

MRCert: Towards Post-deployment Patch Robustness Certification for Adversarially Patched Samples via Type-specific Masking

802 A Formal Proof

803 **Property** (Infeasibility of Re-certifying Adversarially
 804 Patched Samples for Existing Masking-based Recovery
 805 Works). $\forall x' \in \mathbb{A}_{\mathbb{P}}(x), c_r(x) \wedge f(x') \neq f(x) \implies \neg c_r(x')$.

806 *Proof.* We know in [Xiang *et al.*, 2022], $c_r(\hat{x}) :=$
 807 $[\forall M_1, M_2 \in \mathbb{M}, f(\hat{x} \odot M_1 \odot M_2) = f(\hat{x})]$. Then if $c_r(x) =$
 808 *True*, we know $[\forall M_1, M_2 \in \mathbb{M}, f(x \odot M_1 \odot M_2) = f(x)]$.
 809 We also know $\forall P \in \mathbb{P}, \exists M \in \mathbb{M}_P, M \odot P = P$ by definition,
 810 which makes $\forall x' \in \mathbb{A}_{\mathbb{P}}(x), x' \odot M = x \odot M$ (see Fig. 2).
 811 Therefore, $\exists x' \in \mathbb{A}_{\mathbb{P}}(x), f(x' \odot M) = f(x \odot M) = f(x)$.
 812 We have $f(x') \neq f(x)$ from the antecedent; therefore,
 813 $\exists x' \in \mathbb{A}_{\mathbb{P}}(x), f(x' \odot M) \neq f(x')$. Since $c_r(x')$ requires
 814 $[\forall M_1, M_2 \in \mathbb{M}, f(x' \odot M_1 \odot M_2) = f(x')]$, $c_r(x') = \text{False}$
 815 holds. \square

816 **Lemma** (Certification of benign samples). *Given an arbitrary sample \hat{x} , if $[\forall M_1, M_2, M_3 \in \mathbb{M}, f(\hat{x} \odot M_1 \odot M_2 \odot M_3) = f(\hat{x})]$ holds, $\forall \hat{x}' \in \mathbb{A}_{\mathbb{P}}(\hat{x}), g(\hat{x}') = g(\hat{x})$.*

817 *Proof.* We first analyze $g(\hat{x})$. Since $[\forall M_1, M_2, M_3 \in$
 818 $\mathbb{M}, f(\hat{x} \odot M_1 \odot M_2 \odot M_3) = f(\hat{x})] \implies [\forall M_1, M_2 \in$
 819 $\mathbb{M}, f(\hat{x} \odot M_1 \odot M_2) = f(\hat{x})]$, we know its prediction label
 820 $g(\hat{x}) = f(\hat{x})$ output in Case ①. We then analyze $g(\hat{x}')$.
 821 For $\forall \hat{x}' \in \{\hat{x}' \mid \hat{x}' = (J - P) \odot \hat{x} + P \odot \hat{x}' \wedge P \in \mathbb{P}\}$ (i.e.,
 822 $\forall \hat{x}' \in \mathbb{A}_{\mathbb{P}}(\hat{x})$), w.l.o.g., we let $M_1 \odot P = O$. Therefore, we get
 823 $\hat{x}' \odot M_1 = ((J - P) \odot \hat{x} + P \odot \hat{x}') \odot M_1 = \hat{x} \odot M_1$. Then, we get
 824 $[\exists M_1 \in \mathbb{M}, \forall M_2, M_3 \in \mathbb{M}, f(\hat{x}' \odot M_1 \odot M_2 \odot M_3) = f(\hat{x}')]$ (see
 825 Fig. 2). Note that the special cases $M_2 = M_3, M_1 = M_2 = M_3$
 826 are included. Case 1: Suppose the returned label of \hat{x}' output
 827 in Case ① as $f(\hat{x}')$ (i.e., $\forall M_1, M_2 \in \mathbb{M}, f(\hat{x}' \odot M_1 \odot M_2) =$
 828 $f(\hat{x}')$), then we know $g(\hat{x}') = f(\hat{x}') = f(\hat{x}) = g(\hat{x})$.
 829 Case 2: Otherwise, its returned label should be output in
 830 Case ② as $f(\hat{x}' \odot M_1)$, since $[\exists M_1 \in \mathbb{M}, \forall M_2, M_3 \in$
 831 $\mathbb{M}, f(\hat{x}' \odot M_1 \odot M_2 \odot M_3) = f(\hat{x}' \odot M_1) = f(\hat{x})]$, and further
 832 $g(\hat{x}') = f(\hat{x}' \odot M_1) = f(\hat{x}) = g(\hat{x})$. \square

833 **Lemma** (Certification of adversarially patched samples).
 834 *Given an arbitrary sample \hat{x} , if $[\exists M_1 \in \mathbb{M}, \forall M_2, M_3, M_4 \in$
 835 $\mathbb{M}, f(\hat{x} \odot M_1 \odot M_2 \odot M_3 \odot M_4) = f(\hat{x} \odot M_1)]$ holds,
 836 $\forall \hat{x}' \in \mathbb{A}_{\mathbb{P}}(\hat{x}), g(\hat{x}') = g(\hat{x})$.*

837 *Proof.* We first analyze $g(\hat{x})$. Case 1: If \hat{x} meet the condition
 838 of Case ①, we can further get $\forall M_1, M_2, M_3, M_4 \in \mathbb{M}, f(\hat{x} \odot$
 839 $M_1 \odot M_2 \odot M_3 \odot M_4) = f(\hat{x})$. Note that the special case
 840 $M_1 = M_2 = M_3 = M_4$ is included, which means we can get
 841 $g(\hat{x}) = f(\hat{x} \odot M_1)$. Case 2: Otherwise, by the given condition
 842 $[\exists M_1 \in \mathbb{M}, \forall M_2, M_3, M_4 \in \mathbb{M}, f(\hat{x} \odot M_1 \odot M_2 \odot M_3 \odot M_4) =$
 843 $f(\hat{x} \odot M_1)] \implies [\exists M_1 \in \mathbb{M}, \forall M_2, M_3 \in \mathbb{M}, f(\hat{x} \odot M_1 \odot M_2 \odot$
 844 $M_3) = f(\hat{x} \odot M_1)]$, \hat{x} meet the condition in Case ② and its
 845 prediction label $g(\hat{x}) = f(\hat{x} \odot M_1)$, same as Case ①. We then
 846 analyze $g(\hat{x}')$. For $\hat{x}' \in \{\hat{x}' \mid \hat{x}' = (J - P) \odot \hat{x} + P \odot \hat{x}' \wedge P \in \mathbb{P}\}$
 847 (i.e., $\forall \hat{x}' \in \mathbb{A}_{\mathbb{P}}(\hat{x})$), Case 1: Suppose $M_1 \odot P = O$. Then we
 848 can get $\hat{x}' \odot M_1 = ((J - P) \odot \hat{x} + P \odot \hat{x}') \odot M_1 = \hat{x} \odot M_1$,

849 and further get $[\forall M_2, M_3, M_4 \in \mathbb{M}, f(\hat{x}' \odot M_1 \odot M_2 \odot M_3 \odot$
 850 $M_4) = f(\hat{x} \odot M_1)]$, which is the same condition as that
 851 on \hat{x} . Therefore, repeating those analysis for $g(\hat{x})$ above
 852 can get $g(\hat{x}) = g(\hat{x}')$. Case 2: Otherwise, for M_2, M_3, M_4 ,
 853 w.l.o.g., we let $M_2 \odot P = O$ ($M_1 \neq M_2$). Then similarly,
 854 we get $[\exists M_1, M_2 (\neq M_1) \in \mathbb{M}, \forall M_3, M_4 \in \mathbb{M}, f(\hat{x}' \odot M_1 \odot$
 855 $M_2 \odot M_3 \odot M_4) = f(\hat{x} \odot M_1)]$. Note that the special cases
 856 $M_1 = M_3, M_2 = M_4, M_3 = M_4$ are included. Case 2.1: Sup-
 857 pose the prediction label $g(\hat{x}')$ output in Case ①. Then, since
 858 $[\exists M_1, M_2 (\neq M_1) \in \mathbb{M}, f(\hat{x}' \odot M_1 \odot M_2) = f(\hat{x} \odot M_1)]$ (spe-
 859 cial case $M_1 = M_3, M_2 = M_4$, by the condition of Case ①,
 860 we know $g(\hat{x}') = f(\hat{x}') = f(\hat{x} \odot M_1) = g(\hat{x})$. Case 2.2: Sup-
 861 pose the prediction label $g(\hat{x}')$ output in Case ②. Then, since
 862 $[\exists M_1, M_2 (\neq M_1) \in \mathbb{M}, \forall M_3 \in \mathbb{M}, f(\hat{x}' \odot M_1 \odot M_2 \odot M_3) =$
 863 $f(\hat{x} \odot M_1)]$ (special case $M_3 = M_4$), by the condition of Case
 864 ②, we know $g(\hat{x}') = f(\hat{x}' \odot M_1) = f(\hat{x} \odot M_1) = g(\hat{x})$. Case
 865 2.3: Otherwise, the prediction label of \hat{x}' should output in
 866 Case ③ since $[\exists M_1, M_2 (\neq M_1) \in \mathbb{M}, \forall M_3, M_4 \in \mathbb{M}, f(\hat{x}' \odot$
 867 $M_1 \odot M_2 \odot M_3 \odot M_4) = f(\hat{x} \odot M_1) = f(\hat{x}' \odot M_1 \odot M_2)]$
 868 (special case $M_1 = M_3, M_2 = M_4$), and further $g(\hat{x}') =$
 869 $f(\hat{x}' \odot M_1 \odot M_2) = f(\hat{x} \odot M_1) = g(\hat{x})$. \square

870 **Theorem** (Certification of samples). *Given an arbitrary sam-
 871 ple \hat{x} , if $c(\hat{x}) = \text{True}$ holds, $\forall \hat{x}' \in \mathbb{A}_{\mathbb{P}}(\hat{x}), g(\hat{x}') = g(\hat{x})$.*

872 Simply conjoining the antecedent of Lemma 1 and
 873 Lemma 2 can prove this theorem. \square

874 **Theorem** (Round-trip certification of samples). *Given
 875 a benign sample x , if $[\forall M_1, M_2, M_3, M_4 \in \mathbb{M}, f(x \odot$
 876 $M_1 \odot M_2 \odot M_3 \odot M_4) = f(x)]$ (i.e., $c_r^2(x) = \text{True}$), $[\forall x' \in$
 877 $\mathbb{A}_{\mathbb{P}}(x), g(x') = g(x) \wedge c_r(x') = \text{True} \wedge c_r(x) = \text{True}]$.*

878 *Proof.* By the condition $[\forall M_1, M_2, M_3, M_4 \in \mathbb{M}, f(x \odot M_1 \odot$
 879 $M_2 \odot M_3 \odot M_4) = f(x)]$, we know the returned label $g(x) =$
 880 $f(x)$ in Case ① and $c_r(x) = \text{True}$. Still by this condi-
 881 tion, we know $[\forall x' \in \mathbb{A}_{\mathbb{P}}(x), \exists M_1 \in \mathbb{M}_P, \forall M_2, M_3, M_4 \in$
 882 $\mathbb{M}, f(x' \odot M_1 \odot M_2 \odot M_3 \odot M_4) = f(x)]$, since $[\exists M_1 \in \mathbb{M}_P, \forall$
 883 $x' \in \mathbb{A}_{\mathbb{P}}(x), x' \odot M_1 = x \odot M_1]$ (see Fig. 2 for illustration).
 884 Then by Lemma 2, we know $[\forall x' \in \mathbb{A}_{\mathbb{P}}(x), c_r(x') = \text{True}]$.
 885 By Lemma 1, we also know $[\forall x' \in \mathbb{A}_{\mathbb{P}}(x), g(x') = g(x)]$.
 886 Finally, we know $[\forall x' \in \mathbb{A}_{\mathbb{P}}(x), g(x') = g(x) \wedge c_r(x') =$
 887 $\text{True} \wedge c_r(x) = \text{True}]$. \square

888 B The design subtlety of MRCert

889 **The ordering of the Cases.** The sequence of the Cases is cru-
 890 cial. Changing the order may either break the certification
 891 guarantee for adversarially patched samples or degrade clean
 892 accuracy. For example, swapping Case ② and Case ③ would
 893 require the benign counterpart of an adversarially patched
 894 sample to exhibit consistency under four masks (rather than
 895 the current three) to remain certified; otherwise, an attacker
 896 could exploit a fourth-order masked mutant that predicts an-
 897 other label to evade detection. Case 1 is intentionally placed
 898 \square

900 first to prevent benign samples from reaching the recovery
901 stages, thereby improving clean accuracy.

902 *The ordering of mask conditions within a Case.* The internal
903 ordering of masking conditions also matters. Consider
904 Case 2 (indirect testing): if its quantifiers were reordered
905 from “there exists mask 1 such that for all mask 2, mask3
906 …” to “there exist mask 1 and mask 2 such that for all mask 3
907 …”, Then adversarial samples containing two patches could
908 escape certification, even when their benign counterparts are
909 round-trip certified. In this reordered structure, the adversarial
910 patched sample could simply choose two masks (mask 1
911 and mask 2) that expose both patches, i.e., do not cover either
912 patch. Then, even if mask 3 covers one patch, the other
913 patch remains uncovered, allowing the attacker to arbitrarily
914 alter the prediction and evade detection. This demonstrates
915 that the original quantifier ordering is essential for preventing
916 such escape cases.

917 *The number of masks within Cases.* Taking Case 2 as an
918 example, the number of masks is also critical. Reducing the
919 number of masks in Case 2 would allow adversarial samples
920 with two patches to escape, for reasons analogous to the
921 ordering issue discussed above. Conversely, increasing the
922 number of masks (e.g., increasing 1 mask) would enlarge the
923 masking requirements for certification. In that case, certifying
924 a benign sample would require prediction consistency
925 under four masks, instead of the current three, which directly
926 decreases the samples that can be certified.

927 C Extension of MRCert to Recover and 928 Certify Samples with N-patches

929 We can extend the maximum number of patches from 2 to N
930 (called MRCert-N-patch, a variant of MRCert) following the
931 following idea. First, we apply each set of N masks in the
932 covering mask set \mathbb{M} on the input sample \hat{x} to test whether
933 \hat{x} is harmful. If it is not harmful, MRCert-N-patch returns
934 the label $f(\hat{x})$ (marked as N-Case ①). If \hat{x} is detected as
935 harmful, we then test whether all its first-order mutants are
936 harmful by applying each possible subset with N masks se-
937 lected with replacement from \mathbb{M} on each first-order mutant of
938 \hat{x} . If there exists a first-order mutant, whose all $(N + 1)$ th-
939 order mutants generated from the first-order mutant of \hat{x} are
940 predicted with the same label as this first-order sample, then
941 \hat{x} is deemed as a one-patch harmful sample, this first-order
942 mutant is “clean” and the prediction label of this first-order
943 mutant is returned (marked as N-Case ②). If that is not the
944 case, we then test whether all second-order mutants of \hat{x} are
945 harmful in the same manner, and repeat until the N th-order
946 mutants of \hat{x} are tested. For the certification function c_r with
947 the input sample \hat{x} , it should be extended to the condition that
948 all $(N + 1)$ th-order mutants of \hat{x} are predicted with the same
949 label as \hat{x} (for those input samples whose label returned in
950 Case ①), and the condition that there exists a first-order mu-
951 tant of \hat{x} , whose all $(N + 2)$ th-order mutants are predicted
952 with the same label as this first-order mutant of \hat{x} (for those
953 input samples whose label returned in Case ②), and certifying
954 the input samples output in other cases by the condition
955 in the same manner. For the round-trip certification function,
956 it should be the condition that all $2N$ th-order mutants of a

957 benign sample x are predicted with the same label as x . We
958 leave the formal proof and implementation as future work.

959 D Experimental Setup

960 D.1 Environment and Dataset

961 The evaluation is conducted on an Ubuntu 20.04 machine
962 equipped with four Nvidia 3090 GPUs. Following [Xiang *et*
963 *al.*, 2022], we adopt 1000-class ImageNet [Deng *et al.*, 2009],
964 10-class CIFAR10 [Krizhevsky *et al.*, 2009], and 10-class Im-
965 ageNette as our datasets, encompassing both large-scale, di-
966 verse datasets and efficient benchmarks widely used for rapid
967 experimentation.

968 D.2 Models and Baselines

969 We adopt the Vision Transformer (ViT) as our base model,
970 which achieves the state-of-the-art in many patch robustness
971 certification defenders [Xiang *et al.*, 2024; Li *et al.*, 2022;
972 Salman *et al.*, 2022; Xiang *et al.*, 2022; Huang *et al.*, 2023].

973 We adopt the state-of-the-art masking-based defender
974 PatchCURE (**PC**) [Xiang *et al.*, 2024] and state-of-the-
975 art smoothing-based defender VOT-CrossCert (**VOT**) from
976 CrossCert [Cro, 2024; Zhou *et al.*, 2024], which shares the
977 common methodology for smoothing-based recovery with
978 ViP [Li *et al.*, 2022] and S-ViT [Salman *et al.*, 2022]. We
979 arm VOT with the methodology for runtime verification pro-
980 posed by [Yatsura *et al.*, 2023] as our baseline to compare
981 with **MRCert** in our infrastructure.

982 Specifically, MRCert adopts the end-to-end ViT-SRF
983 model (setting $14 \times 1 \times k \times 6^1$) proposed and used by PatchCURE
984 [Xiang *et al.*, 2024] from [Xiang, 2024] with the pre-train
985 weights and setting from MAE [He *et al.*, 2022], so that
986 MRCert can use the same strategy in the experiment as
987 PatchCURE [Xiang *et al.*, 2024] to calculate the covering
988 mask set, generate mutants with their predictions inside the
989 ViT-SRF. We use the training scripts with the settings and
990 hyperparameters from the official repository of PatchCURE
991 [Xiang, 2024] for fine-tuning ViT-SRF, which uses the train-
992 ing samples with their first and second-order mutants only.
993 We then apply the same finetuned ViT-SRF to both MRCert
994 and PatchCURE to ensure fairness. We also adopt this of-
995 ficial repository [Xiang, 2024] to implement PatchCURE in
996 our infrastructure. We adopt the same pre-trained weights
997 and settings from MAE [He *et al.*, 2022] for VOT. We follow
998 the fine-tuning settings and hyperparameters from the state-
999 of-the-art S-ViT [Salman *et al.*, 2022] for VOT (note that
1000 smoothing-based recovery adopts a different notion of mu-
1001 tants, where a mutant is a slice of a sample; therefore, a differ-
1002 ent fine-tuning on the base model is needed [Li *et al.*, 2022]).
1003 We also adopt the column ablation with the ablation size of
1004 19 pixels from [Salman *et al.*, 2022]. We have extracted the

¹Other variants ViT-SRF14x2, ViT-SRF2x2, and BagNet33 cannot handle the situation against two patches, since all their receptive fields would be inherently masked by VIT-SRF [Xiang, 2024] when generating corresponding mutants. k is the parameter to control the position of splitting SRF and LRF in VIT-SRF in the range $[0, 12]$, which can tune the trade-off between computational efficiency and robustness. We adopt the middle one since it is not the focus of this paper.

1005 corresponding results from the original papers of ViP [Li *et* al., 2022] and S-ViT [Salman *et al.*, 2022] to compare with
1006 VOT in Tab. 1 (their results on round-trip certification are not
1007 reported in their papers).
1008

1009 D.3 Metrics

1010 Let x be a benign sample with the true label y_0 in a clean
1011 test dataset \mathbb{D} , and $R = \langle g(x), c(x), c_r^2(x) \rangle$ be a certi-
1012 fied recovery defender. **Clean accuracy** is the fraction
1013 of \mathbb{D} that are correctly predicted, defined as $acc_{clean} =$
1014 $\frac{|\{x \in \mathbb{D} | g(x) = y_0\}|}{|\mathbb{D}|}$, which evaluates the standard performance
1015 of certified recovery as a classifier. **Certified accuracy** is
1016 the fraction of \mathbb{D} that are correctly predicted and certified
1017 robust, whose adversarially patched samples should be pre-
1018 dicted with the same label as the benign sample, defined
1019 as $acc_{cert} = \frac{|\{x \in \mathbb{D} | g(x) = y_0 \wedge c_r(x) = True\}|}{|\mathbb{D}|}$. **Round-trip cer-**
1020 **tified accuracy** is the fraction of \mathbb{D} that are correctly pre-
1021 dicted and round-trip certified, whose adversarially patched
1022 samples should be predicted with the same label as the be-
1023 nign sample and gain a provable robust verdict, defined
1024 as $acc_{cert^2} = \frac{|\{x \in \mathbb{D} | g(x) = y_0 \wedge c_r^2(x) = True\}|}{|\mathbb{D}|}$. The clean accu-
1025 racy of the base classification model f is $\frac{|\{x \in \mathbb{D} | f(x) = y_0\}|}{|\mathbb{D}|}$.
1026 Clean accuracy and certified accuracy are commonly adopted
1027 by peers [Levine and Feizi, 2020a; Salman *et al.*, 2022;
1028 Li *et al.*, 2022; Xiang *et al.*, 2024; Xiang *et al.*, 2022;
1029 Xiang *et al.*, 2021], and round-trip certified accuracy is pro-
1030 posed by [Yatsura *et al.*, 2023].

1031 In RQ1, the test dataset \mathbb{D} is a set of patched samples from
1032 attacking a subset of the test dataset, denoted as \mathbb{D}_{pat} . Clean
1033 accuracy and certified accuracy are measured in this set of
1034 adversarially patched samples to evaluate certified recovery
1035 defenders on their ability to recover predictions and recover
1036 predictions with a provably robust verdict, respectively. In
1037 RQ2, we use the original test dataset as \mathbb{D} .

1038 D.4 Experimental Procedure

1039 We adopt the patch sizes of 16 pixels (measured as a square
1040 patch region with a side length of 16 pixels) and 32 pixels for
1041 all three datasets. All samples are rescaled to 224x224 [Zhou
1042 *et al.*, 2024; Li *et al.*, 2022; Xiang *et al.*, 2022; Xiang *et al.*,
1043 2024].

1044 In RQ1, we perform an actual adversarial patch attack
1045 IFGSM adopted from [Levine and Feizi, 2020a] on PC and
1046 MRCert, which uses their shared base model as the attack
1047 model for gradient-based attacks without the knowledge of
1048 the non-differentiable label recovery function for fairness.
1049 Note that the recent proposed attacks are more toward prac-
1050 tical, such as limiting the access times or can only get the
1051 returned prediction label, while our IFGSM attack has direct
1052 and full access to the base model, which is a powerful exam
1053 against attackers. We set 80 random starts, 150 iterations per
1054 random start, and a step size of 0.05 following [Levine and
1055 Feizi, 2020a]. We randomly select 500 test samples from
1056 each test dataset for attacks. We follow the practice in [Levine
1057 and Feizi, 2020a] to return the worst patched sample for each
1058 benign sample and place it into \mathbb{D}_{pat} .

In both RQ1 (as a follow-up to the attack) and RQ2, for
1059 each defender on each sample in each test dataset, we first
1060 generate and evaluate corresponding mutants using the corre-
1061 sponding base model (PC and MRCert share the same ones),
1062 then apply each defender to the prediction results collected
1063 for these mutants, and measure the metric values.
1064