# ESE 524 - Homework 6

**Database Problems and Solutions**
**Assigned date: 03/26/19**
**Due Date: xx/yy/zz**
Total Points: 100

1) (Hypothesis testing)
Consider the simple binary hypothesis-testing problem

$$f(x|H_0) = \begin{cases} c_0, & |x| \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

versus

$$f(x|H_1) = \begin{cases} c_1(3-|x|), & |x| \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

where $c_0$ and $c_1$ are normalizing constants.

a) Determine $c_0$ and $c_1$.
b) Assuming that the priors are equal, i.e., $\pi_0 = \pi_1 = 0.5$, and $0-1$ loss is used, find the Bayes' decision rule and specify the range of $x$ for accepting $H_0$.
c) Based on the above Bayes' decision rule, find minimum Bayes risk.

**Solution:**

a) As we know,

$$1 = \int f(x|H_0)dx = \int_{-1}^{1} c_0 dx = 2c_0,$$

we have $c_0 = 1/2$. From

$$1 = \int f(x|H_1)dx = \int_{-3}^{3} c_1(3-|x|)dx = 9c_1,$$

we have $c_1 = 1/9$.

b) We know that the decision rule is

$$D(x) = \frac{f(x|H_1)}{f(x|H_0)} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} 1$$

When $1 < |x| \leq 3$, $D(x) = +\infty$, thus $H_1$ is accepted. When $|x| \leq 1$, we can see that

$$D(x) = \frac{1/9(3-|x|)}{1/2} = \frac{6-2|x|}{9} \leq \frac{6}{9} < 1,$$

and we accept $H_0$.

c) Denote $\mathcal{X}_0 = [-1,1]$ and $\mathcal{X}_1 = [-3,-1) \cup (1,3]$. The minimum Bayes risk is

$$\begin{aligned} R &= \int_{\mathcal{X}_1} L(1|0)p(\boldsymbol{x}|H_0)\pi_0 dx + \int_{\mathcal{X}_0} L(0|1)p(\boldsymbol{x}|H_1)\pi_1 dx \\ &= 0.5 * \int_{-3}^{-1} \frac{1}{2}dx + 0.5 * \int_{1}^{3} \frac{1}{2}dx + 0.5 * \int_{-1}^{1} \frac{1}{9}(3-|x|)dx \\ &= \frac{23}{18} \end{aligned}$$

2) (Uniformly most powerful test)
   Consider the hypothesis test

$$H_0 : Y_1, \ldots, Y_n \sim \text{Binomial}(m, \theta_0),\ VS$$
$$H_1 : Y_1, \ldots, Y_n \sim \text{Binomial}(m, \theta), \theta > \theta_0$$

   where $\theta_0 \in (0, 1)$ and $m$ are assumed to be known constants, and $Y_1, \ldots, Y_n$ are iid.

   (a) For a fixed $\theta > \theta_0$, what is the general form of the Neyman-Pearson optimal test?
   **Solution:**
   The likelihood ratio of observations $y \equiv (y_1, \ldots, y_n)$ is

$$\frac{p_1(y)}{p_0(y)} = \prod_{k=1}^{n} \frac{\theta^{y_k}(1-\theta)^{m-y_k}}{\theta_0^{y_k}(1-\theta_0)^{m-y_k}} = (\frac{\theta}{\theta_0})^{s(y)}(\frac{1-\theta}{1-\theta_0})^{mn-s(y)}$$
$$= (\frac{\theta(1-\theta_0)}{\theta_0(1-\theta)})^{s(y)}(\frac{1-\theta}{1-\theta_0})^{mn}$$

   where $s(y) = \sum_{k=1}^{n} y_k$ and the $N-P$ optimal test is to compare this to a suitable threshold, which is equivalent to comparing $s(y) = \sum_{k=1}^{n} y_k$ to a threshold.

   (b) Does there exist a Uniformly Most Powerful (UMP) test for $\theta_0$ vs $\theta > \theta_0$?
   **Solution:**
   Yes. The N-P optimal test for level $\alpha$ is the same for testing $\theta_0$ vs. $\theta$ for each $\theta > \theta_0$, so it follows that this common NP-optimal test is indeed a UMP test for the entire family $(\theta_0, \infty)$.

3) A communication source generates binary digits 0 and 1 and transmits the respective signals $(s_t(i))_{t=0}^{N-1}$ for $i = 0, 1$. A sequence of independently identically distributed noise $(n_t(i))_{t=0}^{N-1}$ is added to produce the measurements $(x_t(i))_{t=0}^{N-1}$ with $x_t \sim \mathcal{N}(s_t(i), \sigma^2)$. Test $\mathcal{H}_0 : i = 0$ vs $\mathcal{H}_1 : i = 1$. Assume $\sum_{t=0}^{N-1} s_t^2(i) = E_s$ and $\sum_{t=0}^{N-1} s_t(0)s_t(1) = \rho E_s$, $-1 < \rho < 1$.

   (a) What is the likelihood detector for the above test?

   **Solution**: Let $\boldsymbol{m}_0 = [s_0(0), \ldots, s_{N-1}(0)]^T$ and $\boldsymbol{m}_1 = [s_0(1), \ldots, s_{N-1}(1)]^T$. Implies

$$\mathcal{H}_0 \sim \mathcal{N}(\boldsymbol{m}_0, \sigma^2 \boldsymbol{I})$$
$$\mathcal{H}_1 \sim \mathcal{N}(\boldsymbol{m}_1, \sigma^2 \boldsymbol{I})$$

   The likelihood ratio is given as

$$l(\boldsymbol{x}) = \frac{f(\boldsymbol{x}; \mathcal{H}_1)}{f(\boldsymbol{x}; \mathcal{H}_0)}$$

   On further simplifying $l(\boldsymbol{x})$ and applying the log we get

$$\ln l(\boldsymbol{x}) = \Lambda(\boldsymbol{x}) = \frac{1}{\sigma^2}(\boldsymbol{m}_1 - \boldsymbol{m}_0)^T(\boldsymbol{x} - \boldsymbol{x}_0)$$

   where $\boldsymbol{x}_0 = 1/2(\boldsymbol{m}_1 + \boldsymbol{m}_0)$.
   Under $\mathcal{H}_0$:

$$\mathbb{E}(\Lambda(\boldsymbol{x})) = -\frac{1}{2\sigma^2}(\boldsymbol{m}_1 - \boldsymbol{m}_0)^T(\boldsymbol{m}_1 - \boldsymbol{m}_0)$$
$$\text{cov}(\Lambda(\boldsymbol{x})) = \frac{1}{\sigma^2}(\boldsymbol{m}_1 - \boldsymbol{m}_0)^T(\boldsymbol{m}_1 - \boldsymbol{m}_0)$$

   Under $\mathcal{H}_1$:

$$\mathbb{E}(\Lambda(\boldsymbol{x})) = \frac{1}{2\sigma^2}(\boldsymbol{m}_1 - \boldsymbol{m}_0)^T(\boldsymbol{m}_1 - \boldsymbol{m}_0)$$
$$\text{cov}(\Lambda(\boldsymbol{x})) = \frac{1}{\sigma^2}(\boldsymbol{m}_1 - \boldsymbol{m}_0)^T(\boldsymbol{m}_1 - \boldsymbol{m}_0)$$

Consider the following expansion

$$\begin{aligned}(\boldsymbol{m}_1-\boldsymbol{m}_0)^T(\boldsymbol{m}_1-\boldsymbol{m}_0) &= \boldsymbol{m}_1^T\boldsymbol{m}_1 - 2\boldsymbol{m}_1^T\boldsymbol{m}_0 + \boldsymbol{m}_0^T\boldsymbol{m}_0 \\ &= \sum_{i=0}^{N-1} s_i^2(1) - 2\sum_{i=0}^{N-1} s_i(0)s_i(1) + \sum_{i=0}^{N-1} s_i^2(0) \\ &= E_s - 2\rho E_s + E_s = 2E_s(1-\rho)\end{aligned}$$

Using the above simplification, the expectation and covariance under each hypothesis is given as:
Under $\mathcal{H}_0$:

$$\begin{aligned}\mathbb{E}(\Lambda(\boldsymbol{x})) &= -\frac{E_s}{\sigma^2}(1-\rho) \\ \mathrm{cov}(\Lambda(\boldsymbol{x})) &= \frac{2E_s}{\sigma^2}(1-\rho)\end{aligned}$$

Under $\mathcal{H}_1$:

$$\begin{aligned}\mathbb{E}(\Lambda(\boldsymbol{x})) &= \frac{E_s}{\sigma^2}(1-\rho) \\ \mathrm{cov}(\Lambda(\boldsymbol{x})) &= \frac{2E_s}{\sigma^2}(1-\rho)\end{aligned}$$

Therefore

$$\begin{aligned}\mathcal{H}_0 : \Lambda(\boldsymbol{x}) &\sim \mathcal{N}\left(-\frac{E_s}{\sigma^2}(1-\rho), \frac{2E_s}{\sigma^2}(1-\rho)\right) \\ \mathcal{H}_1 : \Lambda(\boldsymbol{x}) &\sim \mathcal{N}\left(\frac{E_s}{\sigma^2}(1-\rho), \frac{2E_s}{\sigma^2}(1-\rho)\right)\end{aligned}$$

(b) What are the expressions of $P_{\text{FA}}$ (false alarm) and $P_{\text{D}}$ (detection) for $E_s/\sigma^2 = 10$ and $\rho = 1/2$?
**Solution**: Given $E_s/\sigma^2 = 10$ and $\rho = 1/2$,

$$\begin{aligned}\mathcal{H}_0 : \Lambda(\boldsymbol{x}) &\sim \mathcal{N}(-5, 10) \\ \mathcal{H}_1 : \Lambda(\boldsymbol{x}) &\sim \mathcal{N}(+5, 10)\end{aligned}$$

Therefore, the expressions of probability of false alarm and probability of detection are given as

$$\begin{aligned}P_{\text{FA}} &= \mathcal{Q}\left(\frac{\eta+5}{\sqrt{10}}\right) \\ P_{\text{D}} &= \mathcal{Q}\left(\frac{\eta-5}{\sqrt{10}}\right)\end{aligned}$$

(c) Where is the operating point on the ROC for the Bayes detector when $p_0 = 7/16$, $p_1 = 9/16$, $L_{01} = L_{10} = 1$. (Assume $L_{00} = L_{11} = 0$).
**Solution**: Given $L_{01} = L_{10} = 1$, $p_0 = 7/16$, and $p_1 = 9/16$ the decision function can be written as

$$\Phi(\boldsymbol{x}) = \begin{cases} 1 & \Lambda(\boldsymbol{x}) \geq \eta \\ 0 & \Lambda(\boldsymbol{x}) < \eta \end{cases} \quad \text{where} \quad \eta = \ln\frac{L_{01}p_0}{L_{10}p_1}$$

In this case, $\eta = \ln\frac{7}{9}$, which gives us

$$\begin{aligned}\alpha &= \mathcal{Q}\left(\frac{\ln(7/9)+5}{\sqrt{10}}\right) = 0.066 \\ \beta &= \mathcal{Q}\left(\frac{\ln(7/9)-5}{\sqrt{10}}\right) = 0.9516\end{aligned}$$

4) **Neyman-Pearson detector** :

Let $\mathbf{x} = [x[1], ..., x[N]]$. A Neyman-Pearson detector is to be used to distinguish between two hypotheses based on the log likelihood ratio. The two hypotheses $\boldsymbol{H}_0$ and $\boldsymbol{H}_1$ are:

$$H_0 : p(x[n]|\lambda = \lambda_0) \sim \exp(-\lambda_0)\frac{\lambda_0^x}{x!}$$
$$H_1 : p(x[n]|\lambda = \lambda_1) \sim \exp(-\lambda_1)\frac{\lambda_1^x}{x!}$$

Find the log-likelihood ratio. Compute the probability of false alarm for size $\alpha$,i.e., $P_{FA} = \alpha$. If $\lambda_0 = 1$,$\lambda_1 = e$, and $\alpha = 0.1$, what is the threshold value for the likelihood ratio test? (Using MATLAB (or other software) will help to compute this number)

**Solution**:

The likelihood ratio is

$$l(x) = \frac{P(\mathbf{x}|H_0)}{P(\mathbf{x}|H_1)} = (\frac{\lambda_1}{\lambda_0})^{\sum_{n=1}^{N} x[n]} \exp[-N(\lambda_1 - \lambda_0)] \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \upsilon$$

The log-likelihood ratio

$$\Lambda(x) = ln[\frac{P(x|H_0)}{P(x|H_1)} = \sum_{n=1}^{N} x[n] ln(\frac{\lambda_1}{\lambda_0}) - N(\lambda_1 - \lambda_0) \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} ln(\upsilon)$$

Suppose $\lambda_1 > \lambda_0$. Let

$$T(\mathbf{x}) = \sum_{n=1}^{N} x[n] \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \frac{1}{ln(\frac{\lambda_1}{\lambda_0})}(ln(\upsilon) + N(\lambda_1 - \lambda_0)) = \upsilon'$$

then $T(\mathbf{x}) \sim \text{Poisson}(N\lambda)$

$$P_{FA} = P(T(\mathbf{X}) > \upsilon'|\lambda_0) = \sum_{T(\mathbf{x})=\upsilon'}^{\infty} \frac{N^{T(\mathbf{x})}\lambda_0^{T(\mathbf{x})}}{T(\mathbf{x})!} exp(-N\lambda_0) = \alpha$$
$$= Q_{Poisson,N\lambda_0}(\alpha)$$

And

$$\alpha = 1 - \sum_{T(\mathbf{x})=0}^{\upsilon'} \frac{(N\lambda_0)^{T(\mathbf{x})}}{T(\mathbf{x})!} \exp(-N \cdot 1) = 0.1$$
$$\sum_{T(\mathbf{x})=0}^{\upsilon'} \frac{1}{T(\mathbf{x})!} exp(-N) = 0.9$$

Since $T(x)$ is an integer, $\upsilon'$ has a discrete value, keep increasing the sum until it passes the desired $\alpha$ value (see example code).

$$\upsilon' = 2$$

Matlab Codes:

```
a=0.9;
sum=0;
```

```
for k=0:1:200
temp=1/factorial(k)*exp(-1);
sum=temp+sum;
if (abs(sum-a)¡0.1)
break;
end
end
```

5) **Matlab Problem, Experimenting with ROC Curves**:
   a) Consider the Example "DC Level in AWGN" example in"24_bayesdetex.pdf" on Blackboard. Let $\theta_0 = 5$, $\theta_1 = 10$, and $\sigma_2 = 1$. Recreate Figure 1 from the notes.

   b) From a non-bayesian point of view, $\eta'$ is not a fixed threshold, but is a parameter to choose. Varying $\eta'$ (in equation (3)) from $-\infty$ to $\infty$, compute the probability of detection, $P_D$, the probability of false alarm, $P_F$, and plot the ROC curve. Change $\theta_1$ to $5.5$ and Plot the ROC Curve again on the same plot. Which $\theta_1$ is easier to distinguish from $\theta_0$?

      In Bayesian Detection, $\eta'$ is determined by the ratio $\frac{\pi_0 L(1|0)}{\pi_1 L(0|1)}$. Choose several combinations of $\pi_0, \pi_1, L(0|1), L(1|0)$ and highlight their corresponding $P_D$'s and $P_F$'s on the ROC Curves you created earlier.

   c) Repeat parts **(a)** and **(b)** for the "Deciding between Two Rates for Poisson Measurements" example. Set $\lambda_0 = 5$ with $\lambda_1 = 10$ for the first ROC and $\lambda_1 = 6$ for the second ROC curve.
      **Solutions**: See HW6.m
   d) **Extra Credit**:
      (**20 pts**) Come up with an example and solution illustrating one or more concepts from class so far. This example should be something you believe would be good to present in class to help other students understand a concept from the lectures. MATLAB (or other software) simulations are encouraged. Problems can be inspired by or explore applications from literature, but should not just copy the results of a paper. We will also accept larger "projects" which may take more time to complete but will be worth more points.