ESE 524: Exponential Family and Cramer-Rao Bound Example

January 29, 2019

Holding out for a Hero

- Some probability distributions do not look like they are "exponential", but looks can be deceiving.
- Let $x[n] \in \{1, 2, 3, ...\}$, for $n = 0, ..., N-1$ be integer valued samples from the discrete distribution:

$$p(x[n];\theta) = \frac{1}{1+\theta}\left(\frac{\theta}{1+\theta}\right)^{x[n]-1}$$

- Goal: Find a a sufficient statistic for θ, and unbiased estimator for θ, and the cramer rao bound on estimators of θ.
- Source: Problem 4.18 in *Statistical Methods for Signal Processing* by Alfred Hero.

Step 1: Find a Sufficient Statistic

- The joint distribution is given as the product of the independent distributions:

$$p(\mathbf{x};\theta) = \Pi_{n=0}^{N-1} \frac{1}{1+\theta}\left(\frac{\theta}{1+\theta}\right)^{x[n]-1} = \left(\frac{1}{1+\theta}\right)^{N}\left(\frac{\theta}{1+\theta}\right)^{\sum_{n=0}^{N-1}(x[n])-N}$$

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  - $h(x) = 1,\ \forall x$
  - $\eta(\theta) = \ln\left(\frac{\theta}{1+\theta}\right)$
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  - $B(\theta) = \ln(\theta + 1)$
- By using the fact that $a = \exp(\ln(a))$, we have shown that this distribution is in the exponential family - this is a good trick for distributions involving exponents.
- Since this is an exponential family distribution, $T(\mathbf{x}) = \sum_{n=0}^{N-1}(x[n] - 1)$ is the natural sufficient statistic to use here.

Step 2: Find an Unbiased estimator

- Oftentimes, we can use the sufficient statistic from Slide 2 to easily build an estimator.
- Calculate the expected value of T (See board for details):

$$\mathbb{E}(T(x)) = \mathbb{E}(\sum_{n=0}^{N-1}(x[n]-1)) = \sum_{n=0}^{N-1}\mathbb{E}(x[n]-1) =$$

$$\sum_{n=0}^{N-1}\sum_{j=1}^{\infty}(j-1)p(x[n]=j;\theta) = N\sum_{j=1}^{\infty}(j-1)p(x[0]=j;\theta)$$

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- The sufficient statistic is a "natural" estimator for $N\theta$, so $\hat{\theta}(\mathbf{x}) = \frac{T(\mathbf{x})}{N}$ will be an unbiased estimator of θ!

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$$\ln(p(x[n];\theta)) = \ln(\frac{1}{1+\theta}) + (x[n]-1)\ln(\frac{\theta}{1+\theta}) =$$

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- Then take the expectation:

$$I_1(\theta) = \mathbb{E}[(\frac{\partial}{\partial\theta}(\ln(p(x[n];\theta))))^2] = \sum_{j=1}^{\infty}\left(\frac{j-1-\theta}{(1+\theta)\theta}\right)^2 \frac{1}{1+\theta}\left(\frac{\theta}{1+\theta}\right)^{j-1} =$$

$$\frac{1}{\theta(\theta+1)}$$

Step 3: Find the Cramer-Rao Bound - Cont.

- Now that we know $I_1(\theta)$, $I(\theta) = \frac{N}{\theta(1+\theta)}$.
- Then the Cramer-Rao Bound is:

$$C(\theta) = \frac{\theta(1+\theta)}{N}$$

- Note that as the number of samples N increases, the Cramer-Rao Bound gets lower and theoretically estimators will perform better.
- Does $\hat{\theta} = \frac{T(\mathbf{x})}{N}$ reach the CRB?

Performance of $\hat{\theta}$

- When $N = 1$, the variance of $\hat{\theta}$ is:

$$\mathbb{E}[(x[n]-1)^2]\sum_{j=1}^{\infty}(j-1-\theta)^2 * \frac{1}{1+\theta}\left(\frac{\theta}{1+\theta}\right)^{j-1} = \theta(1+\theta)$$

- So our estimator hit the CRB!
- IF $p(x;\theta)$ belongs to the exponential family, and IF $\mathbb{E}[T(x)] = \theta$, THEN AND ONLY THEN $\mathbb{E}[T(x)]$ will be the MVU Estimator for θ.
- This leads to the Mean Value Parameterization for exponential family distributions (source - 04_sufficiency.pdf on Canvas)

Mean Value Parameterization

- To find an MVU estimator for exponential family distributions, we usually have to change variables.
- Let $\mathbf{x} = [x[0], ..., x[N-1]]$ be samples from an exponential family distribution with functions $h(x), \eta(\theta), T(\mathbf{x}), B(\theta)$.
- Then the Canonical Form is the result of a change of variables from θ to η.
- To do this, solve $\eta(\theta)$ for θ, with our probability distribution:

$$\eta(\theta) = \ln(\frac{1}{1+\theta}),\ \theta(\eta) = \frac{\exp(\eta)}{1-\exp(\eta)}$$

- Then the probability distribution becomes:

$$p(x[n];\eta) = h(x)\exp(\eta T(\mathbf{x}) - B(\theta(\eta)) = \exp(\eta(\sum_{n=0}^{N-1}(x[n]-1)) - \ln(\frac{\exp(\eta)}{1-\exp(\eta)}+1)) =$$

$$\exp(\eta(\sum_{n=0}^{N-1}(x[n]-1)) - (\ln(\exp(\eta)-1)))$$

Mean Value Parameterization Cont.

- To find the Mean Value Parameterization, make a second change of variables.
- We know that $\mathbb{E}[T(\mathbf{x})] = \frac{\partial}{\partial \eta} B(\theta(\eta))$ (here $A(\eta)$ from the notes is $B(\theta(\eta))$.
- Let $\psi = \frac{\partial}{\partial \eta} B(\theta(\eta))$ and rewrite the probability distribution in terms of ψ.
- In our examples $\psi = \frac{\partial}{\partial \eta} - \ln(\exp(\eta) - 1) = \frac{-\exp(\eta)}{\exp(\eta)-1} = \frac{\exp(\eta)}{1-\exp(\eta)}$, so $\eta(\psi) = \ln\left(\frac{\psi}{1+\psi}\right)$.
- Then the probability distribution becomes:

$$p(\mathbf{x}; psi) = h(\mathbf{x}) \exp(\eta(\psi) T(\mathbf{x}) - B(\theta(\eta(\psi)))) = \exp(\ln\left(\frac{\psi}{1+\psi}\right) T(\mathbf{x}) - \ln(\psi + 1)))$$

- Notice that this is the same as the original probability distribution

Comments of Mean Value Parameterization

- This problem was carefully chosen so that the original distribution was already in the MVP, the changes of variables will rarely reproduce the original distribution.
- $\mathbb{E}[T(\mathbf{x})]$ will be the MVU Estimator for ψ, NOT θ. This is VERY IMPORTANT. The transformation from θ to ψ is not necessarily 1-1 so you can't make conclusions about θ based off of this.
- This is our first "complete" problem in the class, where we used sufficient statistics to find an estimator, and then figured out the performance of that estimator.
- Using -$\mathbb{E}[\frac{\partial}{\partial\theta}\ln(p(\mathbf{x};\theta))]$ to find the Fisher Information is usually easier than using the original definition.
- Fisher Information is ubiquitous in statistics, because it represents a metric of how much information the data carries. E.g. if you have two sets of samples, and one has higher fisher information, that set of samples is better for estimating θ.