1. Find the chord of curvature through the cardiode $r = a(1+\cos\theta)$

Sol: Given $r = a(1+\cos\theta)$ —①

diff. w.r.t θ on b.s

$$\frac{dr}{d\theta} = a(-\sin\theta) = -a\sin\theta \quad —②$$

Again we know that $\tan\phi = r\frac{d\theta}{dr}$

by eq ② $\frac{d\theta}{dr} = \frac{-1}{a\sin\theta}$

$$\tan\phi = \frac{a(1+\cos\theta)}{-a\sin\theta} = -\frac{\cancel{a}\,2\cos^2\theta/2}{\cancel{a}\,2\sin\theta/2\,\cancel{\cos\theta/2}}$$

$$\tan\phi = -\cot\theta/2$$

$$\cancel{\tan}\phi = \cancel{\tan}\left(\frac{\pi}{2}+\frac{\theta}{2}\right) \qquad [\because \tan(90+\theta) = -\cot\theta]$$

$$\phi = \frac{\pi}{2}+\frac{\theta}{2}$$

W.K.T $p = r\sin\phi$

$$p = r\sin\left(\frac{\pi}{2}+\frac{\theta}{2}\right) \qquad [\sin(90+\theta) = \cos\theta]$$

$$p = r\cos\frac{\theta}{2} \quad —③$$

but given that $r = a(1+\cos\theta)$

$$r = 2a\cos^2 \theta/2$$

$$\cos^2\frac{\theta}{2} = \frac{r}{2a} \Rightarrow \cos\frac{\theta}{2} = \sqrt{\frac{r}{2a}}$$

$$③ \Rightarrow \quad p = r\sqrt{\frac{r}{2a}} = \frac{r \cdot r^{1/2}}{\sqrt{2a}}$$

$$p = \frac{r^{3/2}}{\sqrt{2a}} \quad — ④$$

diff. eq ④ w.r.t 'r'

$$\frac{dp}{dr} = \frac{1}{\sqrt{2a}} \frac{3}{2} r^{\frac{3}{2}-1}$$

$$= \frac{1}{\sqrt{2a}} \cdot \frac{3}{2} r^{\frac{1}{2}}$$

$$\frac{dr}{dp} = \frac{2}{3} \frac{\sqrt{2a}}{\sqrt{r}}$$

W.K.T $\rho = r\frac{dr}{dp}$

$$\rho = r \frac{2}{3} \frac{\sqrt{2a}}{\sqrt{r}}$$

$$\rho = \frac{2}{3}\sqrt{2a}\, r^1 \cdot r^{-1/2}$$

$$= \frac{2}{3}\sqrt{2a} \cdot \sqrt{r}$$

$$\rho = \frac{2}{3}\sqrt{2ar}$$

Now the chord of curvature through pole

$$L = 2\rho \sin\phi$$

$$L = 2\times\frac{2}{3}\sqrt{2ar}\sin\phi$$

$$L = \frac{4}{3}\sqrt{2ar}\,\sin\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$$

$$L = \frac{4}{3}\sqrt{2ar}\cos\frac{\theta}{2} \qquad \left[\because \cos\frac{\theta}{2} = \sqrt{\frac{r}{2a}}\right.$$

$$L = \frac{4}{3}\sqrt{2ar}\times\sqrt{\frac{r}{2a}}$$

$$L = \frac{4}{3}\cancel{\sqrt{2a}}\sqrt{r}\times\frac{\sqrt{r}}{\cancel{\sqrt{2a}}}$$

$$\boxed{L = \frac{4}{3}r}$$

② Find the radius of curvature of the curve $y^2(a-x) = x^2(a+x)$ at the origin.

Sol Given $y^2(a-x) = x^2(a+x)$

$$y^2 = \frac{x^2(a+x)}{a-x}$$

$$\therefore y = \pm x\frac{(a+x)^{1/2}}{(a-x)^{1/2}}$$

$$y = \frac{\pm x \cancel{a}\left(1+\frac{x}{a}\right)^{1/2}}{\cancel{a}\left(1-\frac{x}{a}\right)^{1/2}}$$

$$= \pm x\left(1+\frac{x}{a}\right)^{\frac{1}{2}} \cdot \left(1-\frac{x}{a}\right)^{-\frac{1}{2}}$$

$$= \pm x\left(1+\frac{x}{2a}+\cdots\right)\left(1+\frac{x}{2a}+\cdots\right)$$

$$\left[\because (1+x)^n = 1+nx+\ldots\right]$$

$$= \pm x\left(1+\frac{x}{2a}+\frac{x}{2a}+\frac{x^2}{4a^2}+\cdots\right)$$

ie, $$y = \pm\left(x+\frac{2x^2}{2a}+\frac{x^3}{4a^2}+\cdots\right)$$

$$y = \pm\left(x+\frac{x^2}{a}+\frac{x^3}{4a^2}+\cdots\right)$$

$$y_1 = \pm\left(1+\frac{2x}{a}+\frac{3x^2}{4a^2}+\cdots\right)$$

and

$$y_2 = \pm\left(\frac{2}{a}+\frac{6x}{4a^2}+\cdots\right)$$

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At $(0,0)$ $y_1 = \pm 1$, $y_2 = \pm \frac{2}{a}$

$$\therefore \rho_{(0,0)} = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \frac{(1+1)^{3/2}}{\pm \frac{2}{a}} = \pm \cancel{2}\sqrt{2}\,\frac{a}{\cancel{2}}$$

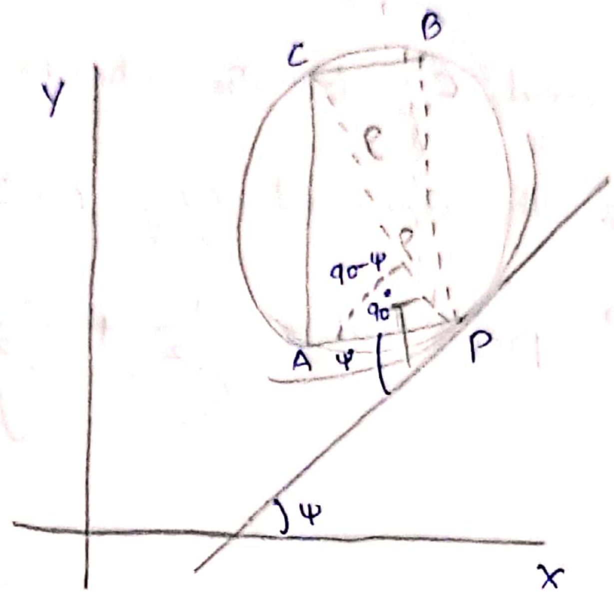
$$= \pm a\sqrt{2}$$

Hence $\rho(0,0)$ is numerically $a\sqrt{2}$

(3)

If C_x and C_y be the chords of curvature ||el to the axes at any point of the curve $y = ae^{x/a}$

Prove that $\frac{1}{C_x^2} + \frac{1}{C_y^2} = \frac{1}{2aC_x}$



Chord Ap || x-axis

Ac || y-axis

From ΔACP, $\angle A = 90^\circ$ angle in semicircle

$\angle APC = 90 - \psi$

$\cos(90-\psi) = \frac{AP}{CP}$

$\sin\psi = \frac{AP}{2\rho}$

$\boxed{AP = 2\rho\sin\psi}$ ||el to x-axis

and $\sin(90-\psi) = \frac{AC}{CP}$

$\cos\psi = \frac{AC}{2\rho}$

$\boxed{AC = 2\rho\cos\psi}$ $\|^{el}$ to y-axis

Let C_x be the chord of curvature $\|^{el}$ to x-axis

C_y be the chord of curvature $\|^{el}$ to y-axis

$$C_x = 2\rho \sin\psi$$

$$= \frac{2\rho}{\operatorname{cosec}\psi}$$

$$= \frac{2\rho}{\sqrt{1+\cot^2\psi}}$$

$$= \frac{2\rho}{\sqrt{1+\frac{1}{\tan^2\psi}}} = \frac{2\rho}{\sqrt{\tan^2\psi+1}} \times \tan\psi$$

$$= 2\rho \frac{\tan\psi}{\sqrt{1+\tan^2\psi}}$$

$$C_x = \frac{2(1+y_1^2)^{3/2}}{y_2} \times \frac{y_1}{\sqrt{1+y_1^2}}$$

$$= 2\left(\frac{y_1}{y_2}\right)(1+y_1^2)^{3/2-1/2}$$

$$C_x = \frac{2y_1}{y_2}(1+y_1^2) \quad \text{——} \ (1)$$

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Consider $y = ae^{x/a}$

$$y_1 = \frac{dy}{dx} = ae^{x/a} \cdot \frac{1}{a}$$

$$y_1 = e^{x/a}$$

and $y_2 = \frac{d^2y}{dx^2} = e^{x/a} \cdot \frac{1}{a}$

$$y_2 = \frac{1}{a} e^{x/a}$$

Substitute the corresponding values in eq ①

$$C_x = \frac{2\cancel{e^{x/a}}}{\left(\frac{1}{a}\right)\cancel{e^{x/a}}} \left[1 + (e^{x/a})^2\right]$$

$$C_x = 2a\left(1 + e^{2x/a}\right) \text{ ——— } ②$$

and C_y be the chord $\parallel^{el}$ to y-axis

$$C_y = 2\rho \cos\psi$$

$$= \frac{2\rho}{\sec\psi}$$

$$= \frac{2\rho}{\sqrt{1+\tan^2\psi}}$$

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$$= \frac{2(1+y_1^2)^{3/2}}{y_2} \times \frac{1}{\sqrt{1+y_1^2}}$$

$$C_y = \frac{2(1+y_1^2)}{y_2}$$

Substitute the corresponding values in eq ③

$$= \frac{2\left(1+\left(e^{x/a}\right)^2\right)}{\frac{1}{a}e^{x/a}}$$

$$C_y = \frac{2a}{e^{x/a}}\left(1+e^{2x/a}\right) \quad \text{——} ④$$

Squaring and adding ② & ④

$$C_x^2 + C_y^2 = \left[2a\left(1+e^{2x/a}\right)\right]^2 + \left[\frac{2a}{e^{x/a}}\left(1+e^{2x/a}\right)\right]^2$$

$$= 4a^2\left(1+e^{2x/a}\right)^2 + \frac{4a^2}{e^{2x/a}}\left(1+e^{2x/a}\right)^2$$

$$\frac{1}{C_x^2} + \frac{1}{C_y^2} = \frac{1}{4a^2\left(1+e^{2x/a}\right)^2} + \frac{e^{2x/a}}{4a^2\left(1+e^{2x/a}\right)^2}$$

$$= \frac{1}{4a^2\left(1+e^{2x/a}\right)^{2}}\left[1+e^{2x/a}\right]$$

$$= \frac{1}{4a^2\left(1+e^{2x/a}\right)}$$

$$= \frac{1}{2ac_x} \qquad \left[\because c_x = 2a\left(1+e^{2x/a}\right)\right]$$

$$\therefore \boxed{\frac{1}{c_x^2} + \frac{1}{c_y^2} = \frac{1}{2ac_x}}$$