Optimal Fund Allocation for State Sports Academies using Game Theory and Mechanism Design

Use Case Report

Aryan Agarwal
Roll No. 2018102024

Shivansh
Roll No. 2018102007

April 5, 2021

Outline

In many states, the funds are allocated to different sports academies either equally or using some improper criteria. Usually good performing academies get more funds, but there is no solid criteria which differentiates between two good performing academies. The difference of the amount of funds received by a good academy and poor academy is not formulated using some algorithm. We propose a new fair scheme to improvise the fund allocation, where the allocated funds are calculated using Game theory. By a "Good Performing Academy", we mean an academy that produces more number of "Recognised Players" with high "Fund Allocation Points (FAP)". Recognised players are those players that have played at least one inter-state game and have played in front of all "Academy representatives". Academy representatives are those people who rate all the recognised players. We assume that each academy has one academy representative, who is not a recognised player. Fund Allocation Points (FAP) are the utility points for recognised players in our scheme that decide fund allocation for their academy. So for i^{th} academy,

$$\text{TAF} = \text{BF} + \text{RF}$$

where BF are Base Funds that denote default funds that are given to every academy for being an active sports academy of the state, RF are Reward Funds which are decided by our scheme and TAF are Total Allocated Funds for an academy. This scheme gives a purpose for academies to observe other academies, work hard and perform well.

Contents

Elements of the Game

- $N = \{1, 2 \ldots L, (L+1) \ldots K\}$ where $L = \Sigma_{j=1}^{n} m_j$, n is the number of academies, m_j is the number of recognized players from j^{th} academy and $K = \Sigma_{j=1}^{n} m_j + n$. So N contains all recognised players and academy representatives. First m_1 players belong to the first academy, next m_2 players belong to the second academy and so on. Last n players are academy representatives. $(L+1)^{th}$ player is the academy representative for first academy, $(L+2)^{th}$ player is the academy representative for second academy and so on.

- Total State Funds (TSF) denote the total funds that the state can allocate completely among the academies. Maximum Academy Funds (MAF) denote the maximum possible sum of RF of all academies.
So, MAF = TSF - $n\times$BF.

- $u_i \;\epsilon [0, M]\ \forall i = 1, 2 \ldots\ L$: The utility of i^{th} player, corresponding to the fund allocation points, where M is the maximum possible FAP.
$u_i \;\epsilon [0,\text{MAF}]\ \forall i = L + j,\ \forall j = 1, 2 \ldots\ n$: The utility of i^{th} player, corresponding to RF allocated to j^{th} academy, where MAF is the maximum possible RF.

- $\Theta(i) = [0,10]\ \forall i \epsilon N$: Type for each player, where $\theta(i)\epsilon\Theta(i)$ is the rating of the i^{th} player out of 10. Ratings are defined only for recognised players and not for academy representatives and thus last n values are 0 in $\theta(i)$. Also recognised players can't rate every other player in the game. So $\theta(i)$ as reported by them would be a vector of zeros. We assume that each academy representative knows how all the recognised players play. Hence, θ_i is not actually private.

- $X = [0, M] \times [0, M] \times \ldots \times [0, M] \times [0,\text{MAF}] \ldots \times [0,\text{MAF}]$: The set of all outcomes. where $x \epsilon X$ denotes the FAP for the i^{th} player $\forall i = 1$ to L and RF allocated to the j^{th} academy $\forall i = L + j,\ \forall j = 1, 2 \ldots\ n$.

- We want to construct a mechanism where, if $\theta' \;\epsilon\; \Theta$ is the reported type of academy representatives, then Nash Equilibrium of the induced Bayesian game is when $\theta' = \theta$, where θ is the true type of academy representatives.

Mechanism Design

Each academy representative is called individually and simultaneously to report the ratings of all the recognised players. Every academy representative knows how each recognised player plays. No prior information is provided to the representatives and thus we can assume that there is no cooperation between academy representatives. Also ratings given by some academy representative are not known to some other academy representative. Our objective is to take these ratings and allocate appropriate FAP to recognised players and funds to academy representatives.

So we ask each academy representative to report ratings for all the recognised players, even for recognised players from his/her own academy. We calculate the average rating for a recognised player and deviation in the reporting for the academy representatives. It is clear that academy representative from a particular academy would prefer his players and thus give them high ratings. Hence we have used a penalisation system that reduces FAP of a recognised player if his/her academy's representative lies. This penalty is calculated using standard deviation. Also it is important to note that a good academy would have many recognised players and thus lying would lead to penalisation for each recognised player and thus heavy penalty in the RF of that academy.

Therefore, lying is harmful. So every academy representative saying the truth is the Nash equilibrium (Proof shown through BIC and examples in further sections).

- Let M be the maximum FAP that can be awarded to a recognised player.
- Let θ_i be the set of ratings reported by i^{th} academy representative, $\forall i \epsilon \{L+1,\ L+2\ ...\ K\}$.
- Let μ_i be the average rating given to a recognised player,

$$\mu_i = \frac{\Sigma_{j=L+1}^{K} \theta_j(i)}{n}, \forall i \epsilon \{1, 2...L\} \tag{1}$$

- Let σ_j be the deviation of values reported by academy representative j, $\forall j \epsilon \{L+1,\ L+2\ ...\ K\}$

$$\sigma_j = \sqrt{\frac{1}{L} \Sigma_{k=1}^{L} (\mu_k - \theta_j(k))^2} \tag{2}$$

We propose the following utility function:

- $\forall i \epsilon \{1,\ 2\ ...\ L\}$

$$u_i(x) = f(\mu_i, \sigma_j)$$

  σ_j is the deviation of academic representative who belongs to the same academy as the recognised player i. So for utility of first m_1 players, we pass σ_1 in the social choice function, for the utility of next m_2 players, we pass σ_2 in the social choice function and so on. We will discuss about f in further sections.

- $\forall i \epsilon \{L+1, L+2...K\}$

$$u_i(x) = \frac{\Sigma_{j=B}^{T} u_j(x)}{\Sigma_{j=1}^{L} u_j(x)} \times \text{MAF}$$

  where $B = \Sigma_{k=1}^{(i-L)-1} m_k + 1$ and $T = \Sigma_{k=1}^{(i-L)} m_k$. We assume $\Sigma_{j=1}^{L} u_j(x) \neq 0$ (We will consider the case with $\Sigma_{j=1}^{L} u_j(x) = 0$ in further sections). This expression sums up the utilities/FAP of recognised players from an academy and normalizes them according to MAF. This is nothing but RF for $(i-L)^{th}$ academy.

Truth Revelation

We will show how truth revelation is the best response for an academy representative in expectation that rest of the aca in demy representatives tell the truth, which is nothing but Bayesian Incentive Compatibility (BIC). Let i be the only academy representative who diverts from truth and reports wrong set of ratings while all other academy representatives, as an expectation, report the true set of ratings. We have already mentioned that θ_i is common knowledge.

- We have $\theta_j = \{\theta_j(1), \theta_j(2)\ ...\ \theta_j(K)\}\ \forall j \epsilon \{L+1,\ L+2\ ...\ K\}$ - $\{i\}$, for some $i\ \epsilon \{L+1,\ L+2\ ...\ K\}$.
  Since all the academy representatives apart from i are reporting true, $\theta_j = \theta = \{\theta(1), \theta(2) ... \theta(K)\}\ \forall j \epsilon \{L+1,\ L+2\ ...\ K\}$ - $\{i\}$, for some $i\ \epsilon \{L+1,\ L+2\ ...\ K\}$
- For academy representative i, we have $\theta_i = \{\theta_i(1), \theta_i(2)\ ...\ \theta_i(K)\}$.
  Let's say $\theta_i = \theta' = \{\theta'(1), \theta'(2)\ ...\ \theta'(K)\}$.
- Now we find the means of ratings of each recognised player given by each academy representative:

$$\mu_j = \frac{\Sigma_{k=L+1}^{K} \theta_k(j)}{n}, \forall j \epsilon \{1, 2...L\}$$
$$= \frac{[(n-1) \times \theta(j) + \theta'(j)]}{n}$$

- Now we find the deviations for each of the academy representatives:
  For the honest players, $\forall j \epsilon \{L+1,\ L+2\ ...\ K\}$ - $\{i\}$, for some $i\ \epsilon \{L+1,\ L+2\ ...\ K\}$,

$$\sigma_j = \sqrt{\tfrac{1}{L} \Sigma_{k=1}^{L} (\mu_k - \theta(k))^2}$$
$$= \sqrt{\tfrac{1}{L} \Sigma_{k=1}^{L} \{\tfrac{1}{n}[(n-1) \times \theta(k) + \theta'(k)] - \theta(k)\}^2}$$
$$= \sqrt{\tfrac{1}{L} \Sigma_{k=1}^{L} \{\tfrac{1}{n}[\theta'(k) - \theta(k)]\}^2}$$
$$= \tfrac{1}{n\sqrt{L}} \sqrt{\Sigma_{k=1}^{L} [\theta'(k) - \theta(k)]^2}$$

  For the dishonest player i,

$$\sigma_i = \sqrt{\tfrac{1}{L} \Sigma_{k=1}^{L} (\mu_k - \theta'(k))^2}$$
$$= \sqrt{\tfrac{1}{L} \Sigma_{k=1}^{L} \{\tfrac{1}{n}[(n-1) \times \theta(k) + \theta'(k)] - \theta'(k)\}^2}$$
$$= \sqrt{\tfrac{1}{L} \Sigma_{k=1}^{L} \{\tfrac{1}{n}[(n-1) \times (\theta(k) - \theta'(k))]\}^2}$$
$$= \tfrac{(n-1)}{n\sqrt{L}} \sqrt{\Sigma_{k=1}^{L} [\theta(k) - \theta'(k)]^2}$$
$$= \tfrac{(n-1)}{n\sqrt{L}} \sqrt{\Sigma_{k=1}^{L} [\theta'(k) - \theta(k)]^2}$$

- We can now actually see how large the deviation is for a dishonest member compared to an honest member. For all $j\epsilon\{L+1, L+2 \ldots K\}$ - $\{i\}$, for some $i \ \epsilon\{L+1, L+2 \ldots K\}$,

$$\frac{\sigma_j}{\sigma_i} = \frac{1}{(n-1)}$$

  Now when $n \geq 3$, we always have the dishonest academy representative having a higher deviation than the rest of the representatives.

Now we want to design a social choice function f that takes in μ_k and σ_j for a recognised player k, academy representative j of the same academy and penalises the recognised player whose academy representative has a higher deviation σ_j. This way we might be able to achieve an incentive compatible mechanism in expectation that rest of the academy representatives report true sets. We will explore different social choice functions in the next section.

Social Choice Functions

In this section we will look at possible social choice functions for our mechanism. Many types of functions are possible, but we will try to formulate some of them. As we previously saw, f takes μ_i and σ_j as input, where i is the recognised player and j is the academy representative of the same academy. We have seen from equations (1) and (2) that μ_i and σ_j are functions of Θ and hence f is a valid social choice function that returns a real number. Let D denote the average rating for recognised players according to the type sets of academy representatives. So,

$$D = \frac{\Sigma_{j=L+1}^{K} \Sigma_{k=1}^{L} \theta_j(k)}{n \times L}$$

Social Choice Function 1 (SCF1): This is a linear function within the range $[0, M]$.

$$f(\mu_i, \sigma_j) = \begin{cases} 0, & a(\mu_i - D) - b\sigma_j < 0 \\ a(\mu_i - D) - b\sigma_j, & 0 \leq a(\mu_i - D) - b\sigma_j < M \\ M, & a(\mu_i - D) - b\sigma_j \leq M \end{cases}$$

Here $a > 0$ and $b > 0$ are parameters decided by the evaluator. Values a and b can be decided according to the contributions average and deviation should have to determine FAP.

Social Choice Function 2 (SCF2): This is a linear function within the range $[0, M]$ that takes care of unfairness added by any academy representative using σ'_j.

$$\sigma'_j = \sigma_j - \mu(\sigma)$$

where $\mu(\sigma)$ denotes mean of deviations for each academy representative.

$$f(\mu_i, \sigma_j) = \begin{cases} 0, & a(\mu_i - D) - b\sigma'_j < 0 \\ a(\mu_i - D) - b\sigma'_j, & 0 \leq a(\mu_i - D) - b\sigma'_j < M \\ M, & a(\mu_i - D) - b\sigma'_j \leq M \end{cases}$$

Here $a > 0$ and $b > 0$ are parameters decided by the evaluator. Generally $b >>> a$, as deviation values are very small as compared to mean values.

Social Choice Function 3 (SCF3): This is a function that awards high FAP if ratings are high. This function has a lesser focus on σ_j or false reporting.

$$f(\mu_i, \sigma_j) = \begin{cases} 0, & a.e^{b(\mu_i - D)} - c\sigma_j < 0 \\ a.e^{b(\mu_i - D)} - c\sigma_j, & 0 \leq a.e^{b(\mu_i - D)} - c\sigma_j < M \\ M, & a.e^{b(\mu_i - D)} - c\sigma_j \leq M \end{cases}$$

Here $a > 0$, $b > 0$ and $c > 0$ are parameters decided by the evaluator.

Bayesian Incentive Compatibility

We want our scheme to be Bayesian Incentive Compatible (BIC) and we have seen how we can use means and deviations to build our mechanism with the use of explained social choice functions.

Now we will try to prove our mechanism is BIC. We restrict ourselves to SCF2 for further analysis, a similar analysis can be extended for other social choice functions as well. We consider 2 cases below when academy representative j reports truthfully and when he/she doesn't in expectation that rest of the academy representatives report truthfully. So, for all recognised players i, $\forall i\epsilon\{1, 2 \ldots L\}$ and academy representatives j, $\forall j\epsilon\{L+1, L+2 \ldots K\}$, where i and j belong to the same academy,

$$f(\mu_i, \sigma_j) = \begin{cases} 0, & a(\mu_i - D) - b\sigma'_j < 0 \\ a(\mu_i - D) - b\sigma'_j, & 0 \leq a(\mu_i - D) - b\sigma'_j < M \\ M, & a(\mu_i - D) - b\sigma'_j \leq M \end{cases}$$

$$a, b > 0$$

When j reports misleading values,

$$\mu_i = \frac{[(n-1) \times \theta(i) + \theta'(i)]}{n}, \forall i\epsilon(j-L)^{th} \text{ academy}$$

$$\sigma_j = \frac{(n-1)}{n\sqrt{L}} \sqrt{\Sigma_{k=1}^{L} [\theta'(k) - \theta(k)]^2}$$

$$\mu(\sigma) = \frac{2(n-1)}{n^2\sqrt{L}} \sqrt{\Sigma_{k=1}^{L} [\theta'(k) - \theta(k)]^2}$$

When j reports truthfully,

$$\mu_i = \theta(i), \forall i\epsilon(j-L)^{th} \text{ academy}$$

$$\sigma_j = 0$$

$$\mu(\sigma) = 0$$

For Bayesian Incentive Compatibility,

$$\mathbb{E}_{\theta_j}[u_j(f(\theta_j, \theta_{-j}), \theta_j)|\theta_j] \geq \mathbb{E}_{\theta_{-j}}[u_j(f(\theta'_j, \theta_{-j}), \theta_j)|\theta_j]$$

$$\forall \theta'_j \epsilon \Theta_j, \forall \theta_j \epsilon \Theta_j, \forall j \epsilon \{L+1, L+2 \ldots K\}$$

So let us find,

$$\mathbb{E}_{\theta_{-j}}[u_j(f(\theta_j,\theta_{-j}),\theta_j)|\theta_j] - \mathbb{E}_{\theta_{-j}}[u_j(f(\theta'_j,\theta_{-j}),\theta_j)|\theta_j]$$

$$= \frac{MAF}{\sum_{p=1}^{L} u_p(x)} \left[\frac{a}{n} \left(\sum_{\substack{i\epsilon(j-L)^{th} \\ academy}} \theta(i) - \sum_{\substack{i\epsilon(j-L)^{th} \\ academy}} \theta'(i) \right) - \frac{bZm_j(n-1)(2-n)}{n^2\sqrt{L}} \right]$$

$$= \frac{MAF}{\sum_{p=1}^{L} u_p(x)} \left[\frac{bZm_j(n-1)(n-2)}{n^2\sqrt{L}} - \frac{a}{n} \left(\sum_{\substack{i\epsilon(j-L)^{th} \\ academy}} \theta'(i) - \sum_{\substack{i\epsilon(j-L)^{th} \\ academy}} \theta(i) \right) \right] \tag{3}$$

where $Z = \sqrt{\Sigma_{k=1}^{L}[\theta'(k) - \theta(k)]^2}$.
We assume $\Sigma_{p=1}^{L} u_p(x) \neq 0$.

Let us analyse the result,

- $$\left(\frac{MAF}{\sum_{p=1}^{L} u_p(x)} \right) \left(\frac{bZm_j(n-1)(n-2)}{n^2\sqrt{L}} \right) \tag{4}$$

  is positive as $b, Z, m_j, \mathrm{MAF}, L, (\Sigma_{p=1}^{L} u_p(x)) > 0$ and also $n \geq 3$ as mentioned earlier.

- $$\frac{MAF}{\sum_{p=1}^{L} u_p(x)} \left(\frac{a}{n} \right) \left(\sum_{\substack{i\epsilon(j-L)^{th} \\ academy}} \theta'(i) - \sum_{\substack{i\epsilon(j-L)^{th} \\ academy}} \theta(i) \right) \tag{5}$$

  is positive when

  $$\sum_{\substack{i\epsilon(j-L)^{th} \\ academy}} \theta'(i) > \sum_{\substack{i\epsilon(j-L)^{th} \\ academy}} \theta(i)$$

  or, the academy representative j reports higher ratings for recognised players of his/her own academy than the actual ratings. If,

  $$\sum_{\substack{i\epsilon(j-L)^{th} \\ academy}} \theta'(i) < \sum_{\substack{i\epsilon(j-L)^{th} \\ academy}} \theta(i)$$

  then it is clear that equation (3) will be positive. We are more interested in the case where the academy representative j reports higher ratings for recognised players of his/her own academy than the actual ratings.

- Now if equations (4) and (5) are positive, we can always choose parameters a and b in such a way that equation (3) is always positive and hence satisfy our condition of truthfulness in expectation that other academy representatives are truthful. Generally, $b >>> a$.

Hence we can adjust the values of a and b to achieve Bayesian Incentive Compatibility. We also need to make sure that b is not arbitrarily greater than a, because in that case, our mechanism will be unfair. We can run many simulations to check which values give the desired results.

Special Case

This scheme is appropriate for every scenario, except when $\Sigma_{j=1}^{L} u_j(x) = 0$, or, rating given to every recognised player from every academy by every academy representative is the same. Practically, we know that this case is not possible as different players have different levels of performance. It is a very rare case that this scenario is a true scenario. Thus to handle this case, we propose a different scheme, where for i^{th} academy,

$$\mathrm{TAF}_i = \left(\frac{D}{10}\right)\left(\frac{m_i}{\Sigma_{j=1}^{n} m_j}\right)\mathrm{TSF} \tag{6}$$

where

$$D = \frac{\Sigma_{j=L+1}^{K}\Sigma_{k=1}^{L}\theta_j(k)}{n \times L}$$

.

Although this case is a very rare case practically, we propose to handle this case separately as mentioned above.

Solved Examples

Example 1: Consider the below table, where the players 1-4 are recognised players, while players 5, 6 and 7 are academy representatives for academy 1, 2 and 3 respectively. Player 1 is from academy 1, player 2 is from academy 2, and players 3 and 4 are from academy 3.

AR	1	2	3	4
5	7	9	4	4
6	7	9	4	4
7	7	8	7	8

Here AR is Academy Representative. In the above table, it is clearly visible that the academy 3 is an average academy (according to ratings of academy 1 and 2), which performs very poorly compared to others. Academy 1 is better than academy 3 but, academy 2 is the best among all. Looking at the values reported by academy 3 representative (player 7), it is clear that he/she has tried to manipulate ratings for recognised players of his academy. This action should result in a bad fund allocation for academy 3 based on our scheme. We can get a clearer view if we calculate mean and deviation based on the mechanism. Following table is obtained,

	1	2	3	4	5	6	7
μ	7.0	8.66	5.0	5.33	0	0	0
σ	0	0	0	0	0.85	0.85	1.7

The above table clearly depicts that the means do not match the reported ratings for academy 3 players and also the standard deviation is too high for academy 3. Thus this proves academy 3 representative misreported ratings of recognised players from his/her own academy.

Now we can compute the utilities for each player. The values are computed using the SCF2, with $M = 25$, MAF = 100, $a = 10$ and $b = 200$.

	1	2	3	4	5	6	7
u	25	25	0	0	50	50	0

Example 2: In the following example, there are 11 recognised players and 4 academy representatives. Players 1 and 2 belong to academy 1, players 3 and 4 belong to academy 2, players 5, 6 and 7 belong to academy 7 and players 8, 9, 10 and 11 belong to academy 4. Academy representatives, i.e., players 12, 13, 14 and 15 have reported the following ratings for recognised players,

AR	1	2	3	4	5	6	7	8	9	10	11
12	10	9	5	6	7	8	7	4	3	5	2
13	9	6	5	6	7	8	7	4	3	5	2
14	9	6	5	6	8	9	8	4	3	5	2
15	9	6	5	6	7	8	7	4	3	5	2

We can see that academy 1 is the best academy, but looking at ratings submitted by academy representative of academy 1 (player 12), it seems like he/she has tried to manipulate ratings given to recognised players from his/her academy to get better RF. Similarly, academy 3 is second best academy and academy representative of academy 3 (player 14) has opted for a similar action of reporting higher ratings for his/her academy's recognised players. Academies 2 and 4 have most likely submitted true ratings. Since we have spotted this, our allocation strategy should penalize academies 1 and 3 accordingly. We can confirm our observations by finding the means and deviations.

	1	2	3	4	5	6	7	8
μ	9.25	6.75	5.0	6.0	7.25	8.25	7.25	4.0
σ	0	0	0	0	0	0	0	0

	9	10	11	12	13	14	15
μ	3.0	5.0	2.0	0	0	0	0
σ	0	0	0	0.72	0.27	0.45	0.27

The means of the recognised players, whose ratings have been exaggerated, have increased. But we can easily find which academy representative manipulated his/her ratings using deviation values. Note the values of σ of academy representatives of academies 1 and 3 (players 12 and 14) exceed other values of σ significantly. Now we can compute the utilities for all the recognised players and academy representatives using SCF2, with $M = 50$, MAF = 100, $a = 30$ and $b = 200$.

	1	2	3	4	5	6	7	8
u	44.70	0	8.22	38.22	38.38	50	38.38	0

	9	10	11	12	13	14	15
u	0	8.22	0	19.76	20.54	56.05	3.63

Queries

1. **What is the significance of Base Funds?**
   Base Funds (BF) make it possible for government to allocate some basic default funds for small or growing sports academies with less or no recognised players.

2. **Why is this scheme based on recognised players and not on all players from an academy?**
   Practically, it is not possible for an academy representative to rate all the players from the state. Also academy representative has to know the players in order to rate them. This scheme being based on recognised players, will serve a purpose for academies to train more players and make them play in big competitions so that academy representatives can see them. This will result in better development of players.

3. **What are advantages of using SCF2 and SCF3 over SCF1?**
   SCF2 depends more on genuineness of the reporting of academy representatives. SCF3 depends more on the ratings of recognised players. Depending on the purpose, evaluator can select the appropriate social choice function.

4. **When will this scheme not work? What is the role of the evaluator to ensure that this scheme works?**
   This scheme will not work when each academy representative reports some different ratings for each recognised player which are not the true ratings. Evaluator has to ensure that academy representatives know that reporting the true ratings is the common knowledge. This scheme will also not work if number of academies are less than 3 in the state. Generally, each state has more than 3 academies, so this scheme is implementable.

5. **What are the drawbacks of this mechanism?**
   The only drawback of this mechanism is that players of the game should not know about this scheme at any point of time. It is clearly possible to manipulate the final RF, if the scheme is known already. Academy representatives can work together and report wrong values for their benefit. Thus, evaluator plays a key role in implementing this scheme by asking all academy representatives to report all the ratings at once, without informing them about the scheme, allowing them no time to discuss.

Conclusion

We have provided a mechanism for optimal allocation of funds to sports academies in a state, based on the reported ratings of recognised players. We have designed the mechanism in such a way that being truthful is always a better strategy in expectation that the other academy representatives are also speaking the truth (BIC). In this scheme, academy representatives reporting true ratings is the Nash Equilibrium. We have provided solved examples and queries regarding the implementation of this scheme. We have also provided a simple code showing the implementation of this scheme.

References

Narahari, Y. (2014). Game Theory and Mechanism Design.