UNIT – II

TOTAL DIFFERENTIATION

Composite function :–

Let $z = f(x, y)$ be a function of two variables then the total derivative is

$$\boxed{dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} \cdot dy}$$

Let $z = f(x, y)$ is a continuous partial derivatives and let $x = \phi(t)$, $y = \psi(t)$ are continuous derivatives.

then $$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}.$$

→ This is known as Differentiation of composite function.

→ Let f be any function of two variables x and y.

then the first order derivative is

$$\boxed{\frac{dy}{dx} = -\frac{f_x}{f_y}}$$

and second order derivative is

$$\boxed{\frac{d^2y}{dx^2} = -\frac{f_{xx}(f_y)^2 - 2f_{yx} \cdot f_x \cdot f_y + f_{yy} \cdot (f_x)^2}{(f_y)^3}}$$

Problems :—

(1) Find $\frac{dz}{dt}$ when $z = xy^2 + x^2y$; $x = at^2$, $y = 2at$.

Sol :— Given that $z = xy^2 + x^2y$ —— (1)

$x = at^2$ —— (2)

$y = 2at$ —— (3)

We know that

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} \quad —— (4)$$

Now, partially derivative of eqn (1) w.r.to 'x':

$$\frac{\partial z}{\partial x} = y^2(1) + y(2x)$$

$$\frac{\partial z}{\partial x} = y^2 + 2xy.$$

Now, partially derivative of eqn (1) w.r.to 'y':

$$\frac{\partial z}{\partial y} = x(2y) + x^2(1).$$

$$\frac{\partial z}{\partial y} = 2xy + x^2$$

Now, derivative of eqn (2) w.r.to 't':

$$\frac{dx}{dt} = a(2t).$$

Now, derivative of eqn (3) w.r.to 't':

$$\frac{dy}{dt} = 2a(1).$$

(4) $\Rightarrow$ $$\frac{dz}{dt} = (y^2 + 2xy)(2at) + (2xy + x^2)(2a)$$

$$\frac{dz}{dt} = 2aty^2 + 4atxy + 4axy + 2ax^2$$

Sub eqn (2) & (3) in above eqn, we get

$$\frac{dZ}{dt} = 2at(2at)^2 + 4at(at^2)(2at) + 4a(at^2)(2at) + 2a(at^2)^2$$

$$= 2at(4a^2t^2) + 4at(2a^2t^3) + 4a(2a^2t^3) + 2a(a^2t^4)$$

$$= 8a^3t^3 + 8a^3t^4 + 8a^3t^3 + 2a^3t^4$$

$$\frac{dZ}{dt} = 16a^3t^3 + 10a^3t^4$$

(Or).

Sol:- Given that $Z = xy^2 + x^2y$ — ①

$x = at^2$ — ②

$y = 2at$ — ③

Sub eqns ② & ③ in eqn ①, we get

$$Z = (at^2)(2at)^2 + (at^2)^2(2at)$$

$$= (at^2)(4a^2t^2) + (a^2t^4)(2at)$$

$$Z = 4a^3t^4 + 2a^3t^5.$$

Now, derivative w.r.to 't'.

$$\frac{dZ}{dt} = 4a^3(4t^3) + 2a^3(5t^4).$$

$$\frac{dZ}{dt} = 16a^3t^3 + 10a^3t^4$$

② Z is a function of x and y. If $x = e^u + e^{-v}$, $y = e^{-u} - e^v$ then prove that $\frac{\partial Z}{\partial u} - \frac{\partial Z}{\partial v} = x\frac{\partial Z}{\partial x} - y\frac{\partial Z}{\partial y}$

Sol:- Given that $x = e^u + e^{-v}$ — ①

$y = e^{-u} - e^v$ — ②

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partially derivative of eqn ① w.r.to u & v

$\frac{\partial x}{\partial u} = e^{u} + 0 \Rightarrow \frac{\partial x}{\partial u} = e^{u}$

$\frac{\partial x}{\partial v} = 0 + e^{-v}(-1) \Rightarrow \frac{\partial x}{\partial v} = -e^{-v}$.

partially derivative of eqn ② w.r.to u & v.

$\frac{\partial y}{\partial u} = e^{-u}(-1) - 0 \Rightarrow \frac{\partial y}{\partial u} = -e^{-u}$.

$\frac{\partial y}{\partial v} = 0 - e^{v} \Rightarrow \frac{\partial y}{\partial v} = -e^{v}$.

Now, We have

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}.$$

$$= \frac{\partial z}{\partial x} \cdot e^{u} + \frac{\partial z}{\partial y}(-e^{-u}).$$

$$\frac{\partial z}{\partial u} = e^{u}\frac{\partial z}{\partial x} - e^{-u}\frac{\partial z}{\partial y}.$$

Now,

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}.$$

$$= \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(-e^{v}).$$

$$\frac{\partial z}{\partial v} = -e^{-v}\frac{\partial z}{\partial x} - e^{v}\frac{\partial z}{\partial y}.$$

Now, Consider

$$L.H.S = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}.$$

$$= e^{u}\frac{\partial z}{\partial x} - e^{-u}\frac{\partial z}{\partial y} + e^{-v}\frac{\partial z}{\partial x} + e^{v}\frac{\partial z}{\partial y}$$

$$= (e^{u} + e^{-v})\frac{\partial z}{\partial x} - (e^{-u} - e^{v})\frac{\partial z}{\partial y}.$$

$$L.H.S = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x \cdot \frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} \quad [\because \text{From } ① \& ②]$$

$$= R.H.S$$

$\therefore$ $L.H.S = R.H.S$

$$\therefore \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$

Hence proved.

③ If $H = f(y-z, z-x, x-y)$ then prove that

$$\frac{\partial H}{\partial x} + \frac{\partial H}{\partial y} + \frac{\partial H}{\partial z} = 0.$$

Sol:- Given that $H = f(y-z, z-x, x-y)$ —— ①

Let $u = y-z$ (i), $v = z-x$ (ii), $w = x-y$. (iii)

① $\Rightarrow$ $H = f(u, v, w)$.

Now, we have

$$\frac{\partial H}{\partial x} = \frac{\partial H}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial H}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial H}{\partial w} \cdot \frac{\partial w}{\partial x} \quad —— ②$$

Now, partially derivative of eqns (i), (ii) & (iii) w.r.to 'x'.

$$\frac{\partial u}{\partial x} = 0 \; ; \; \frac{\partial v}{\partial x} = -1 \; ; \; \frac{\partial w}{\partial x} = 1.$$

$$② \Rightarrow \frac{\partial H}{\partial x} = \frac{\partial H}{\partial u}(0) + \frac{\partial H}{\partial v}(-1) + \frac{\partial H}{\partial w} \cdot (1)$$

$$\frac{\partial H}{\partial x} = -\frac{\partial H}{\partial v} + \frac{\partial H}{\partial w} \quad —— ③$$

Now, we have.

$$\frac{\partial H}{\partial y} = \frac{\partial H}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial H}{\partial v} \cdot \frac{\partial v}{\partial y} + \frac{\partial H}{\partial w} \cdot \frac{\partial w}{\partial y} \quad —— ④$$

Now, partially derivative of eqns (i) & (ii) & (iii) w.r.to y

$$\frac{\partial u}{\partial y} = 1 \ ; \ \frac{\partial v}{\partial y} = 0 \ ; \ \frac{\partial w}{\partial y} = -1$$

$$(4) \Rightarrow \frac{\partial H}{\partial y} = \frac{\partial H}{\partial u}(1) + \frac{\partial H}{\partial v}(0) + \frac{\partial H}{\partial w}(-1)$$

$$\frac{\partial H}{\partial y} = \frac{\partial H}{\partial u} - \frac{\partial H}{\partial w} \quad \text{—— (5)}$$

Now, partially derivative of eqns (i), (ii) & (iii) w.r.to z

$$\frac{\partial u}{\partial z} = -1 \ ; \ \frac{\partial v}{\partial z} = 1 \ ; \ \frac{\partial w}{\partial z} = 0.$$

Now, We have

$$\frac{\partial H}{\partial z} = \frac{\partial H}{\partial u} \cdot \frac{\partial u}{\partial z} + \frac{\partial H}{\partial v} \cdot \frac{\partial v}{\partial z} + \frac{\partial H}{\partial w} \cdot \frac{\partial w}{\partial z}.$$

$$= \frac{\partial H}{\partial u}(-1) + \frac{\partial H}{\partial v}(1) + \frac{\partial H}{\partial w}(0).$$

$$\frac{\partial H}{\partial z} = -\frac{\partial H}{\partial u} + \frac{\partial H}{\partial v} \quad \text{—— (6)}$$

Now, Adding eqns (3), (5) & (6), we get

$$\frac{\partial H}{\partial x} + \frac{\partial H}{\partial y} + \frac{\partial H}{\partial z} = -\cancel{\frac{\partial H}{\partial v}} + \cancel{\frac{\partial H}{\partial w}} + \cancel{\frac{\partial H}{\partial u}} - \cancel{\frac{\partial H}{\partial w}} - \cancel{\frac{\partial H}{\partial u}} + \cancel{\frac{\partial H}{\partial v}}$$

$$\therefore \frac{\partial H}{\partial x} + \frac{\partial H}{\partial y} + \frac{\partial H}{\partial z} = 0.$$

Hence proved.

④ If $z = \frac{\cos y}{x}$ and $x = u^2 - v$, $y = e^v$ then find $\frac{\partial z}{\partial v}$.

Sol:- Given that $z = \frac{\cos y}{x}$ — ① and

$x = u^2 - v$, $y = e^v$

$$① \Rightarrow z = \frac{\cos e^v}{u^2 - v} \quad —②$$

Now, partially derivative of eqn ② w.r.to v:

$$\frac{\partial z}{\partial v} = \frac{(u^2-v)\left[-\sin e^v \cdot e^v (1)\right] - \cos e^v (0 - 1)}{(u^2-v)^2}.$$

$$\frac{\partial z}{\partial v} = \frac{\cos e^v - (u^2 - v) \cdot e^v \cdot \sin e^v}{(u^2-v)^2}.$$

Replace $u^2 - v = x$, $e^v = y$.

$$\therefore \frac{\partial z}{\partial v} = \frac{\cos y - xy \sin y}{x^2}.$$

⑤ If $y^3 - 3ax^2 + x^3 = 0$ then prove that $\frac{d^2y}{dx^2} + \frac{2a^2x^2}{y^5} = 0$.

Sol:- Given that $y^3 - 3ax^2 + x^3 = 0$.

$$\Rightarrow y^3 = 3ax^2 - x^3.$$

Let $f(x, y) = y^3 - 3ax^2 + x^3 = 0$ — ①

partially derivative of eqn ① w.r.to 'x'.

$$\frac{\partial f}{\partial x} = f_x = 3x^2 - 3a(2x) \quad —②$$

$$f_x = 3x^2 - 6ax.$$

Again, partially derivative w.r.to 'x'.

$$\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial x}\left[3x^2 - 6ax\right]$$

$$\frac{\partial^2 f}{\partial x^2} = 3(2x) - 6a(1)$$

$$f_x^2 = 6x - 6a.$$

Now, partially derivative of eqn ① w.r.to 'y'.

$$\frac{\partial f}{\partial y} = 3y^2 - 0 + 0 \quad \text{—} ③$$

$$f_y = 3y^2$$

Again, partially derivative w.r.to 'y'.

$$\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial y}\left[3y^2\right]$$

$$\frac{\partial^2 f}{\partial y^2} = 3(2y)$$

$$f_y^2 = 6y.$$

Again, partially derivative of eqn ③ w.r.to 'x'.

$$\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial x}\left[3y^2\right]$$

$$\frac{\partial^2 f}{\partial x \partial y} = 0$$

$$f_{xy} = 0.$$

Again, partially derivative of eqn ② w.r.to 'y'.

$$\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial y}\left[3x^2 - 6ax\right]$$

$$\frac{\partial^2 f}{\partial y \partial x} = 0$$

$$f_{yx} = 0.$$

Now, we have

$$\frac{d^2y}{dx^2} = -\frac{f_x^2 (f_y)^2 - 2 f_{yx} \cdot f_x \cdot f_y + f_y^2 \cdot (f_x)^2}{(f_y)^3}.$$

$$= -\frac{(6x-6a)(3y^2)^2 - 2(0)(3x^2-6ax)(3y^2) + 6y(3x^2-6ax)^2}{(3y^2)^3}.$$

$$= -\frac{6(x-a)(9y^4) + 54y(x^2-2ax)^2}{27y^6}.$$

$$= -\frac{54(x-a)y^4 + 54y\left[x^4 - 4ax^3 + 4a^2x^2\right]}{27y^6}.$$

$$= -\frac{\overset{2}{\cancel{54}}y\left[(x-a)y^3 + x^4 - 4ax^3 + 4a^2x^2\right]}{\cancel{27}y^6}$$

$$= -\frac{2\left[(x-a)(3ax^2 - x^3) + x^4 - 4ax^3 + 4a^2x^2\right]}{y^5}.$$

$$= \frac{-2}{y^5}\left[\cancel{3ax^3} - \cancel{x^4} - 3a^2x^2 + \cancel{ax^3} + \cancel{x^4} - \cancel{4ax^3} + 4a^2x^2\right]$$

$$= -\frac{2a^2x^2}{y^5}.$$

$$\therefore \frac{d^2y}{dx^2} + \frac{2a^2x^2}{y^5} = 0.$$

Hence proved.

(6) If $x^3+y^3-3axy=0$ then find $\frac{d^2y}{dx^2}$ and prove that

$$\frac{d^2y}{dx^2}\cdot\frac{d^2x}{dy^2}=\frac{4a^6}{xy(xy-2a^2)^3}$$

Sol:- Given that $x^3+y^3-3axy=0 \Rightarrow x^3+y^3=3axy$ — (2)

Let $f(x,y)=x^3+y^3-3axy$ — (1)

$$\frac{\partial f}{\partial x}=3x^2+0-3ay(1) \qquad \frac{\partial f}{\partial y}=3y^2-3ax(1)$$

$$\boxed{f_x=3x^2-3ay} \qquad \boxed{f_y=3y^2-3ax}$$

$$\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right]=\frac{\partial}{\partial x}\left[3x^2-3ay\right] \qquad \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right]=\frac{\partial}{\partial y}\left[3y^2-3ax\right]$$

$$\frac{\partial^2 f}{\partial x^2}=3(2x)-0 \qquad \frac{\partial^2 f}{\partial y^2}=3(2y)-0$$

$$\boxed{f_x^2=6x} \qquad \boxed{f_y^2=6y}$$

Now, $\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]=\frac{\partial}{\partial y}\left[3x^2-3ay\right]$

$$\frac{\partial^2 f}{\partial y\,\partial x}=0-3a(1)$$

$$\boxed{f_{yx}=-3a.}$$

Now, We have.

$$\frac{d^2y}{dx^2}=-\frac{f_x^2\cdot(f_y)^2-2f_{yx}\cdot f_x\cdot f_y+f_y^2(f_x)^2}{(f_y)^3}.$$

$$=-\frac{6x(3y^2-3ax)^2-2(-3a)(3x^2-3ay)(3y^2-3ax)+6y(3x^2-3ay)^2}{(3y^2-3ax)^3}.$$

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$$\frac{d^2y}{dx^2} = -\frac{54x(y^2-ax)^2+54a(x^2-ay)(y^2-ax)+54y(x^2-ay)^2}{27(y^2-ax)^3}$$

$$= -\frac{\cancel{54}^{\,2}\left[x(y^4-2axy^2+a^2x^2)+a(x^2y^2-ax^3-ay^3+a^2xy)+y(x^4-2ax^2y+a^2y^2)\right]}{\cancel{27}(y^2-ax)^3}$$

$$= -\frac{2xy^4-4ax^2y^2+\cancel{2a^2x^3}+2ax^2y^2-\cancel{2a^2x^3}-\cancel{2a^2y^3}+2a^3xy+2x^4y-4ax^2y^2+\cancel{2a^2y^3}}{(y^2-ax)^3}$$

$$= -\frac{2xy^4-4ax^2y^2+2ax^2y^2+2a^3xy+2x^4y-4ax^2y^2}{(y^2-ax)^3}$$

$$= -\frac{2xy(x^3+y^3)-6ax^2y^2+2a^3xy}{(y^2-ax)^3}$$

$$= -\frac{2xy(3axy)-6ax^2y^2+2a^3xy}{(y^2-ax)^3} \qquad [\because \text{From eqn } ②]$$

$$= -\frac{\cancel{6ax^2y^2}-\cancel{6ax^2y^2}+2a^3xy}{(y^2-ax)^3}$$

$$\frac{d^2y}{dx^2} = -\frac{2a^3xy}{(y^2-ax)^3}$$

Similarly,

$$\frac{d^2x}{dy^2} = -\frac{2a^3xy}{(x^2-ay)^3}$$

Now, Consider

$$L \cdot H \cdot S = \frac{d^2y}{dx^2} \cdot \frac{d^2x}{dy^2}$$

$$= \frac{-2a^3xy}{(y^2-ax)^3} \cdot \frac{-2a^3xy}{(x^2-ay)^3}$$

$$= \frac{4a^6x^2y^2}{(y^2-ax)^3(x^2-ay)^3}$$

$$= \frac{4a^6x^2y^2}{[(y^2-ax)(x^2-ay)]^3}$$

$$= \frac{4a^6x^2y^2}{[x^2y^2-ay^3-ax^3+a^2xy]^3}$$

$$= \frac{4a^6x^2y^2}{[x^2y^2-a(x^3+y^3)+a^2xy]^3}$$

$$= \frac{4a^6x^2y^2}{[x^2y^2-a(3axy)+a^2xy]^3} \quad \left[\because \text{From eqn } ②\right]$$

$$= \frac{4a^6x^2y^2}{[x^2y^2-3a^2xy+a^2xy]^3}$$

$$= \frac{4a^6x^2y^2}{[x^2y^2-2a^2xy]^3} = \frac{4a^6x^2y^2}{x^3y^3[xy-2a^2]^3}$$

$$= \frac{4a^6}{xy(xy-2a^2)^3} = R \cdot H \cdot S$$

$\therefore L \cdot H \cdot S = R \cdot H \cdot S$

$$\frac{d^2y}{dx^2} \cdot \frac{d^2x}{dy^2} = \frac{4a^6}{xy(xy-2a^2)^3}$$ Hence proved.

⑦ If $x^4+y^4=4a^2xy$ then find $\frac{d^2y}{dx^2}$.

Sol:- Given that $x^4+y^4=4a^2xy$ —— ①

Let $f(x,y) = x^4+y^4-4a^2xy=0$.

$$\frac{\partial f}{\partial x} = 4x^3-4a^2y \qquad \frac{\partial f}{\partial y} = 4y^3-4a^2x.$$

$$\boxed{f_x = 4x^3-4a^2y} \qquad \boxed{f_y = 4y^3-4a^2x}$$

$$\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial x}\left[4x^3-4a^2y\right] \qquad \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial y}\left[4y^3-4a^2x\right]$$

$$\frac{\partial^2 f}{\partial x^2} = 4(3x^2)-0 \qquad \frac{\partial^2 f}{\partial y^2} = 4(3y^2)-0$$

$$\boxed{f_x^2 = 12x^2} \qquad \boxed{f_y^2 = 12y^2}$$

Now,

$$\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial y}\left[4x^3-4a^2y\right]$$

$$\frac{\partial^2 f}{\partial y \partial x} = 0-4a^2(1)$$

$$\boxed{f_{yx} = -4a^2}$$

Now, We have

$$\frac{d^2y}{dx^2} = -\frac{f_x^2\cdot(f_y)^2 - 2f_{yx}\cdot f_x\cdot f_y + f_y^2\cdot(f_x)^2}{(f_y)^3}.$$

$$= -\frac{12x^2(4y^3-4a^2x)^2 - 2(-4a^2)(4x^3-4a^2y)(4y^3-4a^2x) + 12y^2(4x^3-4a^2y)^2}{(4y^3-4a^2x)^3}.$$

$$\frac{d^2y}{dx^2} = -\frac{192x^2(y^3-a^2x)^2 + 128a^2(x^3-a^2y)(y^3-a^2x) + 192y^2(x^3-a^2y)^2}{64(y^3-a^2x)^3}$$

$$= -\frac{3x^2(y^6-2a^2xy^3+a^4x^2)+2a^2(x^3y^3-a^2x^4-a^2y^4+a^4xy)+3y^2(x^6-2a^2x^3y+a^4y^2)}{(y^3-a^2x)^3}$$

$$= -\frac{3x^2y^6-6a^2x^3y^3+3a^4x^4+2a^2x^3y^3-2a^4x^4-2a^4y^4+2a^6xy+3x^6y^2-6a^2x^3y^3+3a^4y^4}{(y^2-a^2x)^3}$$

$$= -\frac{3x^2y^2(y^4+x^4)+3a^4(x^4+y^4)-10a^2x^3y^3-2a^4(x^4+y^4)+2a^6xy}{(y^2-a^2x)^3}$$

$$= -\frac{3x^2y^2(4a^2xy)+3a^4(4a^2xy)-10a^2x^3y^3-2a^4(4a^2xy)+2a^6xy}{(y^2-a^2x)^3}$$

$$= -\frac{12a^2x^3y^3+12a^6xy-10a^2x^3y^3-8a^6xy+2a^6xy}{(y^2-a^2x)^3}$$

$$= -\frac{2a^2x^3y^3+6a^6xy}{(y^2-a^2x)^3}$$

$$\frac{d^2y}{dx^2} = -\frac{2a^2xy(x^2y^2+3a^4)}{(y^2-a^2x)^3}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{2a^2xy(x^2y^2+3a^4)}{(a^2x-y^2)^3}.$$

⑧ If $x^5+y^5=5a^3xy$ then find $\frac{d^2y}{dx^2}$.

Sol: – Given that $x^5+y^5=5a^3xy$ —— ①

Let $f(x,y) = x^5+y^5-5a^3xy$.

$\frac{\partial f}{\partial x} = 5x^4-5a^3y(1)$

$\boxed{f_x = 5x^4-5a^3y}$

$\frac{\partial f}{\partial y} = 5y^4-5a^3x(1)$

$\boxed{f_y = 5y^4-5a^3x}$

$\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial x}\left[5x^4-5a^3y\right]$

$\frac{\partial^2 f}{\partial x^2} = 5(4x^3)-0$

$\boxed{f_x^2 = 20x^3}$

$\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial y}\left[5y^4-5a^3x\right]$

$\frac{\partial^2 f}{\partial y^2} = 5(4y^3)-0$

$\boxed{f_y^2 = 20y^3}$

Now,

$\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial y}\left[5x^4-5a^3y\right]$

$\frac{\partial^2 f}{\partial y \partial x} = 0-5a^3(1)$

$\boxed{f_{yx} = -5a^3}$

Now, We have

$$\frac{d^2y}{dx^2} = -\frac{f_x^2\cdot(f_y)^2-2f_{yx}\cdot f_x\cdot f_y+f_y^2\cdot(f_x)^2}{(f_y)^3}.$$

$$= -\frac{20x^3(5y^4-5a^3x)^2-2(-5a^3)\cdot(5x^4-5a^3y)(5y^4-5a^3x)+20y^3(5x^4-5a^3y)^2}{(5y^4-5a^3x)^3}.$$

$$\frac{d^2y}{dx^2} = -\frac{500x^3(y^4-a^3x)^2 + 250a^3(x^4-a^3y)(y^4-a^3x) + 500y^3(x^4-a^3y)^2}{125(y^4-a^3x)^3}$$

$$= -\frac{4x^3(y^8-2a^3xy^4+a^6x^2) + 2a^3(x^4y^4-a^3x^5-a^3y^5+a^6xy) + 4y^3(x^8-2a^3x^4y+a^6y^2)}{(y^4-a^3x)^3}$$

$$= -\frac{4x^3y^8-8a^3x^4y^4+4a^6x^5+2a^3x^4y^4-2a^6x^5-2a^6y^5+2a^9xy + 4x^8y^3-8a^3x^4y^4+4a^6y^5}{(y^4-a^3x)^3}$$

$$= \frac{4x^3y^3(x^5+y^5)+4a^6(x^5+y^5)-14a^3x^4y^4 - 2a^6(x^5+y^5) + 2a^9xy}{(a^3x-y^4)^3}$$

$$= \frac{4x^3y^3(5a^3xy)+4a^6(5a^3xy)-14a^3x^4y^4 - 2a^6(5a^3xy)+2a^9xy}{(a^3x-y^4)^3}$$

$$= \frac{20a^3x^4y^4+20a^9xy-14a^3x^4y^4 - 10a^9xy+2a^9xy}{(a^3x-y^4)^3}$$

$$= \frac{6a^3x^4y^4+12a^9xy}{(a^3x-y^4)^3}$$

$$\frac{d^2y}{dx^2} = \frac{6a^3xy(2a^6+x^3y^3)}{(a^3x-y^4)^3}$$

⑨ If $x^5+y^5=5a^3x^2$ then find $\frac{d^2y}{dx^2}$.

Sol:- Given that $x^5+y^5=5a^3x^2$ —— ①

Let $f(x,y)=x^5+y^5-5a^3x^2$.

$\frac{\partial f}{\partial x}=5x^4-5a^3(2x)$ $\qquad$ $\frac{\partial f}{\partial y}=5y^4-0$

$\boxed{f_x=5x^4-10a^3x}$ $\qquad$ $\boxed{f_y=5y^4}$

$\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right]=\frac{\partial}{\partial x}\left[5x^4-10a^3x\right]$ $\qquad$ $\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right]=\frac{\partial}{\partial y}\left[5y^4\right]$

$\frac{\partial^2 f}{\partial x^2}=5(4x^3)-10a^3(1)$ $\qquad$ $\frac{\partial^2 f}{\partial y^2}=5(4y^3)$

$\boxed{f_x^2=20x^3-10a^3}$ $\qquad$ $\boxed{f_y^2=20y^3}$

Now,

$$\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]=\frac{\partial}{\partial y}\left[5x^4-10a^3x\right]$$

$$\frac{\partial^2 f}{\partial y\partial x}=0.$$

$$\boxed{f_{yx}=0.}$$

Now, We have.

$$\frac{d^2y}{dx^2}=-\frac{f_x^2\cdot(f_y)^2-2f_{yx}\cdot f_x\cdot f_y+f_y^2\cdot(f_x)^2}{(f_y)^3}.$$

$$=-\frac{(20x^3-10a^3)(5y^4)^2-2(0)(5x^4-10a^3x)(5y^4)+20y^3(5x^4-10a^3x)^2}{(5y^4)^3}.$$

$$\frac{d^2y}{dx^2} = -\frac{(20x^3-10a^3)(25y^8)+500y^3(x^4-2a^3x)^2}{125y^{12}}$$

$$= -\frac{250y^8(2x^3-a^3)+500y^3(x^8-4a^3x^5+4a^6x^2)}{125y^{12}}$$

$$= -\frac{2y^5(2x^3-a^3)+4x^8-16a^3x^5+16a^6x^2}{y^9}$$

$$= -\frac{4x^3y^5-2a^3y^5+4x^8-16a^3x^5+16a^6x^2}{y^9}$$

$$= -\frac{4x^3(y^5+x^5)-16a^3x^5-2a^3y^5+16a^6x^2}{y^9}$$

$$= -\frac{4x^3(5a^3x^2)-16a^3x^5-2a^3y^5+16a^6x^2}{y^9}$$

$$= -\frac{20a^3x^5-16a^3x^5-2a^3(5a^3x^2-x^5)+16a^6x^2}{y^9}$$

$$= -\frac{4a^3x^5-10a^6x^2+2a^3x^5+16a^6x^2}{y^9}$$

$$= -\frac{6a^3x^5+6a^6x^2}{y^9}$$

$$\frac{d^2y}{dx^2} = -\frac{6a^3x^2(x^3+a^3)}{y^9}$$

(10) If $ax^2+2hxy+by^2+2gx+2fy+c=0$ then prove that $\frac{d^2y}{dx^2}=\frac{abc+2fgh-af^2-bg^2-ch^2}{(hx+by+f)^3}$.

Sol:- Given that $ax^2+2hxy+by^2+2gx+2fy+c=0$

Let $f(x,y)=ax^2+2hxy+by^2+2gx+2fy+c$.

$$\frac{\partial f}{\partial x}=a(2x)+2hy(1)+2g(1) \qquad \frac{\partial f}{\partial y}=2hx(1)+b(2y)+2f(1)$$

$$f_x=2ax+2hy+2g. \qquad f_y=2hx+2by+2f.$$

$$\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right]=\frac{\partial}{\partial x}\left[2ax+2hy+2g\right] \qquad \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right]=\frac{\partial}{\partial y}\left[2hx+2by+2f\right]$$

$$\frac{\partial^2 f}{\partial x^2}=2a(1) \qquad \frac{\partial^2 f}{\partial y^2}=2b(1)$$

$$f_x^2=2a. \qquad f_y^2=2b.$$

Now,

$$\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]=\frac{\partial}{\partial y}\left[2ax+2hy+2g\right]$$

$$\frac{\partial^2 f}{\partial y\partial x}=2h(1)$$

$$f_{yx}=2h.$$

Now, We have

$$\frac{d^2y}{dx^2}=-\frac{f_x^2(f_y)^2-2f_{yx}\cdot f_x\cdot f_y+f_y^2\cdot(f_x)^2}{(f_y)^3}.$$

$$=-\frac{2a(2hx+2by+2f)^2-2(2h)(2ax+2hy+2g)(2hx+2by+2f)+2b(2ax+2hy+2g)^2}{(2hx+2by+2f)^3}.$$

$$\frac{d^2y}{dx^2} = -\frac{8a(hx+by+f)^2 - 16h(ax+hy+g)(hx+by+f) + 8b(ax+hy+g)^2}{8(hx+by+f)^3}$$

$$= -\frac{a(h^2x^2+b^2y^2+f^2+2hbxy+2bfy+2hfx) - 2h(ahx^2+abxy+afx+h^2xy+hby^2+hfy+ghx+gby+gf) + b(a^2x^2+h^2y^2+g^2+2haxy+2hgy+2agx)}{(hx+by+f)^3}$$

$$= -\left\{\frac{ah^2x^2+ab^2y^2+af^2+\cancel{2abhxy}+2abfy+\cancel{2ahfx}-2ah^2x^2-\cancel{2abhxy}-\cancel{2ahfx}-2h^3xy-2h^2by^2-2h^2fy-2gh^2x-\cancel{2hgby}-2hfg+a^2bx^2+bh^2y^2+bg^2+2abhxy+\cancel{2bhgy}+2abgx}{(hx+by+f)^3}\right\}$$

$$= -\frac{\left\{-ah^2x^2+ab^2y^2+af^2+2abfy-2h^3xy-h^2by^2-2h^2fy-2gh^2x-2hfg+a^2bx^2+bg^2+2abhxy+2abgx\right\}}{(hx+by+f)^3}$$

$$= -\frac{\left\{ab(ax^2+by^2+2hxy+2gx+2fy)-h^2(ax^2+by^2+2gx+2fy+2hxy)+(af^2-2hfg+bg^2)\right\}}{(hx+by+f)^3}$$

$$= -\frac{ab(-c)-h^2(-c)+af^2-2hfg+bg^2}{(hx+by+f)^3}$$

$$= -\frac{-abc+h^2c+af^2-2hfg+bg^2}{(hx+by+f)^3}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{abc+2hfg-af^2-bg^2-ch^2}{(hx+by+f)^3}$$

(11) If $x\sqrt{1-y^2}+y\sqrt{1-x^2}=a$ then show that $\frac{d^2y}{dx^2}=\frac{-a}{(1-x^2)^{3/2}}$

Sol:- Given that $x\sqrt{1-y^2}+y\sqrt{1-x^2}=a$ —(1)

Let $f(x,y)=x\sqrt{1-y^2}+y\sqrt{1-x^2}-a$.

$$\frac{\partial f}{\partial x}=\sqrt{1-y^2}.(1)+y.\frac{1}{\cancel{2}\sqrt{1-x^2}}.(-\cancel{2}x)$$

$$f_x=\sqrt{1-y^2}-\frac{xy}{\sqrt{1-x^2}}.$$

and $$\frac{\partial f}{\partial y}=x.\frac{1}{\cancel{2}\sqrt{1-y^2}}(-\cancel{2}y)+\sqrt{1-x^2}.(1).$$

$$f_y=\sqrt{1-x^2}-\frac{xy}{\sqrt{1-y^2}}.$$

Now, We have.

$$\frac{dy}{dx}=-\frac{f_x}{f_y}.$$

$$=-\frac{\sqrt{1-y^2}-\frac{xy}{\sqrt{1-x^2}}}{\sqrt{1-x^2}-\frac{xy}{\sqrt{1-y^2}}}.$$

$$=-\frac{\sqrt{1-y^2}.\cancel{\sqrt{1-x^2}}-xy}{\sqrt{1-x^2}}\Big/\frac{\sqrt{1-x^2}.\cancel{\sqrt{1-y^2}}-xy}{\sqrt{1-y^2}}.$$

$$\frac{dy}{dx}=-\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}.$$

Again, derivative w.r. to 'x',

$$\frac{d^2y}{dx^2} = -\frac{\sqrt{1-x^2}\cdot\frac{1}{\cancel{2}\sqrt{1-y^2}}(-\cancel{2}y)\cdot\frac{dy}{dx} - \sqrt{1-y^2}\cdot\frac{1}{\cancel{2}\sqrt{1-x^2}}(-\cancel{2}x)}{\left(\sqrt{1-x^2}\right)^{\cancel{2}}}$$

$$= -\frac{\frac{\cancel{\sqrt{1-x^2}}}{\cancel{\sqrt{1-y^2}}}(-y)\left[-\frac{\cancel{\sqrt{1-y^2}}}{\cancel{\sqrt{1-x^2}}}\right] + x\cdot\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}}{1-x^2}$$

$$= -\frac{y + x\cdot\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}}{1-x^2}$$

$$= -\frac{y\sqrt{1-x^2} + x\sqrt{1-y^2}}{(1-x^2)(1-x^2)^{1/2}}$$

$$\frac{d^2y}{dx^2} = -\frac{a}{(1-x^2)^{3/2}} \quad [\because \text{From eqn } ①].$$

(12) If u and v are functions of x and y defined by $x = u + e^{-v}\sin u$; $y = v + e^{-v}\cos u$ then prove that $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$.

Sol:- Given that

$$x = u + e^{-v}\sin u \text{ — } ① \qquad y = v + e^{-v}\cos u \text{ — } ②$$

Partially derivative of eqn ① w.r.to x:

$$1 = \frac{\partial u}{\partial x} + e^{-v}\cdot\cos u\cdot\frac{\partial u}{\partial x} + \sin u\cdot e^{-v}(-1)\cdot\frac{\partial v}{\partial x}.$$

$$1 = (1 + e^{-v}\cos u)\frac{\partial u}{\partial x} - e^{-v}\sin u\cdot\frac{\partial v}{\partial x} \text{ — } ③$$

partially derivative of eqn ① w.r.to y.

$$0 = \frac{\partial u}{\partial y} + e^{-v}\cdot\cos u\cdot\frac{\partial u}{\partial y} + \sin u\cdot e^{-v}(-1)\frac{\partial v}{\partial y}.$$

$$0 = (1+e^{-v}\cos u)\frac{\partial u}{\partial y} - e^{-v}\sin u\frac{\partial v}{\partial y} \quad \text{--- (4)}$$

partially derivative of eqn (2) w.r.to 'x'.

$$0 = \frac{\partial v}{\partial x} + e^{-v}(-\sin u)\cdot\frac{\partial u}{\partial x} + \cos u\cdot e^{-v}(-1)\cdot\frac{\partial v}{\partial x}.$$

$$0 = (1-e^{-v}\cos u)\frac{\partial v}{\partial x} - e^{-v}\sin u\cdot\frac{\partial u}{\partial x} \quad \text{--- (5)}$$

partially derivative of eqn (2) w.r.to 'y'.

$$1 = \frac{\partial v}{\partial y} + e^{-v}(-\sin u)\frac{\partial u}{\partial y} + \cos u\cdot e^{-v}(-1)\cdot\frac{\partial v}{\partial y}.$$

$$1 = (1-e^{-v}\cos u)\frac{\partial v}{\partial y} - e^{-v}\sin u\cdot\frac{\partial u}{\partial y} \quad \text{--- (6)}.$$

From eqn (4) $\Rightarrow e^{-v}\sin u\frac{\partial v}{\partial y} = (1+e^{-v}\cos u)\frac{\partial u}{\partial y}$.

$$\frac{\partial v}{\partial y} = \frac{(1+e^{-v}\cos u)\frac{\partial u}{\partial y}}{e^{-v}\sin u} \quad \text{--- (7)}$$

From eqn (5) $\Rightarrow e^{-v}\sin u\cdot\frac{\partial u}{\partial x} = (1-e^{-v}\cos u)\frac{\partial v}{\partial x}$.

$$\frac{\partial u}{\partial x} = \frac{(1-e^{-v}\cos u)\frac{\partial v}{\partial x}}{e^{-v}\sin u} \quad \text{--- (8)}$$

Sub eqn (8) in eqn (3), we get.

$$1 = (1+e^{-v}\cos u)\cdot\frac{(1-e^{-v}\cos u)\frac{\partial v}{\partial x}}{e^{-v}\sin u} - e^{-v}\sin u\cdot\frac{\partial v}{\partial x}.$$

$$1 = \frac{(1)^2-(e^{-v}\cos u)^2-(e^{-v}\sin u)^2}{e^{-v}\cdot\sin u}\cdot\frac{\partial v}{\partial x}.$$

$$\boxed{\begin{array}{l}\text{From eqn (1)}\\ x-u = e^{-v}\sin u\end{array}}$$

$$1 = \frac{1-e^{-2v}\cos^2 u - e^{-2v}\sin^2 u}{x-u}\cdot\frac{\partial v}{\partial x}.$$

$$x-u = \left[1-e^{-2v}(\cos^2 u+\sin^2 u)\right]\frac{\partial v}{\partial x}.$$

$$x-u = \left[1-e^{-2v}(1)\right]\frac{\partial v}{\partial x}$$

$\boxed{\cos^2\theta+\sin^2\theta=1}$

$$\frac{\partial v}{\partial x} = \frac{x-u}{1-e^{-2v}} \quad \text{——} \; (9)$$

Sub eqn (7) in eqn (6), we get

$$1 = (1-e^{-v}\cos u)\cdot\frac{(1+e^{-v}\cos u)\frac{\partial u}{\partial y}}{e^{-v}\sin u} - e^{-v}\sin u\frac{\partial u}{\partial y}.$$

$$1 = \frac{(1)^2-(e^{-v}\cos u)^2 - (e^{-v}\sin u)^2}{e^{-v}\sin u}\cdot\frac{\partial u}{\partial y}.$$

$$1 = \frac{1-e^{-2v}\cos^2 u - e^{-2v}\sin^2 u}{x-u}\cdot\frac{\partial u}{\partial y}.$$

$$x-u = \left[1-e^{-2v}(\cos^2 u+\sin^2 u)\right]\frac{\partial u}{\partial y}$$

$$x-u = \left[1-e^{-2v}(1)\right]\frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{x-u}{1-e^{-2v}} \quad \text{——} \; (10)$$

From eqns (9) & (10), we get

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}.$$

Hence proved.

(13) Find $\frac{dy}{dx}$ of the following functions :—

(i) $x\cdot\sin(x-y)-(x+y)=0.$

(ii) $(\cos x)^y-(\sin y)^x=0.$

(iii) $x^y=y^x.$

Sol:— (i) Given that $x\cdot\sin(x-y)-(x+y)=0.$ — (i)

Let $f(x,y)=x\sin(x-y)-(x+y).$

$$\frac{\partial f}{\partial x}=\sin(x-y)\cdot(1)+x\cdot\cos(x-y)\cdot(1)-1$$

$$f_x=\sin(x-y)+x\cos(x-y)-1.$$

$$\frac{\partial f}{\partial y}=x\cdot\cos(x-y)(-1)-1$$

$$f_y=-x\cos(x-y)-1$$

Now, We have

$$\frac{dy}{dx}=-\frac{f_x}{f_y}.$$

$$=-\frac{\sin(x-y)+x\cos(x-y)-1}{-x\cos(x-y)-1}.$$

$$\frac{dy}{dx}=\frac{\sin(x-y)+x\cos(x-y)-1}{x\cos(x-y)+1}.$$

From eqn (i), we have

$$\sin(x-y)=\frac{x+y}{x}=1+\frac{y}{x}.$$

$$\frac{dy}{dx}=\frac{1+\frac{y}{x}+x\cos(x-y)-1}{x\cos(x-y)+1}.$$

$$\frac{dy}{dx}=\frac{\frac{y}{x}+x\cos(x-y)}{x\cos(x-y)+1}.$$

(ii) Given that $(\cos x)^y - (\sin y)^x = 0$

$$(\cos x)^y = (\sin y)^x$$

Take "log" on both sides

$$\log(\cos x)^y = \log(\sin y)^x$$

$$y\log(\cos x) = x\log(\sin y)$$

Let $f(x,y) = y\log(\cos x) - x\log(\sin y) = 0$.

$$\frac{\partial f}{\partial x} = y\cdot\frac{1}{\cos x}(-\sin x) - \log(\sin y)\cdot(1)$$

$$f_x = \frac{-y\sin x - \log(\sin y)\cdot\cos x}{\cos x}$$

$$\frac{\partial f}{\partial y} = \log(\cos x)\cdot(1) - x\cdot\frac{1}{\sin y}\cdot\cos y$$

$$f_y = \frac{\sin y\cdot\log(\cos x) - x\cos y}{\sin y}$$

Now, we have.

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

$$= -\frac{\dfrac{-y\sin x - \log(\sin y)\cdot\cos x}{\cos x}}{\dfrac{\sin y\cdot\log(\cos x) - x\cos y}{\sin y}}$$

$$\frac{dy}{dx} = \frac{y\cdot\sin x\sin y + \cos x\sin y\cdot\log(\sin y)}{\cos x\sin y\log(\cos x) - x\cos x\cos y}$$

(iii) Given that $x^y = y^x$

Take "log" on both sides

$$\log x^y = \log y^x$$

$$y \log x = x \log y.$$

Let $f(x,y) = y \log x - x \log y = 0.$

$$\frac{\partial f}{\partial x} = y \cdot \frac{1}{x} - \log y \cdot (1).$$

$$f_x = \frac{y}{x} - \log y.$$

$$\frac{\partial f}{\partial y} = \log x \cdot (1) - x \cdot \frac{1}{y}.$$

$$f_y = \log x - \frac{x}{y}.$$

Now, We have.

$$\frac{dy}{dx} = -\frac{f_x}{f_y}.$$

$$= -\frac{\frac{y}{x} - \log y}{\log x - \frac{x}{y}}$$

$$= -\frac{\frac{y - x\log y}{x}}{\frac{y \log x - x}{y}}.$$

$$= -\frac{y(y - x\log y)}{x(y\log x - x)}.$$

$$\frac{dy}{dx} = \frac{y(y - x\log y)}{x(x - y\log x)}.$$

Implicit Function :-

Let $f(x,y)=0$.

We obtain 'f' has a function of 'x' then 'f' is defines 'y' has an implicit function of 'x'.

(14) If $f(x,y)=0$ and $\psi(y,z)=0$ then show that

$$\frac{\partial f}{\partial y}\cdot\frac{\partial \psi}{\partial z}\cdot\frac{dz}{dx}=\frac{\partial f}{\partial x}\cdot\frac{\partial \psi}{\partial y}$$

Sol:- Given that $f(x,y)=0$ and $\psi(y,z)=0$.

Since 'f' is defines 'y' as an implicit function of 'x' then

$$\frac{dy}{dx}=-\frac{f_x}{f_y}=-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \text{ --- (1)}$$

Since 'ψ' is defines 'z' as an implicit function of y. then.

$$\frac{dz}{dy}=-\frac{\psi_y}{\psi_z}=-\frac{\frac{\partial \psi}{\partial y}}{\frac{\partial \psi}{\partial z}} \text{ --- (2)}$$

Multiply eqns (1) & (2), we get.

$$\frac{\cancel{dy}}{dx}\cdot\frac{dz}{\cancel{dy}}=\left[-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}\right]\left[-\frac{\frac{\partial \psi}{\partial y}}{\frac{\partial \psi}{\partial z}}\right]$$

$$\frac{\partial f}{\partial y}\cdot\frac{\partial \psi}{\partial z}\cdot\frac{dz}{dx}=\frac{\partial f}{\partial x}\cdot\frac{\partial \psi}{\partial y}.$$

(15) If A, B, C are the angles of a triangle such that $\sin^2 A + \sin^2 B + \sin^2 C = \text{constant}$. then prove that

$$\frac{dA}{dB} = \frac{\tan B - \tan C}{\tan C - \tan A}.$$

Sol:- Given that

A, B, C are the angles of a triangle such that

$\sin^2 A + \sin^2 B + \sin^2 C = \text{Constant}$. then $A+B+C=\pi$. —(1)

Let $f(A,B) = \sin^2 A + \sin^2 B + \sin^2 C - \text{Constant}$

$f(A,B) = \sin^2 A + \sin^2 B + \sin^2(\pi-(A+B)) - \text{Constant}.$

$f(A,B) = \sin^2 A + \sin^2 B + \sin^2(A+B) - \text{Constant}$ —(2)

Now, $\frac{\partial f}{\partial A} = 2\sin A\cdot(\cos A) + 0 + 2\sin(A+B)\cdot\cos(A+B) - 0.$

$= 2\sin A\cos A + 2\sin(\pi-C)\cdot\cos(\pi-C).$

$= \sin 2A + 2\sin C\, \cancel{\cos C}\, (-\cos C).$

$= \sin 2A - 2\sin C\cos C$

$f_A = \sin 2A - \sin 2C$

and $\frac{\partial f}{\partial B} = 0 + 2\sin B(\cos B) + 2\sin(A+B)\cdot\cos(A+B) - 0$

$= 2\sin B\cos B + 2\sin(\pi-C)\cdot\cos(\pi-C).$

$= \sin 2B + 2\sin C(-\cos C).$

$f_B = \sin 2B - \sin 2C.$

~~We Hence.~~

$\cancel{\frac{dA}{dB} = \frac{dB}{dA} = -\frac{f_B}{f_A}}.$

$\cancel{\Rightarrow \frac{dA}{dB} = -}$

We have

$\frac{dB}{dA} = -\frac{f_A}{f_B}.$

$\frac{dA}{dB} = -\frac{f_B}{f_A}.$

$$\frac{dA}{dB} = -\frac{\sin 2B - \sin 2C}{\sin 2A - \sin 2C}$$

$$= -\left[\frac{-\cancel{2}\cos\left(\frac{2B+2C}{2}\right)\cdot\sin\left(\frac{2B-2C}{2}\right)}{-\cancel{2}\cos\left(\frac{2A+2C}{2}\right)\cdot\sin\left(\frac{2A-2C}{2}\right)}\right]$$

$$= -\left[\frac{\cos(B+C)\cdot\sin(B-C)}{\cos(A+C)\cdot\sin(A-C)}\right]$$

$$= -\left[\frac{\cos(\pi-A)\cdot\sin(B-C)}{\cos(\pi-B)\cdot\sin(A-C)}\right]$$

$$= -\frac{\cancel{\cos A}}{\cancel{\cos B}}\left[\frac{\sin B\cos C - \cos B\sin C}{\sin A\cos C - \cos A\sin C}\right] \times \frac{\cancel{\cos A}\cos\cancel{B}\cos C}{\cancel{\cos A}\cos B\cos C}$$

$$= -\frac{\sin B\cos C - \cos B\sin C}{\cos B\cdot\cos C} \times \frac{\cos A\cos C}{\sin A\cos C - \cos A\sin C}$$

$$= -\left[\frac{\frac{\sin B\cancel{\cos C}}{\cos B\cancel{\cos C}} - \frac{\cancel{\cos B}\sin C}{\cancel{\cos B}\cos C}}{\frac{\sin A\cancel{\cos C}}{\cos A\cancel{\cos C}} - \frac{\cancel{\cos A}\sin C}{\cancel{\cos A}\cos C}}\right]$$

$$= -\left[\frac{\tan B - \tan C}{\tan A - \tan C}\right]$$

$$\frac{dA}{dB} = \frac{\tan B - \tan C}{\tan C - \tan A}.$$

Equality of $f_{xy}(a,b)$ and $f_{yx}(a,b)$:-

① $$f_{xy}(a,b) = \lim_{h \to 0} \frac{f_y(a+h,b) - f_y(a,b)}{h}.$$

Where,

$$f_y(a,b) = \lim_{k \to 0} \frac{f(a,b+k) - f(a,b)}{k}. \quad \text{and}$$

$$f_y(a+h,b) = \lim_{k \to 0} \frac{f(a+h,b+k) - f(a,b)}{k}.$$

② $$f_{yx}(a,b) = \lim_{k \to 0} \frac{f_x(a,b+k) - f_x(a,b)}{k}.$$

Where,

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}. \quad \text{and}$$

$$\cancel{f_y(a+h} \; f_x(a,b+k) = \lim_{h \to 0} \frac{f(a+h,b+k) - f(a,b)}{h}.$$

Problems :-

① Prove that $f_{xy}(0,0) \neq f_{yx}(0,0)$ for the function and. f given by $f(x,y) = \dfrac{xy(x^2-y^2)}{x^2+y^2}$; $(x,y) \neq (0,0)$. and $f(0,0) = 0$.

Sol:- Given that $f(x,y) = \dfrac{xy(x^2-y^2)}{x^2+y^2}$; $(x,y) \neq (0,0)$

and $f(0,0) = 0$.

We know that

$$f_{xy}(a,b) = \lim_{h \to 0} \frac{f_y(a+h,b) - f_y(a,b)}{h}.$$

~~Where~~,

$\cancel{f_y(a,b) = \lim_{k \to 0}}$

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{f_y(0+h,0) - f_y(0,0)}{h} \quad \text{——(1)}$$

Where.

$$f_y(a,b) = \lim_{k\to 0} \frac{f(a,b+k) - f(a,b)}{k}.$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,0+k) - f(0,0)}{k}.$$

$$= \lim_{k\to 0} \frac{f(0,k) - 0}{k}.$$

$$= \frac{0}{k}.$$

$$f_y(0,0) = 0$$

and $$f_y(a+h,0) = \lim_{k\to 0} \frac{f(a+h,b+k) - f(a+h,0)}{k}.$$

$$f_y(0+h,0) = \lim_{k\to 0} \frac{f(0+h,0+k) - f(0+h,0)}{k}.$$

$$= \lim_{k\to 0} \frac{f(h,k) - f(h,0)}{k}.$$

$$= \lim_{k\to 0} \frac{\frac{hk(h^2-k^2)}{h^2+k^2} - 0}{k}.$$

$$= \lim_{k\to 0} \frac{h\cancel{k}(h^2-k^2)}{\cancel{k}(h^2+k^2)}$$

$$= \frac{h^{\cancel{3}}}{\cancel{h^2}}$$

$$f_y(0+h,0) = h$$

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(i) $\Rightarrow f_{xy}(0,0) = \lim_{h\to 0} \frac{h-0}{h}$.

$= \lim_{h\to 0} (1)$

$\therefore f_{xy}(0,0) = 1$.

Now,

$$f_{yx}(a,b) = \lim_{k\to 0} \frac{f_x(a,b+k) - f_x(a,b)}{k}.$$

$$f_{yx}(0,0) = \lim_{k\to 0} \frac{f_x(0,0+k) - f_x(0,0)}{k} \quad \text{--- (2)}$$

Where.

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}.$$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}.$$

$$= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}.$$

$$= \lim_{h\to 0} \frac{0}{h}.$$

$f_x(0,0) = 0$.

and $f_x(a,b+k) = \lim_{h\to 0} \dfrac{f(a+h,b+k) - f(a,b)}{h}$.

$$f_x(0,0+k) = \lim_{h\to 0} \frac{f(0+h,0+k) - f(0,0)}{h}.$$

$$= \lim_{h\to 0} \frac{f(h,k) - f(0,0)}{h}.$$

$$= \lim_{h\to 0} \frac{\frac{hk[h^2-k^2]}{h^2+k^2} - 0}{h}.$$

$$f_x(0,0+k) = \lim_{h\to 0} \frac{hk[h^2-k^2]}{k[h^2+k^2]}$$

$$= \frac{-k^{\cancel{3}}}{\cancel{k^2}}$$

$$f_x(0,0+k) = -k.$$

② $\Rightarrow$ $$f_{yx}(0,0) = \lim_{k\to 0} \frac{-k-0}{k}$$

$$= \lim_{k\to 0} \frac{-\cancel{k}}{\cancel{k}}$$

$$= \lim_{k\to 0} (-1)$$

$$f_{yx}(0,0) = -1.$$

$\therefore$ $f_{xy}(0,0) \neq f_{yx}(0,0)$.

Hence proved.

② Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$ where $f(x,y)=0$ of $xy=0$. and $f(x,y) = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$ of $xy \neq 0$.

Sol:- Given that $f(x,y)=0$ of $xy=0$ and

$f(x,y) = x\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$ of $xy \neq 0$. —— ①

We know that

$$f_{xy}(a,b) = \lim_{h\to 0} \frac{f_y(a+h,b) - f_y(a,b)}{h}.$$

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{f_y(0+h,0) - f_y(0,0)}{h}. \quad \text{—— ②}$$

Where

$$f_y(a,b) = \lim_{k \to 0} \frac{f(a,b+k) - f(a,b)}{k}$$

$$f_y(0,0) = \lim_{k \to 0} \frac{f(0,0+k) - f(0,0)}{k}$$

$$= \frac{0}{k}$$

$$f_y(0,0) = 0$$

$$f_y(\cancel{a}+h,b) = \lim_{k \to 0} \frac{f(a+h,b+k) - f(a,b)}{k}$$

$$f_y(0+h,0) = \lim_{k \to 0} \frac{f(0+h,0+k) - f(0,0)}{k}$$

$$= \lim_{k \to 0} \frac{f(h,k) - f(0,0)}{k}$$

$$= \lim_{k \to 0} \frac{h^2 \tan^{-1}(k/h) - k^2 \tan^{-1}(\frac{h}{k}) - 0}{k}$$

$$= \lim_{k \to 0} \frac{h^2 \tan^{-1}(k/h)}{k} - \lim_{k \to 0} \frac{k^2 \tan^{-1}(h/k)}{k}$$

$$= h \lim_{k \to 0} \frac{\tan^{-1}(k/h)}{(k/h)} - \lim_{k \to 0} h \cdot \frac{\tan^{-1}(h/k)}{(h/k)}$$

$$= h \cdot \lim_{k/h \to 0} \frac{\tan^{-1}(k/h)}{(k/h)} - h \cdot \lim_{\frac{h}{k} \to \infty} \frac{\tan^{-1}(h/k)}{(h/k)}$$

$$= h[1] - h[0]$$

$$f_y(0+h,0) = h$$

$$② \Rightarrow f_{xy}(0,0) = \lim_{h \to 0} \frac{h-0}{h}$$

$$f_{xy}(0,0) = 1.$$

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Now,

$$f_{yx}(a,b) = \lim_{k\to 0} \frac{f_x(a,b+k) - f_x(a,b)}{k}.$$

$$f_{yx}(0,0) = \lim_{k\to 0} \frac{f_x(0,0+k) - f_x(0,0)}{k} \quad \text{——— (3)}$$

Where

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}.$$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}.$$

$$= \lim_{h\to 0} \frac{f(h,0) - 0}{h}.$$

$$= \frac{0}{h}$$

$$f_x(0,0) = 0.$$

and

$$f_x(a,b+k) = \lim_{h\to 0} \frac{f(a+h,b+k) - f(a,b)}{h}.$$

$$f_x(0,0+k) = \lim_{h\to 0} \frac{f(0+h,0+k) - f(0,0)}{h}.$$

$$= \lim_{h\to 0} \frac{f(h,k) - 0}{h}.$$

$$= \lim_{h\to 0} \frac{h^2 \tan^{-1}(k/h) - k^2 \tan^{-1}(h/k)}{h}.$$

$$= \lim_{h\to 0} \frac{h^2 \tan^{-1}(k/h)}{h} - k.\lim_{h\to 0} \frac{\tan^{-1}(h/k)}{h/k}.$$

$$= k.\lim_{h\to 0} \frac{\tan^{-1}(k/h)}{(k/h)} - k.\lim_{h\to 0} \frac{\tan^{-1}(h/k)}{(h/k)}$$

$$f_x(0, 0+k) = k \cdot \lim_{(k/h)\to\infty} \frac{\tan^{-1}(k/h)}{(k/h)} - k \cdot \lim_{h/k\to 0} \frac{\tan^{-1}(h/k)}{(h/k)}$$

$$= k[0] - k[1]$$

$$f_x(0, 0+k) = -k$$

$$(3) \Rightarrow f_{yx}(0,0) = \lim_{k\to 0} \frac{-k-0}{k}$$

$$f_{yx}(0,0) = -1$$

$$\therefore f_{xy}(0,0) \neq f_{yx}(0,0).$$

Hence proved.

Taylor's theorem for a function of two variables :-

Statement :- If $f(x,y)$ possesses continuous partial derivatives of the n^{th} order in a neighbourhood of a point (a,b) and if $(a+h, b+k)$ be a point of this neighbourhood then there exist a positive number θ which is less than 1.
[i.e, $\exists$ $0<\theta<1$] such that.

$$f(a+h, b+k) = f(a,b) + \left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right] f(a,b) + \frac{1}{2!}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^2 f(a,b)$$
$$+ \cdots\cdots + \frac{1}{(n-1)!}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^{n-1} f(a,b) + R_n.$$

Where,

$$R_n = \frac{1}{n!}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^n f(a+\theta h, b+\theta k).$$

Another form :-

The Taylor's expansion of $f(x,y)$ about the point (a,b) in powers of $(x-a)$ and $(y-b)$ is.

$$f(x,y) = f(a,b) + \left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right] f(a,b) + \frac{1}{2!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^2$$
$$f(a,b) + \cdots + \frac{1}{(n-1)!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^{n-1} \cdot f(a,b) + R_n.$$

Where.

$$R_n = \frac{1}{n!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^n f(a+(x-a)\theta, b+(y-b)\theta).$$

Problems :-

① Expand $f(x,y) = x^2 + xy - y^2$ by Taylor's theorem in powers of $(x-1)$ and $(y+2)$.

Sol:- Let $f(x,y) = x^2 + xy - y^2$ and the points $(x-1)$ and $(y+2)$.

Here $a=1$ and $b=-2$

$$f(a,b) = f(1,-2) = (1)^2 + (1)(-2) - (-2)^2$$
$$= 1-2-4$$
$$= -5$$

Now,

$$\frac{\partial f}{\partial x} = 2x + y \Rightarrow \left.\frac{\partial f}{\partial x}\right|_{(1,-2)} = 2(1) - 2 = 0$$

$$\frac{\partial^2 f}{\partial x^2} = 2 \Rightarrow \left.\frac{\partial^2 f}{\partial x^2}\right|_{(1,-2)} = 2$$

and $$\frac{\partial f}{\partial y} = x - 2y \Rightarrow \left.\frac{\partial f}{\partial y}\right|_{(1,-2)} = 1 - 2(-2) = 5$$

$$\frac{\partial^2 f}{\partial y^2} = -2 \Rightarrow \left.\frac{\partial^2 f}{\partial y^2}\right|_{(1,-2)} = -2$$

and $$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial x}\left[x - 2y\right] = 1$$

$$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(1,-2)} = 1.$$

By Taylor's theorem, we have

$$f(x,y) = f(a,b) + \left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right] f(a,b) + \frac{1}{2!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^2 f(a,b).$$

$$f(x,y) = f(a,b) + \left[(x-1)\frac{\partial}{\partial x}f(a,b) + (y+2)\frac{\partial}{\partial y}f(a,b)\right] + \frac{1}{2}\left[(x-1)^2 \cdot \frac{\partial^2}{\partial x^2}f(a,b) + 2(x-1)(y+2)\frac{\partial^2}{\partial x \partial y}f(a,b) + (y+2)^2\frac{\partial^2}{\partial y^2}f(a,b)\right]$$

$$= -5 + (x-1)(0) + (y+2)(5) + \frac{1}{2}\left[(x-1)^2(2) + 2(x-1)(y+2)(1) + (y+2)^2(-2)\right]$$

$$= -5 + 5(y+2) + (x-1)^2 + (x-1)(y+2) - (y+2)^2.$$

$\therefore f(x,y) = x^2 + xy - y^2 = -5 + 5(y+2) + (x-1)^2 + (x-1)(y+2) - (y+2)^2.$

(2) Expand $f(x,y) = x^2 + xy + y^2$ by Taylor's theorem in powers of $(x-2)$ and $(y-3)$.

Sol: - Given that $f(x,y) = x^2 + xy + y^2$. and the points $(x-2)$ and $(y-3)$.

Here $a=2$ and $b=3$.

$$f(a,b) = f(2,3) = (2)^2 + (2)(3) + (3)^2$$
$$= 4+6+9$$
$$f(a,b) = 19.$$

Now,

$$\frac{\partial f}{\partial x} = 2x + y \Rightarrow \left.\frac{\partial f}{\partial x}\right|_{(2,3)} = 2(2)+3 = 7.$$

$$\frac{\partial f}{\partial y} = x + 2y \Rightarrow \left.\frac{\partial f}{\partial y}\right|_{(2,3)} = 2+2(3) = 8$$

$$\frac{\partial^2 f}{\partial x^2} = 2 \Rightarrow \left.\frac{\partial^2 f}{\partial x^2}\right|_{(2,3)} = 2$$

$$\frac{\partial^2 f}{\partial y^2} = 2 \Rightarrow \left.\frac{\partial^2 f}{\partial y^2}\right|_{(2,3)} = 2$$

and $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left[\dfrac{\partial f}{\partial y}\right] = \dfrac{\partial}{\partial x}\left[x+2y\right] = 1$

$$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(2,3)} = 1.$$

By Taylor's theorem, we have

$$f(x,y) = f(a,b) + \left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right] f(a,b) + \frac{1}{2!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^2 f(a,b)$$

$$= f(a,b) + (x-2)\frac{\partial}{\partial x} f(a,b) + (y-3)\frac{\partial}{\partial y} f(a,b) + \frac{1}{2}\left[(x-2)^2 \frac{\partial^2}{\partial x^2} f(a,b) + 2(x-2)(x-3)\frac{\partial^2}{\partial x \partial y} f(a,b) + (y-3)^2 \frac{\partial^2}{\partial y^2} f(a,b)\right]$$

$$= 19 + (x-2)(7) + (y-3)(8) + \frac{1}{2}\left[(x-2)^2(2) + 2(x-2)(y-3)(1) + (y-3)^2(2)\right]$$

$$f(x,y) = x^2+xy+y^2 = 19 + 7(x-2) + 8(y-3) + (x-2)^2 + (x-2)(y-3) + (y-3)^2.$$

③ Expand $x^2y+3y-2$ in powers of $(x-1)$ and $(y+2)$.

Sol:- Given that $f(x,y) = x^2y+3y-2$ and the points $(x-1)$ and $(y+2)$.

Here $a=1$ and $b=-2$

$$f(a,b) = f(1,-2) = (1)^2(-2)+3(-2)-2$$
$$= -2-6-2$$
$$f(a,b) = -10$$

Now,

$$\frac{\partial f}{\partial x} = y(2x) \Rightarrow \left.\frac{\partial f}{\partial x}\right|_{(1,-2)} = 2(1)(-2) = -4$$

$$\frac{\partial f}{\partial y} = x^2+3 \Rightarrow \left.\frac{\partial f}{\partial y}\right|_{(1,-2)} = (1)^2+3 = 4$$

and $\frac{\partial^2 f}{\partial x^2} = 2y \Rightarrow \left.\frac{\partial^2 f}{\partial x^2}\right|_{(1,-2)} = 2(-2) = -4$

$\frac{\partial^2 f}{\partial y^2} = 0 \Rightarrow \left.\frac{\partial^2 f}{\partial y^2}\right|_{(1,-2)} = 0.$

and $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial x}\left[x^2+3\right] = 2x$

$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(1,-2)} = 2(1) = 2.$

By Taylor's theorem, we have

$$f(x,y) = f(a,b) + \left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right] f(a,b) + \frac{1}{2!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^2 f(a,b).$$

$$= f(a,b) + \left[(x-1)\frac{\partial}{\partial x} f(a,b) + (y+2)\frac{\partial}{\partial y} f(a,b)\right] + \frac{1}{2}\left[(x-1)^2\frac{\partial^2}{\partial x^2} f(a,b) + 2(x-1)(y+2)\frac{\partial^2}{\partial x \partial y} f(a,b) + (y+2)^2\frac{\partial^2}{\partial y^2} f(a,b)\right]$$

$$= -10 + (x-1)(-4) + (y+2)(4) + \frac{1}{2}\left[(x-1)^2 \cdot (-4) + 2(x-1)(y+2)(2) + (y+2)^2(0)\right]$$

$$f(x,y) = x^2 y + 3y - 2 = -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2).$$

(4) Expand $\sin xy$ in powers of $(x-1)$ and $\left(y-\frac{\pi}{2}\right)$ upto second degree terms.

Sol:– Given that $f(x,y) = \sin xy$ and the points $(x-1)$ and $\left(y-\frac{\pi}{2}\right)$

Here $a=1$, $b=\frac{\pi}{2}$.

$$f(a,b) = f\left(1,\frac{\pi}{2}\right) = \sin(1)\left(\frac{\pi}{2}\right) = 1$$

$$f(a,b) = 1$$

Now,

$$\frac{\partial f}{\partial x} = \cos xy\,(y) \Rightarrow \left.\frac{\partial f}{\partial x}\right|_{\left(1,\frac{\pi}{2}\right)} = \frac{\pi}{2}\cdot\cos(1)\left(\frac{\pi}{2}\right) = 0$$

$$\frac{\partial f}{\partial y} = \cos xy\,(x) \Rightarrow \left.\frac{\partial f}{\partial y}\right|_{\left(1,\frac{\pi}{2}\right)} = (1)\cdot\cos(1)\left(\frac{\pi}{2}\right) = 0$$

and $$\frac{\partial^2 f}{\partial x^2} = y\cdot\left(-\sin xy\,(y)\right) = -y^2\sin xy$$

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{\left(1,\frac{\pi}{2}\right)} = -\left(\frac{\pi}{2}\right)^2\cdot\sin(1)\left(\frac{\pi}{2}\right) = -\frac{\pi^2}{4}(1) = \frac{-\pi^2}{4}.$$

$$\frac{\partial^2 f}{\partial y^2} = x\cdot\left(-\sin xy\,(x)\right) = -x^2\sin xy.$$

$$\left.\frac{\partial^2 f}{\partial y^2}\right|_{\left(1,\frac{\pi}{2}\right)} = -(1)^2\cdot\sin(1)\left(\frac{\pi}{2}\right) = -1$$

and $$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial x}\left[x\cos xy\right]$$

$$\frac{\partial^2 f}{\partial x\partial y} = \cos xy\,(1) + x\cdot\left(-\sin xy\,(y)\right) = \cos xy - xy\sin xy$$

$$\left.\frac{\partial^2 f}{\partial x\partial y}\right|_{\left(1,\frac{\pi}{2}\right)} = \cos(1)\left(\frac{\pi}{2}\right) - (1)\left(\frac{\pi}{2}\right)\sin(1)\left(\frac{\pi}{2}\right) = 0 - \frac{\pi}{2}(1).$$

$$= -\frac{\pi}{2}.$$

By Taylor's theorem, we have.

$$f(x,y) = f(a,b) + \left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right] f(a,b) + \frac{1}{2!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^2 f(a,b).$$

$$= f(a,b) + (x-1)\frac{\partial}{\partial x} f(a,b) + (y-\tfrac{\pi}{2})\frac{\partial}{\partial y} f(a,b) + \frac{1}{2}\Big[(x-1)^2\frac{\partial^2}{\partial x^2} f(a,b) + 2(x-1)(y-\tfrac{\pi}{2})\frac{\partial^2}{\partial x \partial y} f(a,b) + (y-\tfrac{\pi}{2})^2 \frac{\partial^2}{\partial y^2} f(a,b).$$

$$= 1 + (x-1)(0) + (y-\tfrac{\pi}{2})(0) + \frac{1}{2}\Big[(x-1)^2(-\tfrac{\pi^2}{4}) + 2(x-1)(y-\tfrac{\pi}{2})(-\tfrac{\pi}{2}) + (y-\tfrac{\pi}{2})^2(-1)\Big].$$

$$f(x,y) = 1 - \frac{\pi^2}{8}(x-1)^2 - \frac{\pi}{2}(x-1)(y-\tfrac{\pi}{2}) - \frac{1}{2}(y-\tfrac{\pi}{2})^2.$$

(5) Obtain Taylor's formula for $f(x,y) = \cos(x+y)$; $n=3$ at $(0,0)$.

Sol:- Given that $f(x,y) = \cos(x+y)$ and the points $(0,0)$.

Here $a=0$ and $b=0$.

Now, $f(a,b) = f(0,0) = \cos(0+0) = \cos(0) = 1$

$f(a,b) = 1$.

Now,

$$\frac{\partial f}{\partial x} = -\sin(x+y) \Rightarrow \frac{\partial f}{\partial x}\Big|_{(0,0)} = -\sin(0+0) = 0.$$

$$\frac{\partial f}{\partial y} = -\sin(x+y) \Rightarrow \frac{\partial f}{\partial y}\Big|_{(0,0)} = -\sin(0+0) = 0.$$

and

$$\frac{\partial^2 f}{\partial x^2} = -\cos(x+y) \Rightarrow \frac{\partial^2 f}{\partial x^2}\Big|_{(0,0)} = -\cos(0+0) = -1.$$

$$\frac{\partial^2 f}{\partial y^2} = -\cos(x+y) \Rightarrow \frac{\partial^2 f}{\partial y^2}\Big|_{(0,0)} = -\cos(0+0) = -1.$$

and $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial x}\left[-\sin(x+y)\right] = -\cos(x+y)$

$$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(0,0)} = -\cos(0+0) = -1.$$

By Taylor's theorem for $n=3$, we have.

$$f(x,y) = f(a,b) + \left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]f(a,b) + \frac{1}{2!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^2 f(a,b) + R_3.$$

$$= f(a,b) + (x-0)\frac{\partial}{\partial x}f(a,b) + (y-0)\frac{\partial}{\partial y}f(a,b) + \frac{1}{2}\left[x^2\frac{\partial^2}{\partial x^2}f(a,b) + 2xy\frac{\partial^2}{\partial x \partial y}f(a,b) + y^2\frac{\partial^2}{\partial y^2}f(a,b)\right] + R_3 \quad \text{---} (1)$$

Where,

$$R_3 = \frac{1}{3!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^3 f(a+(x-a)\theta, b+(y-b)\theta).$$

$$R_3 = \frac{1}{3!}\left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right]^3 f(\theta x, \theta y).$$

$$R_3 = \frac{1}{3!}\left[x^3\frac{\partial^3}{\partial x^3}f(\theta x,\theta y) + 3x^2y\frac{\partial^3}{\partial x^2 \partial y}f(\theta x,\theta y) + 3xy^2\frac{\partial^3}{\partial x \partial y^2}f(\theta x,\theta y) + y^3\frac{\partial^3}{\partial y^3}f(\theta x,\theta y)\right].$$

$$f(x,y) = f(\theta x, \theta y) = \cos(\theta x + \theta y) = \cos\theta(x+y)$$

$$\frac{\partial^3 f}{\partial x^3} = \frac{\partial^3 f}{\partial y^3} = \frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial^3 f}{\partial x \partial y^2} = \sin\theta(x+y).$$

Now, $R_3 = \frac{1}{3!}\left[x^3 \sin\theta(x+y) + 3x^2y\sin\theta(x+y) + 3xy^2\sin\theta(x+y) + y^3\sin\theta(x+y)\right]$

$$= \frac{1}{3!}\left[(x^3+3x^2y+3xy^2+y^3)\sin\theta(x+y)\right]$$

$$R_3 = \frac{(x+y)^3}{3!}\sin\theta(x+y).$$

$$(1) \Rightarrow f(x,y) = 1 + x(0) + y(0) + \frac{1}{2}\left[x^2(-1) + 2xy(-1) + y^2(-1)\right] + \frac{(x+y)^3}{3!} \cdot \sin\theta(x+y).$$

$$= 1 - \frac{1}{2}\left(x^2 + 2xy + y^2\right) + \frac{(x+y)^3}{3!} \sin\theta(x+y).$$

$$f(x,y) = \cos(x+y) = 1 - \frac{(x+y)^2}{2!} + \frac{(x+y)^3}{3!} \sin\theta(x+y).$$

(6) Obtain Taylor's formula for the function e^{x+y} at $(0,0)$ for $n=3$.

Sol:- Given that $f(x,y) = e^{x+y}$ and the point $(0,0)$.

Here $a=0$ and $b=0$.

$$f(a,b) = f(0,0) = e^{0+0} = e^0 = 1$$

$$f(a,b) = 1.$$

Now $$\left[\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial x \partial y}\right]_{(0,0)} = e^{x+y} = e^{0+0} = 1$$

and $$\frac{\partial^3 f}{\partial x^3} = \frac{\partial^3 f}{\partial y^3} = \frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial^3 f}{\partial x \partial y^2} = e^{x+y} = e^{0+0} = 1$$

By Taylor's theorem for $n=3$, we have.

$$f(x,y) = f(a,b) + \left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right] f(a,b) + \frac{1}{2!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^2 f(a,b) + R_3.$$

$$= f(a,b) + x \cdot \frac{\partial}{\partial x} f(a,b) + y\frac{\partial}{\partial y} f(a,b) + \frac{1}{2}\left[x^2 \frac{\partial^2}{\partial x^2} f(a,b) + 2xy\frac{\partial^2}{\partial x \partial y} f(a,b) + y^2\frac{\partial^2}{\partial y^2} f(a,b)\right] + R_3. \quad \text{——} (1)$$

$$R_3 = \frac{1}{3!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^3 f\left(a + (x-a)\theta, b + (y-b)\theta\right).$$

$$R_3 = \frac{1}{3!}\left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right]^3 f(\theta x, \theta y).$$

$$R_3 = \frac{1}{3!}\left[x^3\frac{\partial^3}{\partial x^3} f(\theta x, \theta y) + 3x^2y\frac{\partial^3}{\partial x^2\partial y} f(\theta x, \theta y) + 3xy^2\frac{\partial^3}{\partial x\partial y^2} f(\theta x, \theta y)\right.$$
$$\left. + y^3\frac{\partial^3}{\partial y^3} f(\theta x, \theta y)\right].$$

$$f(x,y) = f(\theta x, \theta y) = e^{\theta x + \theta y} = e^{\theta(x+y)}.$$

Now,

$$R_3 = \frac{1}{3!}\left[x^3 e^{\theta(x+y)} + 3x^2y\, e^{\theta(x+y)} + 3xy^2 e^{\theta(x+y)} + y^3 e^{\theta(x+y)}\right]$$

$$= \frac{1}{3!}\left[(x^3 + 3x^2y + 3xy^2 + y^3)e^{\theta(x+y)}\right]$$

$$R_3 = \frac{(x+y)^3}{3!} e^{\theta(x+y)}.$$

$$(1) \Rightarrow f(x,y) = 1 + x(1) + y(1) + \frac{1}{2!}\left[x^2(1) + 2xy(1) + y^2(1)\right] +$$
$$\frac{(x+y)^3}{3!} e^{\theta(x+y)}.$$

$$f(x,y) = 1 + (x+y) + \frac{(x+y)^2}{2!} + \frac{(x+y)^3}{3!} e^{\theta(x+y)}.$$

Maxima and Minima of function of two Variables :-

Maximum Value :-

If a function $f(x,y)$ is said to a maximum value at the point (a,b). if $f(a,b) > f(a+h, b+k)$ for small values of h and k are +ve (or) -ve.

Minimum Value :-

If a function $f(x,y)$ is said to a minimum value at the point (a,b). if $f(a,b) < f(a+h, b+k)$ for small values of h and k are +ve (or) -ve.

Extreme Value :-

A minimum or maximum value of a function is called Extreme Value.

Rules to find extreme values of a function $f(x,y)$:-

Lagranges Method :-

Step (1) :- Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Step (2) :- Solve $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$. Simultaneously.

Let (x_1, y_1) and (x_2, y_2) be the solution of this equations.

Step (3) :- Find. $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$.

Step(4):-

(a) If $rt-s^2>0$ & $r<0$ then $f(x,y)$ has maximum value at (x_1, y_1).

(b) If $rt-s^2>0$ & $r>0$ then $f(x,y)$ has minimum value at (x_1,y_1).

(c) If $rt-s^2<0$ then $f(x,y)$ no extreme values at (x_1,y_1). such a point is called "Suddle points"

(d) If $rt-s^2=0$ then the case is doubtfull.

Problems :-

(1) Show that minimum value of $U=xy+\frac{a^3}{x}+\frac{a^3}{y}$ is $3a^2$.

Sol:- Given that $U=xy+\frac{a^3}{x}+\frac{a^3}{y}$

Let $f(x,y)=xy+\frac{a^3}{x}+\frac{a^3}{y}$ — (1)

$\frac{\partial f}{\partial x}=y(1)+a^3\left(\frac{-1}{x^2}\right)+0$ $\qquad$ $\frac{\partial f}{\partial y}=x(1)+0+a^3\left(\frac{-1}{y^2}\right)$

$\frac{\partial f}{\partial x}=y-\frac{a^3}{x^2}$ $\qquad$ $\frac{\partial f}{\partial y}=x-\frac{a^3}{y^2}$

$\frac{\partial f}{\partial x}=0$ $\qquad$ $\frac{\partial f}{\partial y}=0$

$y-\frac{a^3}{x^2}=0$ $\qquad$ $x-\frac{a^3}{y^2}=0$

$x^2y-a^3=0$ $\qquad$ $xy^2-a^3=0$

$a^3=x^2y$ — (2) $\qquad$ $a^3=xy^2$ — (3)

From eqns (2) & (3), we have

$x^2y=xy^2$

$$\Rightarrow x^2y - xy^2 = 0$$

$$\Rightarrow xy(x-y) = 0.$$

$$\Rightarrow x - y = 0$$

$$\boxed{x = y}$$

$\because x = y = a$ (say) then $f(x,y) = f(a,a)$.

Now,

$$r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial x}\left[y - \frac{a^3}{x^2}\right]$$

$$= 0 - a^3 \cdot \left(\frac{-2}{x^3}\right)$$

$$r = \frac{2a^3}{x^3}$$

$$r\Big|_{(a,a)} = \frac{2\cancel{a^3}}{\cancel{a^3}} = 2$$

$$r = 2 > 0.$$

and $s = \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left[\dfrac{\partial f}{\partial y}\right] = \dfrac{\partial}{\partial x}\left[x - \dfrac{a^3}{y^2}\right]$

$$= 1 - 0$$

$$s\Big|_{(a,a)} = 1 \Rightarrow s = 1$$

and $t = \dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial}{\partial y}\left[\dfrac{\partial f}{\partial y}\right] = \dfrac{\partial}{\partial y}\left[x - \dfrac{a^3}{y^2}\right]$

$$= 0 - a^3\left(\frac{-2}{y^3}\right)$$

$$t = \frac{2a^3}{y^3}.$$

$$t\Big|_{(a,a)} = \frac{2\cancel{a^3}}{\cancel{a^3}} = 2$$

$$t\Big|_{(a,a)} = 2 > 0.$$

Now, we have

$$rt - s^2 = (2)(2) - (1)^2$$
$$= 4 - 1 = 3$$

$rt - s^2 > 0$. and $r > 0$

$\because f(x, y)$ has a minimum value at (a, a).

$\therefore$ The minimum value is.

$$U = (a)(a) + \frac{a^{\cancel{3}}}{\cancel{a}} + \frac{a^{\cancel{3}}}{\cancel{a}}$$
$$= a^2 + a^2 + a^2$$
$$U = 3a^2$$

Hence proved.

② Discuss the maximum or minimum value of U. when $U = x^3 + y^3 - 3axy$.

Sol:- Given that $U = x^3 + y^3 - 3axy$

Let $f(x, y) = x^3 + y^3 - 3axy$.

$$\frac{\partial f}{\partial x} = 3x^2 + 0 - 3ay(1) \qquad \frac{\partial f}{\partial y} = 0 + 3y^2 - 3ax(1)$$

$$\frac{\partial f}{\partial x} = 3x^2 - 3ay \qquad \frac{\partial f}{\partial y} = 3y^2 - 3ax$$

$$\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial y} = 0$$

$$3x^2 - 3ay = 0 \qquad 3y^2 - 3ax = 0$$

$$x^2 - ay = 0 \qquad y^2 - ax = 0$$

$$x^2 = ay \qquad y^2 = ax$$

$$a = \frac{x^2}{y} \text{ --- ①} \qquad a = \frac{y^2}{x} \text{ --- ②}$$

Form eqns ① & ②, we have

$$\cancel{\frac{x^2}{y}} \cancel{=} \cancel{\frac{y^2}{x}}.$$

$$\Rightarrow \frac{x^2}{y} = \frac{y^2}{x}$$

$$\Rightarrow x^3 = y^3$$

$$\Rightarrow x^3 - y^3 = 0$$

$$\Rightarrow (x-y)(x^2+xy+y^2) = 0.$$

$$\Rightarrow x - y = 0.$$

$x = y = a$ (say) then $f(x,y) = (a,a)$.

Now,

$$r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial x}\left[3x^2 - 3ay\right]$$

$$= 3(2x) - 0 = 6x$$

$$r\Big|_{(a,a)} = 6a$$

and $$s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial x}\left[3y^2 - 3ax\right]$$

$$= 0 - 3a(1)$$

$$s\Big|_{(a,a)} = -3a.$$

and $$t = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial y}\left[3y^2 - 3ax.\right]$$

$$= 3(2y) - 0 = 6y$$

$$t\Big|_{(a,a)} = 6a.$$

Now, we have

$$rt - s^2 = (6a)(6a) - (-3a)^2$$

$$= 36a^2 - 9a^2$$

$$= 27a^2 > 0 \quad ; \text{ If } a > 0.$$

$$rt - s^2 > 0$$

If $a<0 \Rightarrow r = 6a < 0$ then $f(x,y)$ has a maximum value at (a,a). and

If $a>0 \Rightarrow r = 6a > 0$ then $f(x,y)$ has a minimum value at (a,a).

Hence maximum/minimum value is

$$U = a^3 + a^3 - 3a(a)(a)$$
$$= 2a^3 - 3a^3$$
$$U = -a^3.$$

(3) Discuss the maximum or minimum value of 'U' given by $U = x^3y^2(1-x-y)$.

Sol:- Given that $U = x^3y^2(1-x-y)$.

Let $f(x,y) = x^3y^2 - x^4y^2 - x^3y^3$.

$$\frac{\partial f}{\partial x} = y^2(3x^2) - y^2(4x^3) - y^3(3x^2).$$

$$\frac{\partial f}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3.$$

$$\frac{\partial f}{\partial x} = 0$$

$$3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0.$$

$$x^2y^2(3-4x-3y) = 0.$$

$$3-4x-3y = 0 \text{ —— } (1)$$

and $\frac{\partial f}{\partial y} = x^3(2y) - x^4(2y) - x^3(3y^2).$

$$\frac{\partial f}{\partial y} = 2x^3y - 2x^4y - 3x^3y^2.$$

$$\frac{\partial f}{\partial y} = 0.$$

$$\Rightarrow 2x^3y - 2x^4y - 3x^3y^2 = 0.$$

$$\Rightarrow x^3y(2-2x-3y) = 0.$$

$$2-2x-3y = 0 \quad \text{——} \quad (2)$$

Solve eqns (1) & (2), we get.

$$(1) \Rightarrow 4x+3y = -3.$$

$$2\times(2) \Rightarrow \cancel{4x} \pm 6y = \underline{\ 4}$$

$$-3y = -$$

$$(1) \Rightarrow -4x - \cancel{3y} + 3 = 0.$$

$$(2) \Rightarrow \underset{+}{-}2x \underset{+}{-}\cancel{3y} \pm 2 = 0$$

$$-2x+1 = 0$$

$$1 = 2x$$

$$\boxed{x = 1/2}$$

$$(2) \Rightarrow 2 - \cancel{2}(1/\cancel{2}) - 3y = 0$$

$$2-1 = 3y \Rightarrow 1 = 3y \Rightarrow \boxed{y = 1/3}$$

$\therefore$ The point $(x,y) = (1/2, 1/3)$.

Now,

$$r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial x}\left[3x^2y^2 - 4x^3y^2 - 3x^2y^3\right]$$

$$= 3y^2(2x) - 4y^2(3x^2) - 3y^3(2x)$$

$$r = 6xy^2 - 12x^2y^2 - 6xy^3.$$

$$r\Big|_{(1/2,1/3)} = 6\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^2 - 12\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right)^2 - 6\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^3$$

$$= \cancel{6}^{3}\left(\frac{1}{\cancel{2}}\right)\left(\frac{1}{\cancel{9}_3}\right) - \cancel{12}^{3}\left(\frac{1}{\cancel{4}}\right)\left(\frac{1}{\cancel{9}_3}\right) - \cancel{6}^{3}\left(\frac{1}{\cancel{2}}\right)\left(\frac{1}{\cancel{27}_9}\right)$$

$$= \cancel{\frac{1}{3}} - \cancel{\frac{1}{3}} - \frac{1}{9}$$

$$r\Big|_{(1/2,1/3)} = -\frac{1}{9} < 0.$$

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and $s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial x}\left[2x^3y - 2x^4y - 3x^3y^2\right].$

$$= 2y(3x^2) - 2y(4x^3) - 3y^2(3x^2).$$

$$s = 6x^2y - 8x^3y - 9x^2y^2$$

$$s\Big|_{(1/2,\,1/3)} = 6\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right) - 8\left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right) - 9\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right)^2$$

$$= \cancel{6}^{\,2}\left(\frac{1}{\cancel{4}_{\,2}}\right)\left(\frac{1}{\cancel{3}}\right) - \cancel{8}\left(\frac{1}{\cancel{8}}\right)\left(\frac{1}{3}\right) - \cancel{9}\left(\frac{1}{4}\right)\cdot\left(\frac{1}{\cancel{9}}\right)$$

$$= \frac{1}{2} - \frac{1}{3} - \frac{1}{4}. = \frac{6-4-3}{12}$$

$$s\Big|_{(1/2,\,1/3)} = \frac{-1}{12}.$$

and $t = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[2x^3y - 2x^4y - 3x^3y^2\right]$

$$= 2x^3(1) - 2x^4(1) - 3x^3(2y).$$

$$t = 2x^3 - 2x^4 - 6x^3y.$$

$$t\Big|_{(1/2,\,1/3)} = 2\left(\frac{1}{2}\right)^3 - 2\left(\frac{1}{2}\right)^4 - 6\left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right).$$

$$= \cancel{2}\left(\frac{1}{\cancel{8}_{\,4}}\right) - \cancel{2}\left(\frac{1}{\cancel{16}_{\,8}}\right) - \cancel{6}^{\,3}\left(\frac{1}{\cancel{8}_{\,4}}\right)\left(\frac{1}{\cancel{3}}\right)$$

$$= \cancel{\frac{1}{4}} - \frac{1}{8} - \cancel{\frac{1}{4}}$$

$$t\Big|_{(1/2,\,1/3)} = -\frac{1}{8}.$$

Now, We have

$$rt - s^2 = \left(\frac{-1}{9}\right)\left(\frac{-1}{8}\right) - \left(\frac{-1}{12}\right)^2.$$

$$= \frac{1}{72} - \frac{1}{144}.$$

$$rt - s^2 = \frac{2-1}{144} = \frac{1}{144}.$$

$rt - s^2 > 0$. and $r < 0$.

$\therefore f(x,y)$ has a maximum value at $(1/2, 1/3)$.

$\therefore$ The maximum value is

$$U = \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{3}\right)^2 \left[1 - \frac{1}{2} - \frac{1}{3}\right]$$

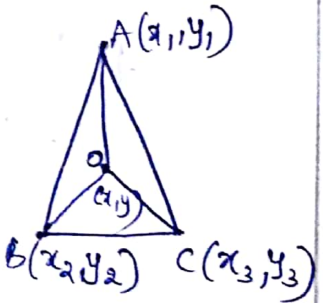
$$= \left(\frac{1}{8}\right)\left(\frac{1}{9}\right)\left[\frac{6-3-2}{6}\right]$$

$$= \frac{1}{72}\left(\frac{1}{6}\right).$$

$$U = \frac{1}{432}.$$

(4) Find a point with in a triangle such that the sum of the squares of its distance from the 3 vertices is minimum.

Sol:- Let $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ are the 3 vertices of the triangle and (x,y) be any point inside of triangle



From the given data.

$$U = (x-x_1)^2 + (y-y_1)^2 + (x-x_2)^2 + (y-y_2)^2 + (x-x_3)^2 + (y-y_3)^2.$$

$$U = \sum_{r=1}^{3} \left[(x-x_r)^2 + (y-y_r)^2\right]$$

Let $f(x,y) = \sum_{r=1}^{3} \left[(x-x_r)^2 + (y-y_r)^2\right]$

Now,

$$\frac{\partial f}{\partial x} = \sum_{r=1}^{3} 2(x-x_r).$$

$$\frac{\partial f}{\partial x} = 0$$

$$\frac{\partial f}{\partial y} = \sum_{r=1}^{3} 2(y-y_r)$$

$$\frac{\partial f}{\partial y} = 0$$

$$2\sum_{r=1}^{3}(x-x_r)=0 \qquad 2\sum_{r=1}^{3}(y-y_r)=0$$

$$\sum_{r=1}^{3}(x-x_r)=0 \qquad \sum_{r=1}^{3}(y-y_r)=0$$

$$(x-x_1)+(x-x_2)+(x-x_3)=0 \qquad (y-y_1)+(y-y_2)+(y-y_3)=0$$

$$3x-(x_1+x_2+x_3)=0 \qquad 3y-(y_1+y_2+y_3)=0$$

$$3x=x_1+x_2+x_3 \qquad 3y=y_1+y_2+y_3$$

$$x=\frac{x_1+x_2+x_3}{3}. \qquad y=\frac{y_1+y_2+y_3}{3}.$$

$\therefore$ The point $(x,y)=\left(\frac{x_1+x_2+x_3}{3},\ \frac{y_1+y_2+y_3}{3}\right)$.

Now,

$$r=\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right]=\frac{\partial}{\partial x}\left[\sum_{r=1}^{3}2(x-x_r)\right]$$

$$=2\frac{\partial}{\partial x}\left[3x-(x_1+x_2+x_3)\right]$$

$$=2(3)$$

$$r=6>0.$$

$$s=\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right]=\frac{\partial}{\partial x}\left[2\sum_{r=1}^{3}(y-y_r)\right]$$

$$=2\cdot\frac{\partial}{\partial x}\left[3y-(y_1+y_2+y_3)\right]$$

$$s=0.$$

$$t=\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right]=\frac{\partial}{\partial y}\left[2\sum_{r=1}^{3}(y-y_r)\right]$$

$$=2\cdot\frac{\partial}{\partial y}\left[3y-(y_1+y_2+y_3)\right]$$

$$=2(3)$$

$$t=6>0$$

Now,

$$rt - s^2 = (6)(6) - (0)^2$$

$$= 36 > 0$$

$\therefore rt - s^2 > 0$ and $r > 0$.

$\therefore f(x,y)$ has a minimum value at $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$.

(5) In a plane triangle find the maximum value of $U = \cos A \cos B \cos C$.

Sol:- Given that $U = \cos A \cos B \cos C$ — (1)

In a plane triangle, we have $A+B+C = \pi$ — (2)

$(1) \Rightarrow U = \cos A \cos B \cos(\pi - (A+B))$.

$U = -\cos A \cos B \cos(A+B) = f(A,B)$

Now,

$$\frac{\partial U}{\partial A} = -\cos B\left[+\cos A(-\sin(A+B)) + \cos(A+B)\cdot(-\sin A)\right]$$

$$= \cos B\left[\cos A \sin(A+B) + \sin A \cos(A+B)\right]$$

$$= \cos B \cdot \sin(A+A+B).$$

$$\frac{\partial U}{\partial A} = \cos B \cdot \sin(2A+B).$$

$$\frac{\partial U}{\partial A} = 0$$

$$\cos B \cdot \sin(2A+B) = 0.$$

If $\cos B = 0$ then $\boxed{B = \pi/2}$

and $\frac{\partial U}{\partial B} = -\cos A\left[\cos B(-\sin(A+B)) + \cos(A+B)(-\sin B)\right]$

$$= \cos A\left[\cos B \cdot \sin(A+B) + \sin B \cdot \cos(A+B)\right].$$

$$\frac{\partial U}{\partial B} = \cos A \cdot \sin(A+2B).$$

$$\frac{\partial U}{\partial B} = 0.$$

$$\cos A \cdot \sin(A+2B) = 0.$$

If $\cos A = 0$ then $\boxed{A = \pi/2}$

$$② \Rightarrow \frac{\pi}{2} + \frac{\pi}{2} + c = \pi$$

$$\Rightarrow \pi + c = \pi.$$

$$\Rightarrow c = \pi - \pi$$

$$\Rightarrow \boxed{c = 0}$$

This is absurd

$\therefore \cos A \neq 0$ and $\cos B \neq 0$.

$\therefore \sin(2A+B) = 0$ and $\sin(A+2B) = 0$.

$2A + B = \pi$ — (i)

$A + 2B = \pi$ — (ii)

Solve (i) & (ii), we get

$$(i) \Rightarrow \cancel{2A} + B = \pi.$$

$$2 \times (ii) \Rightarrow \cancel{2A} + 4B = 2\pi$$

$$-3B = -\pi \Rightarrow \boxed{B = \pi/3}$$

$$(i) \Rightarrow 2A + \frac{\pi}{3} = \pi \Rightarrow 2A = \pi - \frac{\pi}{3}.$$

$$2A = \frac{3\pi - \pi}{3}. \Rightarrow \cancel{2}A = \frac{\cancel{2}\pi}{3}$$

$$\therefore \boxed{A = \pi/3}$$

$\therefore$ The point is $(A, B) = (\pi/3, \pi/3)$.

Now,

$$r = \frac{\partial^2 U}{\partial A^2} = \frac{\partial}{\partial x}\left[\frac{\partial U}{\partial A}\right] = \frac{\partial}{\partial A}\left[\cos B \cdot \sin(2A+B)\right]$$

$$= \cos B \cdot \cos(2A+B) \cdot (2).$$

$$r\Big|_{(\frac{\pi}{3},\frac{\pi}{3})} = \cos\frac{\pi}{3}\left[2\cdot\cos\left(2\cdot\frac{\pi}{3}+\frac{\pi}{3}\right)\right]$$

$$= \left(\frac{1}{2}\right)\left[2\cos\left(\frac{3\pi}{3}\right)\right]$$

$$= \frac{1}{\cancel{2}}(\cancel{2}(-1))$$

$$r = -1 < 0.$$

and $$s = \frac{\partial^2 u}{\partial A\partial B} = \frac{\partial}{\partial A}\left[\frac{\partial u}{\partial B}\right] = \frac{\partial}{\partial A}\left[\cos A\cdot\sin(A+2B)\right]$$

$$= \cos A\cdot\left[\cos(A+2B)\right] + \sin(A+2B)\cdot(-\sin A)$$

$$s\Big|_{(\frac{\pi}{3},\frac{\pi}{3})} = \cos\frac{\pi}{3}\left[\cos\left(\frac{\pi}{3}+2\cdot\frac{\pi}{3}\right)\right] + \sin\left(\frac{\pi}{3}+2\cdot\frac{\pi}{3}\right)\cdot\left(-\sin\frac{\pi}{3}\right).$$

$$= \left(\frac{1}{2}\right)\left[\cos(\pi)\right] + \sin(\pi)\cdot\left(-\frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{2}(-1) - (0)\left(\frac{\sqrt{3}}{2}\right)$$

$$s = \frac{-1}{2}.$$

and $$t = \frac{\partial^2 u}{\partial B^2} = \frac{\partial}{\partial B}\left[\frac{\partial u}{\partial B}\right] = \frac{\partial}{\partial B}\left[\cos A\cdot\sin(A+2B)\right]$$

$$= \cos A\cdot\cos(A+2B)(2)$$

$$t\Big|_{(\frac{\pi}{3},\frac{\pi}{3})} = \cos\frac{\pi}{3}\left[2\cdot\cos\left(\frac{\pi}{3}+2\,\frac{\pi}{3}\right)\right]$$

$$= \left(\frac{1}{2}\right)\left[2\cdot\cos\pi\right]$$

$$= \frac{1}{\cancel{2}}\left[\cancel{2}(-1)\right]$$

$$t = -1.$$

Now, we have.

$$rt - s^2 = (-1)(-1) - \left(-\frac{1}{2}\right)^2$$

$$= 1 - \frac{1}{4} = \frac{3}{4} > 0.$$

$$rt - s^2 > 0 \text{ and } r < 0.$$

$\therefore f(A,B)$ has a ~~minimum~~ maximum value at $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$.

$\therefore$ The maximum value is

$$U = -\cos\left(\frac{\pi}{3}\right)\cdot\cos\left(\frac{\pi}{3}\right)\left[\cos\left(\frac{\pi}{3}+\frac{\pi}{3}\right)\right]$$

$$= -\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left[\cos\left(\frac{2\pi}{3}\right)\right]$$

$$= -\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)$$

$$U = \frac{1}{8}.$$

(6) ~~In a~~ If x, y, z are angles of a triangle, then find the maximum value of $\sin x\cdot\sin y\cdot\sin z$.

Sol:- Let $U = \sin x\cdot\sin y\cdot\sin z$ — (1)

In a triangle, we have $x+y+z=\pi$ — (2)

$$(1) \Rightarrow U = \sin x\cdot\sin y\cdot\sin[\pi-(x+y)].$$

$$U = \sin x\sin y\sin(x+y). = f(x,y)$$

Now,

$$\frac{\partial U}{\partial x} = \sin y\left[\sin x\cdot\cos(x+y)+\sin(x+y)\cdot\cos x\right]$$

$$= \sin y\left[\sin(x+x+y)\right]$$

$$\frac{\partial U}{\partial x} = \sin y\cdot\sin(2x+y).$$

$$\frac{\partial U}{\partial x} = 0$$

$$\Rightarrow \sin y\cdot\sin(2x+y)=0.$$

$$\Rightarrow \sin(2x+y)=0$$

$$2x+y=\pi \text{ — (3)}$$

and $$\frac{\partial U}{\partial y} = \sin x\left[\sin y\cdot\cos(x+y)+\sin(x+y)\cdot\cos y\right]$$

$$= \sin x\left[\sin(x+y+y)\right].$$

$$\frac{\partial u}{\partial y} = \sin x \cdot \sin(x+2y)$$

$$\frac{\partial u}{\partial y} = 0$$

$$\Rightarrow \sin x \cdot \sin(x+2y) = 0$$

$$\Rightarrow \sin(x+2y) = 0$$

$$x + 2y = \pi \quad \text{--- (ii)}$$

Solve eqns (i) & (ii), we get

$$\text{(i)} \Rightarrow \cancel{2x} + y = \pi$$

$$2 \times \text{(ii)} \Rightarrow \cancel{2x} + 4y = 2\pi$$

$$-3y = -\pi \Rightarrow \boxed{y = \pi/3}$$

$$\text{(i)} \Rightarrow 2x + \frac{\pi}{3} = \pi \Rightarrow 2x = \pi - \frac{\pi}{3}$$

$$\Rightarrow 2x = \frac{3\pi - \pi}{3} \Rightarrow \cancel{2}x = \frac{\cancel{2}\pi}{3}$$

$$\Rightarrow \boxed{x = \frac{\pi}{3}}$$

$\therefore$ The point is $(x, y) = \left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

Now,

$$r = \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\right] = \frac{\partial}{\partial x}\left[\sin y \cdot \sin(2x+y)\right]$$

$$= \sin y \cdot \cos(2x+y)(2)$$

$$r\Big|_{\left(\frac{\pi}{3}, \frac{\pi}{3}\right)} = \sin\frac{\pi}{3} \cdot 2 \cdot \cos\left(2\frac{\pi}{3} + \frac{\pi}{3}\right)$$

$$= 2 \cdot \sin\frac{\pi}{3} \cdot \cos\left(\frac{\cancel{3}\pi}{\cancel{3}}\right)$$

$$= \cancel{2} \cdot \left(\frac{\sqrt{3}}{\cancel{2}}\right) \cdot (-1)$$

$$r = -\sqrt{3} < 0.$$

and $s = \dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left[\dfrac{\partial u}{\partial y}\right] = \dfrac{\partial}{\partial x}\left[\sin x \cdot \sin(x+2y)\right]$

$=$

$$s = \sin x \cdot \cos(x+2y) + \sin(x+2y)\cdot \cos x.$$

$$s\Big|_{(\frac{\pi}{3},\frac{\pi}{3})} = \sin\frac{\pi}{3}\cdot\cos\left(\frac{\pi}{3}+2\frac{\pi}{3}\right)+\sin\left(\frac{\pi}{3}+2\cdot\frac{\pi}{3}\right)\cdot\cos\frac{\pi}{3}.$$

$$= \sin\frac{\pi}{3}\cdot\cos(\pi)+\sin(\pi)\cdot\cos\frac{\pi}{3}.$$

$$= \frac{\sqrt{3}}{2}\cdot(-1)+0\left(\frac{1}{2}\right)$$

$$s = -\frac{\sqrt{3}}{2}.$$

and $t = \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial}{\partial y}\left[\dfrac{\partial u}{\partial y}\right] = \dfrac{\partial}{\partial y}\left[\sin x\cdot\sin(x+2y)\right]$

$$= \sin x\cdot\cos(x+2y)(2)$$

$$t\Big|_{(\frac{\pi}{3},\frac{\pi}{3})} = 2\cdot\sin\frac{\pi}{3}\cdot\cos\left(\frac{\pi}{3}+2\cdot\frac{\pi}{3}\right).$$

$$= 2\cdot\sin\frac{\pi}{3}\cdot\cos(\pi).$$

$$= \cancel{2}\left(\frac{\sqrt{3}}{\cancel{2}}\right)(-1)$$

$$t = -\sqrt{3}.$$

Now, We have

$$rt - s^2 = (-\sqrt{3})(-\sqrt{3}) - \left(-\frac{\sqrt{3}}{2}\right)^2$$

$$= 3 - \frac{3}{4}$$

$$= \frac{12-3}{4}. = \frac{9}{4} > 0$$

$$rt - s^2 > 0 \text{ and } r<0.$$

$\therefore$ $f(x,y)$ has a maximum value at $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$.

$\therefore$ The maximum value is

$$u = \sin\frac{\pi}{3}\cdot\sin\frac{\pi}{3}\cdot\sin\left(\frac{\pi}{3}+\frac{\pi}{3}\right) = \sin\frac{\pi}{3}\cdot\sin\frac{\pi}{3}\cdot\sin\frac{2\pi}{3}.$$

$$= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}.$$

$$u = \frac{3\sqrt{3}}{8}.$$

Maxima and minima problem :-

(7) Discuss the maximum and mimum value of $f(x,y) = xy + \frac{9}{x} + \frac{3}{y}$.

Sol:- Given that $f(x,y) = xy + \frac{9}{x} + \frac{3}{y}$ — (1)

$\frac{\partial f}{\partial x} = y + 9\left(\frac{-1}{x^2}\right)$ | $\frac{\partial f}{\partial y} = x + 3\left(\frac{-1}{y^2}\right)$

$\frac{\partial f}{\partial x} = y - \frac{9}{x^2}$. | $\frac{\partial f}{\partial y} = x - \frac{3}{y^2}$

$\frac{\partial f}{\partial x} = 0$ | $\frac{\partial f}{\partial y} = 0$

$y - \frac{9}{x^2} = 0$. | $x - \frac{3}{y^2} = 0$

$x^2 y = 9$ — (2) | $xy^2 = 3$ — (3)

Now. (2) ÷ (3), we get

$$\frac{\cancel{x}^2 \cancel{y}}{\cancel{x}y\cancel{y}} = \frac{\cancel{9}^3}{\cancel{3}} \Rightarrow \frac{x}{y} = \frac{3}{1}.$$

$\therefore x = 3,\ y = 1.$

$\therefore$ The extreme point is $(x,y) = (3,1)$.

Now, $r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[y - \frac{9}{x^2}\right] = 0 - 9\cdot\left(\frac{-2}{x^3}\right) = \frac{18}{x^3}$

$$r\Big|_{(3,1)} = \frac{18}{(3)^3} \Rightarrow r\Big|_{(3,1)} = \frac{\cancel{18}^2}{\cancel{27}_3}$$

$$r\Big|_{(3,1)} = \frac{2}{3} > 0$$

and $s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial x}\left[x - \frac{3}{y^2}\right]$

$$s = 1$$

$$s\Big|_{(3,1)} = 1.$$

Now,

$$t = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial y}\left[x - \frac{3}{y^2}\right]$$

$$= 0 - 3\left(\frac{-2}{y^3}\right)$$

$$t = \frac{6}{y^3}$$

$$t\Big|_{(3,1)} = \frac{6}{(1)^3}$$

$$t = 6.$$

Now, we have.

$$rt - s^2 = \left(\frac{2}{\cancel{3}}\right)\left(\overset{2}{\cancel{6}}\right) - (1)^2$$

$$= 4 - 1$$

$$rt - s^2 = 3.$$

$rt - s^2 > 0$ and $r > 0$

$\therefore$ $f(x,y)$ has a minimum value at $(3,1)$

$\therefore$ The minimum value is

$$f(x,y) = f(3,1) = 3(1) + \frac{\overset{3}{\cancel{9}}}{\cancel{3}} + \frac{3}{1}$$

$$= 3 + 3 + 3$$

$$= 9.$$

Lagrange's method of undetermined multipliers :-

To find maximum (or) minimum values of a function of three or more variables when the variables are not independent are connected by some given relation, we try to convert the given equation to one (or) two having least no. of independent variables with the help of given relation.

Let $u = f(x, y, z)$ — ① be a function of x, y, z. which is to be examine for maxima (or) minima value.

Let the variables x, y, z be connected by relation

$$\phi(x, y, z) \quad \text{—} ②.$$

Lagrange's equations for maxima (or) minima are given by.

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0.$$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0$$

$$\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0.$$

These equations together with eqn ② is gives the values of x, y, z & λ for maxima (or) minima.

Problems :-

① Find the minimum value of

(i) $U = x^2+y^2+z^2$ when $x+y+z=3a$.

(ii) $U = x^2+y^2+z^2$ when $xy+yz+zx=3a^2$

(iii) $U = x^2+y^2+z^2$ when $xyz=a^3$.

Sol:-

(i) Given that $U = f(x,y,z) = x^2+y^2+z^2$ — ①

and $x+y+z=3a$ — ②

Let $\phi = x+y+z-3a$.

Now,

$$\frac{\partial f}{\partial x} = 2x \quad \Big| \quad \frac{\partial \phi}{\partial x} = 1$$

$$\frac{\partial f}{\partial y} = 2y \quad \Big| \quad \frac{\partial \phi}{\partial y} = 1$$

$$\frac{\partial f}{\partial z} = 2z \quad \Big| \quad \frac{\partial \phi}{\partial z} = 1.$$

By the Lagrange's equations are

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0 \Rightarrow 2x+\lambda(1) \Rightarrow 2x+\lambda \Rightarrow x = \frac{-\lambda}{2}.$$

$$\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0 \Rightarrow 2y+\lambda(1) \Rightarrow 2y+\lambda=0 \Rightarrow y = \frac{-\lambda}{2}.$$

$$\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0 \Rightarrow 2z+\lambda(1)=0 \Rightarrow 2z+\lambda=0 \Rightarrow z = \frac{-\lambda}{2}.$$

$$\therefore x = y = z = \frac{-\lambda}{2} \quad \text{— ③}.$$

From eqn ②, we have

$$-\frac{\lambda}{2} - \frac{\lambda}{2} - \frac{\lambda}{2} = 3a \Rightarrow \frac{-\cancel{3}\lambda}{2} = \cancel{3}a$$

$$\boxed{\lambda = -2a}$$

③ $\Rightarrow$ $x = y = z = -\frac{(-2a)}{2}$

$x = y = z = a.$

$\therefore$ The extreme point is $(x, y, z) = (a, a, a)$.

$\therefore$ The minimum value is

$$U = a^2 + a^2 + a^2$$

$$U = 3a^2.$$

(iii) Given that $U = x^2 + y^2 + z^2$ —①

and $xy + yz + zx = 3a^2$ —②.

Let $\phi = xy + yz + zx - 3a^2.$

Now,

$\frac{\partial f}{\partial x} = 2x.$ | $\frac{\partial \phi}{\partial x} = y + z.$

$\frac{\partial f}{\partial y} = 2y$ | $\frac{\partial \phi}{\partial y} = x + z$

$\frac{\partial f}{\partial z} = 2z$ | $\frac{\partial \phi}{\partial z} = y + x.$

By Lagrange's equations are

$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0. \Rightarrow 2x + \lambda(y+z) = 0$ —③

$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \Rightarrow 2y + \lambda(x+z) = 0$ —④

$\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0 \Rightarrow 2z + \lambda(y+x) = 0$ —⑤

Solve eqns ③ & ④, we get

$$2x + \cancel{\lambda z} + \lambda y = 0$$
$$\underline{-2y \cancel{+\lambda z} \underset{-}{+} \lambda x = 0}$$
$$2x - 2y + \lambda y - \lambda x = 0..$$

$\Rightarrow 2(x-y) - \lambda(x-y) = 0$

$\Rightarrow (x-y)(2-\lambda) = 0.$

$x - y = 0 \quad | \quad 2 - \lambda = 0$

$x = y \quad | \quad \lambda = 2$

Now, Solve eqns (4) & (5), we get

$2y + \cancel{\lambda x} + \lambda z = 0$

$\underline{2z \pm \cancel{\lambda x} \pm \lambda y = 0}$

$2y - 2z + \lambda z - \lambda y = 0.$

$2(y-z) - \lambda(y-z) = 0.$

$(y-z)(2-\lambda) = 0$

$y - z = 0 \quad | \quad 2 - \lambda = 0$

$y = z \quad | \quad \lambda = 2$

$\therefore x = y = z.$

From eqn (2), we have.

(2) $\Rightarrow x(x) + x(x) + x(x) = 3a^2$

$x^2 + x^2 + x^2 = 3a^2$

$\cancel{3} x^2 = \cancel{3} a^2$

$x^2 = a^2$

$x = a$

$\therefore x = y = z = a$

$\therefore$ The extrem point is $(x, y, z) = (a, a, a).$

$\therefore$ The minimum value is

$U = a^2 + a^2 + a^2$

$U = 3a^2.$

(iii) Given that $U = x^2 + y^2 + z^2$ — ①

and $xyz = a^3$ — ②

Let $\phi = xyz - a^3$.

Now,

$$\frac{\partial f}{\partial x} = 2x \qquad \frac{\partial \phi}{\partial x} = yz$$

$$\frac{\partial f}{\partial y} = 2y \qquad \frac{\partial \phi}{\partial y} = xz$$

$$\frac{\partial f}{\partial z} = 2z \qquad \frac{\partial \phi}{\partial z} = xy$$

Lagranges eqns are

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \Rightarrow 2x + \lambda(yz) = 0 \Rightarrow 2x = -\lambda yz$$

$$\Rightarrow \frac{x}{yz} = \frac{-\lambda}{2}.$$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \Rightarrow 2y + \lambda(xz) = 0 \Rightarrow 2y = -\lambda xz$$

$$\Rightarrow \frac{y}{xz} = \frac{-\lambda}{2}.$$

$$\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0 \Rightarrow 2z + \lambda(xy) = 0 \Rightarrow 2z = -\lambda xy$$

$$\Rightarrow \frac{z}{xy} = \frac{-\lambda}{2}.$$

$$\therefore \frac{x}{yz} = \frac{y}{xz} = \frac{z}{xy}.$$

$$\frac{x^2}{xyz} = \frac{y^2}{xyz} = \frac{z^2}{xyz}$$

$$\therefore x^2 = y^2 = z^2$$

$$\therefore x = y = z.$$

From eqn ②, we have.

$$② \Rightarrow x(x)(x) = a^3 \Rightarrow x^3 = a^3 \Rightarrow \boxed{x = a}$$

∴ The extreme point is $(x, y, z) = (a, a, a)$

∴ The minimum value is

$$U = a^2 + a^2 + a^2$$

$$U = 3a^2.$$

(2) Find the extreme value of xy when $x^2 + xy + y^2 = a^2$.

Sol:- Given that $U = xy$ —— (1)

and $x^2 + xy + y^2 = a^2$ —— (2)

Let $\phi = x^2 + xy + y^2 - a^2$

Now,

$$\frac{\partial f}{\partial x} = y \qquad \frac{\partial \phi}{\partial x} = 2x + y$$

$$\frac{\partial f}{\partial y} = x \qquad \frac{\partial \phi}{\partial y} = x + 2y$$

The Lagrange's equations are.

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \Rightarrow y + \lambda(2x + y) = 0 \quad \text{—— (3)}$$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \Rightarrow x + \lambda(x + 2y) = 0. \quad \text{—— (4)}$$

Solve (3) & (4), we get

$$y + 2\lambda x + \lambda y = 0$$

$$x + \lambda x + 2\lambda y = 0$$

$$y - x + \lambda x - \lambda y = 0$$

$$(y - x) - \lambda(y - x) = 0$$

$$(y - x)(1 - \lambda) = 0.$$

$$y - x = 0 \qquad 1 - \lambda = 0$$

$$x = y \qquad \lambda = 1$$

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From eqn ②, we have

$$x^2 + x(x) + x^2 = a^2$$

$$3x^2 = a^2 \Rightarrow x^2 = \frac{a^2}{3}$$

$$x = \frac{a}{\sqrt{3}}.$$

$\therefore$ $x = y = \frac{a}{\sqrt{3}}$.

$\therefore$ The extreme point is $(x, y) = \left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right)$

$\therefore$ The extreme value is

$$U = \left(\frac{a}{\sqrt{3}}\right)\left(\frac{a}{\sqrt{3}}\right)$$

$$U = \frac{a^2}{3}.$$

③ Find the minimum value of $x^2 + y^2 + z^2$ given that $ax + by + cz = p$.

Sol:- Given that $U = x^2 + y^2 + z^2$ —①

and $ax + by + cz = p$ —②

Let $\phi = ax + by + cz - p$.

$\frac{\partial f}{\partial x} = 2x$	$\frac{\partial \phi}{\partial x} = a$
$\frac{\partial f}{\partial y} = 2y$	$\frac{\partial \phi}{\partial y} = b$
$\frac{\partial f}{\partial z} = 2z$	$\frac{\partial \phi}{\partial z} = c.$

Now, The Lagrange's eqns are

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \Rightarrow 2x + \lambda(a) = 0 \Rightarrow 2x + \lambda a = 0 \Rightarrow 2x = -\lambda a$$

$$\Rightarrow \frac{x}{a} = \frac{-\lambda}{2}$$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \Rightarrow 2y + \lambda(b) = 0 \Rightarrow 2y + \lambda b = 0 \Rightarrow 2y = -\lambda b$$

$$\Rightarrow \frac{y}{b} = \frac{-\lambda}{2}$$

$$\frac{\partial f}{\partial z}+\lambda\frac{\partial \phi}{\partial z}=0 \Rightarrow 2z+\lambda(c)=0 \Rightarrow 2z=-\lambda c \Rightarrow \frac{z}{c}=\frac{-\lambda}{2}.$$

$$\therefore \frac{x}{a}=\frac{y}{b}=\frac{z}{c}=\frac{-\lambda}{2}.$$

$$x=\frac{-a\lambda}{2},\quad y=\frac{-b\lambda}{2},\quad z=\frac{-c\lambda}{2}.$$

From eqn ②, we have

$$a\left(\frac{-a\lambda}{2}\right)+b\left(\frac{-b\lambda}{2}\right)+c\left(\frac{-c\lambda}{2}\right)=p$$

$$-\lambda\left[a^2+b^2+c^2\right]=2p.$$

$$\frac{-\lambda}{2}=\frac{p}{a^2+b^2+c^2}.$$

$$\therefore \frac{x}{a}=\frac{y}{b}=\frac{z}{c}=\frac{p}{a^2+b^2+c^2}.$$

$$x=\frac{ap}{a^2+b^2+c^2},\quad y=\frac{bp}{a^2+b^2+c^2},\quad z=\frac{cp}{a^2+b^2+c^2}.$$

$\therefore$ The extreme point is $(x,y,z)=\left(\frac{ap}{a^2+b^2+c^2},\frac{bp}{a^2+b^2+c^2},\frac{cp}{a^2+b^2+c^2}\right)$

$\therefore$ The minimum value is

$$U=\left[\frac{ap}{a^2+b^2+c^2}\right]^2+\left[\frac{bp}{a^2+b^2+c^2}\right]^2+\left[\frac{cp}{a^2+b^2+c^2}\right]^2.$$

$$=\frac{a^2p^2}{(a^2+b^2+c^2)^2}+\frac{b^2p^2}{(a^2+b^2+c^2)^2}+\frac{c^2p^2}{(a^2+b^2+c^2)^2}.$$

$$=\frac{\cancel{(a^2+b^2+c^2)}\,p^2}{(a^2+b^2+c^2)^{\cancel{2}}}$$

$$U=\frac{p^2}{a^2+b^2+c^2}.$$