Week 15 Limits and differentiation Reading note

Notebook:	Computational Mathematics
Created:	2020-04-21 2:48 PM
Updated:	2020-07-22 5:24 PM
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URL:	https://www.coursera.org/learn/uol-cm1015-computational-mathematics/home/week/13

Cornell Notes	**Topic:** Limits and differentiation	Course: BSc Computer Science Class: Computational Mathematics[Reading] Date: July 22, 2020

Essential Question:

What are limits and derivatives and how do they relate to the notion of continuity of a function?

Questions/Cues:

- What is the definition of a limit?
- Under which conditions do limits not exist?
- What are some basic limits?
- What are the properties of limits?
- What are the limits of Polynomial and Rational Functions?
- What is the dividing out technique?
- What is the rationalization technique?
- What are one-sided limits?
- How do we confirm the existence of a limit?
- What is the squeeze theorem?
- What is calculus?
- What is the tangent line to a graph?
- What is the definition of the slope of a graph?
- What is the definition of a derivative?
- What is the definition of limits at infinity?
- How are limits at infinity defined for rational functions?
- What is the limit of a sequence?
- What are the summation formulas and properties?
- How do we calculate the exact area of a plane region?

Notes

- Limit =
  If $f(x)$ becomes arbitrarily close to a unique number L as x approaches c from either side, then the limit of $f(x)$ as x approaches c is L. This is written as
  $$\lim_{x \to c} f(x) = L.$$

- Or alternatively as, $f(x) \rightarrow L$ as $x \rightarrow c$, which is read as " $f(x)$ approaches L as x approaches c."

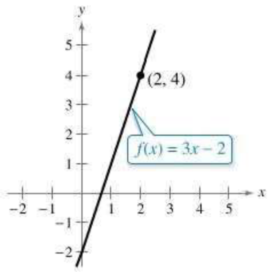


Figure 12.1

EXAMPLE 2 **Estimating a Limit Numerically**

Use a table to estimate the limit numerically: $\lim_{x \to 2} (3x - 2)$.

Solution Let $f(x) = 3x - 2$. Then construct a table that shows values of $f(x)$ for two sets of x-values—one that approaches 2 from the left and one that approaches 2 from the right.

x	1.9	1.99	1.999	2.0	2.001	2.01	2.1
$f(x)$	3.700	3.970	3.997	?	4.003	4.030	4.300

From the table, it appears that the closer x gets to 2, the closer $f(x)$ gets to 4. So, estimate the limit to be 4. Figure 12.1 verifies this conclusion.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

Use a table to estimate the limit numerically: $\lim_{x \to 3} (3 - 2x)$.

In Figure 12.1, note that the graph of $f(x) = 3x - 2$ is continuous. For graphs that are not continuous, finding a limit can be more challenging.

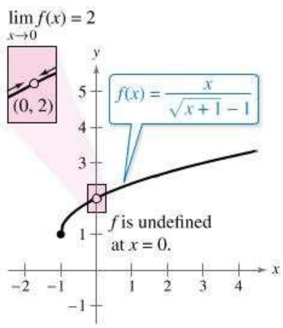


Figure 12.2

EXAMPLE 3 **Estimating a Limit Numerically**

Use a table to estimate the limit numerically.

$$\lim_{x \to 0} \frac{x}{\sqrt{x+1} - 1}$$

Solution Let $f(x) = x/(\sqrt{x+1} - 1)$. Then construct a table that shows values of $f(x)$ for two sets of x-values—one that approaches 0 from the left and one that approaches 0 from the right.

x	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01
$f(x)$	1.99499	1.99950	1.99995	?	2.00005	2.00050	2.00499

From the table, it appears that the limit is 2. Figure 12.2 verifies this conclusion.

REMARK In Example 3, note that $f(0)$ is undefined, so it is not possible to *reach* the limit. In Example 2, note that $f(2) = 4$, so it *is* possible to reach the limit.

In Example 3, note that $f(x)$ has a limit when $x \to 0$ even though the function is not defined when $x = 0$. This often happens, and it is important to realize that *the existence or nonexistence of $f(x)$ at $x = c$ has no bearing on the existence of the limit of $f(x)$ as x approaches c.*

EXAMPLE 4 **Estimating a Limit**

Estimate the limit: $\lim_{x \to 1} \frac{x^3 - x^2 + x - 1}{x - 1}$.

Numerical Solution

Let $f(x) = (x^3 - x^2 + x - 1)/(x - 1)$. Then construct a table that shows values of $f(x)$ for two sets of x-values—one that approaches 1 from the left and one that approaches 1 from the right.

x	0.9	0.99	0.999	1.0
$f(x)$	1.8100	1.9801	1.9980	?

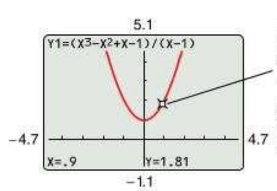
x	1.001	1.01	1.1
$f(x)$	2.0020	2.0201	2.2100

From the table, it appears that the limit is 2.

Graphical Solution

Use a graphing utility to graph

$$f(x) = (x^3 - x^2 + x - 1)/(x - 1).$$



Use the *trace* feature to determine that as x gets closer and closer to 1, $f(x)$ gets closer and closer to 2 from the left and from the right.

From the graph, estimate the limit to be 2. As you use the *trace* feature, notice that there is no value given for y when $x = 1$, and that there is a hole or break in the graph at $x = 1$.

EXAMPLE 5 Using a Graph to Find a Limit

Find the limit of $f(x)$ as x approaches 3.

$$f(x) = \begin{cases} 2, & x \neq 3 \\ 0, & x = 3 \end{cases}$$

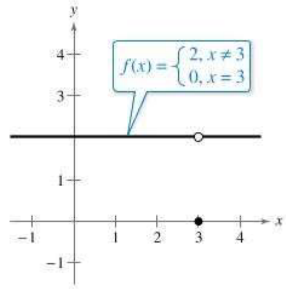


Figure 12.3

Solution Because $f(x) = 2$ for all x other than $x = 3$ and the value of $f(3)$ is immaterial, it follows that the limit is 2 (see Figure 12.3). So, write

$$\lim_{x \to 3} f(x) = 2.$$

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com

Find the limit of $f(x)$ as x approaches 2.

$$f(x) = \begin{cases} -3, & x \neq 2 \\ 0, & x = 2 \end{cases}$$

In Example 5, the fact that $f(3) = 0$ has no bearing on the existence or value of the limit as x approaches 3. For example, if the function were defined as

$$f(x) = \begin{cases} 2, & x \neq 3 \\ 4, & x = 3 \end{cases}$$

then the limit as x approaches 3 would still equal 2.

Limits That Fail to Exist

Next, you will examine some limits that fail to exist.

EXAMPLE 6 Comparing Left and Right Behavior

Show that the limit does not exist.

$$\lim_{x \to 0} \frac{|x|}{x}$$

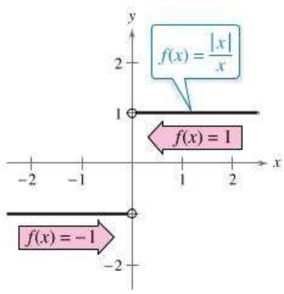


Figure 12.4

Solution Consider the graph of $f(x) = |x|/x$, shown in Figure 12.4. Notice that for positive x-values

$$\frac{|x|}{x} = 1, \quad x > 0$$

and for negative x-values

$$\frac{|x|}{x} = -1, \quad x < 0.$$

This means that no matter how close x gets to 0, there are both positive and negative x-values that yield $f(x) = 1$ and $f(x) = -1$, respectively. This implies that the limit does not exist.

EXAMPLE 7 Unbounded Behavior

Discuss the existence of the limit.

$$\lim_{x \to 0} \frac{1}{x^2}$$

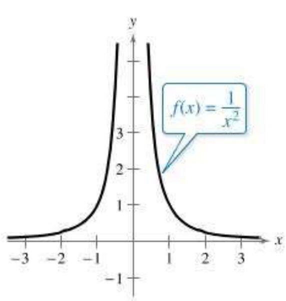


Figure 12.5

Solution Let $f(x) = 1/x^2$. In Figure 12.5, note that as x approaches 0 from either the right or the left, $f(x)$ increases without bound. This means that choosing x close enough to 0 enables you to force $f(x)$ to be as large as you want. For example, $f(x)$ is larger than 100 when you choose x that is within $\frac{1}{10}$ of 0. That is,

$$0 < |x| < \frac{1}{10} \quad \Rightarrow \quad f(x) = \frac{1}{x^2} > 100.$$

Similarly, you can force $f(x)$ to be larger than 1,000,000 by choosing x that is within $\frac{1}{1000}$ of 0, as shown below.

$$0 < |x| < \frac{1}{1000} \quad \Rightarrow \quad f(x) = \frac{1}{x^2} > 1{,}000{,}000$$

Because $f(x)$ is not approaching a unique real number L as x approaches 0, the limit does not exist.

EXAMPLE 8 **Oscillating Behavior**

See LarsonPrecalculus.com for an interactive version of this type of example.

Discuss the existence of the limit.

$$\lim_{x \to 0} \sin \frac{1}{x}$$

Solution Let $f(x) = \sin(1/x)$. Notice in Figure 12.6 that as x approaches 0, $f(x)$ oscillates between -1 and 1. So, the limit does not exist because no matter how close you are to 0, it is possible to choose values of x_1 and x_2 such that $\sin(1/x_1) = 1$ and $\sin(1/x_2) = -1$, as shown in the table.

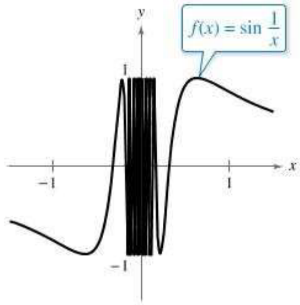


Figure 12.6

x	$-\frac{2}{\pi}$	$-\frac{2}{3\pi}$	$-\frac{2}{5\pi}$	0	$\frac{2}{5\pi}$	$\frac{2}{3\pi}$	$\frac{2}{\pi}$
$\sin \frac{1}{x}$	-1	1	-1	?	1	-1	1

Conditions Under Which Limits Do Not Exist

The limit of $f(x)$ as $x \to c$ does not exist when any of the conditions listed below are true.

1. $f(x)$ approaches a different number from the right side of c than it approaches from the left side of c. Example 6
2. $f(x)$ increases or decreases without bound as x approaches c. Example 7
3. $f(x)$ oscillates between two fixed values as x approaches c. Example 8

Properties of Limits and Direct Substitution

Sometimes, as in Example 2, the limit of $f(x)$ as $x \to c$ is $f(c)$. In such cases, the limit can be evaluated by **direct substitution.** That is,

$$\lim_{x \to c} f(x) = f(c). \qquad \text{Substitute } c \text{ for } x.$$

There are many "well-behaved" functions, such as polynomial functions and rational functions with nonzero denominators, that have this property. The list below includes some basic limits.

Basic Limits

Let b and c be real numbers and let n be a positive integer.

1. $\lim_{x \to c} b = b$ Limit of a constant function
2. $\lim_{x \to c} x = c$ Limit of the identity function
3. $\lim_{x \to c} x^n = c^n$ Limit of a power function
4. $\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$, valid for all c when n is odd and valid for $c > 0$ when n is even Limit of a radical function

Properties of Limits

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the limits

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = K.$$

1. Scalar multiple: $\lim_{x\to c} [bf(x)] = bL$
2. Sum or difference: $\lim_{x\to c} [f(x) \pm g(x)] = L \pm K$
3. Product: $\lim_{x\to c} [f(x)g(x)] = LK$
4. Quotient: $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}, \quad K \neq 0$
5. Power: $\lim_{x\to c} [f(x)]^n = L^n$

EXAMPLE 9 Direct Substitution and Properties of Limits

Find each limit.

a. $\lim_{x\to 4} x^2$ **b.** $\lim_{x\to 4} 5x$ **c.** $\lim_{x\to \pi} \dfrac{\tan x}{x}$

d. $\lim_{x\to 9} \sqrt{x}$ **e.** $\lim_{x\to \pi} (x \cos x)$ **f.** $\lim_{x\to 3} (x + 4)^2$

Solution Use the properties of limits and direct substitution to evaluate each limit.

a. $\lim_{x\to 4} x^2 = (4)^2 = 16$ Use direct substitution.

b. $\lim_{x\to 4} 5x = 5 \lim_{x\to 4} x = 5(4) = 20$ Use the Scalar Multiple Property and direct substitution.

c. $\lim_{x\to \pi} \dfrac{\tan x}{x} = \dfrac{\lim_{x\to\pi} \tan x}{\lim_{x\to\pi} x} = \dfrac{\tan \pi}{\pi} = \dfrac{0}{\pi} = 0$ Use the Quotient Property and direct substitution.

d. $\lim_{x\to 9} \sqrt{x} = \sqrt{9} = 3$ Use direct substitution.

e. $\lim_{x\to \pi} (x \cos x) = \left(\lim_{x\to\pi} x\right)\left(\lim_{x\to\pi} \cos x\right) = \pi(\cos \pi) = -\pi$ Use the Product Property and direct substitution.

f. $\lim_{x\to 3} (x + 4)^2 = \left[\left(\lim_{x\to 3} x\right) + \left(\lim_{x\to 3} 4\right)\right]^2 = (3 + 4)^2 = 49$ Use the Power and Sum Properties and direct substitution.

Limits of Polynomial and Rational Functions

1. If p is a polynomial function and c is a real number, then

$$\lim_{x\to c} p(x) = p(c).$$

2. If r is a rational function $r(x) = p(x)/q(x)$, and c is a real number such that $q(c) \neq 0$, then

$$\lim_{x\to c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

EXAMPLE 10 **Evaluating Limits by Direct Substitution**

Find each limit.

a. $\lim_{x \to -1} (x^2 + x - 6)$ **b.** $\lim_{x \to -1} \frac{x^3 - 5x}{x}$ **c.** $\lim_{x \to -1} \frac{x^2 + x - 6}{x + 3}$

Solution The first function is a polynomial function. The second and third functions are rational functions (with nonzero denominators at $x = -1$). So, you can evaluate the limits by direct substitution.

a. $\lim_{x \to -1} (x^2 + x - 6) = (-1)^2 + (-1) - 6 = -6$

b. $\lim_{x \to -1} \frac{x^3 - 5x}{x} = \frac{(-1)^3 - 5(-1)}{-1} = -\frac{4}{1} = -4$

c. $\lim_{x \to -1} \frac{x^2 + x - 6}{x + 3} = \frac{(-1)^2 + (-1) - 6}{-1 + 3} = -\frac{6}{2} = -3$

Dividing Out Technique

In Section 12.1, you studied several types of functions whose limits can be evaluated by direct substitution. In this section, you will study several techniques for evaluating limits of functions for which direct substitution fails. For example, consider the limit

$$\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3}.$$

Direct substitution produces 0 in both the numerator and denominator.

$(-3)^2 + (-3) - 6 = 0$ Numerator is 0 when $x = -3$.

$-3 + 3 = 0$ Denominator is 0 when $x = -3$.

The resulting fraction, $\frac{0}{0}$, has no meaning as a real number. It is called an **indeterminate form** because you cannot, from the form alone, determine the limit. By using a table, however, it appears that the limit of the function as x approaches -3 is -5.

x	-3.01	-3.001	-3.0001	-3	-2.9999	-2.999	-2.99
$\frac{x^2 + x - 6}{x + 3}$	-5.01	-5.001	-5.0001	?	-4.9999	-4.999	-4.99

When you attempt to evaluate a limit of a rational function by direct substitution and encounter the indeterminate form $\frac{0}{0}$, the numerator and denominator must have a common factor. After factoring and dividing out, use direct substitution again. Examples 1 and 2 show this **dividing out technique.**

EXAMPLE 1 Dividing Out Technique

See LarsonPrecalculus.com for an interactive version of this type of example.

Find the limit: $\lim_{x \to -3} \dfrac{x^2 + x - 6}{x + 3}$.

Solution From the discussion above, you know that direct substitution fails. So, begin by factoring the numerator and dividing out any common factors.

$$\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3} = \lim_{x \to -3} \frac{(x - 2)(x + 3)}{x + 3} \qquad \text{Factor numerator.}$$

$$= \lim_{x \to -3} (x - 2) \qquad \text{Divide out common factor and simplify.}$$

$$= -5 \qquad \text{Direct substitution}$$

EXAMPLE 2 Dividing Out Technique

Find the limit.

$$\lim_{x \to 1} \frac{x - 1}{x^3 - x^2 + x - 1}$$

Solution Begin by substituting $x = 1$ into the numerator and denominator.

$$1 - 1 = 0 \qquad \text{Numerator is 0 when } x = 1.$$

$$1^3 - 1^2 + 1 - 1 = 0 \qquad \text{Denominator is 0 when } x = 1.$$

Both the numerator and denominator are zero when $x = 1$, so direct substitution will not yield the limit. To find the limit, factor the numerator and denominator, divide out any common factors, and then use direct substitution again.

$$\lim_{x \to 1} \frac{x - 1}{x^3 - x^2 + x - 1} = \lim_{x \to 1} \frac{x - 1}{(x - 1)(x^2 + 1)} \qquad \text{Factor denominator.}$$

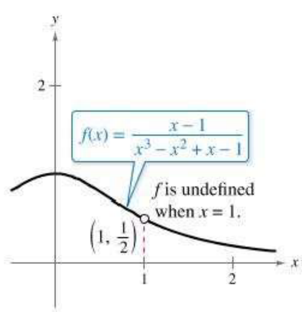
$$= \lim_{x \to 1} \frac{\cancel{x - 1}}{\cancel{(x - 1)}(x^2 + 1)} \qquad \text{Divide out common factor.}$$

$$= \lim_{x \to 1} \frac{1}{x^2 + 1} \qquad \text{Simplify.}$$

$$= \frac{1}{1^2 + 1} \qquad \text{Direct substitution}$$

$$= \frac{1}{2} \qquad \text{Simplify.}$$

The graph below verifies this result.



REMARK To obtain the factorization of the denominator, divide by $(x - 1)$ or factor by grouping.

$$x^3 - x^2 + x - 1$$
$$= x^2(x - 1) + (x - 1)$$
$$= (x - 1)(x^2 + 1)$$

Rationalizing Technique

ALGEBRA HELP To review techniques for rationalizing numerators and denominators, see Appendix A.2.

A way to find the limits of some functions is to first rationalize the numerator of the function. This is the **rationalizing technique.** Recall that to rationalize a numerator of the form $a \pm b\sqrt{m}$ or $b\sqrt{m} \pm a$, multiply the numerator and denominator by the *conjugate* of the numerator. For example, the conjugate of $\sqrt{x}+4$ is $\sqrt{x}-4$.

EXAMPLE 3 **Rationalizing Technique**

Find the limit.

$$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}$$

Solution By direct substitution, you obtain the indeterminate form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}=\frac{\sqrt{0+1}-1}{0}=\frac{0}{0}\qquad\text{Indeterminate form}$$

In this case, rewrite the fraction by rationalizing the numerator.

$$\begin{aligned}\frac{\sqrt{x+1}-1}{x}&=\left(\frac{\sqrt{x+1}-1}{x}\right)\left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right)\\&=\frac{(x+1)-1}{x\left(\sqrt{x+1}+1\right)}&&\text{Multiply.}\\&=\frac{x}{x\left(\sqrt{x+1}+1\right)}&&\text{Simplify.}\\&=\frac{\cancel{x}}{\cancel{x}\left(\sqrt{x+1}+1\right)}&&\text{Divide out common factor.}\\&=\frac{1}{\sqrt{x+1}+1},\quad x\neq 0&&\text{Simplify.}\end{aligned}$$

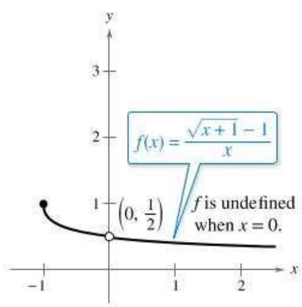


Figure 12.7

Now, evaluate the limit by direct substitution.

$$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}=\lim_{x\to 0}\frac{1}{\sqrt{x+1}+1}=\frac{1}{\sqrt{0+1}+1}=\frac{1}{1+1}=\frac{1}{2}$$

To verify your conclusion that the limit is $\frac{1}{2}$, construct a table, such as the one shown below, or sketch a graph, as shown in Figure 12.7.

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	0.5132	0.5013	0.5001	?	0.4999	0.4988	0.4881

One-Sided Limits

In Section 12.1, you saw that one way in which a limit can fail to exist is when a function approaches a different value from the left side of c than it approaches from the right side of c. This type of behavior can be described more concisely with the concept of a **one-sided limit.**

$$\lim_{x\to c^-}f(x)=L_1 \text{ or } f(x)\to L_1 \text{ as } x\to c^- \qquad \text{Limit from the left}$$

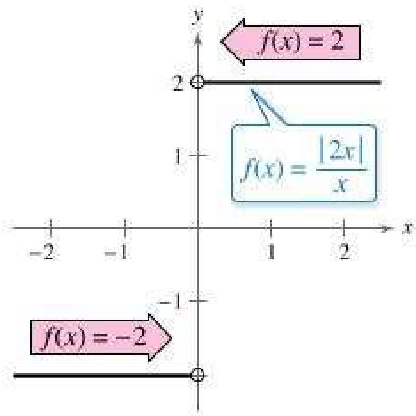
$$\lim_{x\to c^+}f(x)=L_2 \text{ or } f(x)\to L_2 \text{ as } x\to c^+ \qquad \text{Limit from the right}$$

EXAMPLE 6 Evaluating One-Sided Limits

Find the limit of $f(x)$ as x approaches 0 from the left and from the right.

$$f(x) = \frac{|2x|}{x}$$

Solution From the graph of f, shown below, notice that $f(x) = -2$ for all $x < 0$.



So, the limit from the left is

$$\lim_{x \to 0^-} \frac{|2x|}{x} = -2.$$

Limit from the left: $f(x) \to -2$ as $x \to 0^-$

Also from the graph, notice that $f(x) = 2$ for all $x > 0$, so the limit from the right is

$$\lim_{x \to 0^+} \frac{|2x|}{x} = 2.$$

Limit from the right: $f(x) \to 2$ as $x \to 0^+$

In Example 6, note that the function approaches different limits from the left and from the right. In such cases, the limit of $f(x)$ as $x \to c$ does not exist. For the limit of a function to exist as $x \to c$, it must be true that both one-sided limits exist and are equal.

Existence of a Limit

If f is a function and c and L are real numbers, then

$$\lim_{x \to c} f(x) = L$$

if and only if both the left and right limits *exist* and are *equal* to L.

EXAMPLE 7 Evaluating One-Sided Limits

Find the limit of $f(x)$ as x approaches 1.

$$f(x) = \begin{cases} 4 - x, & x < 1 \\ 4x - x^2, & x > 1 \end{cases}$$

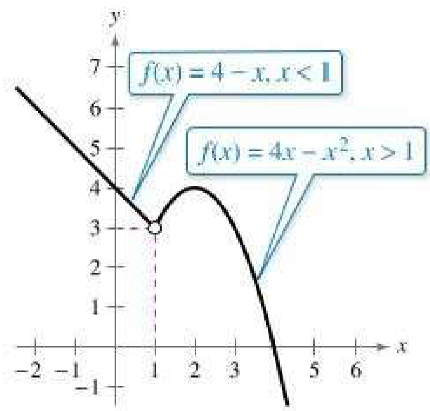
Solution Remember that you are concerned about the value of f *near* $x = 1$ rather than *at* $x = 1$. So, for $x < 1$, $f(x)$ is given by $4 - x$. Use direct substitution to obtain

$$\begin{aligned} \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} (4 - x) \\ &= 4 - 1 \\ &= 3. \end{aligned}$$

For $x > 1$, $f(x)$ is given by $4x - x^2$. Use direct substitution to obtain

$$\begin{aligned} \lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} (4x - x^2) \\ &= 4(1) - 1^2 \\ &= 3. \end{aligned}$$

Both one-sided limits exist and are equal to 3, so it follows that $\lim_{x \to 1} f(x) = 3$. The graph at the right confirms this conclusion.



EXAMPLE 8 Comparing Limits from the Left and Right

For 2-day shipping, a delivery service charges \$24 for the first pound and \$4 for each additional pound or portion of a pound. Let x represent the weight (in pounds) of a package and let $f(x)$ represent the shipping cost. Show that the limit of $f(x)$ as $x \to 2$ does not exist.

$$f(x) = \begin{cases} \$24, & 0 < x \le 1 \\ \$28, & 1 < x \le 2 \\ \$32, & 2 < x \le 3 \end{cases}$$

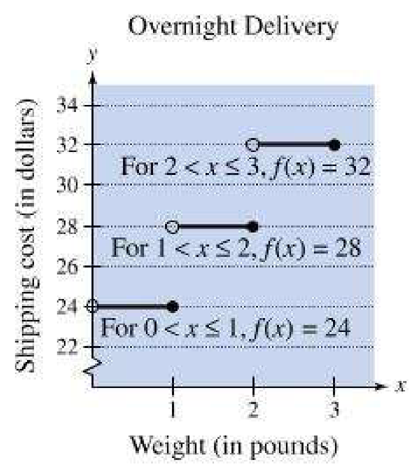
Solution The graph of f is at the right. The limit of $f(x)$ as x approaches 2 from the left is

$$\lim_{x \to 2^-} f(x) = 28$$

whereas the limit of $f(x)$ as x approaches 2 from the right is

$$\lim_{x \to 2^+} f(x) = 32.$$

These one-sided limits are not equal, so the limit of $f(x)$ as $x \to 2$ does not exist.



Limits from Calculus

ALGEBRA HELP To review evaluating difference quotients, see Section 1.4.

In the next example, you will study an important type of limit from calculus—the limit of a *difference quotient*.

EXAMPLE 9 **Evaluating a Limit from Calculus**

For the function

$$f(x) = x^2 - 1$$

find

$$\lim_{h \to 0} \frac{f(3 + h) - f(3)}{h}.$$

Solution Begin by substituting for $f(3 + h)$ and $f(3)$ and simplifying.

$$\lim_{h \to 0} \frac{f(3 + h) - f(3)}{h} = \lim_{h \to 0} \frac{[(3 + h)^2 - 1] - (3^2 - 1)}{h}$$

$$= \lim_{h \to 0} \frac{9 + 6h + h^2 - 1 - 9 + 1}{h}$$

$$= \lim_{h \to 0} \frac{6h + h^2}{h}$$

By factoring and dividing out, you obtain

$$\lim_{h \to 0} \frac{f(3 + h) - f(3)}{h} = \lim_{h \to 0} \frac{\cancel{h}(6 + h)}{\cancel{h}}$$

$$= \lim_{h \to 0} (6 + h)$$

$$= 6 + 0$$

$$= 6.$$

So, the limit is 6.

REMARK Note that for any x-value, the limit of a difference quotient is an expression of the form

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

Direct substitution into the difference quotient always produces the indeterminate form $\frac{0}{0}$.

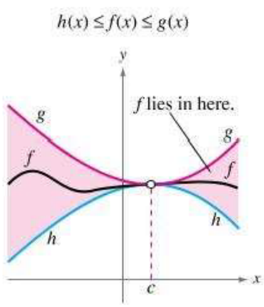
$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

$$= \frac{f(x + 0) - f(x)}{0}$$

$$= \frac{f(x) - f(x)}{0}$$

$$= \frac{0}{0}$$

The theorem below concerns the limit of a function that is squeezed between two other functions, each of which has the same limit at a given x-value, as shown in Figure 12.11.



The Squeeze Theorem
Figure 12.11

The Squeeze Theorem

If $h(x) \le f(x) \le g(x)$ for all x in an open interval containing c, except possibly at c itself, and if

$$\lim_{x \to c} h(x) = L = \lim_{x \to c} g(x)$$

then $\lim_{x \to c} f(x)$ exists and is equal to L.

EXAMPLE 10 A Special Trigonometric Limit

Find the limit: $\lim_{x \to 0} \frac{\sin x}{x}$.

Solution One way to find this limit is to consider a sector of a circle of radius 1 with central angle x, "squeezed" between two triangles (see Figure 12.12), where x is an acute positive angle measured in radians.

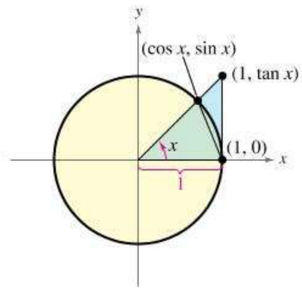
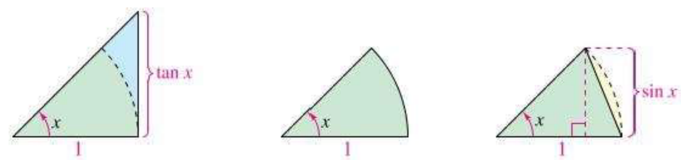


Figure 12.12



Area of triangle	≥	Area of sector	≥	Area of triangle
$\frac{\tan x}{2}$	≥	$\frac{x}{2}$	≥	$\frac{\sin x}{2}$

REMARK Recall from Section 4.1 that the area of a sector of a circle of radius r is given by

$$A = \frac{1}{2}r^2\theta$$

where θ is the measure of the central angle in radians.

Multiplying each expression by $2/\sin x$ produces

$$\frac{1}{\cos x} \geq \frac{x}{\sin x} \geq 1$$

and taking reciprocals and reversing the inequality symbols yields

$$\cos x \leq \frac{\sin x}{x} \leq 1.$$

Because $\cos x = \cos(-x)$ and $(\sin x)/x = [\sin(-x)]/(-x)$, this inequality is valid for *all* nonzero x in the open interval $(-\pi/2, \pi/2)$. Finally, $\lim_{x \to 0} \cos x = 1$ and $\lim_{x \to 0} 1 = 1$, so apply the Squeeze Theorem to conclude that $\lim_{x \to 0} (\sin x)/x = 1$.

- Calculus = branch of mathematics that studies rates of change of functions.
    - The **tangent line** to the graph of a function f at a point $P(x_1, y_1)$ is the line whose slope best approximates the slope of the graph at the point.
        - The problem of finding the slope of a graph at a point is the same as finding the slope of the tangent line at the point

EXAMPLE 1 Visually Approximating the Slope of a Graph

Use Figure 12.13 to approximate the slope of the graph of $f(x) = x^2$ at the point (1, 1).

Solution From the graph of $f(x) = x^2$, notice that the tangent line at (1, 1) rises approximately two units for each unit change in x. So, you can estimate the slope of the tangent line at (1, 1) to be

$$\text{Slope} = \frac{\text{change in } y}{\text{change in } x} \approx \frac{2}{1} = 2.$$

The tangent line at the point (1, 1) has a slope of about 2, so the graph of f has a slope of about 2 at the point (1, 1).

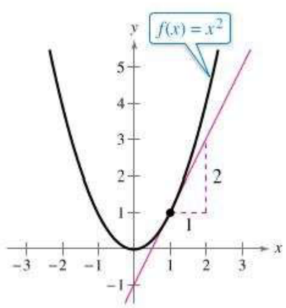
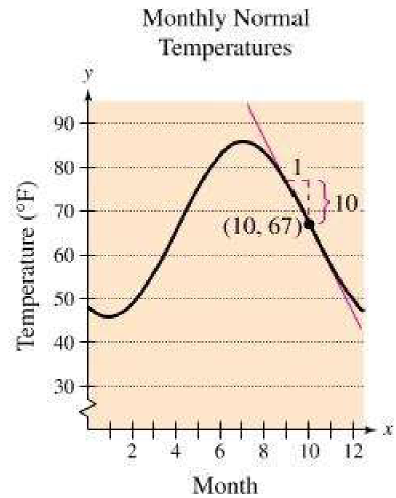


Figure 12.13

EXAMPLE 2 Visually Approximating the Slope of a Graph

The figure at the right graphically depicts the monthly normal temperatures (in degrees Fahrenheit) for Dallas, Texas. Approximate the slope of this graph at the point shown and give a physical interpretation of the result. *(Source: National Climatic Data Center)*



Solution From the graph, the tangent line at the given point falls approximately 10 units for each one-unit change in x. So, you can estimate the slope at the given point to be

$$\text{Slope} = \frac{\text{change in } y}{\text{change in } x}$$

$$\approx \frac{-10}{1}$$

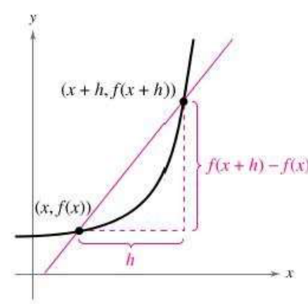
$$= -10 \text{ degrees per month.}$$

This means that the monthly normal temperature in November is about 10 degrees lower than the monthly normal temperature in October.

Slope and the Limit Process

In Examples 1 and 2, you approximated the slope of a graph at a point by creating a graph and then "eyeballing" the tangent line at the point of tangency. A more precise method of approximating tangent lines uses a **secant line** through the point of tangency and a second point on the graph, as shown at the right. If

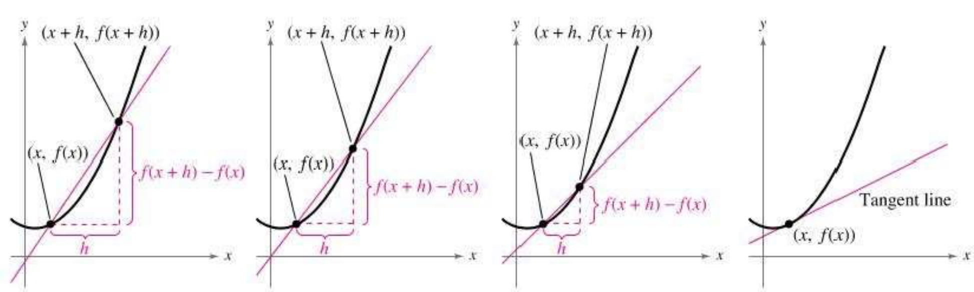
$$(x, f(x)) \quad \text{and} \quad (x + h, f(x + h))$$



are two points on the graph of f, and $(x, f(x))$ is the point of tangency, then the slope of the secant line through the two points is

$$m_{\text{sec}} = \frac{\text{change in } y}{\text{change in } x} = \frac{f(x + h) - f(x)}{h}. \qquad \text{Slope of secant line}$$

Notice that the right side of this equation is a difference quotient. The denominator h is the *change in x*, and the numerator is the *change in y*. Using this method, you obtain more and more accurate approximations of the slope of the tangent line by choosing points closer and closer to the point of tangency, as shown in the figures below.



As h approaches 0, the secant line approaches the tangent line.

Using the limit process, you can find the *exact* slope of the tangent line at $(x, f(x))$.

Definition of the Slope of a Graph

The **slope** m of the graph of f at the point $(x, f(x))$ is equal to the slope of its tangent line at $(x, f(x))$, and is given by

$$\begin{aligned} m &= \lim_{h \to 0} m_{sec} \\ &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \end{aligned}$$

provided this limit exists.

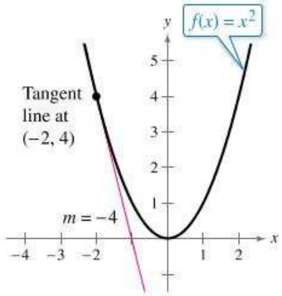


Figure 12.14

EXAMPLE 3 Finding the Slope of a Graph

Find the slope of the graph of $f(x) = x^2$ at the point $(-2, 4)$.

Solution Find an expression that represents the slope of a secant line at $(-2, 4)$.

$$\begin{aligned} m_{sec} &= \frac{f(-2+h) - f(-2)}{h} && \text{Set up difference quotient.} \\ &= \frac{(-2+h)^2 - (-2)^2}{h} && \text{Substitute into } f(x) = x^2. \\ &= \frac{4 - 4h + h^2 - 4}{h} && \text{Expand terms.} \\ &= \frac{-4h + h^2}{h} && \text{Simplify.} \\ &= \frac{\cancel{h}(-4+h)}{\cancel{h}} && \text{Factor and divide out.} \\ &= -4 + h, \quad h \neq 0 && \text{Simplify.} \end{aligned}$$

Next, find the limit of m_{sec} as h approaches 0.

$$\begin{aligned} m &= \lim_{h \to 0} m_{sec} \\ &= \lim_{h \to 0} (-4 + h) \\ &= -4 \end{aligned}$$

The graph has a slope of -4 at the point $(-2, 4)$, as shown in Figure 12.14.

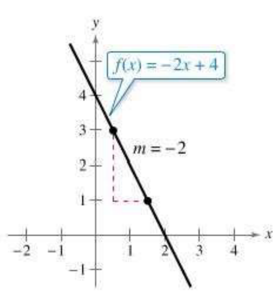


Figure 12.15

EXAMPLE 4 Finding the Slope of a Graph

Find the slope of $f(x) = -2x + 4$.

Solution

$$\begin{aligned} m &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} && \text{Set up difference quotient.} \\ &= \lim_{h \to 0} \frac{[-2(x+h) + 4] - (-2x + 4)}{h} && \text{Substitute into } f(x) = -2x + 4. \\ &= \lim_{h \to 0} \frac{-2x - 2h + 4 + 2x - 4}{h} && \text{Expand terms.} \\ &= \lim_{h \to 0} \frac{-2\cancel{h}}{\cancel{h}} && \text{Divide out.} \\ &= -2 && \text{Simplify.} \end{aligned}$$

You know from your study of linear functions that the line $f(x) = -2x + 4$ has a slope of -2, as shown in Figure 12.15. This conclusion is consistent with that obtained by the limit definition of slope, as shown above.

It is important that you see the difference between the ways the difference quotients were set up in Examples 3 and 4. In Example 3, you found the slope of a graph at a specific point $(c, f(c))$. To find the slope in such a case, use the form of the difference quotient below.

$$m = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h} \qquad \text{Slope at specific point}$$

In Example 4, however, you found a *formula* for the slope at *any* point on the graph. In such cases, you should use x, rather than c, in the difference quotient.

$$m = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \qquad \text{Formula for slope}$$

This form will always produce a function of x, which can then be evaluated to find the slope at any desired point on the graph.

EXAMPLE 5 Finding a Formula for the Slope of a Graph

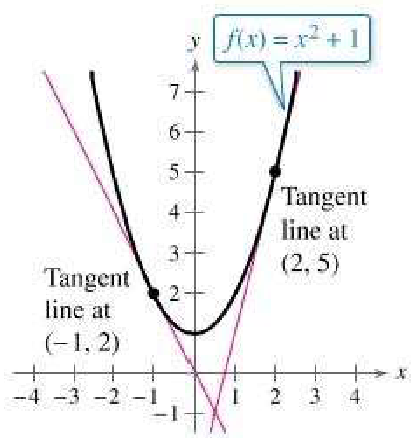
Find a formula for the slope of the graph of $f(x) = x^2 + 1$. What are the slopes at the points $(-1, 2)$ and $(2, 5)$?

Solution

$$\begin{aligned}
m_{\text{sec}} &= \frac{f(x + h) - f(x)}{h} && \text{Set up difference quotient.}\\
&= \frac{[(x + h)^2 + 1] - (x^2 + 1)}{h} && \text{Substitute into } f(x) = x^2 + 1.\\
&= \frac{x^2 + 2xh + h^2 + 1 - x^2 - 1}{h} && \text{Expand terms.}\\
&= \frac{2xh + h^2}{h} && \text{Simplify.}\\
&= \frac{\cancel{h}(2x + h)}{\cancel{h}} && \text{Factor and divide out.}\\
&= 2x + h, \quad h \neq 0 && \text{Simplify.}
\end{aligned}$$

Next, find the limit of m_{sec} as h approaches 0.

$$\begin{aligned}
m &= \lim_{h \to 0} m_{\text{sec}} && \text{Formula for slope}\\
&= \lim_{h \to 0} (2x + h)\\
&= 2x
\end{aligned}$$



Using the formula $m = 2x$ for the slope at $(x, f(x))$, find the slope at the specified points. At $(-1, 2)$, the slope is

$$m = 2(-1) = -2$$

and at $(2, 5)$, the slope is

$$m = 2(2) = 4.$$

The graph of f is at the right.

Definition of a Derivative

The **derivative** of f at x is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

provided this limit exists.

EXAMPLE 6 **Finding a Derivative**

Find the derivative of

$$f(x) = 3x^2 - 2x.$$

Solution

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{[3(x + h)^2 - 2(x + h)] - (3x^2 - 2x)}{h} \\
&= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 2x - 2h - 3x^2 + 2x}{h} \\
&= \lim_{h \to 0} \frac{6xh + 3h^2 - 2h}{h} \\
&= \lim_{h \to 0} \frac{\cancel{h}(6x + 3h - 2)}{\cancel{h}} \\
&= \lim_{h \to 0} (6x + 3h - 2) \\
&= 6x + 3(0) - 2 \\
&= 6x - 2
\end{aligned}$$

So, the derivative of $f(x) = 3x^2 - 2x$ is

$$f'(x) = 6x - 2.$$

EXAMPLE 7 **Using the Derivative**

See LarsonPrecalculus.com for an interactive version of this type of example.

Find $f'(x)$ for $f(x) = \sqrt{x}$. Then find the slopes of the graph of f at the points $(1, 1)$ and $(4, 2)$.

Solution

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

Direct substitution yields the indeterminate form $\frac{0}{0}$, so use the rationalizing technique discussed in Section 12.2 to find the limit.

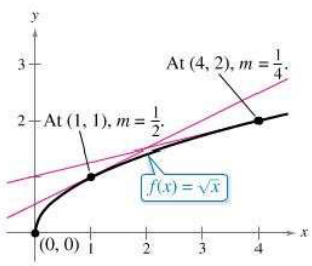
$$f'(x) = \lim_{h \to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h}\right)\left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right)$$

$$= \lim_{h \to 0} \frac{(x+h) - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$



The slope of f at $(x, f(x))$, $x > 0$, is $m = 1/\left(2\sqrt{x}\right)$.

Figure 12.16

At the point $(1, 1)$, the slope is

$$f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2}.$$

At the point $(4, 2)$, the slope is

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

Figure 12.16 shows the graph of f.

Definition of Limits at Infinity

If f is a function and L_1 and L_2 are real numbers, then the statements

$$\lim_{x \to -\infty} f(x) = L_1 \qquad \text{Limit as } x \text{ approaches } -\infty$$

and

$$\lim_{x \to \infty} f(x) = L_2 \qquad \text{Limit as } x \text{ approaches } \infty$$

denote the **limits at infinity.** The first statement is read "*the limit of $f(x)$ as x approaches $-\infty$ is L_1*," and the second is read "*the limit of $f(x)$ as x approaches ∞ is L_2*."

Limits at Infinity

If r is a positive real number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0. \qquad \text{Limit toward the right}$$

Furthermore, if x^r is defined when $x < 0$, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0. \qquad \text{Limit toward the left}$$

Find the limit.

$$\lim_{x\to\infty}\left(4-\frac{3}{x^2}\right)$$

Algebraic Solution

Use the properties of limits listed in Section 12.1.

$$\begin{aligned}\lim_{x\to\infty}\left(4-\frac{3}{x^2}\right) &= \lim_{x\to\infty} 4 - \lim_{x\to\infty}\frac{3}{x^2}\\ &= \lim_{x\to\infty} 4 - 3\left(\lim_{x\to\infty}\frac{1}{x^2}\right)\\ &= 4-3(0)\\ &= 4\end{aligned}$$

So, the limit of $f(x) = 4 - \dfrac{3}{x^2}$ as x approaches ∞ is 4.

EXAMPLE 2 Evaluating Limits at Infinity

See LarsonPrecalculus.com for an interactive version of this type of example.

Find the limit as x approaches ∞ (if it exists) for each function.

a. $f(x) = \dfrac{-2x+3}{3x^2+1}$

b. $f(x) = \dfrac{-2x^2+3}{3x^2+1}$

c. $f(x) = \dfrac{-2x^3+3}{3x^2+1}$

Solution In each case, begin by dividing both the numerator and denominator by x^2, the highest power of x in the denominator.

a. $$\lim_{x\to\infty}\frac{-2x+3}{3x^2+1} = \lim_{x\to\infty}\frac{-\dfrac{2}{x}+\dfrac{3}{x^2}}{3+\dfrac{1}{x^2}} = \frac{-0+0}{3+0} = 0$$

b. $$\lim_{x\to\infty}\frac{-2x^2+3}{3x^2+1} = \lim_{x\to\infty}\frac{-2+\dfrac{3}{x^2}}{3+\dfrac{1}{x^2}} = \frac{-2+0}{3+0} = -\frac{2}{3}$$

c. $$\lim_{x\to\infty}\frac{-2x^3+3}{3x^2+1} = \lim_{x\to\infty}\frac{-2x+\dfrac{3}{x^2}}{3+\dfrac{1}{x^2}}$$

In this case, the limit does not exist because the numerator decreases without bound as the denominator approaches 3.

Limits at Infinity for Rational Functions

Consider the rational function

$$f(x) = \frac{N(x)}{D(x)}$$

where

$$N(x) = a_n x^n + \cdot \cdot \cdot + a_0 \quad \text{and} \quad D(x) = b_m x^m + \cdot \cdot \cdot + b_0.$$

The limit of $f(x)$ as x approaches positive or negative infinity is as follows.

$$\lim_{x\to\pm\infty} f(x) = \begin{cases} 0, & n < m \\ \dfrac{a_n}{b_m}, & n = m \end{cases}$$

If $n > m$, then the limit does not exist.

EXAMPLE 3 **Finding the Average Cost**

You are manufacturing mobile phone protective cases that cost \$4.50 per case to produce. Your initial investment is \$20,000, which implies that the total cost C of producing x cases is given by $C = 4.50x + 20{,}000$. The average cost $\overline{C}$ per case is given by

$$\overline{C} = \frac{C}{x} = \frac{4.50x + 20{,}000}{x}.$$

Find the average cost per case when (a) $x = 1000$, (b) $x = 10{,}000$, and (c) $x = 100{,}000$. (d) What is the limit of $\overline{C}$ as x approaches infinity?

Solution

a. When $x = 1000$,

$$\overline{C} = \frac{4.50(1000) + 20{,}000}{1000}$$

$$= \$24.50.$$

b. When $x = 10{,}000$,

$$\overline{C} = \frac{4.50(10{,}000) + 20{,}000}{10{,}000}$$

$$= \$6.50.$$

c. When $x = 100{,}000$,

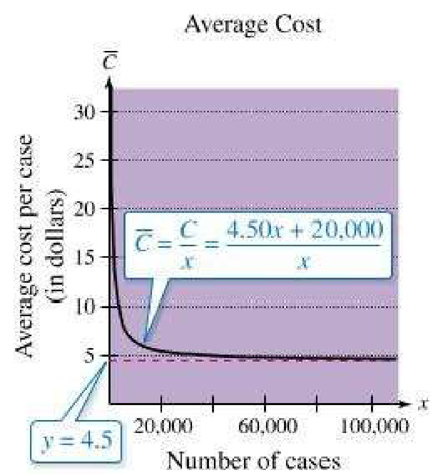
$$\overline{C} = \frac{4.50(100{,}000) + 20{,}000}{100{,}000}$$

$$= \$4.70.$$

d. As x approaches infinity, the limit of $\overline{C}$ is

$$\lim_{x\to\infty} \frac{4.50x + 20{,}000}{x} = \$4.50.$$

The graph of $\overline{C}$ is at the right.



As $x \to \infty$, the average cost per case approaches \$4.50.

Limit of a Sequence

Let f be a function of a real variable such that

$$\lim_{x\to\infty} f(x) = L.$$

If $\{a_n\}$ is a sequence such that $f(n) = a_n$ for every positive integer n, then

$$\lim_{n\to\infty} a_n = L.$$

EXAMPLE 4 Finding the Limit of a Sequence

Write the first five terms of each sequence and find the limit of the sequence. (Assume that n begins with 1.)

a. $a_n = \dfrac{2n+1}{n+4}$ **b.** $a_n = \dfrac{2n+1}{n^2+4}$ **c.** $a_n = \dfrac{2n^2+1}{4n^2}$

Solution

a. The first five terms are $a_1 = \frac{3}{5}$, $a_2 = \frac{5}{6}$, $a_3 = 1$, $a_4 = \frac{9}{8}$, and $a_5 = \frac{11}{9}$. The limit of the sequence is

$$\lim_{n\to\infty} \frac{2n+1}{n+4} = 2.$$

b. The first five terms are $a_1 = \frac{3}{5}$, $a_2 = \frac{5}{8}$, $a_3 = \frac{7}{13}$, $a_4 = \frac{9}{20}$, and $a_5 = \frac{11}{29}$. The limit of the sequence is

$$\lim_{n\to\infty} \frac{2n+1}{n^2+4} = 0.$$

c. The first five terms are $a_1 = \frac{3}{4}$, $a_2 = \frac{9}{16}$, $a_3 = \frac{19}{36}$, $a_4 = \frac{33}{64}$, and $a_5 = \frac{51}{100}$. The limit of the sequence is

$$\lim_{n\to\infty} \frac{2n^2+1}{4n^2} = \frac{1}{2}.$$

EXAMPLE 5 Finding the Limit of a Sequence

Find the limit of the sequence whose nth term is

$$a_n = \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right].$$

Algebraic Solution

Begin by writing the nth term in standard rational function form—as the ratio of two polynomials.

$$a_n = \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] \quad \text{Write original } n\text{th term.}$$

$$= \frac{8(n)(n+1)(2n+1)}{6n^3} \quad \text{Multiply fractions.}$$

$$= \frac{8n^3+12n^2+4n}{3n^3} \quad \text{Write in standard rational form.}$$

This form shows that the degree of the numerator is equal to the degree of the denominator. So, the limit of the sequence is the ratio of the leading coefficients.

$$\lim_{n\to\infty} \frac{8n^3+12n^2+4n}{3n^3} = \frac{8}{3}$$

Numerical Solution

Construct a table that shows the value of a_n as n becomes larger and larger.

n	a_n
1	8
10	3.08
100	2.7068
1000	2.6707
10,000	2.6671
100,000	2.6667
1,000,000	2.6667

Notice from the table that as n approaches ∞, a_n gets closer and closer to

$$2.6667 \approx \frac{8}{3}.$$

Limits of Summations

In Section 9.3, you used the concept of a limit to obtain a formula for the sum S of an infinite geometric series

$$S = a_1 + a_1r + a_1r^2 + \cdot \cdot \cdot = \sum_{i=1}^{\infty} a_1 r^{i-1} = \frac{a_1}{1-r}, \quad |r| < 1.$$

Using limit notation, this sum can be written as

$$\begin{aligned} S &= \lim_{n\to\infty} \sum_{i=1}^{n} a_1 r^{i-1} \\ &= \lim_{n\to\infty} \frac{a_1(1-r^n)}{1-r} && \sum_{i=1}^{n} a_1 r^{i-1} = a_1\left(\frac{1-r^n}{1-r}\right) \\ &= \frac{a_1}{1-r}. && \lim_{n\to\infty} r^n = 0 \text{ for } |r| < 1 \end{aligned}$$

The summation formulas and properties listed below are useful for evaluating finite and infinite summations.

Summation Formulas and Properties

1. $\sum_{i=1}^{n} c = cn$, c is a constant.
2. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
3. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
4. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$
5. $\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$
6. $\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i$, k is a constant.

EXAMPLE 1 Evaluating a Summation

Evaluate the summation.

$$\sum_{i=1}^{200} i = 1 + 2 + 3 + 4 + \cdot \cdot \cdot + 200$$

Solution Using the second summation formula with $n = 200$,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{200} i = \frac{200(200+1)}{2} = \frac{40,200}{2} = 20,100.$$

EXAMPLE 2 Evaluating a Summation

Evaluate the summation

$$S = \sum_{i=1}^{n} \frac{i+2}{n^2} = \frac{3}{n^2} + \frac{4}{n^2} + \frac{5}{n^2} + \cdots + \frac{n+2}{n^2}$$

for $n = 10$, 100, 1000, and 10,000.

Solution Begin by applying summation formulas and properties to simplify S. In the second line of the solution, note that

$$\frac{1}{n^2}$$

factors out of the sum because n is considered to be constant. You cannot factor i out of the first summation in the third line of the solution because i is the index of summation.

$$\begin{aligned}
S &= \sum_{i=1}^{n} \frac{i+2}{n^2} && \text{Write original form of summation.} \\
&= \frac{1}{n^2} \sum_{i=1}^{n} (i+2) && \text{Factor constant } 1/n^2 \text{ out of sum.} \\
&= \frac{1}{n^2}\left(\sum_{i=1}^{n} i + \sum_{i=1}^{n} 2\right) && \text{Write as two sums.} \\
&= \frac{1}{n^2}\left[\frac{n(n+1)}{2} + 2n\right] && \text{Apply Formulas 1 and 2.} \\
&= \frac{1}{n^2}\left(\frac{n^2+5n}{2}\right) && \text{Add fractions.} \\
&= \frac{n+5}{2n} && \text{Simplify.}
\end{aligned}$$

Now, evaluate the sum by substituting the appropriate values of n. The table below shows the results.

n	10	100	1000	10,000
$\sum_{i=1}^{n} \frac{i+2}{n^2} = \frac{n+5}{2n}$	0.75	0.525	0.5025	0.50025

In Example 2, note that the sum appears to approach a limit as n increases. To find the limit of

$$\frac{n+5}{2n}$$

as n approaches infinity, use the techniques from Section 12.4 to write

$$\lim_{n\to\infty} \frac{n+5}{2n} = \frac{1}{2}.$$

EXAMPLE 3 Finding the Limit of a Summation

Find the limit of $S(n)$ as $n \to \infty$.

$$S(n) = \sum_{i=1}^{n} \left(1 + \frac{i}{n}\right)^2 \left(\frac{1}{n}\right)$$

Solution Begin by rewriting the summation in rational form.

$$\begin{aligned}
S(n) &= \sum_{i=1}^{n} \left(1 + \frac{i}{n}\right)^2 \left(\frac{1}{n}\right) && \text{Write original form of summation.} \\
&= \sum_{i=1}^{n} \left(\frac{n^2 + 2ni + i^2}{n^2}\right)\left(\frac{1}{n}\right) && \text{Square } [1 + (i/n)] \text{ and write as a single fraction.} \\
&= \frac{1}{n^3} \sum_{i=1}^{n} (n^2 + 2ni + i^2) && \text{Factor constant } 1/n^3 \text{ out of the sum.} \\
&= \frac{1}{n^3}\left(\sum_{i=1}^{n} n^2 + \sum_{i=1}^{n} 2ni + \sum_{i=1}^{n} i^2\right) && \text{Write as three sums.} \\
&= \frac{1}{n^3}\left\{n^3 + 2n\left[\frac{n(n+1)}{2}\right] + \frac{n(n+1)(2n+1)}{6}\right\} && \text{Use summation formulas.} \\
&= \frac{14n^3 + 9n^2 + n}{6n^3} && \text{Simplify.}
\end{aligned}$$

Now, use this rational form to find the limit as $n \to \infty$.

$$\begin{aligned}
\lim_{n \to \infty} S(n) &= \lim_{n \to \infty} \frac{14n^3 + 9n^2 + n}{6n^3} \\
&= \frac{14}{6} \\
&= \frac{7}{3}
\end{aligned}$$

EXAMPLE 4 Approximating the Area of a Region

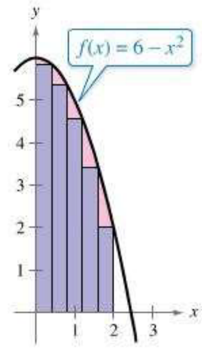


Figure 12.18

See LarsonPrecalculus.com for an interactive version of this type of example.

Use the five rectangles in Figure 12.18 to approximate the area of the region bounded by the graph of $f(x) = 6 - x^2$, the x-axis, and the lines $x = 0$ and $x = 2$.

Solution The length of the interval along the x-axis is 2 and there are five rectangles, so the width of each rectangle is $\frac{2}{5}$. To obtain the height of each rectangle, evaluate $f(x)$ at the right endpoint of each interval. The five intervals are

$$\left[0, \tfrac{2}{5}\right], \quad \left[\tfrac{2}{5}, \tfrac{4}{5}\right], \quad \left[\tfrac{4}{5}, \tfrac{6}{5}\right], \quad \left[\tfrac{6}{5}, \tfrac{8}{5}\right], \quad \text{and} \quad \left[\tfrac{8}{5}, \tfrac{10}{5}\right].$$

Notice that the right endpoint of each interval is $\frac{2}{5}i$ for $i = 1, 2, 3, 4, 5$. The sum of the areas of the five rectangles is

$$\begin{aligned}
\sum_{i=1}^{5} \underbrace{f\left(\frac{2i}{5}\right)}_{\text{Height}}\underbrace{\left(\frac{2}{5}\right)}_{\text{Width}} &= \sum_{i=1}^{5}\left[6 - \left(\frac{2i}{5}\right)^2\right]\left(\frac{2}{5}\right) \\
&= \frac{2}{5}\left(\sum_{i=1}^{5} 6 - \frac{4}{25}\sum_{i=1}^{5} i^2\right) \\
&= \frac{2}{5}\left[6(5) - \frac{4}{25} \cdot \frac{5(5+1)(10+1)}{6}\right] \\
&= \frac{2}{5}\left(30 - \frac{44}{5}\right) \\
&= \frac{212}{25}.
\end{aligned}$$

So, the area of the region is approximately $212/25 = 8.48$ square units.

By increasing the number of rectangles used in Example 4, you obtain a more accurate approximation of the area of the region. For instance, using 25 rectangles each of width $\frac{2}{25}$, you can approximate the area to be $A \approx 9.17$ square units. The table below includes even better approximations.

Number of Rectangles	5	25	100	1000	5000
Approximate Area	8.48	9.17	9.29	9.33	9.33

The Exact Area of a Plane Region

Based on the procedure illustrated in Example 4, the *exact* **area of a plane region** R can be found by using the limit process to increase the number n of approximating rectangles without bound.

Area of a Plane Region

Let f be continuous and nonnegative on the interval $[a, b]$. The area A of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} \underbrace{f\left(a + \frac{(b-a)i}{n}\right)}_{\text{Height}}\underbrace{\left(\frac{b-a}{n}\right)}_{\text{Width}}.$$

EXAMPLE 5 Finding the Area of a Region

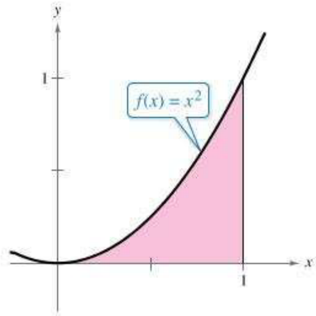


Figure 12.20

Find the area of the region bounded by the graph of $f(x) = x^2$ and the x-axis between $x = 0$ and $x = 1$, as shown in Figure 12.20.

Solution Begin by finding the dimensions of the rectangles.

$$\text{Width: } \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

$$\text{Height: } f\left(a + \frac{(b-a)i}{n}\right) = f\left(0 + \frac{(1-0)i}{n}\right) = f\left(\frac{i}{n}\right) = \frac{i^2}{n^2}$$

Next, approximate the area as the sum of the areas of n rectangles.

$$A \approx \sum_{i=1}^{n} f\left(a + \frac{(b-a)i}{n}\right)\left(\frac{b-a}{n}\right)$$

$$= \sum_{i=1}^{n} \left(\frac{i^2}{n^2}\right)\left(\frac{1}{n}\right) \qquad \text{Summation form}$$

$$= \sum_{i=1}^{n} \frac{i^2}{n^3}$$

$$= \frac{1}{n^3}\sum_{i=1}^{n} i^2$$

$$= \frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \frac{2n^3 + 3n^2 + n}{6n^3} \qquad \text{Rational form}$$

Finally, find the exact area by taking the limit as n approaches ∞.

$$A = \lim_{n \to \infty} \frac{2n^3 + 3n^2 + n}{6n^3} = \frac{1}{3}$$

So, the area of the region is $\frac{1}{3}$ square unit.

EXAMPLE 6 Finding the Area of a Region

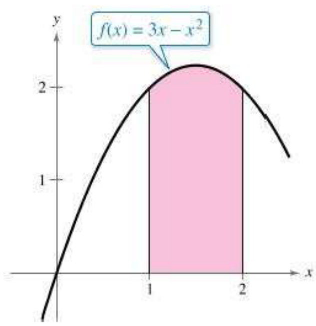


Figure 12.21

Find the area of the region bounded by the graph of $f(x) = 3x - x^2$, the x-axis, and the lines $x = 1$ and $x = 2$, as shown in Figure 12.21.

Solution Begin by finding the dimensions of the rectangles.

$$\text{Width: } \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$$

$$\text{Height: } f\left(a + \frac{(b-a)i}{n}\right) = f\left(1 + \frac{i}{n}\right)$$

$$= 3\left(1 + \frac{i}{n}\right) - \left(1 + \frac{i}{n}\right)^2$$

$$= 2 + \frac{i}{n} - \frac{i^2}{n^2}$$

Next, approximate the area as the sum of the areas of n rectangles.

$$A \approx \sum_{i=1}^{n} f\left(a + \frac{(b-a)i}{n}\right)\left(\frac{b-a}{n}\right)$$

$$= \sum_{i=1}^{n} \left(2 + \frac{i}{n} - \frac{i^2}{n^2}\right)\left(\frac{1}{n}\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n} 2 + \frac{1}{n^2}\sum_{i=1}^{n} i - \frac{1}{n^3}\sum_{i=1}^{n} i^2$$

$$= \frac{1}{n}(2n) + \frac{1}{n^2}\left[\frac{n(n+1)}{2}\right] - \frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \frac{12n^3}{6n^3} + \frac{3n^3 + 3n^2}{6n^3} - \frac{2n^3 + 3n^2 + n}{6n^3}$$

$$= \frac{13n^3 - n}{6n^3}$$

Finally, find the exact area by taking the limit as n approaches ∞.

$$A = \lim_{n \to \infty} \frac{13n^3 - n}{6n^3} = \frac{13}{6}$$

Summary

In this week, we learned about limits, conditions for the existence of limits, properties of limits, one-sided limits, the tangent line to a graph, what a derivative is, limits at infinity, the limit of a sequence, summation formulas/properties, and finally how to calculate the exact area of a plane region.