ESE 524 - Homework 7

Database Problems and Solutions
Assigned date: 03/26/19
Due Date: xx/yy/zz
Total Points: 100

1) **Poisson distribution - Composite Hypothesis Testing**
Suppose $X_1, X_2, \ldots, X_N$ are i.i.d. random samples, and they follow Poisson(λ). Suppose we have the following hypothesis test:

$$\mathcal{H}_0 : \lambda = \lambda_0$$
$$\mathcal{H}_1 : \lambda \neq \lambda_0$$

where λ_0 is a known value.

a) (15 pts) Give the generalized likelihood ratio test (GLRT), Wald test, and Rao test.
b) (5 pts) Using the central limit theorem (CLT) and the Rao test you have in a), show that under $\mathcal{H}_0$, the test statistic for Rao test follows χ_1^2 asymptotically.

(Hint): for Rao test, the restricted estimate of λ (under $\mathcal{H}_0$) is λ_0 in this case.

Solution:

a) The maximum likelihood estimator for λ can be obtained by

$$s(\lambda) = \frac{\partial \log P(X_1, X_2, \ldots, X_N; \lambda)}{\partial \lambda} = \frac{\partial \log \frac{e^{-N\lambda}\lambda^{\sum_{i=1}^N X_i}}{\Pi_{i=1}^N X_i!}}{\partial \lambda} = -N + \sum_{i=1}^{N} \frac{X_i}{\lambda} = 0.$$

Thus, we have $\hat{\lambda} = \frac{1}{N}\sum_{i=1}^N X_i$.
GLRT:

$$\log \frac{P(X_1, X_2, \ldots, X_N; \hat{\lambda})}{P(X_1, X_2, \ldots, X_N; \lambda_0)} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \gamma$$
$$-N(\hat{\lambda} - \lambda_0) + \log\left(\frac{\hat{\lambda}}{\lambda_0}\right) \times \sum_{i=1}^{N} X_i \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \gamma$$

Fisher information $\mathcal{I}(\lambda) = -\mathbb{E}\left[\frac{\partial^2 \log P(X_1,X_2,\ldots,X_N;\lambda)}{\partial \lambda^2}\right] = \mathbb{E}\left[\frac{\sum_{i=1}^N X_i}{\lambda^2}\right] = \frac{N}{\lambda}$.
Wald test:

$$(\hat{\lambda} - \lambda_0)\frac{N}{\hat{\lambda}}(\hat{\lambda} - \lambda_0) \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \gamma$$
$$\frac{N(\hat{\lambda} - \lambda_0)^2}{\hat{\lambda}} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \gamma$$

Rao test:

$$s(\lambda_0)\mathcal{I}(\lambda_0)^{-1}s(\lambda_0) \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \gamma$$
$$\frac{\lambda_0}{N}(-N + \sum_{i=1}^{N} \frac{X_i}{\lambda_0})^2 \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \gamma$$
$$N\frac{(\frac{1}{N}\sum_{i=1}^N X_i - \lambda_0)^2}{\lambda_0} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \gamma$$

b) As we know, under $\mathcal{H}_0$, we have

$$\mathbb{E}(X_i) = \lambda_0, \quad \text{var}(X_i) = \lambda_0.$$

According to central limit theorem,

$$\frac{\sqrt{N}\left(\frac{1}{N}\sum_{i=1}^{N} X_i - \mathbb{E}(X_i)\right)}{\sqrt{\text{var}(X_i)}} = \frac{\sqrt{N}\left(\frac{1}{N}\sum_{i=1}^{N} X_i - \lambda_0\right)}{\sqrt{\lambda_0}} \to \mathcal{N}(0,1), \text{ as } N \to +\infty.$$

Thus, we have

$$\frac{N\left(\frac{1}{N}\sum_{i=1}^{N} X_i - \lambda_0\right)^2}{\lambda_0} \to \chi_1^2, \text{ as } N \to +\infty.$$

2) The number of successes, x, in n trials is to be used to test the null hypothesis that the parameter θ of a binomial population equals $\frac{1}{2}$ against the alternative that it doesn't equal $\frac{1}{2}$.

(a) Find an expression for the likelihood ratio statistic.
Solution:
The test is

$$H_0 : \theta = \theta_0 \quad v.s. \quad H_\alpha : \theta \neq \theta_0$$

where $X \sim Binomial(\theta, n)$. And the likelihood ratio is

$$\Lambda(x) = \frac{\max_{\theta=\theta_0} f_\theta(x)}{\max_{\theta\in(0,1)} f_\theta(x)} = \frac{C_x^n \theta_0^x (1-\theta_0)^{n-x}}{C_x^n (\hat{\theta}_{MLE})^x (1-\hat{\theta}_{MLE})^{n-x}} = \frac{\theta_0^x/(1-\theta_0)^x (1-\theta_0)^n}{(x/n)^x (1-x/n)^{n-x}}$$

(b) Use the result of part (a) to show that the critical region of the likelihood ratio test can be written as

$$x \cdot \ln(x) + (n-x) \cdot \ln(n-x) \geq K$$

where x is the observed number of successes, and K is a constant that depends on the size of the critical region. **Solution:**

$$\begin{aligned} R(\theta_0) &= \{x_0 : \Lambda(x) < k\} = \{x : \ln(\Lambda(x)) < \ln(k)\} \\ &= \{x : x\ln(\frac{\theta_0}{1-\theta_0}) - x\ln(\frac{x}{n}) - (n-x)\ln(1-\frac{x}{n}) < k_1\} \end{aligned}$$

For $\theta_0 = \frac{1}{2}$, then we have:

$$R(\frac{1}{2}) = \{x : x\ln(x) + (n-x)\ln(n-x) > K\}$$

Here, k, k_1, K are some constants.

(c) Study the minimum and the symmetry of the function $f(x) = x \cdot \ln(x) + (n-x) \cdot \ln(n-x)$, and show that the critical region of this likelihood ratio test can also be written as:

$$|x - \frac{n}{2}| \geq K'$$

Solution:
The function $g(x) = x\ln(x) + (n-x)\ln(n-x)$ is symmetric about $\frac{n}{2}$ and also has a minimal at this point. To check this, take derivative:

$$g'(x) = 1 + \ln(x) - \ln(n-x) - 1 = \ln(x) - \ln(n-x)$$

As a result, $g(x)$ decreases over $x \in (0, n/2)$ and increase otherwise. This yields

$$R(1/2) = \{x : |x - \frac{n}{2}| \geq k_0\}$$

for some constant k_0. Given level of significance α, we get

$$P_{\theta=1/2}(|X - n/2| \geq k_0) = \alpha$$

Then use a Binomial Table, we can find the value of k_0.

3) **Stock Market Analysis**
It is desired to detect trend in stock market data. To do so we assume that the data are modeled as

$$x[n] = A + Bn + w[n], \quad n = 0, 1, \ldots, N-1$$

where $w[n]$ is white Gaussian noise with variance σ^2. The average stock price A is unknown but is of no interest to us. More importantly, we wish to test whether $B = 0$ or $B \neq 0$, i.e., that a trend is present. Find the GLRT statistic for this problem.

Hint: Refer to p. 21 of l7.pdf. To get full points, derive the expression of $T(\boldsymbol{x})$ as

$$T(\boldsymbol{x}) = \frac{(N\sum nx[n] - \sum n \sum x[n])^2}{N\sigma^2[N\sum n^2 - (\sum n)^2]}$$

Solution:
For the given problem

$$\boldsymbol{H} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & N-1 \end{bmatrix}, \quad \boldsymbol{\theta} = \begin{bmatrix} A \\ B \end{bmatrix}$$

As parameter A is of no interest, we rewrite the hypothesis as

$$H_0 : B = 0$$
$$H_1 : B \neq 0$$

If $\boldsymbol{A} = [0, 1]$ and $\boldsymbol{b} = [0, 0]^T$, the we can rewrite the above hypothesis as

$$H_0 : \boldsymbol{A\theta} = \boldsymbol{b}$$
$$H_1 : \boldsymbol{A\theta} \neq \boldsymbol{b}$$

Using the result in l7.pdf, we get

$$T(\boldsymbol{x}) = \frac{(\boldsymbol{A}\hat{\boldsymbol{\theta}}_1 - \boldsymbol{b})^T[\boldsymbol{A}(\boldsymbol{H}^T\boldsymbol{H})^{-1}\boldsymbol{A}]^{-1}(\boldsymbol{A}\hat{\boldsymbol{\theta}}_1 - \boldsymbol{b})}{\sigma^2} > \tau$$

where $\hat{\boldsymbol{\theta}}_1 = (\boldsymbol{H}^T\boldsymbol{H})^{-1}\boldsymbol{H}^T\boldsymbol{x}$ is the ML estimator of $\boldsymbol{\theta}$ under H_1. Therefore,

$$T(\boldsymbol{x}) = \frac{[\hat{\boldsymbol{\theta}}_1]_2[\boldsymbol{A}(\boldsymbol{H}^T\boldsymbol{H})^{-1}\boldsymbol{A}]^{-1}[\hat{\boldsymbol{\theta}}_1]_2}{\sigma^2}$$

But $[\boldsymbol{A}(\boldsymbol{H}^T\boldsymbol{H})^{-1}\boldsymbol{A}] = [(\boldsymbol{H}^T\boldsymbol{H})^{-1}]_{2,2}$. Thus, we can further simplify $T(\boldsymbol{x})$ as

$$T(\boldsymbol{x}) = \frac{[\hat{\boldsymbol{\theta}}_1]_2^2}{\sigma^2[(\boldsymbol{H}^T\boldsymbol{H})^{-1}]_{2,2}}$$

We can expand each term and get closed form expression:

$$\boldsymbol{H}^T\boldsymbol{H} = \begin{bmatrix} N & \sum n \\ \sum n & \sum n^2 \end{bmatrix}, \quad (\boldsymbol{H}^T\boldsymbol{H})^{-1} = \frac{1}{N\sum n^2 - (\sum n)^2}\begin{bmatrix} \sum n^2 & -\sum n \\ -\sum n & N \end{bmatrix}, \quad \boldsymbol{Hx} = \begin{bmatrix} \sum x[n] \\ \sum nx[n] \end{bmatrix}$$

Thus

$$T(\boldsymbol{x}) = \frac{(N\sum nx[n] - \sum n \sum x[n])^2}{N\sigma^2[N\sum n^2 - (\sum n)^2]}$$

4) **Uniformly Most powerful test,**
Let x have a density given by:

$$p(x;\theta) = \frac{1+\theta x}{2}$$

for $-1 \leq x \leq 1$.

a) If the hypotheses are:

$$H_0 : \theta = \theta_0$$
$$H_1 : \theta = \theta_1$$

where $\theta_0 \in [-1, 0]$ and $\theta_1 \in [0, 1]$ are known. Find the likelihood ratio test and the threshold corresponding to the level α.

b) Is there a uniformly most powerful test for the hypotheses:

$$H_0 : \theta = 0$$
$$H_1 : \theta > 0$$

If yes, find the UMP test.

c) Find the generalized likelihood ratio for the hypotheses:

$$H_0 : \theta \leq 0$$
$$H_1 : \theta > 0$$

and find the threshold for a level α.

Solution

a) The likelihood ratio with threshold λ is

$$\frac{1+\theta_1 x}{1+\theta_0 x} \gtrless \lambda$$

This implies that $1+\theta_1 x \gtrless \lambda(1+\theta_0 x)$ and solving for x we get $x \gtrless \frac{\lambda-1}{\theta_1-\theta_0\lambda} = \lambda'$ - the inequality doesn't flip because $\theta_1 - \lambda\theta_0 > 0$.
For a level α,

$$\alpha = P_{FA} = \int_{\lambda'}^{1} \frac{1+\theta_0 x}{2} dx = \frac{1-\lambda' + \theta_0(1-\lambda'^2)/2}{2}$$

If $\theta_0 = 0$, then $\alpha = \frac{1-\lambda'}{2}$ and it is easy to solve for λ'.
If $\theta_0 < 0$ then we have that $\frac{1-\lambda'+\theta_0(1-\lambda'^2)/2}{2} - \alpha = 0$, which is a quadratic equation in λ' -

$$\theta_0\lambda'^2 + 2\lambda' + 4\alpha - 2 + \theta_0 = 0$$

The solutions are $\lambda'_{\pm} = \dfrac{-1 \pm \sqrt{4-4\theta_0(4\alpha-2+\theta_0)}}{2\theta_0}$ but only the "+" solution is in the valid interval $[-1, 1]$.

b) For a fixed θ_1, we know that the threshold value is $1 - 2\alpha$. This doesn't depend on the value of θ_1 at all so the likelihood ratio test is always the most powerful test for all $\theta_1 > 0$.

c) The generalized likelihood ratio test is:

$$\frac{\max\limits_{\theta_1\in[0,1]} 1+\theta_1 x}{\max\limits_{\theta_0\in[-1,0]} 1+\theta_0 x} \gtrless \lambda$$

If the measurement $x > 0$, then the maximum value is $\theta_1 = 1$, $\theta_0 = 0$ and vice versa if $x < 0$. Therefore,

$$\frac{\max\limits_{\theta_1\in[0,1]} 1+\theta_1 x}{\max\limits_{\theta_0\in[-1,0]} 1+\theta_0 x} = \begin{cases} 1+x & x > 0 \\ \frac{1}{1-x} & x < 0 \end{cases}$$

Note that for $x < 0$, $\frac{1}{1-x} = \frac{1}{1+|x|}$, so the GLRT is equivalent to $T(x) = (1+|x|)^{\mathrm{sign}(x)} \gtrless \lambda$

If $x > 0$, then the test is $x \gtrless \lambda - 1$, and $T(x) \in [1, 2]$ so

$$P_F A\theta_0 = \int_{\lambda-1}^{1} p(x;\theta_0)dx = \int_{\lambda-1}^{1} 1/2(1+\theta_0 x)dx = 1 - \lambda/2 + \theta_0(1-(1-\lambda)^2)/4$$

.

If $x < 0$, then the test is $\frac{1}{1-x} \gtrless \lambda$ or equivalently $x \gtrless 1 - \frac{1}{\lambda}$, and $T(x) \in [1/2, 1]$. So

$$P_{FA}(\theta_0) = \int_{1-\frac{1}{\lambda}}^{1} 1/2(1+\theta_0 x)dx = \frac{1}{2\lambda} + \theta_0(1-(1-\lambda)^2)/4$$

So the maximum probability of false alarm is

$$P_{FA}(\theta_0) = \alpha = \max_{\theta_0 \in [-1,0]} \begin{cases} 1 - \lambda/2 + \theta_0(1-(1-\lambda)^2)/4 & \lambda \in [1,2] \\ \frac{1}{2\lambda} + \theta_0(1-(1-\lambda)^2)/4 & \lambda \in [1/2,1] \end{cases}$$

Since the term multiplying θ_0 is always non-negative, $\theta_0 = 0$ always maximizes this expression, so the threshold is the same as in part a) and b).

5)