QM, Griffiths:
Chapter 3, Section 1: Hilbert Space

Two constructs of quantum theory:

- wave functions
- operators

Wavefunctions represent the state of a system
Operators represent observables

In math terms, the wave functions are vectors, while the operators are linear transformations.
The vectors (functions) encountered in QM are generally functions, which live in ∞-dimensional space.

The collection of all functions of x is a vector space, but it is too big.
We know, to represent physically possible states, the wave function has to be normalized,

$$\int |\Psi|^2 dx = 1$$

And so the smaller vector space we like is the set of all square-integrable functions on a specific interval (usually $(-\infty, \infty)$)

$\hookrightarrow$ $f(x)$ such that $\int_a^b |f(x)|^2 dx < \infty$

Physicists call this 'Hilbert space' (it is actually a Hilbert space)

Math people call it $L_2(a,b)$

The inner product of two functions f & g

$$\langle f|g\rangle = \int_a^b f(x)^* g(x)\, dx$$ (just like the discrete version!)

If f & g are both square integrable, (both in Hilbert Space) the inner product $\langle f|g\rangle$ or $\langle g|f\rangle$ is guaranteed to exist.

This (is apparent / follows from) the ~~Schwartz~~ Schwarz Inequality

$$\left|\int_a^b f(x)^* g(x)\,dx\right| \le \sqrt{\int_a^b |f(x)|^2 dx \int_a^b |g(x)|^2 dx}$$

Note $\langle g|f\rangle = \langle f|g\rangle^*$

$$\langle f|f\rangle = \int_a^b |f(x)|^2 dx$$

Normalized : A function is normalized when $\langle f|f\rangle = 1$

Orthogonal : ~~A function is 0.~~
Two functions are orthogonal when $\langle f|g\rangle = 0$

Orthonormal : A set of functions $\{f_n\}$ is orthonormal if they are normalized and mutually orthogonal:

$$\langle f_m|f_n\rangle = \delta_{mn}$$

Complete : A set of functions is complete if any other function in Hilbert Space can be expressed as a linear combination of them.

i.e, ~~eg.~~ $$f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$$

if $\{f_n(x)\}$ is orthonormal $(\langle f_m|f_n\rangle = \delta_{mn})$

then the coefficients may be calculated using Fourier's Trick:

$$c_n = \langle f_n|f\rangle = \int_a^b f_n^*(x) f(x)\,dx$$

3.1 Show that the set of all square integrable functions is a vector space

Strategy: Show that the sum of two square integrable functions is itself square-integrable.

Obviously the scalar multiplication requirement will have no effect on a function's square-integrability...

So, we need to tackle the vector addition problem.

if $\int_a^b |f(x)|^2 dx \not\to \infty$, then $f(x)$ is square integrable.

if $\int_a^b |g(x)|^2 dx \not\to \infty$, then $g(x)$ is square integrable.

does $\int_a^b |f(x)+g(x)|^2 dx \longrightarrow \infty$?

$$= \int_a^b \left(f(x)+g(x)\right)^* \left(f(x)+g(x)\right) dx$$

$$= \int_a^b \left(f^*f + g^*f + f^*g + g^*g\right) dx$$

$$= \int_a^b |f|^2 dx + \int_a^b |g|^2 dx + \int_a^b g^* f\, dx + \int_a^b f^* g\, dx$$

$$= \int_a^b |f|^2 dx + \int_a^b |g|^2 dx + \langle g|f\rangle + \langle f|g\rangle$$

By the Schwarz inequality,

$$\langle g|f\rangle \;\&\; \langle f|g\rangle \leq \sqrt{\int_a^b |f|^2 dx \int_a^b |g|^2 dx}$$

~~for~~ from which we know $\langle g|f\rangle$ & $\langle f|g\rangle$ must be finite, since f & g are both square-integrable on their own.

(3.2) For what range of ν is $f(x) = x^{\nu}$ in Hilbert space on the interval $(0,1)$?

(assume ν, real)

To qualify for Hilbert space,

$$\int_a^b |x^{\nu}|^2 dx \not\to \infty$$

$$\int_a^b x^{\nu *} x^{\nu} dx = \int_a^b x^{2\nu} dx, \quad \text{since } \nu \text{ is real.}$$

The interval here is $(0,1)$, so,

$$\int_0^1 x^{2\nu} dx = \frac{1}{2\nu+1}\left[x^{2\nu+1}\right]_0^1 = \langle f|f\rangle$$

$$\cancel{= \frac{x^{2\nu+1}}{2\nu+1}}$$

we need to include $0^{2\nu+1}$!

$$\langle f|f\rangle\Big|_{x\in(0,1)} = \int_0^1 x^{2\nu} dx \cancel{= [}$$

$$= \frac{1}{2\nu+1}\left[x^{2\nu+1} - 0^{2\nu+1}\right]$$

Thus $\langle f|f\rangle \not\to \pm\infty$ as long as $2\nu+1 > 0$.

(For $2\nu+1 < 0$, the integral blows up) $\forall\, \nu \overset{-}{>} \frac{1}{2}$

@ $\nu \overset{-}{=} \frac{1}{2}$?

$$\langle f|f\rangle\Big|_{\nu=1/2} = \int_0^1 x^{-1} dx = \ln x\Big|_0^1 \longrightarrow 0 + \infty$$

$\therefore$ $\boxed{f \text{ is in Hilbert space only for } \nu > -\frac{1}{2}}$

3.1 (b) Are normalized functions a vector space?

Lets disprove:

~~if $\langle f|f\rangle = 1$,~~ if $\langle f|f\rangle = 1$, ~~it is certainly~~,

then $c\langle f|f\rangle$ must also remain normalized:

$$c\langle f|f\rangle = c|f|^2$$

Most simply, there is no normalized $f(x)=0$. not possible!

Even if an exception is made however, scalar multiplication breaks down.

$$\text{if } \langle f|f\rangle = 1, \quad \langle 2f|2f\rangle = 4$$

, which is not normalized, (is outside the space, thus the space is not closed under scalar multiplication.)

(3.2) Show that ~~$\langle f|g\rangle$~~

$\int_a^b f(x)^* g(x)\,dx$ satisfies the conditions for an inner product $\langle f|g\rangle$

① $\langle f|g\rangle^* = \langle g|f\rangle$?

$$\langle f|g\rangle^* = \left[\int_a^b f^* g\,dx\right]^* = \int_a^b g^* f\,dx = \langle g|f\rangle \quad \checkmark$$

② $\langle f|f\rangle \geq 0$?

$$\langle f|f\rangle = \int_a^b f^* f\,dx = \int_a^b |f|^2\,dx \geq 0 \quad \text{(since this is the modulus)}$$

③ ~~Ah~~ $\langle h|(a|f\rangle + b|g\rangle) = a\langle h|f\rangle + b\langle h|g\rangle$?

$$= \int h^*(af + bg)\,dx = \int h^* a f\,dx + \int h^* b g\,dx$$

$$= a\int h^* f\,dx + b\int h^* g\,dx = a\langle h|f\rangle + b\langle h|g\rangle$$

So, with ~~x~~ $f = x^{\nu}$,

for what values of ν does $xf(x)$ live in Hilbert space?

$$\langle xf | xf \rangle = \int_0^1 x^2 x^{2\nu} dx = \int_0^1 x^{2\nu+2} dx$$

$$= \frac{1}{2\nu+3}\left[x^{2\nu+3}\right]_0^1 = \frac{1}{2\nu+3}\left[\cancel{x}^{2\nu+3} - 0^{2\nu+3}\right]$$

ν must be $> \frac{3}{2}$.

but what about $\nu = -\frac{3}{2}$?

$$\langle xf | xf \rangle \Big|_{\nu=\frac{3}{2}} = \int_0^1 x^6 dx.$$

$= 1$ yup.

$$\boxed{xf(x) \text{ is in Hspace for } \nu \geq -\frac{3}{2}}$$

$\left\langle \frac{df}{dx} \middle| \frac{df}{dx} \right\rangle$? ~~$= \frac{1}{2\nu-1}\int_0^1 x^{2\nu-1} d$~~

$$\frac{df}{dx} = \nu - 1\, x^{\nu-1}$$

~~$= 2\nu - 1 \int_0^1 x^{2\nu-1} dx = 2\nu - 1$~~

$$\left|\frac{df}{dx}\right|^2 = (\nu-1)^2 x^{2\nu-2} \longrightarrow (\nu-1)^2\int_0^1 x^{2\nu-2} dx$$

$$= \frac{(\nu-1)^2}{2\nu \cancel{+} 1}\left[x^{2\nu-1}\right]_0^1$$

Simple operations can carry a function <u>OUT</u> of Hilbert space!!

$$\boxed{\text{WAVE FUNCTIONS LIVE IN HILBERT SPACE!}}$$

3.2: Observables:

Expectation value of an observable in inner-product notation:

$$\langle Q \rangle = \int \Psi^* \hat{Q} \Psi \, dx = \langle \Psi | \hat{Q} \Psi \rangle$$

Remember; $\hat{Q}\Psi$ isn't actually an operator operating on a vector... this is just a convenient notation that gives a label to the new vector

$$\hat{Q} | \Psi \rangle$$

Again, $\hat{Q}$ is constructed from $Q(x,p)$ by replacing

$$\cancel{p \rightarrow \left(\frac{\hbar}{i}\right) \frac{d}{dx}}.$$

$$p \rightarrow \hat{p} =: \frac{\hbar}{i} \frac{d}{dx}$$

Since the outcome of a measurement is real, so too must the average of many measurements:

Thus $\langle Q \rangle = \langle Q \rangle^*$

However, the complex conjugate of an inner product reverses its order:

So, for $\cancel{\langle Q \rangle^* =}$ $\langle Q \rangle = \langle Q \rangle^*$,

this suggests that $\langle \hat{Q}\Psi | \Psi \rangle = \langle \Psi | \hat{Q}\Psi \rangle$

Which must be true for any wavefunction Ψ

Thus, operators representing observables have a special property:

$$\langle f | \hat{Q} f \rangle = \langle \hat{Q} f | f \rangle \quad \text{for all } f(x).$$

→ We call these **Hermitian Operators**

OBSERVABLES ARE REPRESENTED BY HERMITIAN OPERATORS!

A Check:

$$\langle f|\hat{p}g\rangle = \int_{-\infty}^{\infty} f^*\left(\frac{\hbar}{i}\frac{d}{dx}\right)g\,dx = \frac{\hbar}{i}\int_{-\infty}^{\infty} f^*\frac{dg}{dx}\,dx$$

$$\cancel{= \frac{\hbar}{i}\left[f^*g\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} g\right.}$$

$$= \frac{\hbar}{i}f^*g\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} g\left(\frac{\hbar}{i}\frac{df}{dx}\right)^* dx$$

$$= \frac{\hbar}{i}f^*g\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} g\left(\frac{\hbar}{i}\right)\left(\frac{df}{dx}\right)^* dx = \langle\hat{p}f|g\rangle$$

0

(3.3) Show that if $\langle h|\hat{Q}h\rangle = \langle\hat{Q}h|h\rangle$ for all functions h in Hilbert space, then $\langle f|\hat{Q}g\rangle = \langle\hat{Q}f|g\rangle$ for all f & g

$$\langle h|\hat{Q}h\rangle = \cancel{\int_a^b h^*\hat{Q}h\,dx}$$

let, $h = f+g$

~~so $h^* = f$~~

$$\cancel{= \int (f+g)^*\hat{Q}(f+g)\,dx}$$

$$\langle f+g|\hat{Q}(f+g)\rangle = \cancel{\langle f+g|(|\hat{Q}f\rangle + |}$$

$$= \langle (f+g)|(|\hat{Q}f\rangle + |\hat{Q}g\rangle)\rangle$$

$$= \langle f|\hat{Q}f\rangle + \langle f|\hat{Q}g\rangle + \langle g|\hat{Q}f\rangle + \langle g|\hat{Q}g\rangle$$

$$\langle\hat{Q}h|h\rangle = \langle\hat{Q}(f+g)|(f+g)\rangle = \langle (f+g)((\ $$

$$= \langle\hat{Q}f|f\rangle + \langle\hat{Q}f|g\rangle + \langle\hat{Q}g|f\rangle + \langle\hat{Q}g|g\rangle$$

We want to prove that a hermitian operator is so if

$$\langle f|\hat{Q}f\rangle = \langle \hat{Q}f|f\rangle ;$$

which is an equivalent definition/requirement to $\langle f|\hat{Q}g\rangle = \langle \hat{Q}f|g\rangle$

~~Let~~ $\{$ let $h = f+g$, or $h = f+ig$. So, $h = f+cg$

$$\langle h|\hat{Q}h\rangle = \langle (f+cg)|\hat{Q}(f+cg)\rangle$$
$$= \langle f|\hat{Q}f\rangle + c^*\langle g|\hat{Q}f\rangle + c\langle f|\hat{Q}g\rangle + |c|^2\langle g|\hat{Q}g\rangle$$

$$\langle \hat{Q}h|h\rangle = \langle \hat{Q}(f+cg)|(f+cg)\rangle$$
$$= \langle \hat{Q}f|f\rangle + c\langle \hat{Q}f|g\rangle + c^*\langle \hat{Q}g|f + |c|^2\langle \hat{Q}g|g\rangle$$

Since g & f are subject to a hermitian operator,

$$\langle f|\hat{Q}f\rangle = \langle \hat{Q}f|f\rangle \quad \text{and} \quad \langle g|\hat{Q}g\rangle = \langle \hat{Q}g|g\rangle$$

if $\langle h|\hat{Q}h\rangle = \langle \hat{Q}h|h\rangle$

then $c\langle f|\hat{Q}g\rangle + c^*\langle g|\hat{Q}f\rangle = c\langle \hat{Q}f|g\rangle + c^*\langle \hat{Q}g|f\rangle$

if $c=1$, $\langle f|\hat{Q}g\rangle + \langle g|\hat{Q}f\rangle = \langle \hat{Q}f|g\rangle + \langle \hat{Q}g|f\rangle$

and if $c=i$, $\langle f|\hat{Q}g\rangle - \langle g|\hat{Q}f\rangle = \langle \hat{Q}f|g\rangle - \langle \hat{Q}g|f\rangle$

$+$

$$\langle f|\hat{Q}g\rangle = \langle \hat{Q}f|g\rangle \quad \text{if}$$

$$\langle h|\hat{Q}h\rangle = \langle \hat{Q}h|h\rangle$$

3.4 Show that the sum of two hermitian operators is hermitian:

(a) $\cancel{\langle f|(\hat{H}+\hat{G})g\rangle}$

$$\langle f|(\hat{Q}+\hat{K})g\rangle = \langle f|\hat{Q}g\rangle + \langle f|\hat{K}g\rangle$$

if $\hat{Q}$ & $\hat{K}$ are hermitian, then,

$$= \langle \hat{Q}f|g\rangle + \langle \hat{K}f|g\rangle$$

$$= \langle (\hat{Q}+\hat{K})f|g\rangle$$

(b) Suppose $\hat{Q}$ is hermitian and α is a complex #. Under what conditions on α is $\alpha\hat{Q}$ hermitian?

$\langle f|\alpha\hat{Q}g\rangle$ must $= \langle \alpha\hat{Q}f|g\rangle$.

$$= \alpha\langle f|\hat{Q}g\rangle$$

and, $\langle \alpha\hat{Q}f|g\rangle = \alpha^*\langle \hat{Q}f|g\rangle$. So,

$\alpha\hat{Q}$ is hermitian, if α is real.

(c) When is the product of two Hermitian operators hermitian?

If hermitian product,

$\langle f|\hat{K}\hat{Q}g\rangle$ would $= \langle \hat{K}f|\hat{Q}g\rangle$, which would

$$= \langle \hat{Q}\hat{K}f|g\rangle$$

Thus to remain Hermitian, $\hat{K}\hat{Q}$ would have to $= \hat{Q}\hat{K}$

or, the product is Hermitian if $[\hat{K},\hat{Q}]=0$

(d) Show that the position operator $\hat{x} = x$ and the Hamiltonian operator $\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$ are hermitian:

For $$\langle f|\hat{x}g\rangle = \int f^* x g\,dx = \int (xf)^* g\,dx = \langle \hat{x} f|g\rangle$$

(This is so, since x is real in all cases)

$$\langle f|\hat{H}g\rangle = \int f^*\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V\right) g\,dx$$

$$= \underbrace{\frac{-\hbar^2}{2m}\int f^*\frac{d^2g}{dx^2}\,dx}_{\text{IBP to solve:}} - \frac{\hbar^2}{2m}\int f^* V g\,dx \quad \left(\begin{array}{c}\text{here we assume} \\ V \text{ is real}\end{array}\right)$$

$$\left[f^*\frac{dg}{dx}\Big|_{-\infty}^{\infty} - \int \frac{df^*}{dx}\frac{dg}{dx}\,dx = \underbrace{f^*\frac{dg}{dx}\Big|_{-\infty}^{\infty}}_{0} - \underbrace{\frac{df^*}{dx}g\Big|_{-\infty}^{\infty}}_{0} + \int \frac{d^2f^*}{dx^2}g\,dx \right]$$

in Hilbert space.

$$\cancel{= \int\left(f\left(\frac{-\hbar^2}{2m}\,d\right.\right.} = \int\left[\left(\frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + V\right)f\right]^* g\,dx = \langle \hat{H} f|g\rangle$$

(3.5) The Hermitian conjugate of an operator $\hat{Q}$ is the operator $\hat{Q}^\dagger$ such that

$$\cancel{\langle \quad = \langle f|\hat{Q}^\dagger g\rangle} \qquad \langle f|\hat{Q}g\rangle = \langle \hat{Q}^\dagger f|g\rangle$$

for all f & g.

ie $\hat{Q} = \hat{Q}^\dagger$.

(a) ~~find~~ $\cancel{\hat{x}^\dagger, \hat{i}^\dagger, \ldots}$

$$\langle f|\hat{x}g\rangle = \int f^*(xg)\,dx = \int (xf)^* g\,dx, \text{ so, } \cancel{\hat{x}^\dagger = x}$$

$$x^\dagger = x$$

$$\langle f|ig\rangle = \int f^*(ig)\,dx = \int (-if)^* g\,dx$$

$$\therefore \quad i^\dagger = -i$$

$$\langle f|\tfrac{d}{dx}g\rangle = \int f^*\left(\frac{dg}{dx}\right)dx = f^*g\Big|_{-\infty}^{\infty} - \int \frac{df^*}{dx}g\,dx$$

$$\therefore \quad \frac{d}{dx}^\dagger = -\frac{d}{dx}$$

(b) Construct the hermitian conjugate of the harmonic oscillator raising operator: a_+

$$a_+ = \frac{1}{\sqrt{2\hbar m\omega}}(-ip + m\omega x)$$

$$\langle f|a_+ g\rangle = \int f^*\left(\frac{1}{\sqrt{2\hbar m\omega}}(ip + m\omega x)\right)g\,dx$$

$$= \int f^*\left(\frac{1}{\sqrt{2\hbar m\omega}}(ip + m\omega x)\right)^* g\,dx$$

$$a_+^\dagger = \frac{1}{\sqrt{2m\omega\hbar}}(ip + m\omega x) = a_-$$

(c) Show that $(\hat{Q}\hat{R})^\dagger = \hat{R}^\dagger\hat{Q}^\dagger$

$$\langle f|\hat{Q}\hat{R}g\rangle = \langle \hat{Q}^\dagger f|\hat{R}g\rangle = \cancel{\langle \hat{Q}^\dagger \hat{R}} \langle \hat{R}^\dagger\hat{Q}^\dagger f|g\rangle = \langle (\hat{Q}\hat{R})^\dagger f|g\rangle$$ ✓

3.2.2: Determinate States

Is it possible to prepare a state such that every measurement of Q is certain to return the same value? (call this value q).

i.e. For an observable Q, is there a determinate state.

eg. Stationary states are the determinate states of the Hamiltonian.

(A measure of the total energy of a particle in stationary state Ψ_n must yield the corresponding ~~one~~ allowed energy E_n

In such a determinate state, the SD must $= 0$.

$$\cancel{\sigma^2 = \langle (\hat{Q})^2 - \langle Q}$$

~~since $\sigma^2 = 0$, $\sigma^2 = \langle (\hat{Q} - \langle Q \rangle)^2 \rangle$~~

$$\sigma^2 = \langle (\Delta Q)^2 \rangle = \langle (\hat{Q} - \langle Q \rangle)^2 \rangle$$

$$= \langle (\hat{Q} - q)^2 \rangle$$

$$= \int \Psi^* (\hat{Q} - q)^2 \Psi \, dx$$

$$= \langle \Psi | (\hat{Q} - q)^2 \Psi \rangle$$

$$= \langle (\hat{Q} - q)\Psi | (\hat{Q} - q)\Psi \rangle$$

remember, we can only do this cause ~~$q - \theta$~~ $\hat{Q} - q$ is a hermitian operator.

eventually, we get the eigenvalue equation:

$$\boxed{\hat{Q}\Psi = q\Psi}$$

$\boxed{\text{Determinate States are eigenfunctions of } \hat{Q}\,!}$

Thus, a measurement Q on such a state will necessarily yield the/an eigenvalue q.

Remember, zero can be an eigenvalue ... just not an eigenvector or an eigenfunction.

Now, as an example:

determinate states of the total energy are eigenfunctions of the Hamiltonian:

$$\hat{H}\psi = E\psi ,$$

The different allowed values of E would be called the spectrum of $\hat{H}$.

eg. 3.1:

Consider $\hat{Q} = i \frac{d}{d\phi}$

is $\hat{Q}$ hermitian?

The interval in this case is finite: $\phi \in [0, 2\pi]$

Thus $f(\phi) = f(\phi + 2\pi)$

Testing if hermitian:

$$\langle f|\hat{Q}g\rangle = \int_0^{2\pi} f^* i \frac{dg}{d\phi} d\phi$$

$$= i f^* g \Big|_0^{2\pi} - \int_0^{2\pi} \left(-i \frac{d}{d\phi} f\right)^* g \, d\phi$$

$$= 0 + \int_0^{2\pi} \left(i \frac{df}{d\phi}\right)^* g \, d\phi = \langle \hat{Q} f | g \rangle$$

Yup, $\hat{Q}$ sure is hermitian

Solving for eigen values, $i \frac{d}{d\phi} f = q f$

$$\frac{df}{d\phi} = -iqf$$

solving this ODE,

$$f(\phi) = Ae^{-iq\phi}$$

But the only way for $f(\phi) = f(\phi + 2\pi)$ consistently is if $e^{-iq\phi} = 1$

this occurs for all $\phi = 2\pi$, so, q has to be $= 0, \pm 1, \pm 2 \ldots \pm n$.

So the spectrum of this operator is the set of all integers.

And where $f = Ae^{-iq\phi}$ is the eigenfunction for $\hat{Q}$, f is unique for every q, thus the ~~operator~~ spectrum is non-degenerate

3.6 Consider, in polar coordinates again, the operator $\hat{Q} = \frac{d^2}{d\phi^2}$

Is $\hat{Q}$ hermitian?

~~No, there is no factor of i in the operator;~~

$$\langle f|\hat{Q}g\rangle = \int_0^{2\pi} f^* \frac{d^2g}{d\phi^2}\,d\phi = f^*\frac{dg}{d\phi}\Big|_0^{2\pi} - \int_0^{2\pi}\frac{df^*}{d\phi}\frac{dg}{d\phi}\,d\phi$$

$$= \left(\frac{df}{d\phi}\right)^*\frac{dg}{d\phi}\Big|_0^{2\pi} + \int_0^{2\pi}\left(\frac{d^2}{d\phi^2}f\right)^* g\,d\phi$$

$$= \langle \hat{Q}f|g\rangle$$

Yes, $\hat{Q}$ is hermitian

So, $\frac{d^2f}{d\phi^2} = q f$ ~~$f = Ae^{ikx} + Be^{-ikx}$~~

$$f_\pm(\phi) = Ae^{\pm\sqrt{q}\,\phi}$$

To satisfy $f(\phi) = f(\phi+2\pi)$, $\sqrt{q}(2\pi)$ must $= 2n\pi i$

So, $\sqrt{q} = ni$

and ~~$|q| = -n^2$~~ $q = -n^2$, $n = 0, 1, 2, \ldots$

The spectrum is $q = -n^2$ for all integers.

This spectrum is degenerate in two ways; First, $|n| = \pm n$, ~~and s~~ ie there are two functions for each value of n. Second, there is the $\pm$ in the original function. 'Double-degenerate'.

However, the spectrum is <u>not</u> degenerate for $n=0$.

3.3: Eigenfunctions of a Hermitian Operator

So we now focus on the eigenfunctions of hermitian operators that represent physically determinate states of observables.

These come in two categories:

Discrete spectrums have eigenfunctions that live in Hilbert space, and are normalizable — physically realizable.

Continuous ~~spec~~ spectrums have eigenfunctions laying outside Hilbert space — These eigenfunctions are not normalizable, and they **DO NOT** represent possible wave functions.

— However, in some case, linear combinations of these eigenfunctions (w/ continuous spectra) may be normalizable

└ (Under the condition though that the resulting functions represent a ~~f~~ spread of eigenvalues)

(1) Discrete Spectra:

Normalizable eigenfunctions of a hermitian operator have two important properties:

1) Their eigenvalues are Real:

— Say $\hat{Q}f = qf$ (f is eigenfunction of $\hat{Q}$ with eigenvalue q)

~~and $\langle f|\hat{Q}f\rangle = \langle f$~~

and $\langle f|\hat{Q}f\rangle = \langle \hat{Q}f|f\rangle$ ($\hat{Q}$ is hermitian)

$$\downarrow$$

— Then, $\langle f|qf\rangle = \langle qf|f\rangle$

$$\downarrow$$

$$q\langle f|f\rangle = q^*\langle f|f\rangle$$

And since $f \neq 0$ by assumption, $q = q^*$,

thus q must be real.

2) Eigenfunctions belonging to distinct eigenvalues are orthogonal.

— Suppose: $\hat{Q}f = qf$ and $\hat{Q}g = sg$

~~then~~ and $\hat{Q} = \hat{Q}^{\dagger}$, ... $\langle f|\hat{Q}g\rangle = \langle \hat{Q}f|g\rangle$

then $\langle f|\hat{Q}g\rangle = \langle f|sg\rangle = s\langle f|g\rangle$

and $\langle \hat{Q}f|g\rangle = \langle qf|g\rangle = q^{*}\langle f|g\rangle$

$\therefore\ s\langle f|g\rangle = q^{*}\langle f|g\rangle$

but $q^{*} = q$,

so $s\langle f|g\rangle = q\langle f|g\rangle$

s & q are distinct, $\therefore$ $s \neq q$,

~~also~~ so, either, $s = q$ – which they cannot, or

$$\boxed{\langle f|g\rangle = 0}$$

3)

The 'Axiom' (the restriction on Hermitian operators that can represent observables)

The eigenfunctions of an observable operator are complete: i.e. Any function in Hilbert space can be represented as a linear combination of the eigenfunctions.

$\boxed{\text{This effectively a } \underline{\text{third}} \text{ property of eigenfunctions with discrete spectra}}$

(3.7) Suppose $f(x)$ & $g(x)$ are two eigenfunctions of an operator $\hat{Q}$ with the same eigen value q.

(a) Show that any linear combination of f & g is itself an eigenfunction of $\hat{Q}$ w/ eigen value q.

Form a linear combination, $af + bg = h$

$$\hat{Q}(af + bg) = a\hat{Q}f + b\hat{Q}g = aqf + bqg$$
$$\hat{Q}h = \quad = q(af + bg)$$
$$= qh$$

(b) Check that ~~$f(x) = \exp$~~ $f(x) = e^x$ and $g(x) = e^{-x}$

are eigenfunctions of the same operator $\frac{d^2}{dx^2}$

$$\frac{d^2}{dx^2} e^x = e^x, \quad \text{where } q = 1$$

$$\frac{d^2}{dx} e^{-x} = -\frac{d}{dx} e^{-x} = e^{-x}, \quad q = 1.$$

An orthogonal linear combination is $ae^x + be^{-x}$, letting $a = b = \frac{1}{2}$,

$$\frac{1}{2}\left[e^x + e^{-x}\right] = \cosh(x)$$

or choose $a = \frac{1}{2}$ & $b = -\frac{1}{2}$,

so $ae^x + be^{-x} = \frac{1}{2}(e^x - e^{-x}) = \sinh(x)$

cosh is even, and sinh is odd, so these two combinations are (must be) orthogonal.

3.8 Check that the eigenvalues of the hermitian operator $\hat{Q} = i\frac{d}{d\phi}$ are real.

$$\hat{Q}f = qf \rightarrow i\frac{df}{d\phi} = qf$$

$$\langle f|\hat{Q}f\rangle = \langle f|qf\rangle = \langle f|i\frac{df}{d\phi}\rangle = i\langle f|\frac{df}{d\phi}\rangle$$

$$= i$$

Wait, we already calculated the eigenvalues here:
→ they are all integers, (definitely real).

Are the corresponding eigenfunctions orthogonal?

To check, we consider any two eigenfunctions: $f = A_q e^{-iq\phi}$
and $g = A_{q'} e^{-iq'\phi}$

$$\langle f|g\rangle = A_q^* A_{q'} \int_0^{2\pi} e^{iq\phi} e^{-iq'\phi} \, d\phi$$

$$= A_q^* A_{q'} \int_0^{2\pi} e^{i(q-q')\phi} \, d\phi = \frac{A_q^* A_{q'}}{i(q-q')} e^{i(q-q')\phi} \Big|_0^{2\pi}$$

$$= \frac{A_q^* A_{q'}}{i(q-q')} \left[e^{i(q-q')2\pi} - 1 \right]$$

But since q & q' are integers, $e^{i(q-q')2\pi} = 1$

$\therefore$ $\langle f|g\rangle = 0$ & we have shown different eigenfunctions are orthogonal.

(b) And how about $\hat{Q} = \frac{d^2}{d\phi^2}$ from 3.6?

↳ here q were integers $\pm n$ again, which are real.
Checking orthogonality...:

From $f(\phi) = A e^{\pm\sqrt{q}\phi}$, $\sqrt{q} = n i$,

Take two eigenfunctions $f = A_q e^{\pm i n \phi}$

and $g = A_{q'} e^{\pm i n' \phi}$;

$$\langle f | g \rangle = A_q^* A_{q'} \int_0^{2\pi} e^{\mp i n \phi} e^{\pm i n' \phi} \, d\phi$$

$$= A_q^* A_{q'} \int_0^{2\pi} e^{\pm i (n'-n)\phi} \, d\phi = \frac{A_q^* A_{q'}}{\pm i (n-n')} \left[e^{\pm i (n'-n)\phi} \right]_0^{2\pi}$$

$$= \cancel{\frac{A_q^* A_{q'}}{\pm i (n-n')}} = \frac{A_q^* A_{q'}}{\pm i (n'-n)} \left[e^{\pm i (n'-n) 2\pi} - 1 \right]$$

And since n & n' are integers & $n \neq n'$, $\langle f | g \rangle = 0$.

However; for each eigenvalue, we note there are in fact two eigenfunctions, ... and so $\langle f | g \rangle = 0$ is refuted as a condition for orthogonality... f & g are <u>not</u> orthogonal

Continuous Spectra:

The difficulties w/ continuous spectra are that because eigenfunctions are not normalizable, their inner products are not guaranteed to exist.

However, in a limited sense, the properties of realness, orthogonality, & completeness still occur...

We will explore this via examples:

e.g. 3.2: Find the eigenvalues & eigenfunctions of the momentum operator.

$\hat{P} = \frac{\hbar}{i}\frac{d}{dx}$, call f_p the momentum eigenfunction, and call p the eigenvalue.

$$\frac{\hbar}{i}\frac{df_p}{dx} = p f_p \xrightarrow{\text{solving}} \underbrace{f_p(x) = Ae^{\frac{ipx}{\hbar}} = Ae^{i\frac{p}{\hbar}x}}_{\text{Not square integrable.}}$$

$$\frac{df_p}{dx} = \frac{ip}{\hbar} f_p$$

But restricting p to real values...

$$\int_{-\infty}^{\infty} f_{p'}^* f_p \, dx = |A|^2 \int_{-\infty}^{\infty} e^{\frac{i(p-p')}{\hbar}x} dx = \cancel{|A|^2 \frac{2\pi}{\hbar}\delta(p-p')} = |A|^2 2\pi\hbar\,\delta(p-p')$$

A convenient choice for A here would naturally be $A = \frac{1}{\sqrt{2\pi\hbar}}$

so that $\langle f_{p'} | f_p \rangle = \delta(p-p')$ ← this is very similar in appearance to the true definition of orthonormality ($\langle f_m | f_n \rangle = \delta_{mn}$)

Call this...

Dirac Orthonormality.

Furthermore, the ~~of~~ original definition of 'completeness' condition

$$f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$$ (form any function w/ a linear combination ....)

Extends in the same way to this continuous spectra case...
... by applying a continuous sum (integral.)

$$\cancel{f(x) = \int_{-\infty}^{\infty} c_n(p)} \qquad f(x) = \int_{-\infty}^{\infty} c(p) f_p(x)\, dp$$

f is square-integrable now...

showing us that any square integrable function may be constructed out of these 'inferior' ersatz eigenfunctions!

And where $f_p = \frac{1}{\sqrt{2\pi\hbar}} e^{i\frac{p}{\hbar}x}$,

$$f(x) = \int_{-\infty}^{\infty} c(p) f_p(x)\, dp = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} c(p) e^{i\frac{p}{\hbar}x}\, dp$$

Now, the expansion coefficient is a function of p, but can still be determined using 'Fourier's Trick'...

$$c(p') \rightarrow \langle f_{p'} | f \rangle = \int_{-\infty}^{\infty} c(p) \langle f_{p'} | f \rangle\, dp = \int_{-\infty}^{\infty} c(p)\, \delta(p-p')\, dp$$

(This is a function of p'!)

$$= c(p)\Big|_{p'} = c(p')$$

Now since $f_p = \frac{1}{\sqrt{2\pi\hbar}} e^{i \frac{p}{\hbar} x}$ is sinusoidal ...
(Euler's formula ...)

ω is apparently $= \frac{p}{\hbar}$

and $\omega = \frac{2\pi}{\lambda}$ $\xrightarrow{\therefore}$ $\lambda = \frac{2\pi\hbar}{p}$

Which is DeBroglie's relationship

And so a deep subtlety emerges ... the DeBroglie formula has some twists, since we just demonstrated it is impossible to have a particle with a determinate momentum!
(i.e. there is no such thing.)

Onward:

eg. 3.3: Find the eigenfunctions & eigenvalues of the position operator.

call g_y the eigenfunction, & y the eigenvalue:

$\hat{X} = x$.

$x g_y = y g_y$

So, what is g_y? — y must be a fixed # for any given eigenfunction.

— The problem is however, that x is a continuous variable...

— What function, when multiplied by x is the same as multiplying by a constant?

g_y must be the delta function.

$g_y(x) = A\,\delta(x-y)$ solves $x g_y = y g_y$

For some reason beyond me, the eigenvalues <u>must</u> be real, for $x g_y = y g_y$, $g_y = A\delta(x-y)$

Checking for Dirac orthonormality —

$$\langle g_{y'} | g_y \rangle = \int_{-\infty}^{\infty} g_{y'}^* g_y \, dx = |A|^2 \int_{-\infty} \delta(x-y')\delta(x-y)\,dx$$

$$= |A|^2 \delta(x-y')\Big|_y = |A|^2 \delta(y-y')$$

Choosing A to $=1$, $\langle g_{y'} | g_y \rangle = \delta(y-y')$

✓ yup, dirac orthonormal!

And completeness,

$$f(x) = \int_{-\infty}^{\infty} c(y) g_y(x)\,dy = \int_{-\infty}^{\infty} c(y)\,\delta(x-y)\,dy$$

For which we find trivially just through integration,

$$\cancel{\langle g_{y'} | g_y \rangle = \int_{-\infty}^{\infty} c(y) \langle g_{y'} | g_y \rangle dy = \int_{-\infty} c(y)}$$

$$\cancel{c(y) = \int_{-\infty}^{\infty} g_y(x)^* f(y)\,dy = \int_{-\infty}^{\infty} \delta(x-y) f(y)\,dy}$$

$$c(y) = \langle g_y | g \rangle$$

Completeness of $\hat{X} = x$:

$$g_y = \delta(x-y)$$

Completeness:

$$f(x) = \int_{-\infty}^{\infty} c(y) g_y(x)\,dy = \int_{-\infty}^{\infty} c(y)\,\delta(x-y)\,dy$$

~~let $-y = u$~~

Evaluating $c(y)$:

$$\cancel{c(y) = \langle f_{y'} | f \rangle} \qquad c(y) = \langle g_{y'} | g \rangle$$

$$\langle g_{y'} | f \rangle = \left\langle \delta(x-y') \middle| \int_{-\infty}^{\infty} c(y)\,\delta(x-y)\,dy \right\rangle$$

$$= \int_{-\infty}^{\infty} \left(\delta(x-y')\right)^{*} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} \delta(x-y')\, f(x)\,dx$$

If a hermitian operator's spectrum is continuous, the

- eigenfunctions are not normalizable.
- they are not in Hilbert space
- they do not represent physically possible states.

However, those eigenvalues (continuous) of real values are at least

- Dirac orthonormal
- complete ... And this is just enough.

(3.9) (a)