Reduction of Wave Equations in Random Media and Periodic Media

Wei Li

DePaul University

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Overview

- Motivations
  - Effectively study wave phenomena.
  - Random media - medical imaging in human tissues, imaging through the atmosphere
  - Periodic media - materials design
- Challenges in computational cost
  - scale separation
  - resonances

Overview

- Ways to overcome using analysis
  - Asymptotic analysis
  - Operator theory and spectral theory
  - Partial differential equations
  - Probability
- A few aspects that I've worked on
  - Structured media [Lai-Li-Li-L. 2018]
  - Random geometric optics [Borcea-L.-Mamonov-Schotland 2017]
  - Resonant homogenization correctors [L.-Lipton-Matthias 2020]
  - Corner resonances [L.-Shipman 2019, L.-Perfekt-Shipman 2020]

- Sound - pressure wave
- Radio, light - electromagnetic waves
- Earth quakes - elastic waves
- LISA - gravitational waves

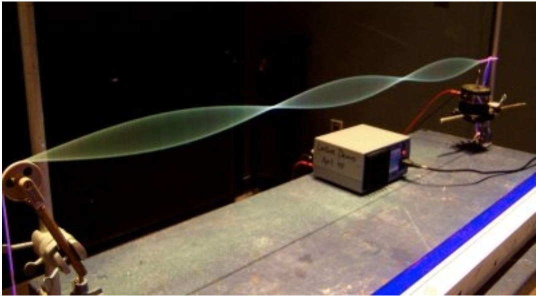


Figure credit:
https://sciencedemonstrations.fas.harvard.edu/presentations/vibrating-string

Wave scattering

The shape of wave becomes irregular when there are obstacles. This is called wave scattering.

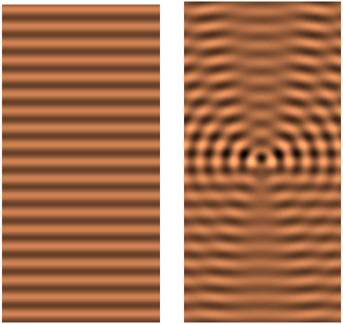


Figure credit: Wiki

Scattering can give intriguing phenomena.
Lycurgus Cup, Roman, 4th Centry.

Figure credit: Wiki

Waves in random media

The relative strength of random scattering changes the behavior of light.

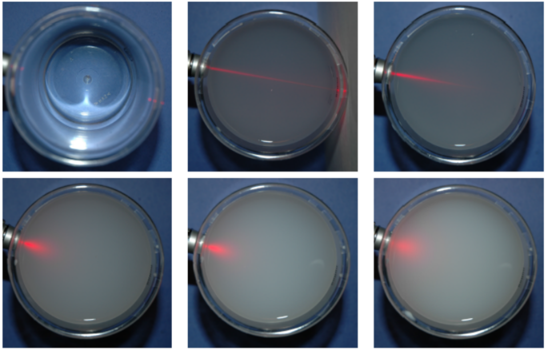


Figure credit: Nina Schotland

There are no pigments.
The color is due to resonant scattering from a periodic structure.

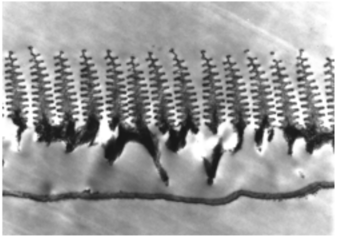



Figure credit: P. Ball

Wave equation

Example of a wave equation: a plucked string with fixed ends.

$$
\begin{aligned}
&u_t(x,t) = u_{xx}(x,t) \qquad 0 < x < \pi, t > 0, \\
&u(x,0) = \phi(x) \quad 0 < x < \pi, \\
&u(0,t) = 0 \quad t > 0, \\
&u(\pi,t) = 0 \quad t > 0,
\end{aligned}
$$

Numerical solutions

Approximate derivatives

$$u'(x) = \frac{u(x) - u(x-\Delta x)}{\Delta x} + O(\Delta x)$$

$$u'(x) = \frac{u(x+\Delta x) - u(x-\Delta x)}{2\Delta x} + O\left((\Delta x)^2\right)$$

$$u''(x) = \frac{u(x+\Delta x) - 2u(x) + u(x-\Delta x)}{(\Delta x)^2} + O\left((\Delta x)^2\right)$$

Numerical solutions

Challenge: Separation of scales (e.g. object much bigger than obstacles, or object much bigger than wavelength) leads to high computational cost.

Nnumber of grid points needed

- In one dimension
  Resolve the obstacles (need even more if the obstacles have corners)
  Resolve the oscillation of the wave: 8 per wavelength
- In n-dimension
  (number of points in one-dimension)^n
- Example: organ 5cm$= 5 \times 10^{-1}$m, infrared light wavelength 1000nm$= 10^{-6}$m. Need $(8 \times 5 \times 10^5)^3 = 6.4 \times 10^{19}$ points.

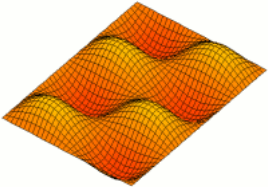


Figure credit: Wiki

Random media

Reductions in random media

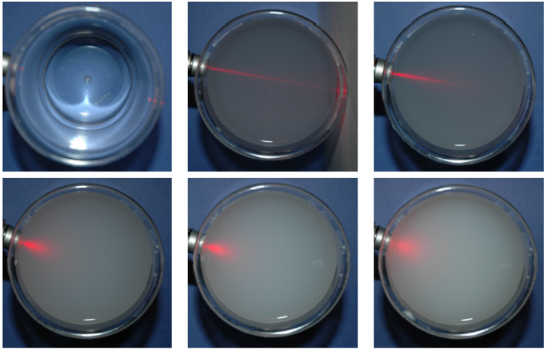


Figure credit: Nina Schotland

Wave equation $\rightarrow$ Random phase regime
$\rightarrow$ Radiative transport equation $\rightarrow$ Diffusion equation

Reductions in random media

Random geometric optics:

- Speckle imaging of stars through the atmosphere (practiced).

star.gif

- Theory was clarified by Borcea-Papniclaou-Toska
- Important elements: scaling regimes (relative size) and signal to noise ratio (SNR).

Length scales of a scattering medium

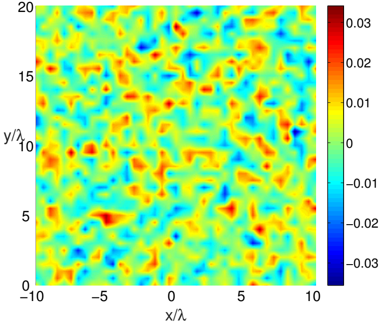
- The scattering mean free path / mean free path l^s: the average distance between two scattering events of a photon.
- The transport mean free path / transport length l^t: the average distance after which the travel direction of a photon is randomized (the wave does not have a well defined propagation direction).
- Propagation distance L: the size of the medium.

Model of the scattering background

- The random medium consists of small, mean zero fluctuations which lacks long range correlations and has an integrable autocorrelation function:

$$4\pi\eta(\boldsymbol{x}) = \sigma\mu\left(\frac{\boldsymbol{x}}{l}\right).$$

- μ is a bounded, mean zero and stationary random process, which lacks long range correlation.
- l is the correlation length, the spatial scale over which the autocorrelation decays.
- σ is the amplitude of the fluctuations.
- For convenience, suppose the autocorrelation $\mathbb{E}[\mu(\boldsymbol{h})\mu(0)] = e^{-\frac{|\boldsymbol{h}|^2}{2}}$.



- Random geometrical optics wave propagation model

$$\lambda \ll l \ll L,$$

$$\sigma \ll (l/L)^{3/2}, \qquad \sigma \ll \sqrt{\lambda l}/L.$$

- In this regime,
  - the waves propagate along straight lines,
  - the variance of the amplitude of the Green function is negligible,
  - and the phase becomes random.

- The Green function is

$$G(\boldsymbol{x},\boldsymbol{y};\omega) = G_0(\boldsymbol{x},\boldsymbol{y};\omega)e^{ik\nu(\boldsymbol{x},\boldsymbol{y})}, \quad \text{for } \boldsymbol{x} \in A \text{ and } \boldsymbol{y} \in R,$$

where the deterministic Green function is

$$G_0(\boldsymbol{x},\boldsymbol{y};j\omega) = \frac{1}{4\pi|\boldsymbol{x}|}e^{ik|\boldsymbol{x}|}$$

and the random phase is

$$\nu(\boldsymbol{x},\boldsymbol{y}) = \frac{\sigma|\boldsymbol{x}-\boldsymbol{y}|}{2}\int_0^1 dt\,\mu\Big(\frac{(1-t)\boldsymbol{y}}{l} + \frac{t\boldsymbol{x}}{l}\Big).$$

- Large wavefront distortions

$$\sigma \gg \frac{\lambda}{\sqrt{lL}}.$$

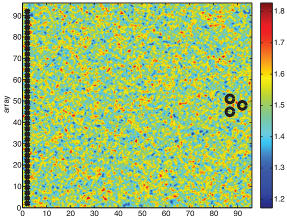
  consistent when $l \gg \sqrt{\lambda L}$.
- Paraxial regime

$$a \ll (\lambda L^3)^{1/4} \ll L,$$

  and

$$r \ll \frac{\lambda L^2}{a^2} \ll a.$$

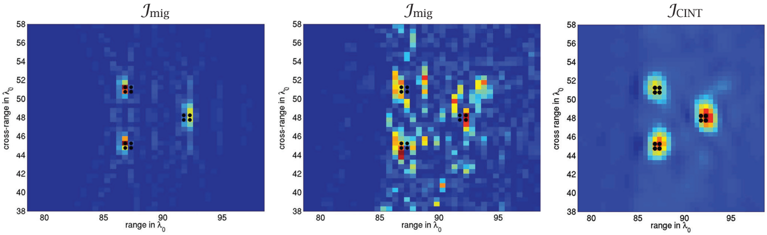
Imaging



- This regime specified as above is referred to as random geometric optics with large wave front distortions.
- Waves are sent into the medium and scattered waves are measured at the boundary.
- Goal is to determine the locations of the scatterers with high SNR.

Imaging

In this regime,



[Borcea 2019 Review of work by Borcea, Papanicolau, Tsogka and Et. Al.]

- Imaging methods for deterministic media fails: not interpretable and unstable
- Taking the random phases into account: coherent interferometric (CINT) imaging. Images become interpretable and stable.

A application to detecting nonlinear scatterers [Borcea-L.-Mamonov-Schotland 2017]

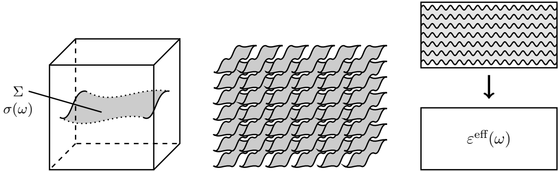
Remark

- Where is the info encoded?
    - In a homogeneous medium, info is in $G_0(\boldsymbol{x}, \boldsymbol{y}; \omega)$.
    - In out regime, info is NOT in $G(\boldsymbol{x}, \boldsymbol{y}; \omega)$ (low SNR). Instead, info is in the auto correlations $G(\boldsymbol{x}, \boldsymbol{y}; \omega) G^{\star}(\boldsymbol{x}', \boldsymbol{y}; \omega')$ (high SNR).
- The SNR of f, defined as $\frac{\mathbb{E}[f]}{\mathrm{StDev}[f]}$, characterizes the reliability in a random media.
- This is why the imaging method is called coherent "interferometry".

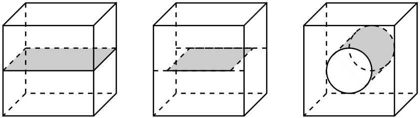
Periodic media

Periodic media

2D plasmonic crystals



Y: cell or period
Σ: inclusion surface inside Y
Σ may or may not have internal edges in Y



Periodic media

Nonmagnetic materials of the following form

Bulk: isotropic constant dielectric tensor

$$\varepsilon(\mathbf{x},\omega) = \varepsilon(\omega)$$

Inclusion: constant surface Drude conductivity

$$\frac{1}{i\omega}\sigma(\mathbf{x},\omega) = \frac{1}{i\omega}\sigma(\omega)P_T \otimes P_T = \eta(\omega)P_T \otimes P_T; \quad \eta(\omega) = \frac{\omega_p}{\omega(\omega + i/\tau)}$$

where P_T projects a vector to its components tangential to the surface Σ.

Homogenization

When the wave length if much larger then the period, the effective dielectric tensor is given by

$$\varepsilon_{ij}^{\text{eff}}(\omega) = \varepsilon\,\delta_{ij} - \eta(\omega)\int_\Sigma \left\{P_T(\boldsymbol{e}_j) + \nabla_T\chi_j(\omega,\boldsymbol{x})\right\}\cdot P_T(\boldsymbol{e}_i)\,\mathrm{d}o_x, \quad i,j = 1,2,3$$

where

$\boldsymbol{e}_j$: jth Euclidean basis vector
P_T: projects a vector to its components tangential to the surface Σ
∇_T: tangential derivative

[Amirat-Shelukhin 2017, Maier-Margetis-Mellet 2020, Wellander 2001 2002, Wellander-Kristensson 2003]

The corrector $\chi_j(\boldsymbol{x}) \in \mathscr{H}$: the Y-periodic corrector fields, is the unique solution to

$$\begin{cases} \Delta\chi_j(\boldsymbol{x})) = 0 & \text{in } Y \setminus \Sigma\ , \\ \left[\chi_j(\boldsymbol{x}))\right]_\Sigma = 0 & \text{on } \Sigma\ , \\ \varepsilon\left[\boldsymbol{\nu}\cdot\nabla\chi_j(\boldsymbol{x})\right]_\Sigma = \dfrac{\sigma}{i\omega}\nabla_T\cdot\left(P_T\boldsymbol{e}_j + \nabla_T\chi_j(\boldsymbol{x})\right) & \text{on } \Sigma\ . \end{cases}$$

Idea

Analogue in linear algebra

- Solve for x

$$Ax = Bx + f,$$

  where A and B are matrices, x and f are vectors.

- If

$$\lambda_n A x_n = B x_n, \quad x_n^T A x_m = \delta_{n,m}, \quad f = b_1 x_1 + \cdots + b_N x_N, \quad x = a_1 x_1 + \cdots + a_N x_N,$$

  then we can find

$$a_n = \frac{b_n}{\lambda_n - 1}$$

  by plugging the expansions of f and x in the equation.

Spectral problem

Auxilary spectral problem

$$\begin{cases} \Delta\varphi_n(\boldsymbol{x})) = 0 & \text{in } Y \setminus \Sigma\,, \\ [\varphi_n(\boldsymbol{x}))]_\Sigma = 0 & \text{on } \Sigma\,, \\ \lambda_n[\boldsymbol{\nu}\cdot\nabla\varphi_n(\boldsymbol{x})]_\Sigma = \nabla_T\cdot\nabla_T\varphi_n(\boldsymbol{x}) & \text{on } \Sigma\,. \end{cases}$$

Eigenmode expansion

$$\chi_j = \sum_n \alpha_j^n \varphi_n + C.$$

Find the coefficients α_j^n by substitute into the defining equations of χ_j.

Spectral problem

Effective tensor in resonant form [L.-Lipton-Matthias 2020]

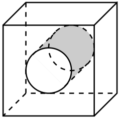
$$\varepsilon_{ij}^{\mathrm{eff}} = \varepsilon\,\delta_{ij} - \eta(\omega)\int_{\Sigma} P_T(\boldsymbol{e}_j)\cdot P_T(\boldsymbol{e}_i)\,\mathrm{d}o_x - \sum_n \frac{\lambda_n \eta^2(\omega)}{\varepsilon - \lambda_n\,\eta(\omega)} \int_{\Sigma} P_T(\boldsymbol{e}_j)\cdot \nabla_T \overline{\xi}_n\,\mathrm{d}o_x \int_{\Sigma} \nabla_T \xi_n \cdot P_T(\boldsymbol{e}_i)\,\mathrm{d}o_x.$$

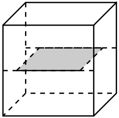
Geometry and frequency are decoupled!

Convergence rate depends on $\int_{\Sigma} P_T(\boldsymbol{e}_j)\cdot\nabla_T\overline{\varphi}_n(\boldsymbol{x})\,\mathrm{d}o_x$

Lorenztian shaped resonances when the real part of $\varepsilon - \lambda_n\,\eta(\omega)$ is zero.

Efficient Computation





order k	λ_k	$\left\lvert \int_\Sigma P_T(\boldsymbol{e}_1) \cdot \nabla_T \overline{\xi}_n \, \mathrm{d}o_x \right\rvert$
1	0.5924	1.1158
2	3.726	0.1077
3	6.289	0.008194
4	8.763	0.003574
5	11.26	0.0002755
6	13.76	0.00008546
7	16.27	0.000009443

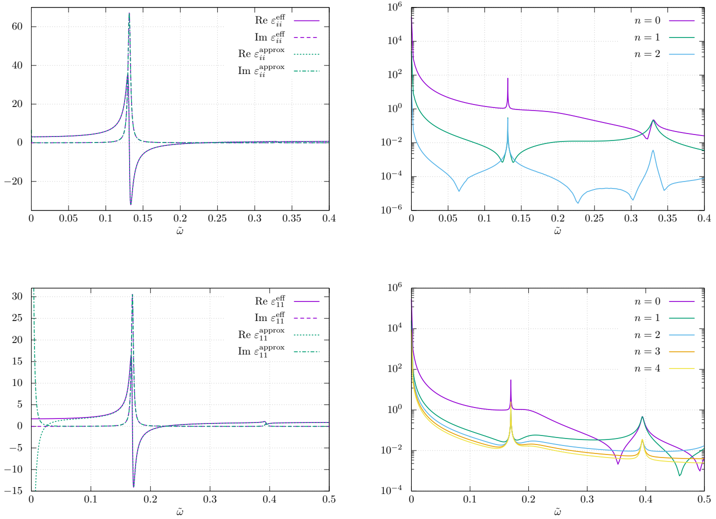
(a) Nanotubes

λ_k	$\left\lvert \int_\Sigma P_T(\boldsymbol{e}_1) \cdot \nabla_T \overline{\xi}_n \, \mathrm{d}o_x \right\rvert$
0.9873	0.8543
5.314	0.1811
9.283	0.1097
13.22	0.07913
17.16	0.06194
25.02	0.04322
28.96	0.03755

(b) Nanoribbons

$\left\lvert \int_\Sigma P_T(\boldsymbol{e}_j) \cdot \nabla_T \overline{\varphi}_k(\boldsymbol{x}) \mathrm{d}o_x \right\rvert$ decays slower in the presence of internal edges

Comparison with FEM and relative error
(top: nanotubes, bottom: nano ribbons)



Remark

The expansion is enabled by

- an underlying compact self-adjoint operator (analogue of symmetric matrices in infinite dimension),
- on properly defined Sovolev spaces (generalized vectors and generalized inner products)

Similar decompositions for different operators appeared in

- Bergman 1978, McPhedran-Milton 1981
  Effective dielectric response
- Milton 1981 2002, Golden-Papanicolaou 1983
  Bounds on effective dielectric response
- Chen-Lipton 2013
  Non-magnetic double negative metamaterials
- Lipton-Viator 2017
  Photonic band gap materials

A lot of fun wave phenomena to explore,
and a lot of tools need to be developed!