Hassani, Ch 9, Complex Calc.

9.1 Complex Functions

eg 9.5.1

(a) how does $f(z) = w = z^2$ map line $y = mx$ onto w-plane?

$$z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy \qquad \left(\text{ie. } (x,y) \xmapsto{f} (u,v)\right)$$

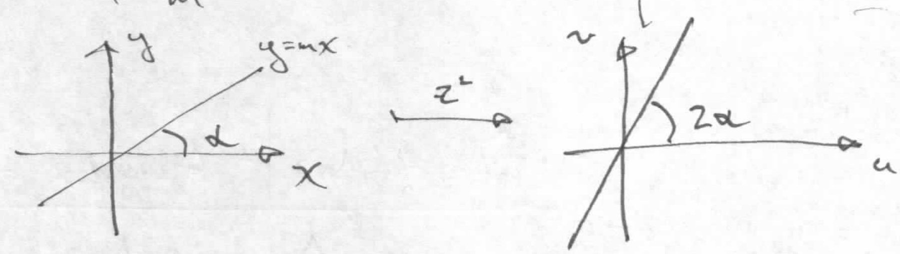
so $u(x,y) = x^2 - y^2$ & $v(x,y) = 2xy$

Sub. $y = mx \rightarrow u = x^2(1-m^2)$ & $v = 2mx^2$

we want an equation for v in terms of u...

$$\frac{u}{1-m^2} = x^2 \qquad \therefore \left[v = \frac{2m}{1-m^2} u \right]$$

show that



$$\tan(\alpha) = \frac{m}{1}$$

$$\frac{2m}{1-m^2} = \frac{2m}{(1+m)(1-m)} = \frac{2\tan(\alpha)}{1-\tan^2(\alpha)}$$

$$= \tan(2\alpha)$$

(b) $f(z) = e^z$

~~$u = \cos($~~ $\qquad e^z = e^{x+iy} = e^x(\cos(y) + i\sin(y))$

$$\left.\begin{aligned} u &= e^x\cos(y) = e^x\cos(mx) \\ v &= e^x\sin(y) = e^x\sin(mx) \end{aligned}\right\} \text{ parametric equations}$$

9.2 ; Analytic Functions

9.2.2 : examine derivative of $f(z) = x^2 + 2iy^2$ @ $z = 1 + i$

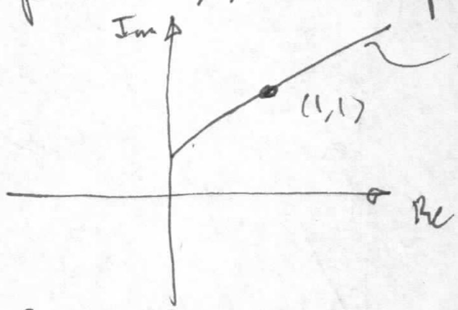
$$\left.\frac{df}{dz}\right|_{z=1+i} = \lim_{\Delta z \to 0} \frac{f(1+i+\Delta z) - f(1+i)}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{(1+\Delta x)^2 + 2i(1+\Delta y)^2 - 1 - 2i}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{2\Delta x + (\Delta x)^2 + 4i\Delta y + 2i(\Delta y)^2}{\Delta x + i\Delta y}$$

Since we are concerned about $z = 1+i$,

we want to appraise the derivative from all directions on the line crossing point $(1,1)$ in the z-plane



$(y-1) = m(x-1)$ for which $\Delta y = m\Delta x$

Substitute into derivative:

$$\left.\frac{df}{dz}\right|_{z_0=1+i} = \lim_{\Delta x \to 0} \frac{2\Delta x + (\Delta x)^2 + 4im\Delta x + 2im^2(\Delta x)^2}{\Delta x + im\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} \cdot \frac{2 + 4im + \Delta x + 2im^2\Delta x}{1 + im}$$

$$\left.\frac{df}{dz}\right|_{z_0=1+i} = \frac{2+4im}{1+im}$$

This shows that there are infinitely many values for $\left.\frac{df}{dz}\right|_{z_0}$ depending on the choice of m (the direction of the test)

The derivative of $f(z)$ does not exist at ~~$z_0 = 1+i$~~ $z = 1+i$

What are the conditions under which a complex function is differentiable? for $f(z) = u(x,y) + i v(x,y)$.

$$\left.\frac{df}{dz}\right|_{z_0 = x+iy} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left\{ \frac{u(x_0+\Delta x, y_0+\Delta y) - u(x_0,y_0)}{\Delta x + i\Delta y} + i\frac{v(x_0+\Delta x, y_0+\Delta y) - v(x_0,y_0)}{\Delta x + i \Delta y} \right.$$

Since Δx & Δy are linearly independent if the limit exists along $\Delta x = 0$ & $\Delta y = 0$ then the limit exists for any combination of Δx & Δy (i.e. $m = \frac{\Delta y}{\Delta x}$)

So the conditions for differentiability become

$\Delta x = 0$: $$\left.\frac{df}{dz}\right|_{z_0} = \lim_{\Delta y \to 0} \left\{ \frac{u(x_0, y_0+\Delta y) - u(x_0,y_0)}{i\Delta y} + i\frac{v(x_0, y_0+\Delta y) - v(x_0,y_0)}{i\Delta y} \right\}$$

$$\cancel{\left.\frac{df}{dz}\right|_{z_0} = \frac{\partial u}{\partial y}}$$

$$\boxed{\left.\frac{df}{dz}\right|_{z_0} = \left.\frac{\partial v}{\partial y}\right|_{(x_0,y_0)} - i\left.\frac{\partial u}{\partial y}\right|_{(x_0,y_0)}}$$

$\Delta y = 0$: $$\left.\frac{df}{dz}\right|_{z_0} = \lim_{\Delta x \to 0} \left\{ \frac{u(x_0+\Delta x, y_0) - u(x_0,y_0)}{\Delta x} + i\frac{v(x_0+\Delta x, y_0) - v(x_0,y_0)}{\Delta x} \right\}$$

$$\boxed{\left.\frac{df}{dz}\right|_{z_0} = \left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)} + i\left.\frac{\partial v}{\partial x}\right|_{(x_0,y_0)}}$$

$$\left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)} + i\left.\frac{\partial v}{\partial x}\right|_{(x_0,y_0)} = \left.\frac{\partial v}{\partial y}\right|_{(x_0,y_0)} - i\left.\frac{\partial u}{\partial y}\right|_{(x_0,y_0)}$$

Hence $\boxed{\text{Cauchy-Riemann (CR) conditions} \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ \& \ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$

$$x = \frac{1}{2}(z+z^*) \qquad y = \frac{1}{2i}(z-z^*)$$

$$\frac{\partial}{\partial x}\left(\frac{1}{2i}(z-z^*)\right) = \cancel{\frac{\partial}{\partial y}} \cancel{\frac{\partial}{\partial x}} \frac{\partial}{\partial y}\left(\frac{1}{2}(z+z^*)\right)$$

$$\frac{\partial}{\partial y}\left(\frac{1}{2i}(z-z^*)\right) = \frac{\partial}{\partial x}\left(\frac{1}{2}(z+z^*)\right)$$

$$\cancel{\frac{1}{2i}\frac{\partial}{\partial x}\Big(} \quad \cancel{\frac{1}{2i}\left[\frac{\partial z}{\partial x} - \frac{\partial z^*}{\partial x}\right] = \frac{1}{2}\left[\frac{\partial z}{\partial y} + \frac{\partial z^*}{\partial y}\right]}$$

$$\cancel{= -i\Big[} \qquad -i\,\cancel{\frac{\partial z}{\partial z}}\,\cancel{\frac{\partial z}{\partial y}}$$

$$\cancel{\frac{\partial f}{\partial z^*} = \frac{\partial u}{\partial z^*} + i\frac{\partial v}{\partial z^*}}$$

let $\quad u = \frac{1}{2i}(z-z^*) \quad$ & $\quad v = \frac{1}{2}(z+z^*)$

$$\cancel{\frac{\partial u}{\partial x} = \frac{1}{2i}\left(\frac{\partial u}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial u}{\partial z^*}\frac{\partial z^*}{\partial y} - \frac{\partial u}{\partial z^*}\frac{\partial z^*}{\partial x} - \frac{\partial u}{\partial z^*}\frac{\partial z^*}{\partial y}\right)}$$

$$\cancel{\frac{\partial v}{\partial x} = \frac{1}{2}\left(\frac{\partial v}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial v}{\partial z^*}\frac{\partial z^*}{\partial x} + \frac{\partial v}{\partial z^*}\frac{\partial z^*}{\partial y}\right)}$$

from CS conditions, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\frac{\partial u}{\partial x} = \frac{1}{2i}\left(\frac{\partial u}{\partial z}\frac{\partial z}{\partial x} - \frac{\partial u}{\partial z^*}\frac{\partial z^*}{\partial x}\right) = \frac{1}{2}\left(\frac{\partial v}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial v}{\partial z^*}\frac{\partial z^*}{\partial y}\right)\frac{\partial}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{1}{2i}\left(\frac{\partial u}{\partial z}\frac{\partial z}{\partial y} - \frac{\partial u}{\partial z^*}\frac{\partial z^*}{\partial y}\right) = -\frac{1}{2}\left(\frac{\partial v}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial v}{\partial z^*}\frac{\partial z^*}{\partial x}\right)$$

$$\cancel{i = i} \qquad f(z) = u(x,y) + iv(x,y)$$

$$\cancel{\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial z}}$$

$$\cancel{\frac{\partial u}{\partial x} =}$$

IF f is differentiable, it must be independent of z^*

$$\cancel{\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial y}} \qquad \cancel{\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial y}}$$

$$\cancel{\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} = \partial u}$$

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}$$

$$\left.\frac{df}{dz}\right|_{z_0} = \lim_{\substack{\Delta x \to 0 \\ \cancel{\Delta y \to 0}}} \frac{u\left(\frac{1}{2}(z+z^*) + \Delta x, \frac{1}{2i}(z-z^*)\right) - u\left(\frac{1}{2}(z+z^*), \frac{1}{2i}(z-z^*)\right)}{\Delta x + i(0)}$$

$$\cancel{= \partial}$$

$$\cancel{+ i\, v} \qquad i\, v(x,y \ldots$$

$$\cancel{= \frac{\partial}{\partial x}\left(\frac{1}{2}(z+z^*)\right) + i\frac{\partial}{\partial x}\left(\right.}$$

$$= \frac{\partial}{\partial x}\left(\frac{1}{2}(z+z^*)\right) + i\frac{\partial v}{\partial x}\left(\frac{1}{2}(z+z^*)\right)$$

9.2.4: Determine differentiability:

(a) $f(z) = x^2 + 2iy^2$ $\dots\to$ $u = x^2$, $v = 2y^2$

$$\frac{\partial u}{\partial x} = 2x \neq 4y \frac{\partial v}{\partial y}$$

so not differentiable

using alternate C-R conditions

$$f(z) = \left[\frac{1}{2}(z+z^*)\right]^2 + 2i\left[\frac{1}{2i}(z-z^*)\right]^2$$

$$= \frac{1}{4}(z+z^*)^2 + \frac{1}{2i}(z-z^*)^2$$

$$= \frac{1}{4}(z^2+z^{*2}+2zz^*) + \frac{1}{2i}(z^2+z^{*2}-2zz^*)$$

$$= \left(\frac{1}{2}+i\right)zz^* + \left(\frac{1}{4}+\frac{1}{2i}\right)(z^2+z^{*2})$$

$\Rightarrow$ f has explicit dependence on z^* & $\therefore$ undiff.

(b) $f(z) = \underbrace{x^2-y^2}_{u} + \underbrace{2ixy}_{v}$

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$

$\therefore$ since C-R conditions are satisfied, $f(z)$ is differentiable

$$f(z) = \left[\frac{1}{2}(z+z^*)\right]^2 - \left[\frac{1}{2i}(z-z^*)\right]^2 + 2i\left[\frac{1}{2}(z+z^*)\right]\left[\frac{1}{2i}(z-z^*)\right]$$

$$= \frac{1}{4}(z^2+2zz^*+z^{*2}) + \frac{1}{4}(z^2-2zz^*+z^{*2}) + \frac{1}{2}(z^2-zz^*\cancel{+}z^{*2})$$

$$= z^2, \quad \therefore \frac{\partial f}{\partial z^*} = 0 \Rightarrow \text{differentiability}$$

(c) $u(x,y) = e^x\cos(y)$

$v(x,y) = e^x\sin(y)$

$$\frac{\partial u}{\partial x} = e^x\cos(y) = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = \cancel{e^x\cos(y)} \; -e^x\sin y = -\frac{\partial v}{\partial x}$$

✓

also, $f(z) = e^x\cos y + ie^x\sin y = e^x(\cos y + i\sin y)$

$$= e^x e^{iy} = e^{x+iy} = e^z$$

$\to \dfrac{\partial f}{\partial z^*} = 0 \quad \therefore f$ is differentiable

F; $f(z) = u(x,y) + i v(x,y)$ is differentiable in a complex region iff C-R conditions are satisfied, and all first partial derivatives are continuous.

T: $f(z) = u + iv$ is differentiable in a region on the complex plane iff C-R conditions are met, and iff all partial derivatives are continuous in that region.

∴ proved C-R, now ~~OO~~ d

if, of iff:

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

or, by D,

$$\left| \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} - \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\right| < \varepsilon$$

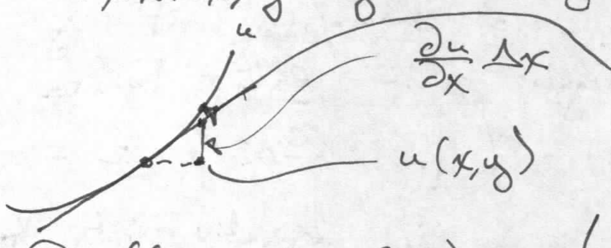
whenever $|\Delta z| < \delta$

S, by D, $f(z+\Delta z) - f(z) = u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)$
$- u(x,y) - iv(x,y)$

∴ since u & v have — partial derivatives

$$u(x+\Delta x, y+\Delta y) = u(x,y) + \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \epsilon_1 \Delta x + \delta_1 \Delta y$$

$$v(x+\Delta x, y+\Delta y) = u(x,y) + \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \epsilon_2 \Delta x + \delta_2 \Delta y$$



apply C-R condition here

$$\therefore,\ f(z+\Delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\Delta y$$
$$+ (\epsilon_1 + i\epsilon_2)\Delta x + (\delta_1 + i\delta_2)\Delta y$$
$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(\Delta x + i\Delta y) + \epsilon \Delta x + \delta \Delta y$$

recall, $\Delta z = \Delta x + i\Delta y$,

so div. by Δz:

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} - \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) = \epsilon \frac{\Delta x}{\Delta z} + \delta \frac{\Delta y}{\Delta z}$$

By the triangle inequality for complex #s

$$\left|\epsilon \frac{\Delta x}{\Delta z} + \delta \frac{\Delta y}{\Delta z}\right| \leq |\epsilon_1 + i\epsilon_2| + |\delta_1 + i\delta_2|$$

~~ues $\Rightarrow$ $\left|\frac{\Delta x}{\Delta z}\right|$ & $\left|\frac{\Delta y}{\Delta z}\right|$ must ≤ 1, otherwise the inequality would fail.~~

$\Leftarrow$ | fact that $\left|\frac{\Delta x}{\Delta z}\right|$ & $\left|\frac{\Delta y}{\Delta z}\right|$ are ≤ 1

D: a function $f : \mathbb{C} \to \mathbb{C}$ is called <u>analytic</u> at z_0 if it is differentiable at z_0 and at all other points in some neighborhood of z_0.

such a point is =: a <u>regular point</u>; the $\sim$, is called a <u>singular point</u> or <u>singularity</u> of f

a f | $\forall$ point in $\mathbb{C}$ is regular is := <u>entire function</u>

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e.g. 9.8.6: f $f(z) = z$ $\left(| u = x \text{ \& } v = y \right)$

(a) $\left(\frac{df}{dz} = \frac{\partial x}{\partial x} + i\frac{\partial y}{\partial x} = 1\right)$ $\therefore$ derivative exists for all points of complex plane, & so $f(z) = z$ is <u>entire</u>

(b) $f(z) = z^2$; $\Rightarrow$ $u = x^2 - y^2$ & $v = \cancel{2ixy}\ 2xy$

By C-R conditions, $\cancel{\frac{\partial u}{\partial x} = 2x =}$ $\frac{df}{dz} = 2x + 2iy = 2z$

$\therefore$ $f(z)$ is regular for $\forall$ $\mathbb{C}$.

© $f(z) = z^n, \quad n \geq 1$

$n = 1 \to f(z) = z$ which is entire, & differentiable

~~n+1, for n=1 is n=2~~ ... $\frac{df}{dz} = 1$

$f(z) = z^{n+1}$

Assuming z^n is true,

$$z^{n+1} = (x+iy)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} x^{n+1-k} (iy)^k$$

of, binomial

$$= \sum_{k=0}^{n+1} \frac{(n+1)!}{k!(n+1-k)!}$$

$$z^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} (iy)^k$$

$$z^n = (x+iy)^n(x+iy) = (x+iy) \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^{n-k} (iy)^k$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k+1} (iy)^k + \sum_{k=0}^{n} \binom{n}{k} x^{n-k} (iy)^{k+1}$$

$$= x^{n+1} + \sum_{k=1}^{n} \binom{n}{k} x^{n-k+1} (iy)^k + \sum_{k=0}^{n-1} \binom{n}{k} x^{n-k} (iy)^{k+1} + (iy)^{n+1}$$

$$= x^{n+1} + \sum_{k=1}^{n} \binom{n}{k} x^{n-k+1} (iy)^k + \sum_{j=1}^{n} \binom{n}{j-1} x^{n-j+1} (iy)^j + (iy)^{n+1}$$

$$= x^{n+1} + \sum_{k=1}^{n} \left[\binom{n}{k} + \binom{n}{k-1}\right] x^{n-k+1} (iy)^k + (iy)^{n+1}$$

$$= x^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} x^{n-k+1} (iy)^k + (iy)^{n+1}$$

$$= \sum_{k=1}^{n+1} \binom{n+1}{k} x^{n-k+1} (iy)^k = z^{n+1}$$

$$\frac{d}{d\zeta} z^n = \frac{d}{d\zeta}(x+iy)^n = \frac{d}{d\zeta}\sum_{k=0}^{n}\binom{n}{k}x^{n-k}(iy)^k$$

$f(z) = n$ .... $\Rightarrow$ is entire,

$\not\frac{d}{d\zeta}x \Rightarrow \int, (x+iy)$:

$$\frac{d}{d\zeta}\left[\sum_{k=0}^{n}\binom{n}{k}x^{n-k+1}(iy)^k + \sum_{k=0}^{n}\binom{n}{k}x^{n-k}(iy)^{k+1}\right]$$

$$= \frac{d}{d\zeta}x^{n+1} + \frac{d}{d\zeta}\sum_{k=1}^{n}\binom{n}{k}x^{n-k+1}(iy)^k + \frac{d}{d\zeta}\sum_{k=0}^{n-1}\binom{n}{k}x^{n-k}(iy)^{k+1} + \frac{d}{d\zeta}(iy)^{n+1}$$

$$= \frac{d}{d\zeta}\left(x^{n+1} + (iy)^{n+1}\right) + \frac{d}{d\zeta}\sum_{k=1}^{n}\binom{n}{k}x^{n-k}(iy)^k + \frac{d}{d\zeta}\sum_{k=1}^{n}\binom{n}{k-1}x^{n-k+1}(iy)^k$$

$$(n+1)x^n + (n+1)(iy)^n$$

i dunno.

eg: 9.2.8; Now, Find unique Function $f: \mathbb{C} \to \mathbb{C} \,|$:

A $f(z)$ is single valued & analytic for all ~~G~~, z

B $\frac{df}{dz} = f(z)$

C $f(z_1 + z_2) = f(z_1) f(z_2)$

$z_1 = 0 = z_2 \implies f(0) = [f(0)]^2 \implies f(0) = 1$ or $f(0) = 0$

and $\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z) f(\Delta z) - f(z)}{\Delta z}$

$= f(z) \lim_{\Delta z \to 0} \frac{f(\Delta z) - 1}{\Delta z}$.

$f(z) = u(x, y) + i v(x, y)$

by B, $\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = u + iv \implies \frac{\partial u}{\partial x} = u$ & $\frac{\partial v}{\partial x} = v$

whose general solutions are ~~$\frac{\partial u}{\partial x} = u$~~ $u(x, y) = a(y) e^x$

& C-R conditions produce $v(x, y) = b(y) e^x$

$\frac{da}{dy} = -b(y)$

& $\frac{db}{dy} = a(y)$

for which the general solution is $a(y) = A\cos(y) + B\sin(y)$

& $b(y) = A\sin(y) - B\cos(y)$

Now since $f(0) = 1$, $u(0,0) = 1$, $v(0,0) = 0 \implies a(0) = 1$, $b(0) = 0 \implies A = 1$, $B = 0$

$\therefore$ ~~$f(z) = A$~~ $f(z) = a(y) e^x + i b(y) e^x = e^x (a(y) + i b(y))$

$= e^x (\cos(y) + i \sin(y)) = e^x e^{iy} = e^{x + iy}$

$= e^z$

Since e^x & e^{iy} are both entire, and the product of two entire functions is likewise entire,

so must e^z also be entire.

We already found that any polynomial in z is also entire. Thus any product/or sum of entire functions is also entire:

$$\therefore \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \& \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

are entire,

$$\text{and also} \quad \sinh z = \frac{e^{z} - e^{-z}}{2} \quad \& \quad \cosh z = \frac{e^{z} + e^{-z}}{2}$$

Ratios do not behave the same tho,
(re: entirety)

$\hookrightarrow \quad \dfrac{f(z)}{g(z)}$ (2 polynomials) will not have a derivative at the zeros of $g(z)$

An interesting property of analytic $u(x,y)$ & $v(x,y)$

The family of curves, $u(x,y) = \text{constant}$, is $\perp$ to the family of curves, $v(x,y) = \text{constant}$ at each point of the complex plane where $f(z) = u + iv$ is analytic... lets look at the normal to the curves:

$$\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) \quad \& \quad \nabla v = \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \right)$$

$$(\nabla u) \cdot (\nabla v) = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) \cdot \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \right)$$

$$= \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}\left(-\frac{\partial u}{\partial y} \right) + \frac{\partial u}{\partial y}\left(\frac{\partial u}{\partial x} \right) = 0$$

(by C-R conditions)

9.3: Conformal Maps

The real & imaginary parts of an analytical function satisfy the two dimensional Laplace equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

For 3-D Laplace problems

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

where ϕ is the electrostatic potential

for cases of cylindrical symmetry, the term $\frac{\partial^2 \phi}{\partial z^2} \to 0$, $\therefore$ The Laplace equation reduces to a two dimensional problem.

For cylindrical symmetry,

this ϕ is $\phi = 2\lambda \ln \rho = 2\lambda \ln((x^2+y^2)^{1/2}) = \ln|z|$

$w(z)$ could be the complex function for which ϕ is the real part

called the complex potential

$$w(z) = 2\lambda \ln \rho + i 2\lambda \varphi = 2\lambda(\ln \rho + \ln(e^{i\varphi}))$$

$$= 2\lambda \ln(\rho e^{i\varphi}) = 2\lambda \ln z$$

for the complex potential representing multiple filaments of charge:

$$w(z) = 2\sum_{k=1}^{n} \lambda_k \ln(z - z_k)$$

$\therefore$ the effect here is to transform the circular electrostatic potential lines ~~into~~ (from the z-plane) into straight lines $\parallel$ to axes in w-plane

A mapping that preserves the angle between two curves is called <u>conformal mapping</u>.

P ; let, γ_1 & γ_2 be curves in z-plane that intersect at a point z_0, angle α,

& let, $f: \mathbb{C} \to \mathbb{C}$ be a mapping given by $f(z) = z' = x' + iy'$ that is analytic at z_0.

let, γ_1' & γ_2' be images of γ_1 & γ_2 under this mapping, which intersect at angle α'.

$\Rightarrow$ S|-3 ; $\alpha' = \alpha$ (f mapping is conformal) if $\left.\frac{dz'}{dz}\right|_{z_0} \neq 0$

f harmonic in $(x,y) \Rightarrow f$ harmonic in (x', y')

~~oo~~ overview:

$$\hat{e}_i = \frac{\hat{e}_x \Delta x_i + \hat{e}_y \Delta y_i}{\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}}, \quad i = 1, 2$$

$$\therefore \quad \hat{e}_1 \cdot \hat{e}_2 = \frac{\Delta x_1 \Delta x_2 + \Delta y_1 \Delta y_2}{\sqrt{(\Delta x_1)^2 + (\Delta y_1)^2}\sqrt{(\Delta x_2)^2 + (\Delta y_2)^2}}$$

likewise in prime plane: $\Big|_{''}$

$\hat{e}_1' \cdot \hat{e}_2' = $ " ... (where $x' = u(x,y)$ & $y' = v(x,y)$)

Using: $\Delta x_i' = \frac{\partial u}{\partial x} \Delta x_i + \frac{\partial u}{\partial y} \Delta y_i$

& $\Delta y_i' = \frac{\partial v}{\partial x} \Delta x_i + \frac{\partial v}{\partial y} \Delta y_i$, $\quad i = 1, 2$

and C-R conditions $\left(\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \right)$

$$\hat{e}_1' \cdot \hat{e}_2' = \frac{\left(\frac{\partial u}{\partial x}\Delta x_1 + \frac{\partial u}{\partial y}\Delta y_1\right)\left(\frac{\partial u}{\partial x}\Delta x_2 + \frac{\partial u}{\partial y}\Delta y_2\right) + \cdots}{\sqrt{\left(\frac{\partial u}{\partial x}\Delta x_1 + \frac{\partial u}{\partial y}\Delta y_1\right)^2 + \cdots}\sqrt{\cdots +}}$$

or $|z' - a'| = r'$ & $a' = a_r' + i a_i' = \frac{a^*}{|a|^2 - r^2}$

OR, $|r| = a$, wherein we have

$$a_r x' - a_i y' = \frac{1}{2}, \text{ which is a line.}$$

Types of conformal mapping

translation; $z' = z + a$

dilation; $z' = bz$

inversion; $z' = \frac{1}{z}$

The general form is:

$$z' = \frac{az + b}{cz + d}$$

which is conformal if

$$cz + d \neq 0 \neq \frac{dz'}{dz},$$

$$|||$$

$$ad - bc \neq 0.$$

These are called <u>homographic</u> transformations.

$$\underbrace{x'^2 + \frac{2a_r x'}{r^2-|a|^2} + \frac{a_r^2}{(r^2-|a|^2)^2}} + \underbrace{y'^2 - \frac{2a_i y'}{r^2-|a|^2} + \frac{a_i^2}{(r^2-|a|^2)^2}} - \frac{1}{r^2-|a|^2} = 0$$

$$\left(x' + \frac{a_r}{(r^2-|a|^2)}\right)^2 + \left(y' - \frac{a_i}{(r^2-|a|^2)}\right)^2 \quad \vdots$$

$$- \frac{(a_r^2+a_i^2)}{(r^2-|a|^2)^2} - \frac{1}{r^2-|a|^2} = 0$$

Now, define, $a_r' := \frac{-a_r}{(r^2-|a|^2)}$ & $a_i' := \frac{a_i}{(r^2-|a|^2)}$

$$\left(x' + a_r'\right)^2 + \left(y' - a_i'\right) \qquad \& \quad r' := \frac{r}{|r^2-|a|^2|}$$

$$= \frac{a_r^2+a_i^2+r^2-|a|^2}{(r^2-|a|^2)^2}$$

$(r^2-|a|^2)(r^2-|a|^2)^*$

$\cancel{(r^2-a_ra_i)(r^2-a_ra_i) =}$ $= (r^2-(a_r^2+a_i^2))(r^2-(a_r^2+a_i^2))$

$r^4 - 2r^2(a_r^2+a_i^2) +$

$$a_r'^2 + a_i'^2 + \frac{r^2-|a|^2}{r^2-a_r^2-a_i^2(r^2-|a|^2)}$$

$$\frac{r^2-|a|^2+a_r^2+a_i^2}{|r^2-|a|^2|}$$

$$(x' + a_r')^2 + (y' + a_i') = r'^2$$

9.3.2: A circle, whose radius r is at a in the z-plane is described by $|z-a| = r$

when transformed to the ~~w~~ z'-plane under inversion, this equation becomes $\left|\frac{1}{z'} - a\right| = r$, or $|1 - az'| = r|z'|$

... Squaring & simplifying yields

$$\cancel{(r^2 - |a|^2)(x'^2} \quad (r^2 - |a|^2)|z'|^2 + 2\,\mathrm{Re}\{az'\} - 1 = 0$$

$$\cancel{\left|\frac{1}{z'} - a\right| = \sqrt{\tfrac{1}{}}} \qquad |1-az'| = \sqrt{(1-az')(1-a^*z'^*)}$$

$$1 + |a|^2|z'|^2 - az' - a^*z'^*$$

$$= 1 + |a|^2|z'|^2 - 2\,\mathrm{Re}\{az'\}$$

In terms of cartesian (x, y),

$$-$$

$$(r^2 - |a|^2)(x'^2 + y'^2) + 2(a_r x' + a_i y') - 1 = 0$$

$$(a := a_r + i a_i)$$

For $r \neq |a|$; Divide by $r^2 - |a|^2$, & complete the squares

$$\rightarrow (x'^2 + y'^2) + \frac{2(a_r x' - a_i y')}{r^2 - |a|^2} - \frac{1}{r^2 - |a|^2} = 0$$

$$x'^2 + \frac{2a_r x'}{r^2 - |a|^2} + y'^2 - \frac{2a_i y'}{r^2 - |a|^2} - \frac{1}{r^2 - |a|^2} = 0 \qquad \left(\frac{b}{2a}\right)^2$$

$$+ \left(\frac{2a_r x'}{2x'(r^2 - |a|^2)}\right)^2 - \ldots + \left(\frac{2a_i y'}{2y'(r^2 - |a|^2)}\right)^2 - \ldots$$

$$= \frac{\partial^2 u}{\partial x}\Delta x_1 \Delta x_2 + \frac{\partial^2 u}{\partial y^2}\Delta y_1 \Delta y_2 + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\left(\Delta x_1 \Delta y_2 + \Delta y_1 \Delta x_2\right)$$

$$+ \frac{\partial^2 v}{\partial x^2}\Delta x_1 \Delta x_2 + \frac{\partial^2 v}{\partial y^2}\Delta y_1 \Delta y_2 + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\left(\Delta x_1 \Delta y_2 + \Delta y_1 \Delta x_2\right)$$

by C-R

$$= \cancel{\frac{\partial^2 y}{\partial x^2}\Delta x_1 \Delta x_2}$$

$$= \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\Delta x_1 \Delta x_2 + \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)\Delta y_1 \Delta y_2$$

$$+ \left(\frac{\partial u^2}{\partial x \partial y} + \frac{\partial v^2}{\partial x \partial y}\right)\left(\Delta x_1 \Delta y_2 + \Delta y_1 \Delta x_2\right)$$

by harmonic f conditions (assuming f to be harmonic)

$$\ldots = \frac{\partial^2 u}{\partial x \partial y}\Delta x_1 \Delta y_2 + \frac{\partial^2 v}{\partial x \partial y}\Delta y_1 \Delta x_2$$