: We are not asking you to compute formulas for these estimators, but to get the answers from your plots).

(c) (**5 pts**) While you've been computing these posteriors, your production line has cranked out another 50 products. Generate another set of N random samples based on the true value. Using your results from (ii) for the Jeffreys' prior, treat this as a sequential Bayesian inference problem. Plot the original posterior and the second posterior together. Then repeat this process several times. Do the posterior distributions appear to converge?