3 Integer Functions

3.1 FLOORS AND CEILINGS

Define: $\lceil x \rceil$ is the least integer greater than or equal to x, and $\lfloor x \rfloor$ is the greatest integer less than or equal to x. Basic rules:

$$\lfloor x \rfloor \leq x; \tag{1}$$
$$\lceil x \rceil \geq x. \tag{2}$$

The two functions are equal precisely at the integer points:

$$\lfloor x \rfloor = x \iff x \text{ is an integer} \iff \lceil x \rceil = x. \tag{3}$$

The two functions are inequal if not at the integer points:

$$\lceil x \rceil - \lfloor x \rfloor = [x \text{ is not an integer}]. \tag{4}$$

The two functinos can be converted:

$$\lceil -x \rceil = -\lfloor x \rfloor; \tag{5}$$
$$\lfloor -x \rfloor = -\lceil x \rceil. \tag{6}$$

Integers can be easily removed or added in the two functions:

$$\lfloor x + n \rfloor = \lfloor x \rfloor + n; \qquad \{\text{integer n}\} \tag{7}$$
$$\lceil x + n \rceil = \lceil x \rceil + n. \qquad \{\text{integer n}\} \tag{8}$$

For important rules:

$$\lfloor x \rfloor = n \iff x - 1 < n \leq x < n + 1; \tag{9}$$
$$\lceil x \rceil = n \iff n - 1 < x \leq n < x + 1. \tag{10}$$

There are many situations in which floor and ceiling brackets are redundant:

$$x < n \iff \lfloor x \rfloor < n; \tag{11}$$
$$n < x \iff n < \lceil x \rceil; \tag{12}$$
$$x \leq n \iff \lceil x \rceil \leq n; \tag{13}$$
$$n \leq x \iff n \leq \lfloor x \rfloor. \tag{14}$$

Define: $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x, then $\lfloor x \rfloor$ is the integer part of x. A simple notation is $x = n + \theta$.

$$\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor + \lfloor \{x\} + \{y\} \rfloor. \tag{15}$$

3.2 FLOOR/CEILING APPLICATIONS

Problem 1: what is the bit number to express n in binary?

$$2^{m-1} \leq x < 2^m \iff \text{the bit number is } m; \tag{16}$$
$$m - 1 \leq \lg x < m \tag{17}$$
$$m = \lfloor \lg x \rfloor + 1. \qquad \{x > 0\} \tag{18}$$

To support $x = 0$, another better solution is $\lceil \lg(x + 1) \rceil$.
Problem 2: what is $m = \lfloor \sqrt{\lfloor x \rfloor} \rfloor$ when $x \geq 0$?

$$m \leq \sqrt{\lfloor x \rfloor} < m + 1; \tag{19}$$
$$m^2 \leq \lfloor x \rfloor < (m + 1)^2; \tag{20}$$
$$m^2 \leq x < (m + 1)^2; \tag{21}$$
$$m \leq \sqrt{x} < m + 1; \tag{22}$$
$$m = \lfloor \sqrt{x} \rfloor. \tag{23}$$

Problem 3: what is $m = \lceil \sqrt{\lceil x \rceil} \rceil$ when $x \geq 0$?

$$m - 1 < \sqrt{\lceil x \rceil} \leq m; \tag{24}$$

$$(m-1)^2 < \lceil x \rceil \leq m^2; \tag{25}$$

$$(m-1)^2 < x \leq m^2; \tag{26}$$

$$m - 1 < \sqrt{x} \leq m; \tag{27}$$

$$m = \lceil \sqrt{x} \rceil. \tag{28}$$

A general theorem: let $f(x)$ be any continuous, monotonically increasing function with the property that

$$f(x) = \text{integer} \implies x = \text{integer}. \tag{29}$$

Then there is:

$$\lfloor f(x) \rfloor = \lfloor f(\lfloor x \rfloor) \rfloor; \tag{30}$$

$$\lceil f(x) \rceil = \lceil f(\lceil x \rceil) \rceil. \tag{31}$$

A special case of the theorem:

$$\left\lfloor \frac{x + m}{n} \right\rfloor = \left\lfloor \frac{\lfloor x \rfloor + m}{n} \right\rfloor; \tag{32}$$

$$\left\lceil \frac{x + m}{n} \right\rceil = \left\lceil \frac{\lceil x \rceil + m}{n} \right\rceil. \tag{33}$$

Problem levels: **level 1** prove a given statement for a number; **level 2** prove a given statement for a set of numbers; **level 3** prove or disprove a given statement for a set of numbers; **level 4** find a necessary and suffcient condition that a statement is true; **level 5** find an interesting property given a set of numbers.
Consider the integer inside a range:

$$\alpha \leq n < \beta \iff \lceil \alpha \rceil \leq n < \lceil \beta \rceil; \tag{34}$$

$$\alpha < n \leq \beta \iff \lfloor \alpha \rfloor < n \leq \lfloor \beta \rfloor. \tag{35}$$

Then

$$[\alpha, \beta) \text{ contains } \lceil \beta \rceil - \lceil \alpha \rceil \text{ elements}; \qquad \{\alpha \leq \beta\} \tag{36}$$

$$(\alpha, \beta] \text{ contains } \lfloor \beta \rfloor - \lfloor \alpha \rfloor \text{ elements}; \qquad \{\alpha \leq \beta\} \tag{37}$$

$$(\alpha, \beta) \text{ contains } \lceil \beta \rceil - \lfloor \alpha \rfloor - 1 \text{ elements}; \qquad \{\alpha < \beta\} \tag{38}$$

$$[\alpha, \beta] \text{ contains } \lfloor \beta \rfloor - \lceil \alpha \rceil + 1 \text{ elements}. \qquad \{\alpha \leq \beta\} \tag{39}$$

Example 1:

$$W = \sum_{1 \leq n \leq 1000} [\lfloor \sqrt[3]{n} \rfloor \setminus n] \tag{40}$$

$$= \sum_{k,n} [k = \lfloor \sqrt[3]{n} \rfloor][1 \leq n \leq 1000][k \setminus n] \tag{41}$$

$$= \sum_{k,n,m} [k^3 \leq n < (k+1)^3][n = km][1 \leq n \leq 1000] \tag{42}$$

$$= 1 + \sum_{k,m} [k^3 \leq km < (k+1)^3][1 \leq k < 10] \tag{43}$$

$$= 1 + \sum_{k,m} [k^2 \leq m < (k+1)^3/k][1 \leq k < 10] \tag{44}$$

$$= 1 + \sum_{1 \leq k < 10} (\lceil (k+1)^3/k \rceil - \lceil k^2 \rceil) \tag{45}$$

$$= 1 + \sum_{1 \leq k < 10} (3k + 4) = 172. \tag{46}$$

General case:

$$W = \sum_{1 \leq n \leq N} [\lfloor \sqrt[3]{n} \rfloor \setminus n] \tag{47}$$

$$= \sum_{k,n} [k = \lfloor \sqrt[3]{n} \rfloor][1 \leq n \leq N][k \setminus n] \tag{48}$$

$$= \sum_{k,n,m} [k^3 \leq n < (k+1)^3][n = km][1 \leq n \leq N] \tag{49}$$

$$= \sum_{k,m} [k^3 \leq km < (k+1)^3][1 \leq k < K] + \sum_{k,m} [K^3 \leq Km \leq N] \tag{50}$$

$$= \sum_{k,m} [k^2 \leq m < (k+1)^3/k][1 \leq k < K] + \sum_{k,m} [K^2 \leq m \leq N/K] \tag{51}$$

$$= \sum_{1 \leq k < K} (3k+4) + \sum_{m} [m \in [K^2, N/K]] \tag{52}$$

$$= (7 + 3K + 1)(K-1)/2 + \lfloor N/K \rfloor - \lceil K^2 \rceil + 1 \tag{53}$$

$$= \frac{1}{2} K^2 + \frac{5}{2} K - 3 + \lfloor N/K \rfloor. \qquad \{K = \lfloor \sqrt[3]{N} \rfloor\} \tag{54}$$

Define $Spec(\alpha) = \{\lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, ...\}$ then $Spec(\sqrt{2})$ and $Spec(\sqrt{2}+2)$ forms a partition of positive integers. Define $N(\alpha, n)$ is the number of elements in $Spec(\alpha)$ that are $\leq n$.

$$N(\alpha, n) = \sum_{k>0} [\lfloor \alpha k \rfloor \leq n] \tag{55}$$

$$= \sum_{k>0} [\lfloor \alpha k \rfloor < n + 1] \tag{56}$$

$$= \sum_{k>0} [\alpha k < n + 1] \tag{57}$$

$$= \sum_{k>0} [0 < k < (n+1)/\alpha] \tag{58}$$

$$= \lceil (n+1)/\alpha \rceil - 1. \tag{59}$$

Then $N(\sqrt{2}, n) + N(\sqrt{2}+2, n) = n$. And it is easy to prove that if $\alpha \neq \beta$ then $Spec(\alpha) \neq Spec(\beta)$.

3.3 FLOOR/CEILING RECURRENCES

Knuth numbers:

$$K_0 = 1; \tag{60}$$
$$K_{n+1} = 1 + min(2K_{\lfloor n/2 \rfloor}, 3K_{\lfloor n/3 \rfloor}). \tag{61}$$

The Josephus problem:

$$J(1) = 1; \tag{62}$$
$$J(n) = 2J(\lfloor n/2 \rfloor) - (-1)^n. \tag{63}$$

Consider the more authentic Josephus problem in which every third person is eliminated:

$$J_3(n) = \left\lceil \frac{3}{2} J_3(\left\lfloor \frac{2}{3} n \right\rfloor) + a_n \right\rceil \mod (n+1). \tag{64}$$

where $a_n = -2, 1, -\frac{1}{2}$ when $n \mod 3 = 0, 1, 2$.
Another method: Whenever a person is passed over, we can assign a new number.

1	2	3	4	5	6	7	8	9	10
11	12		13	14		15	16		17
18			19	20			21		22
			23	24					25
			26						27
			28						
			29						
			30						

Any number $3k+1$ has a next value $10+3k+1-k$, and $3k+2$ has a next value $10+3k+2-k$. More general, there are n people at first and some person has a current number N. For this person his last number should be $3k+1$ or $3k+2$, and this current number is $N = n + 2k + 1$ or $N = n + 2k + 2$. This means

$$k = \left\lfloor \frac{N - n - 1}{2} \right\rfloor. \tag{65}$$

And his last number can be converted into

$$3k + (N - n - 2k) = k + N - n = \left\lfloor \frac{N - n - 1}{2} \right\rfloor + N - n. \tag{66}$$

For the last one to be terminated, his number should be $3n$. Use the method we can always find his last number until the number is smaller than n which is his initial number.

```python
def J3(n):
    N = 3 * n
    while N > n:
        N = int((N - n - 1)/2) + N - n
    return N
```

Listing 1: Method 0

Let $D = 3n + 1 - N$, then $D = 1$ when $N = 3n$ and $D > 2n + 1$ when $N < n$. D can also be updated as N:

$$D = 3n + 1 - N \tag{67}$$

$$= 3n + 1 - \left(\left\lfloor \frac{(3n + 1 - D) - n - 1}{2} \right\rfloor + (3n + 1 - D) - n \right) \tag{68}$$

$$= n + D - \left\lfloor \frac{2n - D}{2} \right\rfloor \tag{69}$$

$$= D - \left\lfloor \frac{-D}{2} \right\rfloor \tag{70}$$

$$= D + \left\lceil \frac{D}{2} \right\rceil \tag{71}$$

$$= \left\lceil \frac{3D}{2} \right\rceil. \tag{72}$$

```python
import math
def J3(n):
    D = 1
    while D <= 2*n:
        D = math.ceil(D*3/2)
    return 3*n + 1 - D
```

Listing 2: Method 1

More general:

```
1  import math
2  def J(n,q):
3      D = 1
4      while D <= (q-1)*n:
5          D = math.ceil(D*q/(q-1))
6      return q*n + 1 - D
```

Listing 3: Method 2

Write it into a recurrence:

$$D_0^{(q)} = 1; \tag{73}$$

$$D_n^{(q)} = \left\lceil \frac{q}{q-1} D_{n-1}^{(q)} \right\rceil. \tag{74}$$

3.4 'MOD': THE BINARY OPERATION

Define operator 'mod':

$$x \bmod y = x - y\lfloor x/y \rfloor. \qquad\qquad \{y \neq 0\} \tag{75}$$

Based on the defination, there are some attributes:

$$0 \leq x \bmod y < y; \qquad\qquad \{y > 0\} \tag{76}$$
$$0 \geq x \bmod y > y. \qquad\qquad \{y < 0\} \tag{77}$$

To complete the defination, we can let $x \bmod y = x$ when $y = 0$.
The 'mod' operator can be used to show the fractional part of a number:

$$x = \lfloor x \rfloor + x \bmod 1. \tag{78}$$

A similar 'mumble' operator can be defined:

$$x \text{ mumble } y = y\lceil x/y \rceil - x. \qquad\qquad \{y \neq 0\} \tag{79}$$

The 'mod' operator follows the distributive law:

$$c(x \bmod y) = (cx) \bmod (cy). \tag{80}$$

Problem: how to partition n things into m groups as equally as possible?
There will be $n \bmod m$ groups contains $\lceil n/m \rceil$ things and the rest contains $\lfloor n/m \rfloor$ things. It also can be converted into:

$$n = \left\lceil \frac{n}{m} \right\rceil + \left\lceil \frac{n-1}{m} \right\rceil + ... + \left\lceil \frac{n-m+1}{m} \right\rceil. \tag{81}$$

and if change n to $km + n \bmod m$, the equation can be converted into:

$$n = \left\lfloor \frac{n}{m} \right\rfloor + \left\lfloor \frac{n+1}{m} \right\rfloor + ... + \left\lfloor \frac{n+m-1}{m} \right\rfloor. \tag{82}$$

If $n = \lfloor mx \rfloor$:

$$\lfloor mx \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{m} \right\rfloor + ... + \left\lfloor x + \frac{m-1}{m} \right\rfloor. \tag{83}$$

3.5 FLOOR/CEILING SUMS

Example 1, let $a = \lfloor \sqrt{n} \rfloor$:

$$\sum_{0 \le k < n} \lfloor \sqrt{k} \rfloor = \sum_{k,m \ge 0} m[m = \lfloor \sqrt{k} \rfloor][k < n] \tag{84}$$

$$= \sum_{k,m \ge 0} m[k < n][m \le \sqrt{k} < m + 1] \tag{85}$$

$$= \sum_{k,m \ge 0} m[k < n][m^2 \le k < (m+1)^2] \tag{86}$$

$$= \sum_{k,m \ge 0} m[m^2 \le k < (m+1)^2 \le n] + \sum_{k,m \ge 0} m[m^2 \le k < n < (m+1)^2] \tag{87}$$

$$= \sum_{m \ge 0} m((m+1)^2 - m^2)[m+1 \le a] + \sum_{m \ge 0} m(a^2 \le k < n) \tag{88}$$

$$= \sum_{m \ge 0} (2m^2 + m)[m+1 \le a] + a(n - a^2) \tag{89}$$

$$= \sum_{m \ge 0} (2m^{\underline{2}} + 3m^{\underline{1}})[m < a] - a^3 + an \tag{90}$$

$$= \sum_0^a (2m^{\underline{2}} + 3m^{\underline{1}})\delta m - a^3 + an \tag{91}$$

$$= \frac{2}{3} m^{\underline{3}} + \frac{3}{2} m^{\underline{2}} - a^3 + an \tag{92}$$

$$= na - \frac{1}{3} a^3 - \frac{1}{2} a^2 - \frac{1}{6} a. \tag{93}$$

Anothe method is le $\lfloor x \rfloor = \sum_j [1 \le j \le x]$:

$$\sum_{0 \le k < n} \lfloor \sqrt{k} \rfloor = \sum_{j,k} [1 \le j \le \sqrt{k}][0 \le k \le a^2] \tag{94}$$

$$= \sum_{1 \le j < a} \sum_k [j^2 \le k < a^2] \tag{95}$$

$$= \sum_{1 \le j < a} (a^2 - j^2) \tag{96}$$

$$= na - \frac{1}{3} a^3 - \frac{1}{2} a^2 - \frac{1}{6} a. \tag{97}$$

Equidistribution theorem:

$$\lim_{x \to \infty} \frac{1}{n} \sum_{0 \le k < n} f(\{k\alpha\}) = \int_0^1 f(x) dx. \tag{98}$$

for all irrational α and all functions f that are continuous almost everywhere.
Example 2, let $d = gcd(m, n)$:

$$\sum_{0 \le k < m} \left\lfloor \frac{nk + x}{m} \right\rfloor = d\left\lfloor \frac{x}{d} \right\rfloor + \frac{m-1}{2} n + \frac{d-m}{2} \tag{99}$$

$$= d\left\lfloor \frac{x}{d} \right\rfloor + \frac{mn}{2} - \frac{n}{2} - \frac{m}{2} + \frac{d}{2} \tag{100}$$

$$= \sum_{0 \le k < n} \left\lfloor \frac{mk + x}{n} \right\rfloor. \tag{101}$$

3.6 Exercises

Warmups 3.1: Let

$$n = 2^m + l. \qquad \{0 \le l < 2^m\} \tag{102}$$

Then:

$$m = \lceil \lg(n+1) \rceil; \tag{103}$$

$$l = n - 2^{\lceil \lg(n+1) \rceil}. \tag{104}$$

Warmups 3.2:

$$round_{down}(x) = \lfloor x + 0.5 \rfloor; \tag{105}$$

$$round_{up}(x) = \lceil x - 0.5 \rceil. \tag{106}$$

Warmups 3.3:

$$\left\lfloor \frac{\lfloor m\alpha \rfloor n}{\alpha} \right\rfloor = \left\lfloor \frac{m\alpha n - \{m\alpha\}n}{\alpha} \right\rfloor = mn - \left\lceil \frac{\{m\alpha\}n}{\alpha} \right\rceil = mn - 1. \tag{107}$$

Warmups 3.4: Pass.

Warmups 3.5:

$$\lfloor n\lfloor x \rfloor + n\{x\} \rfloor = n\lfloor x \rfloor \iff \lfloor n\{x\} \rfloor = 0 \iff 0 \le n\{x\} < 1 \iff \{x\} < \frac{1}{n}. \tag{108}$$

Warmups 3.6:

$$\lfloor f(\lceil x \rceil) \rfloor = A \iff A \le f(\lceil x \rceil) < A+1 \tag{109}$$

$$\iff f^{-1}(A) \ge \lceil x \rceil > f^{-1}(A+1) \tag{110}$$

$$\iff f^{-1}(A) \ge x > f^{-1}(A+1) \tag{111}$$

$$\iff \lfloor f(\lceil x \rceil) \rfloor = \lfloor f(x) \rfloor. \tag{112}$$

Warmups 3.7:

$$X_n = n; \qquad\qquad \{0 \le n < m\} \tag{113}$$

$$X_n = X_{n-m} + 1. \qquad\qquad \{n \ge m\} \tag{114}$$

Solution:

$$X_n = \left\lfloor \frac{n}{m} \right\rfloor + n \bmod m. \tag{115}$$

Warmups 3.8: If m boxes contains $< \lceil n/m \rceil$ elements:

$$n \le m(\lceil n/m \rceil - 1) \iff n/m + 1 \le \lceil n/m \rceil. \tag{116}$$

If m boxes contains $> \lfloor n/m \rfloor$ elements:

$$n \ge m(\lfloor n/m \rfloor + 1) \iff n/m - 1 \ge \lfloor n/m \rfloor. \tag{117}$$

These two statements contradict function 9 and 11.

Warmups 3.9: Because

$$\frac{m}{n} - \frac{1}{q} = \frac{m\lceil \frac{n}{m} \rceil - n}{nq} = \frac{n \text{ mumble } m}{nq}. \tag{118}$$

Then n mumble m is smaller than m and nq is larger than q. This means 1) it is possible to splite a fractional number into a number series; 2) The number series $1/x_1, 1/x_2, ...$ has distinct numbers; 3) This is a finite number series.

Basics 3.10:

$$\left\lceil \frac{2x+1}{2} \right\rceil - \left\lceil \frac{2x+1}{4} \right\rceil + \left\lfloor \frac{2x+1}{4} \right\rfloor = \lceil x + 0.5 \rceil - [x \neq 2k - 0.5]. \qquad \{k \text{ is a integer}\} \qquad (119)$$

This means:

$$\left\lceil \frac{2x+1}{2} \right\rceil - \left\lceil \frac{2x+1}{4} \right\rceil + \left\lfloor \frac{2x+1}{4} \right\rfloor = \lceil x \rceil; \qquad \{x = 2k - 0.5 \text{ or } \{x\} > 0.5\} \qquad (120)$$

$$= \lfloor x \rfloor. \qquad \{\text{else}\} \qquad (121)$$

Basics 3.11:

$$\alpha < n < \beta \iff \lfloor \alpha \rfloor < n < \lceil \beta \rceil. \qquad (122)$$

The number of possible n is

$$(\lceil \beta \rceil - \lfloor \alpha \rfloor - 1)[\lceil \beta \rceil > \lfloor \alpha \rfloor]. \qquad (123)$$

If $\alpha = \beta = $ integer then $\lceil \beta \rceil = \lfloor \alpha \rfloor$, this reults a wrong answer.

Basics 3.12: Prove

$$\left\lceil \frac{n}{m} \right\rceil = \left\lfloor \frac{n+m-1}{m} \right\rfloor \iff \left\lceil \frac{km + n \bmod m}{m} \right\rceil = \left\lfloor \frac{km + n \bmod m + m - 1}{m} \right\rfloor \qquad (124)$$

$$\iff k + \left\lceil \frac{n \bmod m}{m} \right\rceil = k + 1 + \left\lfloor \frac{n \bmod m - 1}{m} \right\rfloor \qquad (125)$$

$$(126)$$

If $n \bmod m = 0$

$$\left\lceil \frac{n}{m} \right\rceil = \left\lfloor \frac{n+m-1}{m} \right\rfloor \iff k = k + 1 - 1. \qquad (127)$$

Else

$$\left\lceil \frac{n}{m} \right\rceil = \left\lfloor \frac{n+m-1}{m} \right\rfloor \iff k + 1 = k + 1 - 0. \qquad (128)$$

Basics 3.13: Because

$$N(\alpha, n) + N(\beta, n) = N(\frac{\beta}{\beta - 1}, n) + N(\beta, n) \qquad (129)$$

$$= n + 1 + \left\lceil \frac{n+1}{\beta} \right\rceil - \left\lfloor \frac{n+1}{\beta} \right\rfloor - 2 \qquad (130)$$

$$= n. \qquad (131)$$

And it is easy to prove that if $\alpha \neq \beta$ then $Spec(\alpha) \neq Spec(\beta)$.

Basics 3.14:

$$(x \bmod ny) \bmod y = x - \left\lfloor \frac{x}{ny} \right\rfloor ny - \left\lfloor \frac{x - \left\lfloor \frac{x}{ny} \right\rfloor ny}{y} \right\rfloor y \qquad (132)$$

$$= x - y(\left\lfloor \frac{x}{ny} \right\rfloor n - \left\lfloor \frac{x}{y} \right\rfloor + \left\lfloor \frac{x}{ny} \right\rfloor n) \qquad (133)$$

$$= x - \left\lfloor \frac{x}{y} \right\rfloor y = x \bmod y. \qquad (134)$$

Basics 3.15:

$$\lceil mx \rceil = \lceil x \rceil + \left\lceil x - \frac{1}{m} \right\rceil + ... + \left\lceil x - \frac{m-1}{m} \right\rceil. \tag{135}$$

Basics 3.16: Prove

$$n \bmod 2 = (1 - (-1)^n)/2. \tag{136}$$

It is true in both even and odd cases.
Solve

$$n \bmod 3 = a + bw^n + cw^{2n}. \qquad \{w = (-1 + i\sqrt{3})/2\} \tag{137}$$

Try $n = 0, 1, 2$:

$$a + b + c = 0; \tag{138}$$
$$a + bw + cw^2 = a + bw + c(-1 - w) = 1; \tag{139}$$
$$a + bw^2 + cw^4 = a + b(-1 - w) + cw = 2. \tag{140}$$

Solution is $a = 1$, $b = (-1 - w)/(1 + 2w) = (w - 1)/3$, $c = -1 - b = -(w + 2)/3$.
To prove this is easy, because $w^3 = 1$.

Basics 3.17:

$$\sum_j \sum_k [0 \le k < m][1 \le j \le x + k/m] \tag{141}$$

$$= \sum_j \sum_k [0 \le k < m][1 \le j \le \lceil x \rceil][j \le x + k/m] \tag{142}$$

$$= \sum_j \sum_k [0 \le k < m][1 \le j \le \lceil x \rceil][k \ge m(j - x)] \tag{143}$$

$$= \sum_{1 \le j \le \lceil x \rceil} \sum_k [0 \le k < m] - \sum_{j = \lceil x \rceil} \sum_k [0 \le k < m(j - x)] \tag{144}$$

$$= m\lceil x \rceil - \lceil m(\lceil x \rceil - x) \rceil \tag{145}$$

$$= \lfloor mx \rfloor. \tag{146}$$

Basics 3.18: pass.

Homework exercises 3.19: To let $f(x) = \lfloor log_b(x) \rfloor = \lfloor log_b(\lfloor x \rfloor) \rfloor$:

$$f(x) = \text{integer} \implies x = \text{integer}. \tag{147}$$

Then b should be an integer.

Homework exercises 3.20:

$$\sum_k [\alpha \le kx \le \beta]xk = x \sum_k [\alpha/x \le k \le \beta/x]k \tag{148}$$

$$= x \sum_k [\lceil \alpha/x \rceil \le k \le \lfloor \beta/x \rfloor]k \tag{149}$$

$$= \frac{1}{2}x(\lfloor \beta/x \rfloor \lfloor \beta/x \rfloor + \lfloor \beta/x \rfloor - \lceil \alpha/x \rceil \lceil \alpha/x \rceil + \lceil \alpha/x \rceil). \tag{150}$$

Homework exercises 3.21: Small cases $1, 16, 128, 1024, ...$ show that for a number which has k digits in decimal notation, there always exactly one number 2^m has leading 1. This is true because for any number n:

$$log_{10}(n) < \lfloor log_{10}(n) + 1 \rfloor \iff 2n < n + 10^{\lfloor log_{10}(n) + 1 \rfloor}. \tag{151}$$

Then solution is $\lfloor M log_{10}2 \rfloor + 1$.

Homework exercises 3.22:

$$S_n = \sum_{k \geq 1} \left\lfloor \frac{n}{2^k} + \frac{1}{2} \right\rfloor. \tag{152}$$

Small cases show that $S_n = n$. Let $n = m_k 2^k + d$

$$\left\lfloor \frac{n}{2^k} + \frac{1}{2} \right\rfloor \neq \left\lfloor \frac{n-1}{2^k} + \frac{1}{2} \right\rfloor \iff \tag{153}$$

$$m_k + \left\lfloor \frac{d}{2^k} + \frac{1}{2} \right\rfloor \neq m_k + \left\lfloor \frac{d-1}{2^k} + \frac{1}{2} \right\rfloor \iff \tag{154}$$

$$d = 2^{k-1}. \tag{155}$$

Consider S_n and S_{n-1}, there always one and only one k to make $n \bmod (2^k) = 2^{k-1}$. Let $n = 2^{k'}q$ where q is a odd, here k' is the one. Then $S_n = S_{n-1} + 1 = n$.

$$T_n = \sum_{k \geq 1} 2^k \left\lfloor \frac{n}{2^k} + \frac{1}{2} \right\rfloor^2. \tag{156}$$

From the observation above, let $n = 2^{k'}q$ where q is a odd. The different terms between T_n and T_{n-1} is $\left\lfloor \frac{2^{k'}q}{2^{k'+1}} + \frac{1}{2} \right\rfloor$ and $\left\lfloor \frac{2^{k'}q-1}{2^{k'+1}} + \frac{1}{2} \right\rfloor$. Then:

$$T_n - T_{n-1} = 2^{k'+1} \left(\left\lfloor \frac{2^{k'}q}{2^{k'+1}} + \frac{1}{2} \right\rfloor^2 - \left\lfloor \frac{2^{k'}q-1}{2^{k'+1}} + \frac{1}{2} \right\rfloor^2 \right) \tag{157}$$

$$= 2^{k'+1} \left(\left\lfloor \frac{q}{2} + \frac{1}{2} \right\rfloor^2 - \left\lfloor \frac{q}{2} - \frac{1}{2^{k'+1}} + \frac{1}{2} \right\rfloor^2 \right) \tag{158}$$

$$= 2^{k'+1} \left(\left\lfloor \frac{q}{2} + \frac{1}{2} \right\rfloor^2 - \left\lfloor \frac{q}{2} \right\rfloor^2 \right) \tag{159}$$

$$= 2^{k'+1} \left(\left\lfloor \frac{q}{2} + \frac{1}{2} \right\rfloor - \left\lfloor \frac{q}{2} \right\rfloor \right) \left(\left\lfloor \frac{q}{2} + \frac{1}{2} \right\rfloor + \left\lfloor \frac{q}{2} \right\rfloor \right) \tag{160}$$

$$= 2^{k'+1}q = 2n. \tag{161}$$

Then $T_n = n(n+1)$.

Homework exercises 3.23: The nth elements should be $X_n = m$

$$\frac{1}{2}m(m-1) < n \leq \frac{1}{2}m(m+1) \iff \tag{162}$$

$$m(m-1) < 2n \leq m(m+1) \iff \tag{163}$$

$$m(m-1) + \frac{1}{4} < 2n < m(m+1) + \frac{1}{4} \iff \tag{164}$$

$$m - \frac{1}{2} < \sqrt{2n} < m + \frac{1}{2} \tag{165}$$

$$m < \sqrt{2n} + \frac{1}{2} \tag{166}$$

$$m = \left\lfloor \sqrt{2n} + \frac{1}{2} \right\rfloor. \tag{167}$$

Homework exercises 3.24: $Spec(\alpha/(\alpha + 1)) = Spec(\alpha) + n + 1$.

Homework exercises 3.25: Can prove $K_n > n$ with induction.

The initial cases are true:

$$K_0 = 1; \tag{168}$$

$$K_1 = 3. \tag{169}$$

Assume that $K_i > i$ when $i \le n$, then:

$$K_{n+1} = 1 + min(2K_{\lfloor n/2 \rfloor}, 3K_{\lfloor n/3 \rfloor}) \tag{170}$$

$$\ge 1 + min(2(\lfloor n/2 \rfloor + 1), 3(\lfloor n/3 \rfloor + 1)) \tag{171}$$

$$\ge 1 + min(n + 1, n + 1) \tag{172}$$

$$> n + 1. \tag{173}$$

Proved.

Homework exercises 3.26: For the left side:

$$D_n^{(q)} = \left\lceil \frac{q}{q-1} D_{n-1}^{(q)} \right\rceil \iff \tag{174}$$

$$\frac{q}{q-1} D_{n-1}^{(q)} \le D_n^{(q)} \tag{175}$$

$$(\frac{q}{q-1})^n D_0^{(q)} \le D_n^{(q)} \tag{176}$$

$$(\frac{q}{q-1})^n \le D_n^{(q)} \tag{177}$$

For the right side, can prove with induction:

$$D_n^{(q)} \le q(\frac{q}{q-1})^n - q + 1. \quad \text{\{this is wired but can be guessed from the initial case.\}} \tag{178}$$

The initial case is true:

$$D_0^{(q)} = 1 \ge q - q + 1. \tag{179}$$

Assume that it is true when $i \le n$, then:

$$D_{n+1}^{(q)} = \left\lceil \frac{q}{q-1} D_{n-1}^{(q)} \right\rceil \tag{180}$$

$$\le \left\lceil q(\frac{q}{q-1})^{n+1} - q \right\rceil \tag{181}$$

$$\le q(\frac{q}{q-1})^{n+1} - q + 1. \tag{182}$$

Proved.

Homework exercises 3.27: Let $D_n^{(3)} = 2^m b - a$ where $a = 0$ or 1 and b is odd, then:

$$D_{n+1}^{(3)} = \left\lceil \frac{3}{2}(2^m b - a) \right\rceil = 3 * 2^{m-1} b - \left\lfloor \frac{3}{2}a \right\rfloor = 3 * 2^{m-1} b - a; \tag{183}$$

$$D_{m+n}^{(3)} = 3^m b - a. \tag{184}$$

This formula shows another way to generate the next number in the $D_n^{(3)}$. This way does not generate the next number one by one which means the formula only generates a subset.

Let the seed number is $D_0^{(3)} = 2^1 - 1 = 1$, then $a = 1$. The numbers generated from the seed contians infinite numbers and these numbers are arranged in the order: odd, even, odd, even, ...

Homework exercises 3.28: Not solved yet.

From an initial number $a_n = m^2$, all numbers can be generated:

$$a_{n+2k+1} = (m + k)^2 + m - k; \quad \{0 \le k \le m\} \tag{185}$$

$$a_{n+2k+2} = (m + k)^2 + 2m; \quad \{0 \le k \le m\} \tag{186}$$

$$a_{n+2m+1} = 4m^2. \tag{187}$$

Then a_{n+2m+1} is the next initial number.

Homework exercises 3.29: pass.

Homework exercises 3.30: Small cases show that

$$X_n = \alpha^{2^n} + \alpha^{-2^n}. \tag{188}$$

It is easy to prove this with the induction.
$\alpha^{-2^n} < 1$ because $\alpha > 1$ which means the integer X_n should be $\lceil \alpha^{2^n} \rceil$.

Homework exercises 3.31: A really smart solution:

$$\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor = \lfloor x + \lfloor y \rfloor \rfloor + \lfloor x + y \rfloor \tag{189}$$

$$\leq \left\lfloor x + \frac{1}{2}\lfloor 2y \rfloor \right\rfloor + \left\lfloor x + \frac{1}{2}\lfloor 2y \rfloor + \frac{1}{2} \right\rfloor \tag{190}$$

$$= \lfloor 2x + \lfloor 2y \rfloor \rfloor = \lfloor 2x \rfloor + \lfloor 2y \rfloor. \tag{191}$$

Homework exercises 3.32:

$$f(x) = \sum_k 2^k ||\frac{x}{2^k}||^2. \qquad \{||x|| = \min x - \lfloor x \rfloor, \lceil x \rceil - x\} \tag{192}$$

Firstly, consider some basic rules.
$f(x) = f(-x)$ because $||x|| = ||-x||$. Also:

$$f(2x) = \sum_k 2^k ||\frac{2x}{2^k}||^2 = 2 \sum_k 2^{k-1} ||\frac{x}{2^{k-1}}||^2 = 2f(x). \tag{193}$$

Secondly, consider x in $0 \leq x < 1$.
Let $f(x) = l(x) + r(x)$, where $l(x)$ is the sum when $k \leq 0$ and $r(x)$ is the sum when $k > 0$.

$$l(x+1) = \sum_{k \leq 0} 2^k ||\frac{x+1}{2^k}||^2 = \sum_{k \leq 0} 2^k ||\frac{x}{2^k} + 2^{-k}||^2 = \sum_{k \leq 0} 2^k ||\frac{x}{2^k}||^2 = l(x). \tag{194}$$

Because $||x|| \leq 0.5$ then:

$$l(x) \leq \sum_{k \leq 0} 2^k (\frac{1}{2})^2 = \frac{1}{2}. \tag{195}$$

Then $r(x+1) = r(x) + 1$ because:

$$r(x) = \frac{x^2}{2} + \frac{x^2}{4} + \frac{x^2}{8} + ... = x^2. \tag{196}$$

$$r(x+1) = 2||\frac{x+1}{2}||^2 + \frac{x^2}{4} + \frac{x^2}{8} + ... \tag{197}$$

$$= \frac{(x-1)^2}{2} + \frac{(x+1)^2}{4} + \frac{(x+1)^2}{8} + ... \tag{198}$$

$$= \frac{(x-1)^2}{2} + \frac{(x+1)^2}{2} = x^2 + 1. \tag{199}$$

Then $f(x+1) = l(x+1) + r(x+1) = l(x) + r(x) + 1 = f(x) + 1$ when $0 \leq x < 1$. It also shows that $f(x+n) = f(x) + n$ and $f(0) = 0$ means $f(n) = n$. Finally small cases show that $f(x) = |x|$, try to prove that when $0 \leq x < 1$:

$$f(x) = 2^{-m} f(2^m x) \tag{200}$$

$$= 2^{-m}(f(\lfloor 2^m x \rfloor + \{2^m x\}) \tag{201}$$

$$= 2^{-m}\lfloor 2^m x \rfloor + 2^{-m} f(\{2^m x\}). \qquad \{m \text{ is any integer}\} \tag{202}$$

Because

$$f(\{2^m x\}) = l(\{2^m x\}) + r(\{2^m x\}) = l(\{2^m x\}) + (\{2^m x\})^2 \le 0.5 + 1 = 1.5. \tag{203}$$

Then

$$|f(x) - x| = |2^{-m}\lfloor 2^m x \rfloor + 2^{-m} f(\{2^m x\}) - x| \tag{204}$$

$$= |2^{-m}\lfloor 2^m x \rfloor - x| + 2^{-m} f(\{2^m x\}) \tag{205}$$

$$\le |2^{-m}\lfloor 2^m x \rfloor - x| + 2^{-m}\frac{3}{2} \tag{206}$$

$$= |2^{-m}(\lfloor 2^m x \rfloor - 2^m x)| + 2^{-m}\frac{3}{2} \tag{207}$$

$$\le 2^{-m}\frac{5}{2}. \tag{208}$$

Because m could be any integer, then there is $|f(x) - x| = 0$ which means $f(x) = |x|$ when $0 \le x < 1$. In summary, because $f(x) = f(-x)$ and $f(x) + n = f(x + n)$, then $f(x) = |x|$ for all real x.

Exam problems 3.33: a.Split the area into for parts, and consider the top right square. Because r is a fractional number, corners are not crossed. The circle edge could be treated as a path go from the top to the right. Because no left or up steps, then the step number is $r + r = 2r = 2n - 1$. Then the number of cells of the board contain a segment of the circle is $4(2n - 1) = 8n - 4$.
b.This is the Guass's Circle Problem, and I have no idea why:

$$f(n, k) = \lfloor \sqrt{r^2 - k^2} \rfloor. \tag{209}$$

Exam problems 3.34: a.Small cases $0, 1, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 4, 4, 4, 4, 5, \dots$ show that it is a good idea to estimate $f(n)$ when n reached the last number in its level. Let $m = \lceil \lg n \rceil$, and there will be 2^m numbers if the last level is full.

$$f(n) + (2^m - n)m = \sum_{k=1}^{2^m} \lceil \lg k \rceil \tag{210}$$

$$= \sum_{j,k} j[j = \lceil \lg k \rceil][1 \le k \le 2^m] \tag{211}$$

$$= \sum_{j,k} j[2^{j-1} < k \le 2^j][1 \le j \le m] \tag{212}$$

$$= \sum_{1 \le j \le m} j2^{j-1} = m2^m - 2^m + 1. \tag{213}$$

Then $f(n) = mn + 2^m - 1$.
b.When $n = 2k$:

$$f(k) = \lceil \lg k \rceil k + 2^{\lceil \lg k \rceil} - 1; \tag{214}$$

$$f(2k) = \lceil \lg 2k \rceil 2k + 2^{\lceil \lg 2k \rceil} - 1 \tag{215}$$

$$= \lceil \lg k + 1 \rceil 2k + 2^{\lceil \lg k+1 \rceil} - 1 \tag{216}$$

$$= 2\lceil \lg k \rceil k + 2k + 2 * 2^{\lceil \lg k \rceil} - 1; \tag{217}$$

$$f(2k) = 2f(k) + 2k - 1; \tag{218}$$

$$f(n) = n - 1 + f(\lceil n/2 \rceil) + f(\lfloor n/2 \rfloor). \tag{219}$$

When $k = 2^m + 1$ and $n = 2k - 1$. Then $m = \lceil \lg k - 1 \rceil$ and $m + 1 = \lceil \lg k \rceil$:

$$f(\lfloor \frac{2k-1}{2} \rfloor) = f(k-1) = m(k-1) - 2^m + 1; \tag{220}$$

$$f(\lceil \frac{2k-1}{2} \rceil) = f(k) = (m+1)k - 2 * 2^m + 1; \tag{221}$$

$$f(2k-1) = 2mk - m + 4k - 4 * 2^m - 1. \tag{222}$$

Then

$$f(2k-1) = f(\left\lfloor \frac{2k-1}{2} \right\rfloor) + f(\left\lceil \frac{2k-1}{2} \right\rceil) + 2k - 1 - 1. \tag{223}$$

When $k \neq 2^m + 1$ and $n = 2k - 1$. Then $m = \lceil \lg k - 1 \rceil = \lceil \lg k \rceil$:

$$f(\left\lfloor \frac{2k-1}{2} \right\rfloor) = m(k-1) - 2^m + 1; \tag{224}$$

$$f(\left\lceil \frac{2k-1}{2} \right\rceil) = mk - 2^m + 1; \tag{225}$$

$$f(2k-1) = 2mk - m + 2k - 2 * 2^m. \tag{226}$$

Then

$$f(2k-1) = f(\left\lfloor \frac{2k-1}{2} \right\rfloor) + f(\left\lceil \frac{2k-1}{2} \right\rceil) + 2k - 1 - 1. \tag{227}$$

Proved.

Exam problems 3.35:

$$\lfloor (n+1)^2 n! e \rfloor = \left\lfloor (n+1)^2 n! \sum_{k=0}^{\infty} \frac{1}{k!} \right\rfloor \tag{228}$$

$$= \left\lfloor \left(\frac{n!}{0!} + \frac{n!}{1!} + ... + \frac{n!}{n!} + \frac{n!}{(n+1)!} + ... + \frac{n!}{\infty!} \right)(n+1)^2 \right\rfloor \tag{229}$$

$$= \left\lfloor (n+1)^2 n((n-1)! + ... + 1) + (n+1)^2 + (n+1) + \frac{n+1}{n+2} + ... + \frac{n!(n+1)^2}{\infty!} \right\rfloor \tag{230}$$

$$\tag{231}$$

Add the mod operator:

$$\lfloor (n+1)^2 n! e \rfloor \bmod n = \left\lfloor (n+1)^2 + (n+1) + \frac{n+1}{n+2} + ... + \frac{n!(n+1)^2}{\infty!} \right\rfloor \bmod n \tag{232}$$

$$= \left\lfloor 2 + \frac{n+1}{n+2} + ... + \frac{n!(n+1)^2}{\infty!} \right\rfloor \bmod n \tag{233}$$

Because

$$\frac{n+1}{n+2} + ... + \frac{n!(n+1)^2}{\infty!} = \frac{n+1}{n+2}\left(1 + \frac{1}{n+3} + \frac{1}{(n+3)(n+4)} + ...\right) \tag{234}$$

$$< \frac{n+1}{n+2}\left(1 + \frac{1}{n+3} + \frac{1}{(n+3)(n+3)} + ...\right) \tag{235}$$

$$= \frac{(n+1)(n+3)}{(n+2)^2} < 1 \tag{236}$$

Then result is 2 mod n.

Exam problems 3.36:

$$\sum_{1<k<2^{2^n}} \frac{1}{2^{\lfloor \lg k \rfloor} 4^{\lfloor \lg \lg k \rfloor}} = \sum_{k,l,m} 2^{-l} 4^{-m} [m = \lfloor \lg l \rfloor][l = \lfloor \lg k \rfloor][1 < k < 2^{2^n}] \tag{237}$$

$$= \sum_{k,l,m} 2^{-l} 4^{-m} [2^m \leq l < 2^{m+1}][2^l \leq k < 2^{l+1}][0 \leq m < n] \tag{238}$$

$$= \sum_{l,m} 4^{-m} [2^m \leq l < 2^{m+1}][0 \leq m < n] \tag{239}$$

$$= \sum_m 2^{-m} [0 \leq m < n] \tag{240}$$

$$= 1 + \frac{1}{2} + ... + \frac{1}{2^{n-1}} = 2(1 - 2^{-n}). \tag{241}$$

Exam problems 3.36:

Exam problems 3.37:

Exam problems 3.38:

Exam problems 3.39:

Exam problems 3.40: