EE 324: Midterm Exam 1

Due on March 2, 2016

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1. As depicted in Figure 1, there is a perfectly conducting track with a resistive sliding bar resting across the two rails. At $t = 0$, a switch closes across the DC voltage source shown to complete the circuit. Current flows, and a force results, as given by the familiar

$$\mathbf{F} = I\ell \times \mathbf{B}.$$

This causes the bar to accelerate rightward.

Now as the bar speeds up from rest, so too does the magnetic flux through the loop change from zero to non-zero. By Lenz' Law, this induces a current in the circuit that creates B-field change in opposition to the prevailing trend, thus the induced current begins to slow the rate of acceleration down to zero. Where is this equilibrium point?

For the constant B-field given in the figure, determine the velocity of the bar as $t \to \infty$, in terms of V_0, B_0, and ℓ.

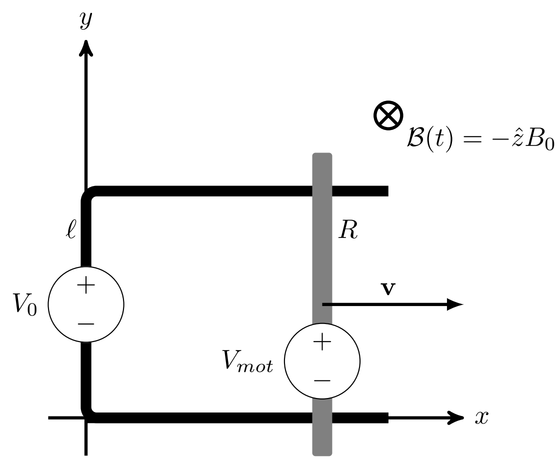

Figure 1: The ol' sliding bar trick.

$$\begin{aligned} \text{at } t \to \infty V_0 &= V_{mot} \\ &= \oint_C (\mathbf{v} \times \mathcal{B}) \cdot \mathrm{d}\ell \\ &= \ell B_0 v \hat{v} \end{aligned}$$

I think it could be easy to overthink things at this point, and I am trying to avoid this if at all possible: I will insist to myself hoever, that I can rest easy by my sanity check concluding that the bar's resistance has no bearing on the final speed attained. For example, if R were very large, it acceleration due to **F** would be very small, but in the limit should still be the same velocity.

$$\boxed{v \to \frac{V_o}{\ell B_0} \text{ as } t \to \infty}$$

2. Once upon a time, there was a linearly polarized uniform plane wave propagating through the ocean in the $+z$-direction. The seawater has the following characteristics: $\epsilon_r = 80$, $\mu_r = 1$, $\sigma = 4$ S/m. At a particular point—call it $z = 0$—the electric field of the wave has an intensity given by

$$\mathcal{E}(t) = \hat{x} 10 \sin(2\pi \times 10^6 t - \pi/8).$$

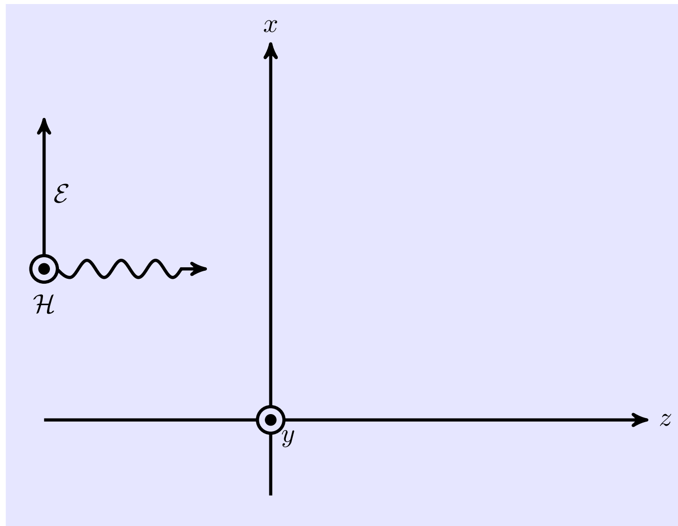

Figure 2: Drowning in the waves.

(a) Determine the attenuation constant α, the phase constant β, the intrinsic impedance η, the phase velocity, and the wavelength.

First, by inspection of $\mathcal{E}(t)$ we need to recognize that $\omega = 2 \times 10^6$ rad/s.

(Also recall that $\epsilon_0 = 8.854 \times 10^{-12}$F/m, and $\mu_0 = 4\pi \times 10^{-7}$H/m.)

we know the propagation constant, $$k = \omega\sqrt{\epsilon_r\mu_r}\sqrt{\epsilon_0\mu_0}\sqrt{1 - j\frac{\sigma}{\omega\epsilon_r\epsilon_0}}$$

$$\omega\sqrt{\frac{\mu\epsilon}{2}} \approx 0.0596$$

$$\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} \approx 2823.58$$

$$\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1\right]^{1/2}$$

$$\left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1\right]^{1/2} \approx 53.128$$

$$\beta = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1\right]^{1/2}$$

$$\left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1\right]^{1/2} \approx 53.147$$

$$\eta = \frac{\omega\mu}{k} = \sqrt{\frac{\mu}{\epsilon}}\left(1 - j\frac{\sigma}{\omega\epsilon}\right)^{-1/2}$$

$$\sqrt{\frac{\mu}{\epsilon}} = j9.204 \times 10^4$$

$$\left(1 - j\frac{\sigma}{\omega\epsilon}\right)^{-1/2} = 2.297 \times 10^{-06} + j2.297 \times 10^{-06}$$

Phase velocity $v_p = \frac{\omega}{\beta}$, and the wavelength is $\lambda f = \frac{c}{\sqrt{\epsilon_r \mu_r}}$

$$\boxed{\beta = 53.147 \qquad \alpha = 53.128}$$

$$\boxed{\eta = -0.2114 + j0.2114}$$

$$\boxed{v_p = 1.822 \times 10^5 m/s \qquad \lambda = 5.34m}$$

(b) Find the distance that this lonely wave must travel for its magnitude of **E** to decay to 0.01 V/m.

$$\mathcal{E}(t) = \hat{x}10e^{-53.128z}\cos(2\pi \times 10^6 - 53.147z + 3\pi/8)$$

we want to find where
$$\frac{1}{e^{-\alpha d}} = 1000$$
$$e^{\alpha d} = 1000$$
$$d = \frac{\ln(1000)}{\alpha}$$

$$\boxed{d = 0.13m}$$

murky water. (or Im wrong.)

(c) Write expressions for $\mathcal{E}(z,t)$ and $\mathcal{H}(z,t)$.

$$\boxed{\mathcal{E}(z,t) = \hat{x}10e^{-53.127z}\cos(2\pi \times 10^6 t - 53.147z + 3\pi/8)}$$

$\mathcal{H}(z,t) = \hat{y}\frac{10}{-23.65-j23.65}\cos(2\pi \times 10^6 t - 53.147z + 3\pi/8)$ $\mathcal{H}(z,t) = \hat{y}0.4228e^{j}2.3564e^{-53.127z}\cos(2\pi \times 10^6 t - 53.147z + 1.178)$

$$\boxed{\mathcal{H}(z,t) = \hat{y}0.4228e^{-53.127z}\cos(2\pi \times 10^6 t - 53.147z + 3.53)}$$

3. Consider a uniform plane-wave in free space ($\mu = \mu_0, \epsilon = \epsilon_0$), incident to a layered slab of dielectric material. The first layer has thickness d with permittivity ϵ_2, and the second layer extends forever with permittivity ϵ_3. Assume $\mu = \mu_0$ throughout. I am also going to assume this is the problem that was missing a frequency: let it be $f = 200MHz$.

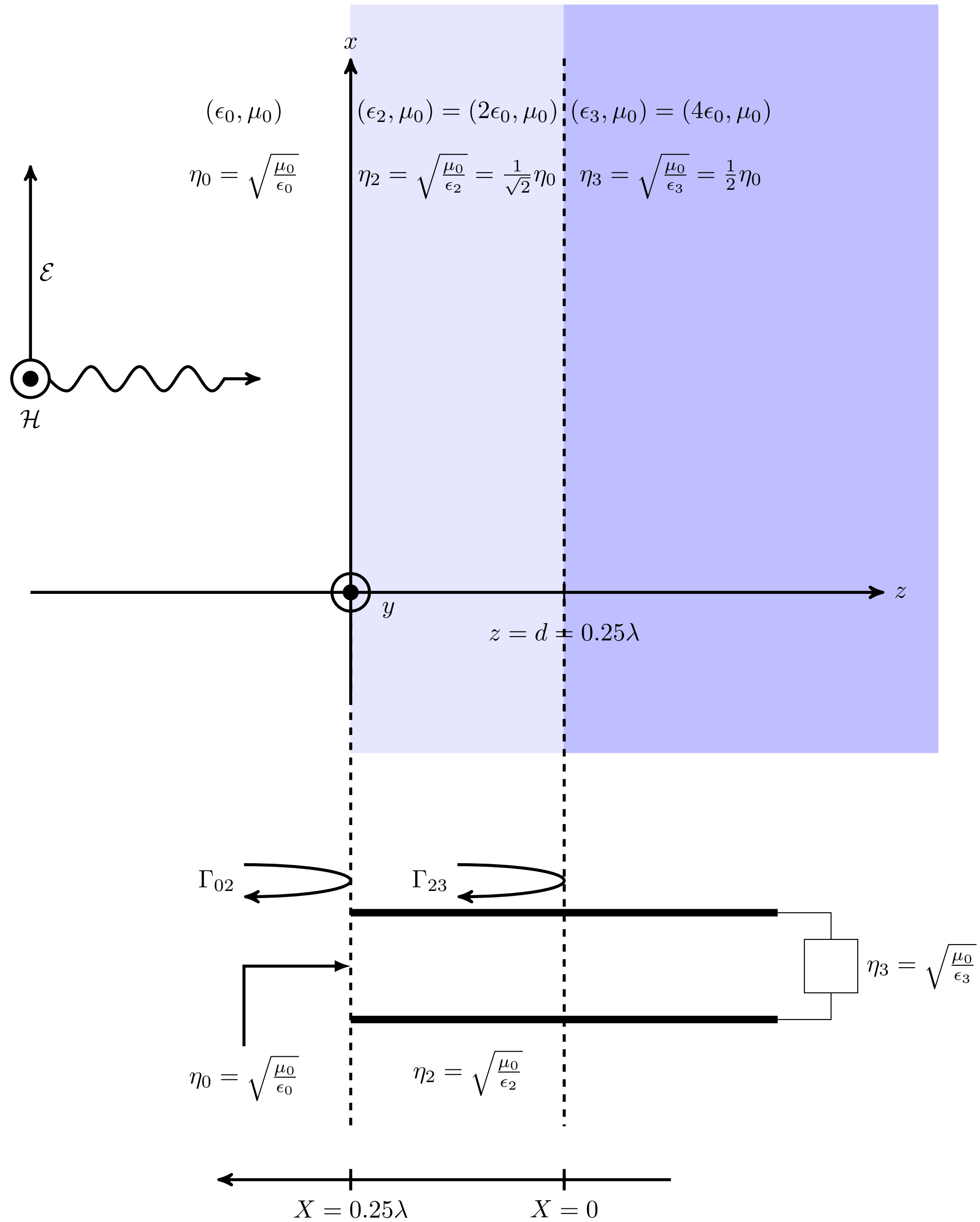

Figure 3: Double-Dielectrics and the Transmission line model equivalent.

(a) What is an expression for the reflection coefficient at $x = 0$? First, calculate reflection coefficient at the second interface.

However much we have reflecting

$$
\begin{aligned}
\Gamma_{23} &= \frac{\eta_3 - \eta_2}{\eta_3 + \eta_2} \\
&= \frac{\frac{1}{2}\eta_0 - \frac{1}{\sqrt{2}}\eta_0}{\frac{1}{2}\eta_0 + \frac{1}{\sqrt{2}}\eta_0} \\
\Gamma_{23} &= \frac{1 - \sqrt{2}}{1 + \sqrt{2}} \approx -0.172 \\
&= \frac{\frac{1}{\sqrt{2}} - 1}{\frac{1}{\sqrt{2}} + 1} \\
\Gamma_{02} &= \frac{\sqrt{2} - 2}{\sqrt{2} - 2} \approx -0.172 \\
\Gamma_(d) &= \frac{\eta_2(1 + \Gamma_{23}e^{-j2\beta d}) - \eta_1(1 - \Gamma_{23}e^{-j2\beta d})}{\eta_2(1 + \Gamma_{23}e^{-j2\beta d}) + \eta_1(1 - \Gamma_{23}e^{-j2\beta d})} \\
\Gamma_{02} &= \frac{\eta_2 - \eta_0}{\eta_2 + \eta_0} \\
\Gamma &= \frac{\Gamma_{02} + \Gamma_{23}e^{-j2\beta d}}{1 + \Gamma_{02}\Gamma_{23}e^{-j2\beta d}}
\end{aligned}
$$

$$\boxed{\Gamma_{23} = \frac{\Gamma_{02} + \Gamma_{23}e^{-j2\beta d}}{1 + \Gamma_{02}\Gamma_{23}e^{-j2\beta d}} \approx \frac{-0.172 - 0.172e^{-j2\beta d}}{1 + 0.0296e^{-j2\beta d}}}$$

zx

(b) If $d = 0.25\lambda$, what is the reflection coefficient in polar form at $x = 0$.

$$
\begin{aligned}
\beta &= \frac{2\pi f}{c}\sqrt{\mu_r \epsilon_r} \\
&= \frac{2\pi f}{c}\sqrt{\mu_r \epsilon_r} \\
-j2\beta X &= -j\frac{4\pi f}{c}\sqrt{\mu_r \epsilon_r}X \\
X &= \lambda/4 \\
\lambda f &= c \\
-j2\beta X\Big|_{X=\lambda/4} &= -j\frac{4\pi f}{c}\sqrt{\mu_r \epsilon_r}\frac{\lambda}{4} \\
&= -j\frac{\pi c}{c}\sqrt{\mu_r \epsilon_r} = -j\pi\sqrt{\mu_r \epsilon_r} \\
\Gamma(X) &= \frac{1 - 1.41e^{-j2\beta X}}{1 + 1.41e^{-j2\beta X}} \\
\Gamma(X)\Big|_{X=\lambda/4} &= \frac{1 - 1.41e^{-j\pi\sqrt{\mu_r \epsilon_r}}}{1 + 1.41e^{-j\pi\sqrt{\mu_r \epsilon_r}}} \\
\pi\sqrt{\mu_r \epsilon_r} &\approx 8.8858
\end{aligned}
$$

$$\boxed{\Gamma_{02} \approx -1.74 + j2.55 = 3.09\angle 124^\circ}$$

This is a nonsense answer. I will try to keep it under 1 next time.

(c) Verify this result using a Smith Chart.

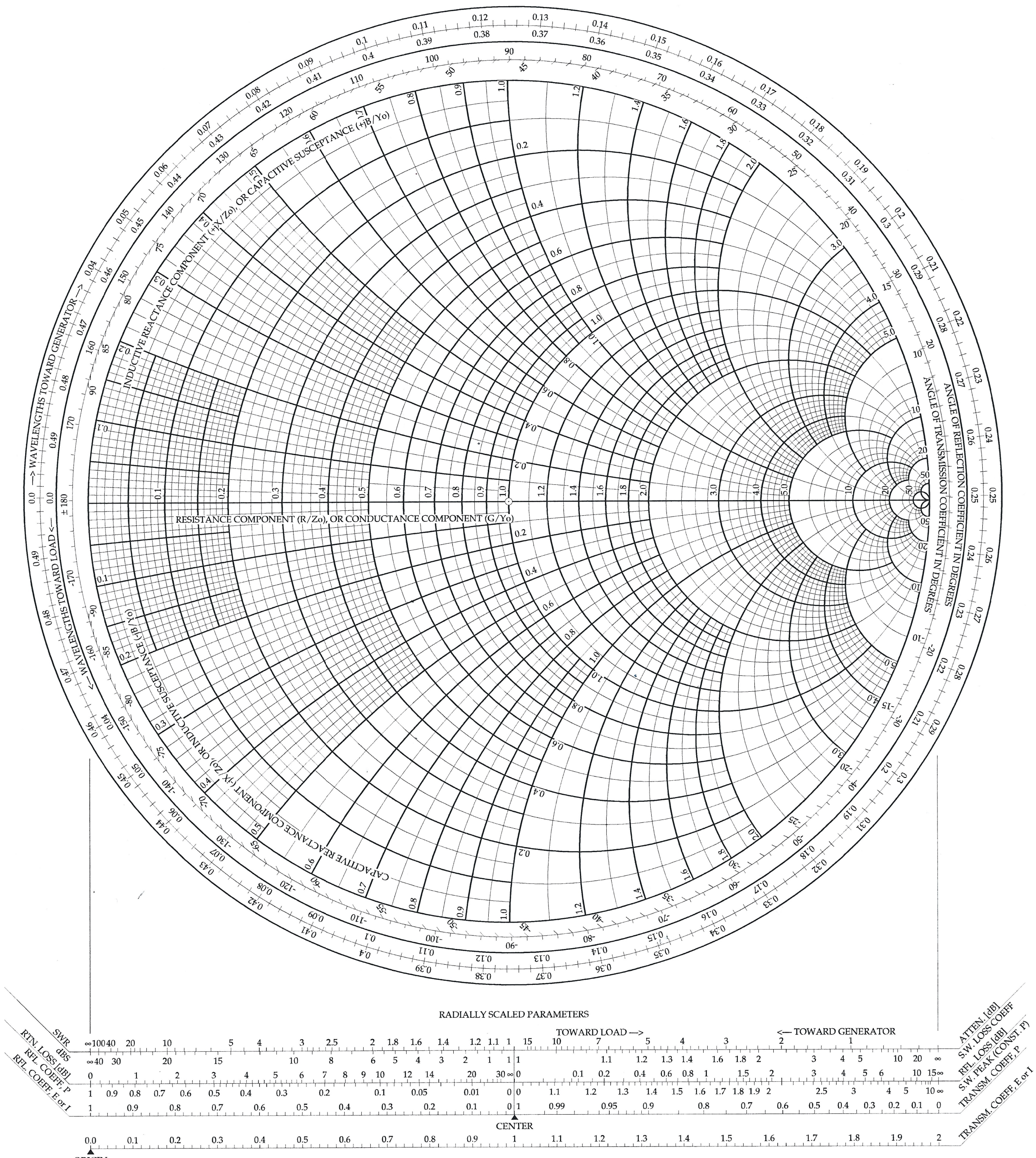

Figure 4: Smith Chart

4. Derive the following general expressions for the attenuation and phase constant $k = \beta - j\alpha$ for a conducting medium:

$$\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}-1\right]^{1/2}$$

$$\beta = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}+1\right]^{1/2}$$

After doing so, obtain expressions for the limiting cases wherein $\omega \to \infty$ and $\omega \to 0$. For lossy media, we express the propagation constant using

$$k = \omega\sqrt{\mu\epsilon}\sqrt{1-j\frac{\sigma}{\omega\epsilon}} \equiv \ \beta - j\alpha$$

To determine alpha and beta, we need to find the real and imaginary part of a number which is the root of a complex quantity, to do so, we make our lives easier by simplifying notation with a change in variables:

$$\omega\sqrt{\mu\epsilon}\sqrt{1-j\frac{\sigma}{\omega\epsilon}} = \beta - j\alpha$$

change variables, let $\qquad (B+jA)^2 = 1 - j\dfrac{\sigma}{\omega\epsilon}$

(solving for A and B)$A^2 - B^2 + j2AB =$

$$\begin{cases} A^2 - B^2 & = & 1 \\ 2AB & = & -\frac{\sigma}{\omega\epsilon} \end{cases}$$

we get two quadratics

$$A = \sqrt{1+B^2}$$

$$B\sqrt{1+B^2} = -\frac{\sigma}{2\omega\epsilon} \qquad 0 = B^4 + B^2 + \left(\frac{\sigma}{2\omega\epsilon}\right)^2$$

$$B\sqrt{1+B^2} = -K \qquad 0 = B^4 + B^2 - K^2$$

Here note that the quantity in the RHS above is an attentuation factor, which by its definition earlier must be positive. We are going to square things here is a second, and this will change the values B can take on, unless we are clear to define our terms: let,

$K = \frac{\sigma}{2\omega\epsilon}K > 0$

Now by inspection, we see that Since the RHS must have the negative term, the only way for this to be possible is if B takes on a negative value. We must meet this condition from hereon out to reach a valid solution.

$$B = \sqrt{A^2 - 1}$$

$$A\sqrt{A^2-1} = \frac{\sigma}{2\omega\epsilon} \qquad 0 = A^4 - A^2 - \left(\frac{\sigma}{2\omega\epsilon}\right)^2$$

use the quadratic

$$B^2 = \frac{-1 \pm \sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2}}{2}$$

$$A^2 = \frac{1 \pm \sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2}}{2}$$

require condition that A and B are real

radicand always>0, so choose(+)

$$B = \left[\frac{-1 + \sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2}}{2}\right]^{1/2}$$

radicand always >1, so choose(+)

$$A = \left[\frac{1 + \sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2}}{2}\right]^{1/2}$$

relate back to k

$$A = \sqrt{1 + B^2}$$

$$= \left(1 + \frac{(-1 + \sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2}}{2}\right)^{1/2}$$

$$= \frac{1}{\sqrt{2}}\left(1 + \sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2}\right)^{1/2}$$

$$\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}}\left(1 + \sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2}\right)^{1/2}$$

5. There is a plane wave in a lossy medium (with material properties ϵ, μ, σ). This wave propagates in the $+\hat{z}$-direction, and its electric field is given by

$$\mathbf{E}(z) = \hat{x}E_0 e^{-\alpha z} e^{-j\beta z}.$$

The propagation constant here is $k = \omega\sqrt{\mu\epsilon}\sqrt{1 - j\frac{\sigma}{\omega\epsilon}} = \beta - j\alpha$. Use the following methods to determine the total power absorbed in a circular cylindrical portion of material (w/ cross sectional area A, and length d along the z-axis) as depicted in Figure 3.

(a) Do this by evaluating the Poynting vector

$$\mathbf{P} = \frac{1}{2}\Re e[\mathbf{E} \times \mathbf{H}^*]$$

at the two cross-sectional interfaces, then take the difference in real power. (i.e. enforce conservation of energy.)

$$
\begin{aligned}
&\text{E-field} && \mathbf{E}(z) = \hat{x}E_0 e^{-\alpha z}e^{-j\beta z} \\
&\text{H-field} && \mathbf{H}(z) = \frac{1}{\eta}\hat{z}\times\mathbf{E} \\
& && = E_0\frac{(\beta - j\alpha)}{\mu\omega}\hat{y} \\
&\text{since material impedance} && \eta = \frac{\omega\mu}{k} = \frac{\omega\mu}{\beta - j\alpha} \\
&\text{power-area density} && \mathbf{P} = \frac{1}{2}\Re e[\mathbf{E}\times\mathbf{H}^*] \\
& && \mathbf{E}\times\mathbf{H}^* = \hat{z}E_0^2 e^{-\alpha z}e^{-j\beta z}\frac{1}{\eta^*}e^{-\alpha z}e^{j\beta z}(\hat{x}\times\hat{y}) \\
& && = \frac{E_0^2}{\eta^*}e^{-2\alpha z} \\
& && \mathbf{P} = \frac{1}{2}\Re e\left[\frac{E_0^2}{\eta^*}e^{-2\alpha z}\right] \\
& && = \frac{1}{2}\Re e\left[E_0^2\frac{(\beta + j\alpha)}{\mu\omega}e^{-2\alpha z}\right] \\
& && = \frac{1}{2}E_0^2\frac{\beta}{\mu\omega}e^{-2\alpha z} \\
& && = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}+1\right]^{1/2}\frac{E_0^2}{2\mu\omega}e^{-2\alpha z} \\
& && = \frac{E_0^2}{2}\sqrt{\frac{\epsilon}{2\mu}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}+1\right]^{1/2}e^{-2\alpha z} \\
&\text{interface 1 power} && P_1 = \frac{E_0^2 A}{2}\sqrt{\frac{\epsilon}{2\mu}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}+1\right]^{1/2} \\
&\text{interface 2 power} && P_{z=d} = \frac{E_0^2 A}{2}\sqrt{\frac{\epsilon}{2\mu}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}+1\right]^{1/2}e^{-2\alpha d} \\
&\text{real power difference} && P_{absorbed} = \frac{E_0^2 A}{2}\sqrt{\frac{\epsilon}{2\mu}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}+1\right]^{1/2}(1-e^{-2\alpha d})
\end{aligned}
$$

$$
\boxed{P_{absorbed} = \frac{E_0^2 A}{2}\sqrt{\frac{\epsilon}{2\mu}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}+1\right]^{1/2}(1-e^{-2\alpha d})}
$$

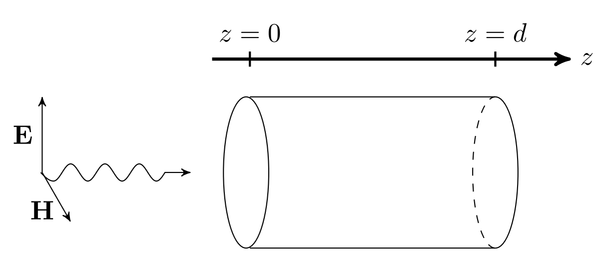

Figure 5: The ol' lossy bar.

(b) Confirm this result by directly calculating the power absorbed due to conductivity, as typically calculated by

$$\frac{1}{2}\int_V \sigma|\mathbf{E}|^2\,\mathrm{d}v.$$

Remember, V is the volume of the cylinder of material, and $|\mathbf{E}|^2 = \mathbf{E}\cdot\mathbf{E}^*$.

$$\begin{aligned}
\text{take conjugate}\quad \mathbf{E}\cdot\mathbf{E}^* &= E_0 e^{-\alpha z}e^{-j\beta z}E_0 e^{-\alpha z}e^{j\beta z}(\hat{x}\cdot\hat{x}) \\
&= E_0^2 e^{-2\alpha z} \\
\text{integral}\qquad P_{absorbed} &= \frac{1}{2}\int_V \sigma|\mathbf{E}|^2\,\mathrm{d}v \\
&= \frac{E_0^2\sigma \mathrm{A}}{2}\int_0^d e^{-2\alpha z}\,\mathrm{d}z \\
&= -\frac{E_0^2\sigma \mathrm{A}}{4\alpha}(1-e^{-2\alpha d})
\end{aligned}$$

$$\boxed{P_{absorb} = -\frac{E_0^2\sigma \mathrm{A}}{4\alpha}(1-e^{-2d\alpha})}$$

$$\begin{aligned}
\text{can we show}\qquad -\frac{E_0^2\sigma\mathrm{A}}{4\alpha} = P_{absorbed} &= \frac{E_0^2 A}{2}\sqrt{\frac{\epsilon}{2\mu}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}+1\right]^{1/2} \\
-\frac{\sigma}{2}\sqrt{\frac{2\mu}{\epsilon}} &= \left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}+1\right]^{1/2}\alpha \\
-\frac{\sigma}{2}\sqrt{\frac{2\mu}{\epsilon}} &= \left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}+1\right]^{1/2}\omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}-1\right]^{1/2} \\
-\frac{\sigma}{2\omega}\sqrt{\frac{2\mu}{\epsilon}}\sqrt{\frac{2}{\mu\epsilon}} &= \left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}+1\right]^{1/2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}-1\right]^{1/2} \\
-\frac{\sigma}{\omega\epsilon} &= \left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}+1\right]^{1/2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}-1\right]^{1/2} \\
-\frac{\sigma}{\omega\epsilon} &= \left[\left(1+\left(\frac{\sigma}{\omega\epsilon}\right)^2\right)-1\right]^{1/2} \\
\left(\frac{\sigma}{\omega\epsilon}\right)^2 &= \left(1+\left(\frac{\sigma}{\omega\epsilon}\right)^2\right)-1 \\
\left(\frac{\sigma}{\omega\epsilon}\right)^2 &= \left(\frac{\sigma}{\omega\epsilon}\right)^2
\end{aligned}$$

Yes, we can.