Week 9 Graph Sketching & Kinematics Reading Note 3

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Cornell Notes	**Topic:** Graph Sketching & Kinematics Reading	Course: BSc Computer Science Class: Computational Mathematics[Reading] Date: May 21, 2020

Essential Question:

What is a function and what are its applications to kinematics (simple motion)?

Questions/Cues:

- How do we calculate average speed?
- What are distance-time graphs?
- What is instantaneous speed?
- What is average velocity?
- What are displacement-time graphs?
- What is acceleration?
- What are velocity-time graphs?
- What is the area underneath the graph and how is it related to an object's displacement?
- What are the suvat equations?
- What is stopping distance?
- What is acceleration due to gravity or the acceleration of free fall?

Notes

Calculating average speed

The **average speed** v of an object can be calculated from the distance x travelled and the time t taken using the equation

$$\text{average speed} = \frac{\text{distance travelled}}{\text{time taken}}$$

You can write this in algebraic form as

$$v = \frac{\Delta x}{\Delta t}$$

The Greek capital letter Δ (delta) means 'change in'. From the SI base units for distance and time, the unit of speed is $\mathrm{m\,s^{-1}}$.

Worked example: Fine or not?

A car travels 2.5 km in 1 minute 22 seconds. The average speed limit for the road is 50 mph ($22\,\mathrm{m\,s^{-1}}$). Has the driver exceeded this average speed limit?

Step 1: Identify the equation and list the known values.

$$v = \frac{\Delta x}{\Delta t}$$

$$\Delta x = 2500\,\mathrm{m},\ \Delta t = 1\ \text{minute}\ 22\ \text{seconds} = 60 + 22 = 82\,\mathrm{s}$$

Step 2: Substitute the values into the equation and calculate the answer.

$$v = \frac{2500}{82} = 30.49\,\mathrm{m\,s^{-1}}$$
$$v = 30\,\mathrm{m\,s^{-1}}\ (2\ \text{s.f.})$$

Yes, the driver has exceeded the average speed limit.

Distance–time graphs

Graphs of distance against time are used to represent the motion of objects.

- Distance is plotted on the y-axis (vertical axis).
- Time is plotted on the x-axis (horizontal axis).

In a distance–time graph, a stationary object is represented by a horizontal straight line. An object moving at a **constant speed** is represented by a straight, sloping line. The **gradient** of that line is equal to the distance travelled divided by the time taken, $\Delta x/\Delta t$, in other words, to the speed of the object.

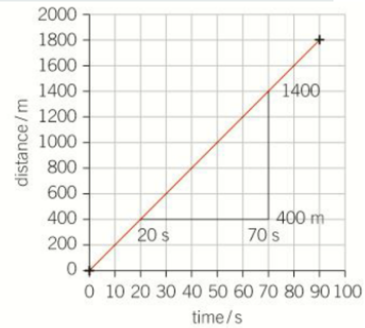
- speed = gradient of a distance–time graph

In Figure 2

change in distance $\Delta x = 1400 - 400 = 1000\,\mathrm{m}$

change in time $\Delta t = 70 - 20 = 50\,\mathrm{s}$

speed $v = \frac{\Delta x}{\Delta t} = \frac{1\,000}{50} = 20\,\mathrm{m\,s^{-1}}$



▲ **Figure 2** *A distance–time graph for an object moving at constant speed – the gradient of the graph represents the speed*

The vector nature of velocity

Displacement s is a vector quantity, unlike distance, which is scalar. Displacement has both magnitude and direction. Speed is a scalar quantity calculated from distance, but velocity is a vector quantity calculated from displacement. The **average velocity** v of an object can be calculated from the change in displacement and the time taken.

$$\text{average velocity} = \frac{\text{change in displacement}}{\text{time taken}}$$

You can write this in algebraic form as

$$v = \frac{\Delta s}{\Delta t}$$

where Δs is the change in displacement and Δt is the time taken. The SI unit for velocity is $\mathrm{m\,s^{-1}}$.

The worked example below shows that average speed and average velocity are very different quantities.

Worked example: Speed and velocity

a Leeds is about 21 km south of Harrogate, but the distance by road is about 24 km (Figure 2). It takes 37 minutes to travel from Harrogate to Leeds by road. Calculate the average speed and the average velocity.

Step 1: Identify the equations needed.

$$\text{average speed } v = \frac{\Delta x}{\Delta t}, \text{ average velocity } v = \frac{\Delta s}{\Delta t}$$

Step 2: Substitute the values in SI units into the equations and calculate the answers.

$$\text{time taken } \Delta t = 60 \times 37 = 2220\,\mathrm{s}$$

$$\text{average speed } v = \frac{\Delta x}{\Delta t} = \frac{24\,000}{2220} = 10.8\,\mathrm{m\,s^{-1}} = 11\,\mathrm{m\,s^{-1}} \text{ (2 s.f.)}$$

$$\text{average velocity } v = \frac{\Delta s}{\Delta t} = \frac{21\,000}{2220} = 9.5\,\mathrm{m\,s^{-1}} \text{ (2 s.f.)}$$

The magnitude of the average velocity is $9.5\,\mathrm{m\,s^{-1}}$ and its direction is due south from Harrogate.

b What would happen to the magnitude of the average velocity if the journey was from Harrogate to Leeds and then back to Harrogate?

The overall change in displacement would be zero and therefore the average velocity would be zero.

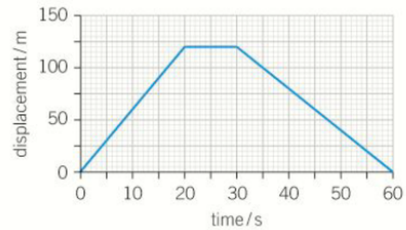
Displacement–time graphs

Graphs of displacement against time are used to represent the motion of objects.

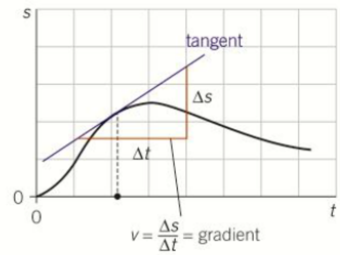
- Displacement is plotted on the y-axis (vertical axis).
- Time is plotted on the x-axis (horizontal axis).

Figure 3 shows the displacement–time graph for a car travelling along a straight road. The car is travelling at a constant velocity between $t = 0$ and $t = 20\,\text{s}$, as can be seen from the first straight-line section of the graph. The horizontal section of the graph between $t = 20\,\text{s}$ and $t = 30\,\text{s}$ shows that the displacement of the car remains constant. Therefore, the car must be stationary. After $t = 30\,\text{s}$, the graph is still a straight line but has a negative slope. The displacement of the car is getting smaller with time. The car must therefore be returning at a constant velocity.

You can determine the velocity of an object from the gradient of its displacement–time (s–t) graph. If the graph is not a straight line, draw a tangent to the graph, then calculate the gradient of this tangent for the instantaneous velocity, as illustrated in Figure 4.



▲ **Figure 3** *A displacement–time graph for a car journey*



▲ **Figure 4** *Velocity can be determined from the gradient of the displacement–time graph*

Worked example: Forwards and backwards

Use Figure 3 to determine the velocity of the car at $t = 10\,\text{s}$ and $t = 40\,\text{s}$.

Step 1: Identify the equation needed and how to obtain the values from the graph.

$$\text{velocity } v = \frac{\Delta s}{\Delta t}$$

The right-hand terms are equivalent to the change on the y-axis (s) divided by the change on the x-axis (t).

Step 2: Substitute the values into the equation and calculate the answer.

The velocity at $t = 10\,\text{s}$ can be determined from the gradient of the straight-line graph between $t = 0$ and $t = 20\,\text{s}$.

$$\text{velocity } v = \frac{\Delta s}{\Delta t} = \frac{120-0}{20-0} = 6.0\,\text{m}\,\text{s}^{-1}$$

The velocity at $t = 40\,\text{s}$ can be determined from the gradient of the straight-line graph between $t = 30\,\text{s}$ and $t = 60\,\text{s}$.

$$\text{velocity } v = \frac{\Delta s}{\Delta t} = \frac{0-120}{60-30} = -4.0\,\text{m}\,\text{s}^{-1}$$

The negative sign for the velocity shows that the car is travelling in the opposite direction to its motion between 0 and 20 s.

Determining acceleration

The **acceleration** of an object is defined as the rate of change of velocity. In mathematical form, the acceleration a is

$$a = \frac{\Delta v}{\Delta t}$$

where Δv is the change in velocity and Δt is the time taken for the change. The unit of acceleration is $\mathrm{m\,s^{-2}}$. Since acceleration is determined from velocity, it too is a vector quantity – it has magnitude and direction. A negative acceleration is often called deceleration.

Acceleration can be determined by calculation, or from a velocity–time (v–t) graph.

Calculating acceleration

You can calculate acceleration if you know the change in velocity of an object and the time taken for this change.

Worked example: 0 to 62, then slam the brakes on

a A Bugatti Veyron can accelerate from 0 to 100 km/h ($27.8\,\mathrm{m\,s^{-1}}$) in 2.46 s. Calculate its average acceleration.

Step 1: Identify the equation needed.

$$a = \frac{\Delta v}{\Delta t}$$

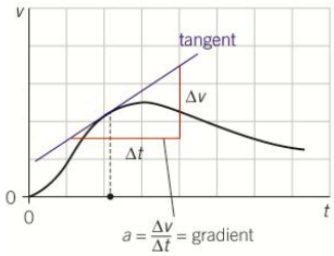
Step 2: Substitute the values into the equation and calculate the answer.

$$a = \frac{\Delta v}{\Delta t} = \frac{27.8-0}{2.46} = 13.3\,\mathrm{m\,s^{-2}}$$

b The car takes 2.34 s to stop from 100 km/h under braking. Calculate its acceleration, assuming that this is constant during braking.

$$a = \frac{\Delta v}{\Delta t} = \frac{0-27.8}{2.34} = -11.9\,\mathrm{m\,s^{-2}}$$

The negative sign means that the velocity of the car is decreasing over time – it is decelerating.



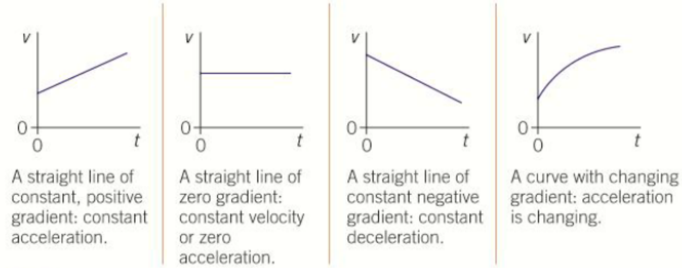
▲ **Figure 3** *Acceleration can be determined from the gradient of the velocity–time graph*

Velocity–time graphs

Since $a = \frac{\Delta v}{\Delta t}$, it follows that the acceleration of an object can be determined from the gradient of a velocity–time graph. You have seen how to determine gradients for straight-line graphs and for non-linear graphs in the previous topics in this chapter; the only difference in this case is that the y-axis of the graph represents velocity v rather than displacement s.

- acceleration = gradient of velocity–time graph

Figure 4 shows how the motion of an object can be deduced from the velocity–time graph.



▲ **Figure 4** *Interpreting velocity–time graphs*

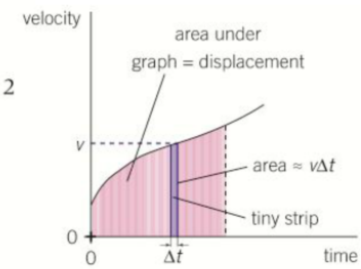
Area under the graph

In the previous topic you saw that acceleration can be determined from the gradient of a velocity–time graph. In addition, we can read the displacement of the object from the area under the graph. Figure 2 shows why this is so.

You will recall that the average velocity v is given by the equation

$$v = \frac{\Delta s}{\Delta t}$$

where Δs is the displacement in the time interval Δt. For the instantaneous velocity, assume that Δt is very small indeed – the velocity of the object is not going to change much. The change in the displacement $\Delta s \approx v\Delta t$. If you look at the graph in Figure 2, this is the area of the very thin rectangular strip marked under the graph. Therefore, the change in displacement is equal to the area of this strip. If you add similar strips for a longer interval of time, then clearly the area under the velocity–time graph is the total displacement of the object.



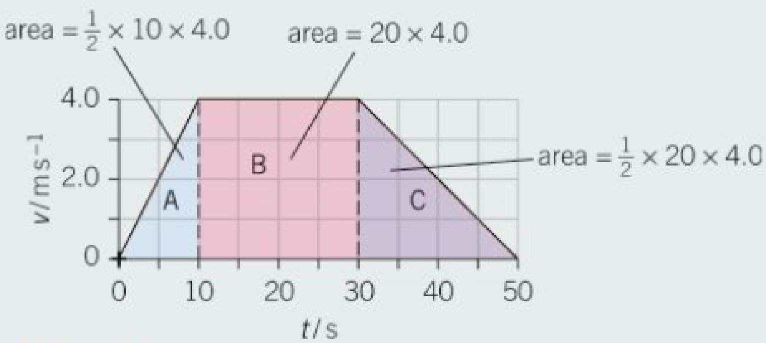
▲ **Figure 2** *The area under the velocity–time graph is equal to displacement*

Calculating displacement for constant accelerations

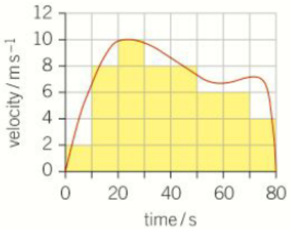
Displacement is easy to calculate when the acceleration is constant, because the areas can be broken down into rectangles and right-angled triangles. This is illustrated in the worked example below for the short journey of a cyclist.

Worked example: Displacement of a cyclist

The velocity–time graph for a cyclist travelling along a straight road is shown in Figure 3.



▲ **Figure 3**



▲ **Figure 4** *Calculating the area under a non-linear velocity–time graph*

Calculate the total distance travelled by the cyclist (the cyclist's displacement) in the period of 50 s.

Step 1: Identify the method needed.

distance travelled = area between the graph and the time axis

Step 2: Calculate the answer.

distance travelled = area of triangle **A** + area of rectangle **B** + area of triangle **C**

$$= \left(\frac{1}{2} \times 10 \times 4.0\right) + \left(20 \times 4.0\right) + \left(\frac{1}{2} \times 20 \times 4.0\right)$$

$$= 20 + 80 + 40$$

$$= 140\,\text{m}$$

Step 3: There is an alternative method to determine the area under the graph. Calculate the area of the trapezium using the formula

area = $\frac{1}{2}$ × (sum of the parallel sides) × vertical height

$$\text{distance} = \frac{1}{2} \times (50 + 20) \times 4.0 = 140\,\text{m}$$

Calculating displacement for changing accelerations

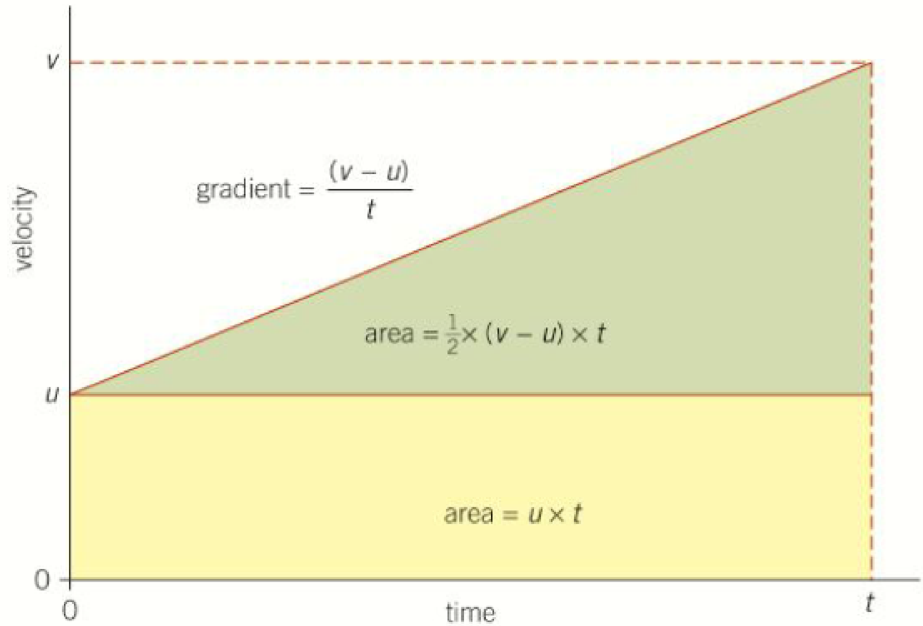
For non-linear velocity–time graphs, you can determine the area under the graph by counting squares. Taking Figure 4 as an example, you would start by counting the squares that are complete or nearly complete (yellow). Then count the remaining squares that lie mostly beneath the graph. Omit squares that are mostly above the graph.

Equations of motion: the *suvat* equations

You need four equations to calculate quantities involving motion in a *straight line* at a constant acceleration. These equations of motion are often informally referred to as the '*suvat* equations' after the symbols for the quantities involved.

Deriving the equations of motion

Figure 2 shows the velocity–time graph for an accelerating object. The initial velocity of the object is u. After a time t the final velocity of the object is v. The object has a constant acceleration a, as you can see from the straight-line graph.



▲ **Figure 2** *Velocity–time graph showing how the* suvat *equations are derived*

▼ **Table 1** *The* suvat *quantities*

Symbol	Quantity
s	displacement (or distance travelled)
u	initial velocity
v	final velocity
a	acceleration
t	time taken for the change in velocity

Equation without *s*

From the graph in Figure 2

$$a = \frac{\Delta v}{\Delta t} = \frac{v-u}{t}$$

This can be rearranged to give

$$\boxed{v = u + at} \qquad \textbf{(1)}$$

Equation without *v*

You will recall that the area under the velocity–time graph is equal to the displacement s.

- the rectangular area = ut
- the triangular area = $\frac{1}{2} \times (v-u) \times t$

From Equation **1** above, $(v - u) = at$. If you substitute this into the expression for the area of the triangle, you get $\frac{1}{2} \times at \times t$. With ut for the area of the rectangle, this gives the total area s.

$$\boxed{s = ut + \frac{1}{2}at^2} \qquad (2)$$

Study tip

The four *s u v a t* equations are

$v = u + at$

$s = ut + \frac{1}{2}at^2$

$s = \frac{1}{2}(u + v)t$

$v^2 = u^2 + 2as$

Equation without *a*

If you treat the area under the graph as the area of a trapezium (with u and v as the parallel sides of the trapezium, and t as the perpendicular separation between them), this becomes

$$\boxed{s = \frac{1}{2}(u + v)t} \qquad (3)$$

In other words, the displacement s is the average velocity, $\left(\frac{u+v}{2}\right)$, multiplied by the time t.

Equation without *t*

You need Equations **1** and **3** to derive the last useful equation of motion.

According to Equation **1** the time t is given by the equation

$$t = \frac{(v - u)}{a}$$

This equation for t can be substituted into Equation **3** to give

$$s = \frac{1}{2}(u + v) \times \frac{(v - u)}{a}$$

Rearranging this gives

$$(u + v)(v - u) = 2as$$

$$v^2 - u^2 = 2as$$

$$\boxed{v^2 = u^2 + 2as} \qquad (4)$$

Equation without *u*

There is one more equation, $s = vt - \frac{1}{2}at^2$.

See if you can derive it from the velocity–time graph.

Worked example: Car accelerating from a standing start

A car on a straight road accelerates from rest to a velocity of $12\,\mathrm{m\,s^{-1}}$ in a time of 9.0 s. Calculate the acceleration of the car and the distance travelled in this time.

Step 1: Write down the quantities given in the order *suvat* and identify the equations needed.

$$s = ?,\ u = 0,\ v = 12\,\mathrm{m\,s^{-1}},\ a = ?,\ t = 9.0\,\mathrm{s}$$

Use Equation **1** to calculate the acceleration a and Equation **3** to calculate s.

Step 2: Substitute the values into the equations and calculate the answers.

Acceleration can be calculated from Equation **1**.

$$v = u + at$$

$$12 = 0 + a \times 9.0$$

$$a = \frac{12 - 0}{9.0} = 1.33\,\text{m}\,\text{s}^{-2}$$

The distance travelled is the displacement *s* along the straight road. Equation **3** can be used to calculate *s*.

$$s = \frac{1}{2}(v + u)t$$

$$s = \frac{1}{2} \times (12 + 0) \times 9.0$$

$$s = 54\,\text{m}$$

Worked example: Particle accelerating

A particle travels a distance of 16 m as it accelerates from $4.0\,\text{m}\,\text{s}^{-1}$ to $12\,\text{m}\,\text{s}^{-1}$. Calculate its acceleration.

Step 1: Again, start with the *suvat* values and identify the equation needed.

$s = 16\,\text{m}$, $u = 4.0\,\text{m}\,\text{s}^{-1}$, $v = 12\,\text{m}\,\text{s}^{-1}$, $a = ?$, $t = ?$

The equation for *v*, *u*, and *s* is Equation **4.** We can use this to calculate the acceleration *a*.

Step 2: Substitute the values into the equation and calculate the answer.

$$v^2 = u^2 + 2as$$

$$12^2 = 4.0^2 + 2 \times a \times 16$$

$$a = \frac{12^2 - 4.0^2}{2 \times 16} = 4.0\,\text{m}\,\text{s}^{-2}$$

Worked example: Falling to Earth

An apple falls from rest in a tree towards soft ground from a height of 1.50 m. Objects falling to Earth have an acceleration (free fall) of $9.81\,\mathrm{m\,s^{-2}}$. Calculate the time taken for the apple to reach the ground. Assume air resistance has negligible effect on the motion.

Step 1: List the *suvat* values and identify the equation needed.

$$s = 1.50\,\mathrm{m},\ u = 0\,\mathrm{m\,s^{-1}},\ a = 9.81\,\mathrm{m\,s^{-2}},\ t = ?$$

Use Equation **2** to calculate the time t.

Step 2: Substitute the values into the equation and calculate the answer.

$$s = ut + \frac{1}{2}at^2$$

$$1.50 = (0 \times t) + \frac{1}{2} \times 9.81 \times t^2$$

$$t^2 = \frac{2 \times 1.50}{9.81} = 0.306\ (3\ \text{s.f.})$$

$$t = 0.553\,\mathrm{s}$$

In this calculation, the equation is effectively $s = \frac{1}{2}at^2$, because $u = 0$.

Worked example: What goes up

A paper clip is flicked vertically up in the air at $6.0\,\mathrm{m\,s^{-1}}$ (Figure 4). Calculate its maximum height. Assume air resistance has negligible effect on the motion.

Step 1: List the known *suvat* values and identify the equation needed.

The paper clip will decelerate as it moves vertically, therefore, a must be *negative*. At maximum height it will stop momentarily, therefore, $v = 0$.

$$s = ?,\ u = 6.0\,\mathrm{m\,s^{-1}},\ v = 0\,\mathrm{m\,s^{-1}},\ a = -9.81\,\mathrm{m\,s^{-2}}$$

The equation for v, u, and s is Equation **4.** We can use this to calculate the height s.

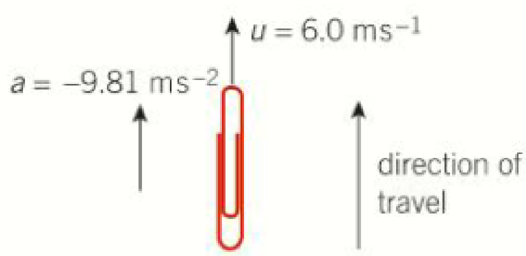
Step 2: Substitute the values into the equation and calculate the answer.

$$v^2 = u^2 + 2as$$

$$0 = u^2 + 2as$$

$$s = -\frac{u^2}{2a} = -\frac{6.0^2}{2 \times -9.81}$$

$$s = 1.83\,\mathrm{m} = 1.8\,\mathrm{m}\ (2\ \text{s.f.})$$



▲ Figure 4

Components of stopping distances

The **stopping distance** is the total distance travelled from when the driver first sees a reason to stop, to when the vehicle stops. It has two components:

- **thinking distance**, the distance travelled between the moment when you first see a reason to stop, to the moment when you use the brake
- **braking distance**, the distance travelled from the time the brake is applied until the vehicle stops.

Many factors influence these distances, including the speed of the vehicle, the condition of the brakes, tyres, and road, the weather conditions, and the alertness of the driver.

Thinking distance

It takes time for a driver to react to a need to stop. For a vehicle moving at constant speed

$$\text{thinking distance} = \text{speed} \times \text{reaction time}$$

Worked example: Reaction time

In the UK Highway Code, the thinking distance at 30 mph ($13.4\,\mathrm{m\,s^{-1}}$) is shown as 9.0 m. Calculate the corresponding reaction time.

Step 1: Identify the equation needed.

$$\text{reaction time} = \frac{\text{thinking distance}}{\text{speed}}$$

Step 2: Substitute the values into the equation and calculate the answer.

$$\text{reaction time} = \frac{9.0}{13.4} = 0.67\,\mathrm{s}\ (2\ \text{s.f.})$$

The greater the speed or the reaction time, the further a vehicle will travel before its driver applies the brakes. Assuming a constant reaction time of 0.67 s, the thinking distance will be about 21 m at the UK national speed limit of 70 mph ($31.1\,\mathrm{m\,s^{-1}}$). This is equivalent to the total length of five average cars lined up.

Braking distance

In the UK Highway Code, the braking distance at 30 mph ($13.4\,m\,s^{-1}$) is shown as 14.0 m. If you assume constant deceleration from $13.4\,m\,s^{-1}$ to $0\,m\,s^{-1}$, you can use one of the equations of motion to determine the magnitude of the deceleration.

▼ **Table 1** *Thinking, braking, and overall stopping distances according to the Highway Code*

Speed / mph	20	30	40	50	60	70
Speed / $m\,s^{-1}$	8.9	13.4	17.8	22.2	26.7	31.1
Thinking distance / m	6	9	12	15	18	21
Braking distance / m	6	14	24	38	55	75
Stopping distance / m	12	23	36	53	73	96

Worked example: Braking distance

Step 1: Once again, start with the *suvat* quantities and identify the equation you need.

$s = 14.0\,\text{m}$

$u = 13.4\,\text{m}\,\text{s}^{-1}$

$v = 0$

$a = ?$

Use the equation $v^2 = u^2 + 2as$.

Step 2: Substitute the values into the equation and calculate the answer.

$$a = \frac{v^2 - u^2}{2s}$$

$v = 0$

Therefore

$$a = -\frac{u^2}{2s}$$

$$= -\frac{13.4^2}{2 \times 14.0}$$

$$= -6.4\,\text{m}\,\text{s}^{-2} \text{ (2 s.f.)}$$

The magnitude of the deceleration is about $6.4\,\text{m}\,\text{s}^{-2}$.

Acceleration due to gravity

Objects with mass exert a gravitational force on each other. The Earth is so massive that its gravitational pull is enough to keep us on its surface. An object released on the Earth will accelerate vertically downwards towards the centre of the Earth. When an object is accelerating under gravity, with no other force acting on it, it is said to be in **free fall**. The **acceleration of free fall** is denoted by the label g (not g, which means grams). Since g is an acceleration, it has the unit m s^{-2}.

Value for *g* close to Earth's surface

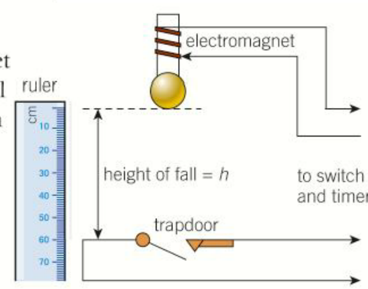
The value for g varies depending upon factors including altitude, latitude, and the geology of an area. For example, g is $9.825\,m\,s^{-2}$ in Helsinki, $9.816\,m\,s^{-2}$ in London, but only $9.776\,m\,s^{-2}$ in Singapore. A value of $9.81\,m\,s^{-2}$ is generally used.

Determining *g*

The basic idea behind determining g in the laboratory is to drop a heavy ball over a known distance and time its descent. The problem is that it all happens very quickly, about 0.45 s for a 1.0 m fall. Methods for measuring g are described here.

Electromagnet and trapdoor

An electromagnet holds a small steel ball above a trapdoor (Figure 2). When the current is switched off, a timer is triggered, the electromagnet demagnetises, and the ball falls. When it hits the trapdoor, the electrical contact is broken and the timer stops. The value for g is calculated from the height of the fall and the time taken.



▲ **Figure 2** *Determining* g *using an electromagnet and timer*

Worked example: An experimental value for g

A ball drops 85.6 cm from an electromagnet to a trapdoor in 0.421 s. Use this information to determine a value for g.

Step 1: List the *suvat* values and identify the equation needed.

$s = 0.856\,\text{m}$, $u = 0\,\text{m}\,\text{s}^{-1}$, $a = g = ?$, $t = 0.421\,\text{s}$ (the distance must be converted into the SI unit m.)

Using $s = ut + \frac{1}{2}at^2$, we have

$$s = \frac{1}{2}at^2 \text{ because } u = 0$$

Step 2: Substitute the values into the equation and calculate the answer.

$$a = \frac{2s}{t^2} = \frac{2 \times 0.856}{0.421^2}$$

$$a = 9.66\,\text{m}\,\text{s}^{-2} \text{ (3 s.f.)}$$

The inaccuracy in this experiment is caused by the presence of air resistance and the slight delay in the release of the steel ball because of the finite time taken for the magnet to demagnetise. The accuracy may be improved by using a heavier ball and a much longer drop.

Summary

In this week, we learned about average/instantaneous speed, average velocity, distance/displacement time graphs, velocity-time graphs, the suvat equations, the acceleration due to gravity and so much more.