# Week 16 Limits and differentiation continued Lecture note

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| **Notebook:** | Computational Mathematics | | |
| **Created:** | 2020-04-21 2:48 PM | **Updated:** | 2020-07-23 8:34 PM |
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| **URL:** | https://www.coursera.org/learn/uol-cm1015-computational-mathematics/home/week/13 | | |

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| **Cornell Notes** | **Topic:**<br>**Limits and differentiation continued** | Course: BSc Computer Science<br>Class: Computational Mathematics[Lecture]<br>Date: July 23, 2020 |

**Essential Question:**

What are limits and derivatives and how do they relate to the notion of continuity of a function?

**Questions/Cues:**

- What are some properties of the derivative?
- What is L'Hopital Rule?
- How does the derivative of a function relate to its minima and maxima?
- What are inflection points?

Notes

Properties:
1)derivative of sum of two functions

Let f(x)=g(x)+h(x)

$f'(x)=g'(x)+h'(x)$

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{g(x+\Delta x)+h(x+\Delta x)-g(x)-h(x)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x}+\frac{h(x+\Delta x)-h(x)}{\Delta x}=g'(x)+h'(x)$$

2)n^th derivative

$$f''(x)=\lim_{\Delta x\to 0}\frac{f'(x+\Delta x)-f'(x)}{\Delta x}\to f^{n}(x)=\lim_{\Delta x\to 0}\frac{f^{n-1}(x+\Delta x)-f^{n-1}(x)}{\Delta x}$$

# Derivative of product of two functions:

$f(x)=g(x)h(x)$

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{g(x+\Delta x)h(x+\Delta x)-g(x)h(x)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{g(x+\Delta x)h(x+\Delta x)\overbrace{-g(x)h(x+\Delta x)+g(x)h(x+\Delta x)}^{=0}-g(x)h(x)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}h(x+\Delta x)\frac{g(x+\Delta x)-g(x)}{\Delta x}+\lim_{\Delta x\to 0}\frac{g(x)h(x+\Delta x)-g(x)h(x)}{\Delta x}$$

$$=h(x)g'(x)+\lim_{\Delta x\to 0}g(x)\frac{h(x+\Delta x)-h(x)}{\Delta x}$$

$$=h(x)g'(x)+g(x)h'(x)$$

# Derivative from first principles

Example: $y=f(x)=x^2$

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

Let $\Delta x=h$

$$\lim_{h\to 0}\frac{(x+h)^2-(x)^2}{h}=\lim_{h\to 0}\frac{(x^2+2xh+h^2)-x^2}{h}$$
$$=\lim_{h\to 0}\frac{(2xh+h^2)}{h}=\lim_{h\to 0}(2x+h)=2x$$

$f'(x)=2x$

# Derivative of ratio of two functions:

$f(x)=g(x)/h(x)$

$$f'(x)=\frac{g'(x)h(x)-g(x)h'(x)}{h^2(x)}$$

Example: $f(x)= x^2/(2x+1) =g(x)/h(x)$

$g(x)=x^2 \quad \rightarrow \quad g'(x)=2x$
$h(x)=(2x+1) \quad \rightarrow \quad h'(x)=2$

$$f'(x)=\frac{2x(2x+1)-x^2\,2}{(2x+1)2}=\frac{4x^2+2x-2x^2}{(2x+1)2}=\frac{2x(x+1)}{(2x+1)2}$$

# Derivative of composed function:

Composed function: two functions g and h applied in succession
Example:
$h(x)$ $x \rightarrow log(x)$
$g(x)$: $x \rightarrow sin(x)$
Composed functions → $g(h(x))= sin(log(x))$ $h(g(x))=log(sin(x))$

$f(x)=g(h(x))$ $\rightarrow f'(x) = g'(h(x))h'(x)$

Example: f(x)= $e^{x^2}$

$h(x)=x^2$ → $h'(x)= 2x$
$g(x)=e^x$ → $g'(x)= e^x$

In g'(h(x)) need to replace h(x) in place of x in g'(x)

$\rightarrow g'(h(x))=e^{h(x)}=e^{x^2}$

$\rightarrow f'(x)=e^{x^2}h'(x)=2xe^{x^2}$

1) $f(x) = g(x) + h(x)$ $\rightarrow f'(x) = g'(x) + h'(x)$

$g(x) = x^2$ $h(x) = e^x$ $g'(x) = 2x$
$f(x) = x^2 + e^x$ $h'(x) = e^x$

$f'(x) = 2x + e^x$

2) $f(x) = h(x)\,g(x)$

$$f'(x) = h'(x)\,g(x) + h(x)\,g'(x)$$

$$f(x) = \underbrace{x}_{h(x)}\overbrace{(\sin x + 1)}^{g(x)} \qquad \begin{cases} h'(x) = 1 \\ g'(x) = \cos x \end{cases}$$

$$f'(x) = 1 \cdot (\sin x + 1) + x \cdot \cos x =$$

$$= \sin x + 1 + x\cos x$$

$$f(x) = \frac{g(x)}{h(x)} \rightarrow f'(x) = \underline{\frac{g'(x)h(x) - g(x)h'(x)}{(h(x))^2}}$$

$$f(x) = \frac{\cos x}{x^2+3} \qquad \begin{array}{l} g(x) = \cos x \rightarrow g'(x) = -\sin x \\ h(x) = \underline{x^2+3} \rightarrow h'(x) = 2x \end{array}$$

$$f'(x) = \frac{-\sin x\,(x^2+3) - \cos x\,(2x)}{(x^2+3)^2} =$$

$$= -\frac{\sin x}{x^2+3} - \frac{2x\cos x}{(x^2+3)^2}$$

$$f(x) = g(h(x)) \rightarrow f'(x) = g'(h(x))\,h'(x)$$

$$f(x) = \cos(\ln x) \qquad \begin{array}{l} g(x) = \cos x \\ h(x) = \ln x \\ g'(x) = -\sin(x) \\ h'(x) = \frac{1}{x} \end{array}$$

$$f'(x) = -\frac{\sin(\ln x)}{x}$$

| $f(x)$ | $f'(x)$ |
|---|---|
| $x^{\alpha}$ | $\alpha x^{\alpha-1}$ |
| $e^{\alpha x}$ | $\alpha e^{\alpha x}$ |
| $\log(x)$ | $\frac{1}{x}$ |
| $\sin(x)$ | $\cos(x)$ |
| $\cos(x)$ | $-\sin(x)$ |
| $\text{ten}(x)$ | $\frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) = \frac{\overbrace{\cos x \cdot \cos x - (-\sin x)\cdot \sin x}^{\cos^2 x + \sin^2 x = 1}}{\cos^2 x} = \frac{1}{\cos^2 x}$ |

L'Hopital's rule

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} \quad \text{if} \quad \lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = \begin{cases} 0 \\ \pm\infty \end{cases}$$

$$g'(x) \neq 0 \quad x \in I/\{x_0\}$$

$$\Rightarrow \boxed{\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}}$$

1) $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = \lim\limits_{x \to 0} \dfrac{\cos x}{1} = 1$

2) $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x} = \lim\limits_{x \to 0} \dfrac{e^x}{1} = 1$

3) $\lim\limits_{x \to 0} \dfrac{\ln(1+x)}{x} = \lim\limits_{x \to 0} \dfrac{\frac{1}{1+x}}{1} = 1$

4) $\lim\limits_{x \to 0} \dfrac{\cos x - 1}{x^2} = \lim\limits_{x \to 0} \dfrac{-\sin x}{2x} = -\dfrac{1}{2} \underbrace{\lim\limits_{x \to 0} \dfrac{\sin x}{x}}_{=1}$

$= -\dfrac{1}{2}$

$$\lim_{x \to 0} \frac{e^x - 1}{x^2 + \sin x} = \lim_{x \to 0} \frac{e^x}{2x + \cos x} = \frac{1}{1} = 1$$

$$\lim_{x \to 0} x^x = \lim_{x \to 0} e^{\log(x^x)} \qquad \boxed{x^x = e^{\log(x^x)}}$$

$$= \lim_{x \to 0} e^{x \log x} \qquad \lim_{x \to 0} e^{f(x)} = e^{\lim_{x \to 0} f(x)}$$

$$\lim_{x \to 0} x \log x = \lim_{x \to 0} \frac{\log x}{\frac{1}{x}} =$$

$$= \lim_{x \to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to 0} -x^2 \cdot \frac{1}{x} = -x \to 0$$

$\lim\limits_{x \to 0} x^x = 1$

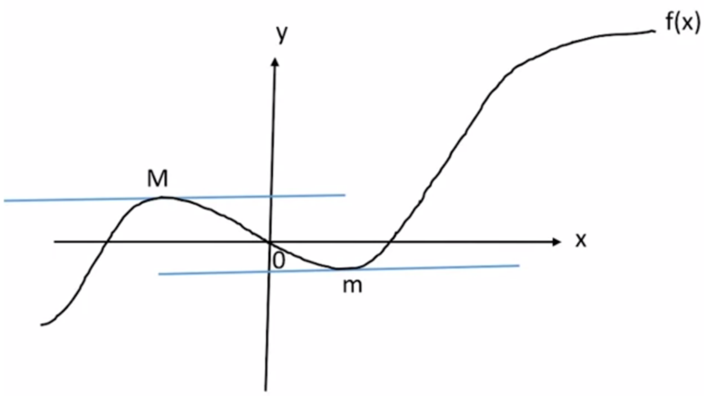
$$\lim_{n \to \infty} \frac{n^3}{e^n} \to \frac{x^3}{e^x}$$

$$\lim_{x \to \infty} \frac{x^3}{e^x} = \lim_{x \to \infty} \frac{3x^2}{e^x} =$$

$$= \lim_{x \to \infty} \frac{6x}{e^x} = \lim_{x \to \infty} \frac{6}{e^x} = 0$$

$$\boxed{\lim_{n \to \infty} \frac{n^3}{e^n} = 0}$$

## Maxima and minima of function f(x)

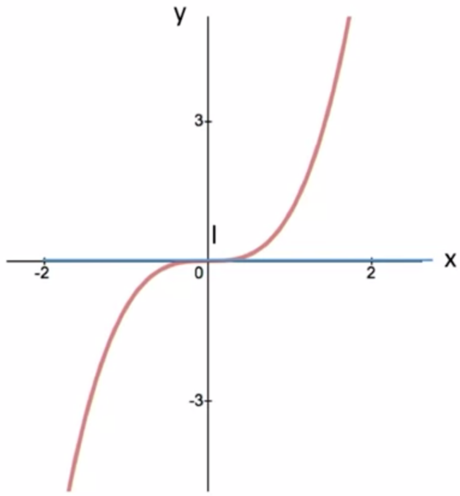



The slope of tangent line in M and m is zero (line is horizontal -> slope=tan(0)=0)

→ f'(x)=0 in points of local Max and min

- Because the derivative alone equaling zero not sufficient enough to determine the minima and maxima of the function

# Inflection points



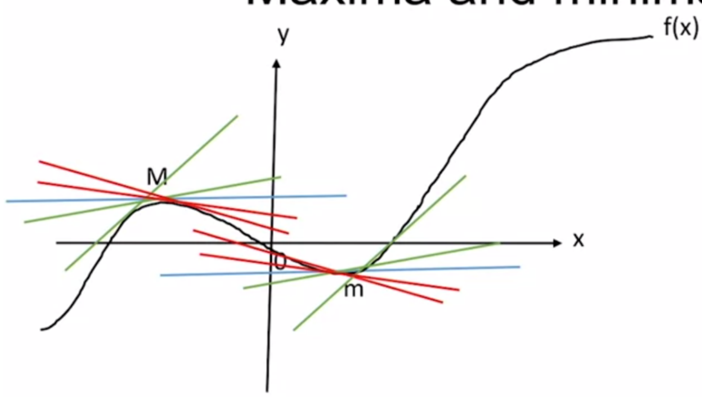

$y=f(x)=x^3 \quad (d(x^n)/dx \to nx^{n-1})$
$f'(x)=3x^2$
$f'(x)=0 \; 3x^2=0 \to x=0$
$f''(x)=6x \to \; f''(x=0)=0$

Inflection point
neither Max nor min
marks change of concavity

$f'(x)=0$ necessary but not sufficient condition
for x to be either local Max or min of the function

Points where $f''(x)=0$ are called inflection points
In which the concavity of the function changes

# Maxima and minima of function f(x)



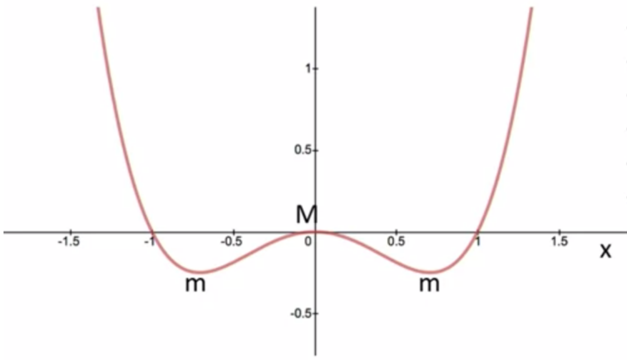

Close to M slope of tangent is positive
first (green) then negative (red)
$\rightarrow$ the slope decreases

$\rightarrow f''(x)<0$ in a Maximum

In m slope of tangent is negative first
(red) then positive(green)
$\rightarrow$ the slope increases

$\rightarrow f''(x)>0$ in a Minimum

# Maxima and minima of function f(x)




$y=f(x)=x^4-x^2$

$f'(x)=4x^3-2x$

$f'(x)=2x(2x^2-1) \rightarrow x=0\ ;\ x=\pm 1/\sqrt{2}$

$f''(x)=12x^2-2$

$f''(x=0)=-2<0 \rightarrow$ Maximum

$f''(x=1/\sqrt{2})=f''(x=-1/\sqrt{2})=12/2-2=4>0$

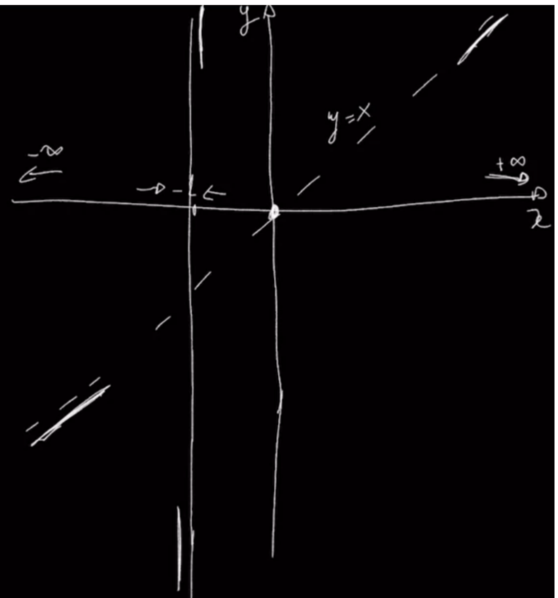
minima

$$f(x)=\frac{x^2}{x+1}$$

$$\begin{cases} \lim\limits_{x\to+\infty} \dfrac{x^2}{x+1} \approx \dfrac{x^2}{x}=x & x+1\approx x \\ \lim\limits_{x\to-\infty} \dfrac{x^2}{x+1} \approx \dfrac{x^2}{x}=x & x+1\approx x \end{cases}$$

$x=-1$ V. asym.

$(x=0, y=0)$




$$f'(x)=\frac{d}{dx}\left(\frac{x^2}{x+1}\right)$$

$$f(x)=\frac{h(x)}{g(x)} \rightarrow f'(x)=\boxed{\frac{h'(x)g(x)-h(x)g'(x)}{(g(x))^2}}$$

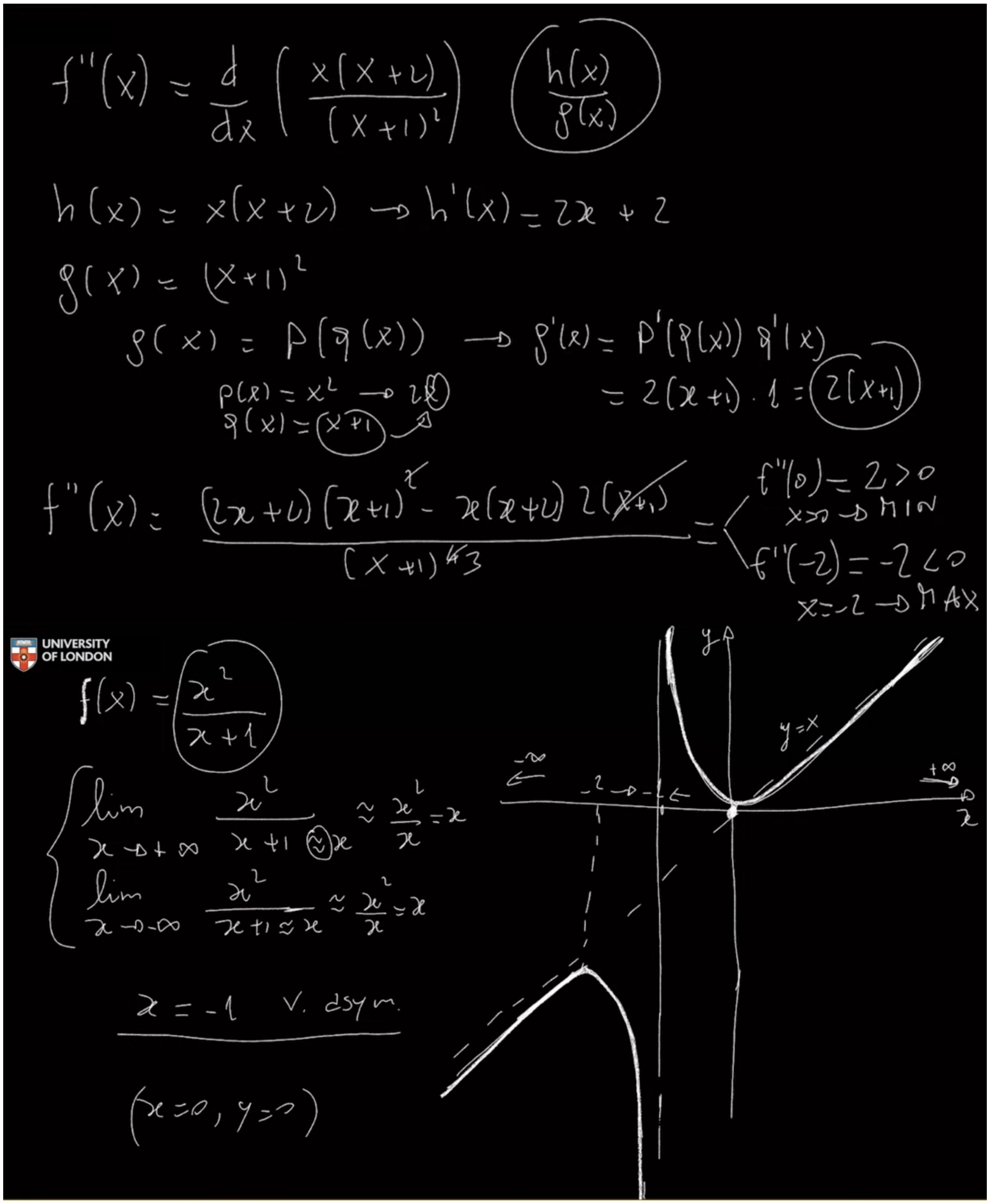
$h(x)=x^2 \longrightarrow h'(x)=2x$

$g(x)=x+1 \longrightarrow g'(x)=1$

$$f'(x)=\frac{2x\cdot(x+1)-x^2\cdot 1}{(x+1)^2}=\frac{x^2+2x}{(x+1)^2}=0\,?$$

$x(x+2)=0 \begin{cases} x=0 \\ x=-2 \end{cases}$

MAX?
MIN?
INFLECTION?




## Summary

In this week, we learned about the properties of the derivative, L'Hopital's Rule, the relation of the minima/maxima of a function to its first and second derivative, and finally points of inflection.