Two probability problems

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Suppose the joint density function of random variables (X,Y) is given by

$$f(x,y) = \begin{cases} \frac{2}{\pi}(x^2+y^2) & x^2+y^2 \leq 1 \\ 0 & \text{else} \end{cases}$$

1) Find the Covariance of X and Y
2) Determine whether X and Y are independent
3) Find the density for $Z = X^2 + Y^2$

Solution:

1) To find $Cov(X,Y)$, we simply calculate $E[X], E[Y], E[XY]$

$$E[X] = \iint_{x^2+y^2\leq 1} x \cdot \frac{2}{\pi}(x^2+y^2)\,dx\,dy = 0$$
$$E[Y] = \iint_{x^2+y^2\leq 1} y \cdot \frac{2}{\pi}(x^2+y^2)\,dx\,dy = 0$$
$$E[XY] = \iint_{x^2+y^2\leq 1} xy \cdot \frac{2}{\pi}(x^2+y^2)\,dx\,dy = 0$$

$Cov(X,Y) = E[XY] - E[X]E[Y] = 0$

2)

$$\begin{aligned} f_X(x) &= \int_{=\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{\pi}(x^2+y^2)\mathrm{d}y \\ &= \frac{4}{3\pi}(1+2x^2)\sqrt{1-x^2},\ -1 \leq x \leq 1 \end{aligned}$$

similarly

$$f_Y(y) = \frac{4}{3\pi}(1+2y^2)\sqrt{1-y^2},\ -1 \le y \le 1$$

therefore, $f(x,y) \neq f_X(x) \cdot f_Y(y)$, thus X and Y are not independent. (notice that though $E[XY] = E[X]E[Y]$ is necessary for X,Y's independence, it is not sufficient)

3) The distribution of Z is given by

$$\begin{aligned} F_Z(z) &= P(Z \le z) = P(X^2 + Y^2 \le z) \\ &= \int_0^{2\pi} \int_0^{\sqrt{2}} \frac{2}{\pi} r^2 \cdot r\, dr\, d\theta = z^2,\ 0 \le z \le 1 \end{aligned}$$

Thus $f_Z(z) = \frac{d}{dz} F_Z(z) = 2z$

Suppose X be the random variable with density function

$$f(x) = \frac{e^x}{(1+e^x)^2}, x \in R$$

Let $Y = e^x$.

1) Find the distribution function for X.
2) Find the density function for Y.
3) Determine whether the expectation of Y exist.

Solution:

1) $F_X(x) = \int_{-\infty}^{x} \frac{e^x}{(1+e^x)^2} \mathrm{d}x = \frac{e^x}{1+e^x}$, $x \in R$
2) Since

$$\begin{aligned} F_Y(y) &= P(Y \le y) = P(e^x \le y) = P(x \le \ln y) \\ &= \int_{-\infty}^{\ln y} \frac{e^x}{(1+e^x)^2} \mathrm{d}x = \frac{y}{1+y},\ y > 0 \end{aligned}$$

Thus $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{1}{(1+y)^2}$

3) Since

$$\begin{aligned} E[Y] &= \int_0^\infty y \cdot f_Y(y)\mathrm{d}y \\ &= \int_0^\infty \frac{y}{(1+y)^2}\mathrm{d}y \\ &= \frac{1}{1+y} + \ln(1+y)\Big|_0^\infty \end{aligned}$$

is divergent. Thus E[Y] do not exist