1 Recurrent Problems

1.1 THE TOWER OF HANOI

T_n is the minimum number of steps that can move n disks from one peg to another.

1.1.1 Three Towers

There are three towers A, B and C. At the begining all disks are on A, and C is the target. Move top n-1 from A to B (T_{n-1}), move the last one from A to C, move n-1 from B to C.

$$T_0 = 0; \tag{1}$$
$$T_n = 2T_{n-1}. \qquad \{n \geq 1\} \tag{2}$$

Then combine these two:

$$T_n = 2^n - 1. \qquad \{n \geq 0\} \tag{3}$$

1.2 Lines In The Plane

L_n is the maximum region number defined by n lines in the plane. The nth line at most crosses n-1 lines if not parallels to any other lines. The nth line at most splits n new spaces if not goes through any existing intersection point.

$$L_0 = 1; \tag{4}$$
$$L_n = L_{n-1} + n. \qquad \{n \geq 1\} \tag{5}$$

Then combine these two:

$$L_n = \frac{1}{2}n(n+1) + 1. \qquad \{n \geq 0\} \tag{6}$$

1.2.1 Zig Lines

Z_n is the maximum region number defined by n zig lines in the plane. The nth zig line corresponds to the 2nth line in the last problems. Each zig line generates 2 less spaces than 2 lines.

$$Z_n = L_{2n} - 2n. \qquad \{n \geq 0\} \tag{7}$$

Which is:

$$Z_n = 2n^2 - n + 1. \qquad \{n \geq 0\} \tag{8}$$

1.3 The Josephus Problem

n people numbered 1 to n stand around a circle. Every second remaining person are eliminated until only one survives. $J(n)$ is the survivor's number. Case $2n$: after the first round (1, 2, 3, ..., $2n$) becomes (1, 3, 5, ..., $2n-1$); Case $2n+1$: after the first round (1, 2, 3, ..., $2n+1$) becomes (3, 5, ..., $2n+1$); In the case $2n$, rename the left n people using the map $(n+1)/2$ to (1, 2, ..., n) and continue play this game. In the case $2n+1$, rename the left n people using the map $(n-1)/2$ to (1, 2, ..., n) and continue play this game.

$$J(1) = 1; \tag{9}$$
$$J(2n) = 2J(n) - 1; \qquad \{n \geq 1\} \tag{10}$$
$$J(2n+1) = 2J(n) + 1. \qquad \{n \geq 1\} \tag{11}$$

Solution:

$$J(2^m + l) = 2l + 1. \qquad \{m \geq 0 \text{ and } 0 \leq l < 2^m\} \tag{12}$$

1.3.1 Binary Solution

Let $n = (1b_{m-1}...b_1b_0)_2$:

$$l = (0b_{m-1}b_{m-2}...b_1b_0)_2; \tag{13}$$
$$J(n) = (b_{m-1}b_{m-2}...b_1b_01)_2. \tag{14}$$

Can get $J(n)$ from n with a one-bit cyclic shift left!

1.3.2 The Repertoire Method

For example:

$$f(1) = \alpha; \tag{15}$$
$$f(2n) = 2f(n) + \beta; \qquad \{n \geq 1\} \tag{16}$$
$$f(2n+1) = 2f(n) + \gamma. \qquad \{n \geq 1\} \tag{17}$$

Solution should be

$$f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma \tag{18}$$

Some pairs of $(f(n), \alpha, \beta\gamma)$ are need to solve $A(n)$, $B(n)$ and $C(n)$.
$(1, 1, -1, -1)$, $(n, 1, 0, 1)$ and $(2^m, 2^m, 0, 0)$:

$$1 = A(n) - B(n) - C(n); \qquad \{n \geq 1\} \tag{19}$$
$$n = A(n) + C(n); \qquad \{n \geq 1\} \tag{20}$$
$$A(n) = 2^m. \qquad \{n \geq 1 \text{ and } 2^m + l = n \text{ and } 0 \leq l < 2^m\} \tag{21}$$

I guess this method (the repertoire method) is not used for solving $A(n)$ in this example because the 2^m is hard to guess. The $A(n)$ is solved by intuition. However $B(n)$ and $C(n)$ can be easily solved by the repertoire method.

1.3.3 Generalized Josephus Recurrence

$$f(1) = \alpha; \tag{22}$$
$$f(2n+j) = 2f(n) + \beta_j. \qquad \{j = 0, 1 \text{ and } n \geq 1\} \tag{23}$$

Solution is $f((b_mb_{m-1}...b_1b_0)_2) = (\alpha\beta_{b_{m-1}}\beta_{b_{m-2}}...\beta_{b_1}\beta_{b_0})_2$.

$$f(j) = \alpha_j; \qquad \{1 \leq j < d\} \tag{24}$$
$$f(dn+j) = cf(n) + \beta_j. \qquad \{0 \leq j < d \text{ and } n \geq 1\} \tag{25}$$

Solution is $f((b_mb_{m-1}...b_1b_0)_d) = (\alpha_{b_m}\beta_{b_{m-1}}\beta_{b_{m-2}}...\beta_{b_1}\beta_{b_0})_c$.

1.4 Exercises

Warmups 1.1: cannot prove the 1st horse and the 2nd one has one same color when $n = 2$.

Warmups 1.2: move n-1 from A via C to B, move the last one from A to C. Move n-1 from B via C to A, move the last one from C to B. Move n-1 from A via C to B.

$$f(1) = 2; \tag{26}$$
$$f(n) = 3f(n-1) + 2. \qquad \{n \geq 2\} \tag{27}$$

Solution is $f(n) = 3^n - 1$ where $n \geq 1$.

Warmups 1.3: the result of **warmups 1.2** is $3^n - 1$, which is the minimal step number. The minimal step number means no two arrangements generated by any two different steps are same. Then 3^n different arrangements have been encountered (plus the begining one). There are at most 3^n different arrangements in this case, because for each disk there are 3 possible needles. Then all arrangements have been encountered.

Warmups 1.4: prove by induction that from any disk arrangement the minimal step number of moving all disks to one needle (B) is $2^n - 1$ when $n \geq 1$.

This state is true when $n = 1$.

If this state is true when $n = k$ where $k \geq 1$, there will be two cases when $n = k + 1$. Case 1 the max disk is on B. Case 2 the max disk is not on B (but on A).

In the case 1, the minimal step is $2^{n-1} - 1$ when move the rest n-1 disks to B (use the assumption). In the case 2, the minimal step is also $2^{n-1} - 1 + 1 + 2^{n-1} - 1 = 2^n - 1$ when move rest n-1 disks to C, move the last one to B, move the rest n-1 disk from C to B.

Then the state is when $n \geq 1$.

Warmups 1.5: the nth circle generates $2(n-1)$ new intersection points. All points are on the nth circle, and every two connected points create a new area. Then $n-1$ new areas are generated.

$$f(1) = 2; \tag{28}$$
$$f(n) = f(n-1) + 2(n-1). \qquad \{n \geq 2\} \tag{29}$$

Solution is $f(n) = n^2 - n + 2$ where $n \geq 1$.

Warmups 1.6: the nth line generates n-1 new intersection points. These n-1 points creates n-2 bounded areas.

$$f(3) = 1; \tag{30}$$
$$f(n) = f(n-1) + n - 2. \qquad \{n \geq 4\} \tag{31}$$

Solution is $f(n) = \frac{1}{2}(n^2 - 3n + 2)$ where $n \geq 3$.

Warmups 1.7: the state is false when $n = 1$:

$$H(1) = J(2) - J(1) = 0. \tag{32}$$

Homework exercises 1.8: some small cases shows a loop:

$$Q_0 = \alpha; Q_1 = \beta; \tag{33}$$
$$Q_2 = \frac{1+\beta}{\alpha}; \tag{34}$$
$$Q_3 = \frac{1+\alpha+\beta}{\alpha\beta}; \tag{35}$$
$$Q_4 = \frac{1+\alpha}{\beta}; \tag{36}$$
$$Q_5 = \alpha; Q_6 = \beta. \tag{37}$$

Solution is:

$$Q_{5n} = \alpha; Q_{5n+1} = \beta; \tag{38}$$
$$Q_{5n+2} = \frac{1+\beta}{\alpha}; \tag{39}$$
$$Q_{5n+3} = \frac{1+\alpha+\beta}{\alpha\beta}; \tag{40}$$
$$Q_{5n+4} = \frac{1+\alpha}{\beta}. \tag{41}$$

Homework exercises 1.9: (a) Rewrite x_n in the right:

$$x_1 ... x_{n-1} x_n \leq (\frac{x_1 + ... + x_n}{n})^n = (\frac{x_1 + ... + x_{n-1}}{n-1})^n. \tag{42}$$

Then rewrite x_n in the left proves the state $P(n-1)$:

$$x_1...x_{n-1} \leq (\frac{x_1 + ... + x_{n-1}}{n-1})^{n-1}. \tag{43}$$

(b) Combine following two inequations:

$$x_1 x_2 \leq (\frac{x_1 + x_2}{2})^2; \tag{44}$$
$$x_1...x_n \leq (\frac{x_1 + ... + x_n}{n})^n. \tag{45}$$

implies $P(2n)$:

$$(x_1...x_n)(x_{n+1}...x_{2n}) \leq (\frac{x_1 + ... + x_n}{n})^n (\frac{x_{n+1} + ... + x_{2n}}{n})^n \leq (\frac{x_1 + ... + x_{2n}}{2n})^{2n}. \tag{46}$$

Homework exercises 1.10: Q_n: move n-1 from A to C, move the last one from A to B, move n-1 from C to B. R_n: move n-1 from B to A, move the last one from B to C, move n-1 from A to B, move the last one from C to A, move the n-1 from B to A.

Homework exercises 1.11: (a) This is simlar to the single tower of hanoi, and every step in the single tower becauses two steps. so minimal step number is $2T_n$. (b) Consider the last two disks α covers β at the begining. Move top 2n-2 from A to B, move α from A to C, move 2n-2 from B to C, move β from A to B, move 2n-n from C to A, move α from C to B, move 2n-2 from A to B.

$$A_0 = 0; \tag{47}$$
$$A_{2n} = 4A_{2n-2} + 3. \qquad \{n > 0\} \tag{48}$$

Solution is $A_{2n} = 4^n - 1$ where $n \geq 0$.

Homework exercises 1.12:

$$A(m_1) = m_1; \tag{49}$$
$$A(m_1, ..., m_n) = 2A(m_1, .., m_{n-1}) + m_n. \qquad \{n > 1\} \tag{50}$$

Solution is $A(m_1, ..., m_n) = (m_1, ..., m_n)_2$ where $n \geq 1$.

Homework exercises 1.13: every three lines can generate 7 planes at most, however one zig-zag line will only generate 2 planes at most. Each zig-zag line generates 5 less planes than three lines:

$$ZZ_n = L_{3n} - 5n. \qquad \{n \geq 0\} \tag{51}$$

Solution is $ZZ_n = \frac{1}{2}(9n^2 - 7n) + 1$ where $n \geq 0$.

Homework exercises 1.14: put the last plane added plane in a table, to achieve maximum space number, each other plane must have a intersection line with it. New spaces below the table correspond to the planes segmented by the intersection lines on the table.

$$P_0 = 1; \tag{52}$$
$$P_n = P_{n-1} + L_{n-1}. \qquad \{n \geq 1\} \tag{53}$$

Solution is $P_n = 1 + \sum_{i=0}^{n-1} L_i$ where $n \geq 0$.

Homework exercises 1.15: the process is un-changed, so the function is unchanged. But the initial value has changed.

$$I(2) = 2; \tag{54}$$
$$I(3) = 1; \tag{55}$$
$$I(2n) = 2I(n) - 1; \qquad \{n > 2\} \tag{56}$$
$$I(2n+1) = 2I(n) + 1. \qquad \{n > 2\} \tag{57}$$

Solution is $I((b_m b_{m-1}...b_0)_2) = (\alpha_{(b_m b_{m-1})_2}\beta_{b_{m-2}}\beta_{b_{m-3}}...\beta_{b_0})$ where $\alpha_{(10)_2} = 2$, $\alpha_{(11)_2} = 1$, $\beta_0 = -1$ and $\beta_1 = 1$.

Homework exercises 1.16: given

$$g(1) = \alpha; \tag{58}$$
$$g(2n+j) = 3g(n) + \gamma n + \beta_j. \qquad \{\text{for } j = 0,1 \text{ and } n \geq 1\} \tag{59}$$

Solution should be

$$g(n) = A(n)\alpha + B(n)\beta_0 + C(n)\beta_1 + D(n)\gamma. \qquad \{n \geq 1\} \tag{60}$$

When $\gamma = 0$ there is

$$g((1b_{m-1}...b_0)_2) = A(n)\alpha + B(n)\beta_0 + C(n)\beta_1 = (\alpha\beta_{b_{m-1}}...\beta_{b_0})_3. \tag{61}$$

Use $g(n) = n$

$$g(1) = \alpha = 1; \tag{62}$$
$$0 = n + \gamma n + \beta_0; \qquad \{n \geq 1\} \tag{63}$$
$$1 = n + \gamma n + \beta_1. \qquad \{n \geq 1\} \tag{64}$$

We have $\gamma = -1$, $\beta_0 = 0$ and $\beta_1 = 1$, which means

$$n = A(n) + C(n) - D(n). \qquad \{n \geq 1\} \tag{65}$$

$A(n)$ and $C(n)$ is required to solve $D(n)$.
$\alpha = 1$ and $\beta_0 = \beta_1 = \gamma = 0$ can solve $A(n) = g(1b_{m-1}...b_0)_2) = 3^m$.
$\beta_1 = 1$ and $\alpha = \beta_0 = \gamma = 0$ can solve $D(n) = g(1b_{m-1}...b_0)_2) = (b_{m-1}...b_0)_3$.
Use function 61, 65 with $A(n)$ and $B(n)$:

$$g((1b_{m-1}...b_0)_2) = (\alpha\beta_{b_{m-1}}...\beta_{b_0})_3 + \gamma(3^m + (b_{m-1}...b_0)_3 - n) \tag{66}$$
$$= (\alpha\beta_{b_{m-1}}...\beta_{b_0})_3 + \gamma((1b_{m-1}...b_0)_3 - n). \tag{67}$$

where $n = (1b_{m-1}...b_0)_2$.

Exam problems 1.17: Set $g(n) = \frac{1}{2}n(n+1)$ where $n \geq 0$:

$$W_{g(n)} \leq T_n + 2W_{g(n-1)} \tag{68}$$
$$\leq T_n + 2(T_{n-1} + W_{g(n-2)}) \tag{69}$$
$$... \tag{70}$$
$$\leq T_n + 2^1 T_{n-1} + ... + 2^{n-2}T_2 + 2^{n-1}W_{g(1)} \tag{71}$$
$$\leq (n-1)2^n - (2^{n-1} - 1) + 2^{n-1} \tag{72}$$
$$\leq (n-1)2^n + 1. \tag{73}$$

Exam problems 1.18: If there are n zag lines, two rays of the ith $(1 \leq i \leq n)$ line is:

$$y = -\frac{1}{n^i}(x - n^{2i}); \qquad \{x <= n^{2i}\} \tag{74}$$
$$y = -\frac{1}{n^i + n^{-n}}(x - n^{2i}). \qquad \{x <= n^{2i}\} \tag{75}$$

To prove the state, following states should be proved: A. Each ray of the ith zag line should have $2(n-1)$ intersection points with other rays. B. All intersection points are different.

In the case $1 \leq i < j \leq n$:

$$-\frac{1}{n^i} < -\frac{1}{n^i + n^{-n}} < -\frac{1}{n^j} < -\frac{1}{n^j + n^{-n}} < 0 \tag{76}$$

State A can be proved by the induction method.

There are four kinds of intersections, given $(1 \le i < j \le n)$

$$y = -\frac{1}{n^i}(x - n^{2i}); \qquad \{x \le n^{2i}\} \tag{77}$$
$$y = -\frac{1}{n^j}(x - n^{2j}); \qquad \{x \le n^{2j}\} \tag{78}$$

Solution is $x_0 = -n^{i+j}$ and $y_0 = n^i + n^j$.

$$y = -\frac{1}{n^i + n^{-n}}(x - n^{2i}). \qquad \{x \le n^{2i}\} \tag{79}$$
$$y = -\frac{1}{n^j + n^{-n}}(x - n^{2j}). \qquad \{x \le n^{2j}\} \tag{80}$$

Solution is $x_1 = -n^{i+j} - n^{-n}(n^i + n^j)$ and $y_1 = n^i + n^j$.

$$y = -\frac{1}{n^i}(x - n^{2i}); \qquad \{x \le n^{2i}\} \tag{81}$$
$$y = -\frac{1}{n^j + n^{-n}}(x - n^{2j}). \qquad \{x \le n^{2j}\} \tag{82}$$

Solution is $x_2 = \frac{n^{2i+j}+n^{2i-n}-n^{i+2j}}{n^j+n^{-n}-n^i}$ and $y_2 = \frac{n^{2j}-n^{2i}}{n^j+n^{-n}-n^i}$.

$$y = -\frac{1}{n^i + n^{-n}}(x - n^{2i}). \qquad \{x \le n^{2i}\} \tag{83}$$
$$y = -\frac{1}{n^j}(x - n^{2j}); \qquad \{x \le n^{2j}\} \tag{84}$$

Solution is $x_3 = \frac{n^{i+2j}+n^{2j-n}-n^{2i+j}}{n^i+n^{-n}-n^j}$ and $y_3 = \frac{n^{2i}-n^{2j}}{n^i+n^{-n}-n^j}$.
$x_0 \ne x_1$ for any i and j because $n^i + n^j \ne 0$.
$y_0 \ne y_2$ for any i and j because $y_0 = \frac{n^{2i}-n^{2j}}{n^i-n^j}$ and $n^i - n^j \ne n^i - n^j - n^{-n}$.
$y_0 \ne y_3$ for any i and j because $y_0 = \frac{n^{2i}-n^{2j}}{n^i-n^j}$ and $n^i - n^j \ne n^i - n^j + n^{-n}$.
$y_2 \ne y_3$ for any i and j because $n^i - n^j + n^n \ne n^i - n^j + n^{-n}$. Then the state is proved.

Exam problems 1.19: Z_n means any two rays have a intersection points, and this means $n \le 11$.

Exam problems 1.20: given

$$h(1) = \alpha; \tag{85}$$
$$h(2n + j) = 4g(n) + \gamma_j n + \beta_j. \qquad \{\text{for } j = 0, 1 \text{ and } n \ge 1\} \tag{86}$$

Solution should be

$$h(n) = A(n)\alpha + B(n)\beta_0 + C(n)\beta_1 + D(n)\gamma_0 + E(n)\gamma_1. \qquad \{n \ge 1\} \tag{87}$$

when $\gamma_0 = \gamma_1 = 0$ there is

$$h((1b_{m-1}...b_0)_2) = A(n)\alpha + B(n)\beta_0 + C(n)\beta_1 = (\alpha\beta_{b_{m-1}}...\beta_{b_0})_4. \tag{88}$$

Use $h(n) = n$

$$h(1) = \alpha = 1; \tag{89}$$
$$2n + 0 = 4n + \gamma_0 n + \beta_0; \tag{90}$$
$$2n + 1 = 4n + \gamma_1 n + \beta_1. \tag{91}$$

We have $\alpha = 1$, $\gamma_0 = \gamma_1 = -2$, $\beta_0 = 0$ and $\beta_1 = 1$ which means:

$$n = A(n) + C(n) - 2D(n) - 2E(n). \tag{92}$$

Use $h(n) = n^2$

$$h(1) = \alpha = 1; \tag{93}$$
$$4n^2 = 4n^2 + \gamma_0 n + \beta_0; \tag{94}$$
$$4n^2 + 4n + 1 = 4n^2 + \gamma_1 n + \beta_1. \tag{95}$$

We have $\alpha = 1$, $\gamma_0 = \beta_0 = 0$, $\gamma_1 = 4$ and $\beta_1 = 1$ which means:

$$n^2 = A(n) + C(n) + 4E(n). \tag{96}$$

$A(n)$ and $C(n)$ is required to solve $D(n) = (3A(n) + 3C(n) - n^2 - 2n)/4$ and $E(n) = (n^2 - A(n) - C(n))/4$.
$\alpha = 1$ and $\beta_0 = \beta_1 = \gamma_0 = \gamma_1 = 0$ can solve $A((1b_{m-1}...b_0)_2) = 4^m$.
$\beta_1 = 1$ and $\beta_0 = \alpha = \gamma_0 = \gamma_1 = 0$ can solve $C((1b_{m-1}...b_0)_2) = (1b_{m-1}...b_0)_4$.
Use function $A(n)$, $C(n)$, (96) and (92):

$$g((1b_{m-1}...b_0)_2) = (\alpha\beta_{b_{m-1}}...\beta_{b_0})_4 \tag{97}$$
$$+ \gamma_0(3 * (2b_{m-1}...b_0)_4 - n^2 - 2n)/4 \tag{98}$$
$$+ \gamma_1(n^2 - (2b_{m-1}...b_0)_4)/4. \tag{99}$$

Exam problems 1.21: excute last people every time can excute bad peole firstly. Then m could be any common multiple of n+1, n+2, ..., n+n.

Bonus problems 1.22: can use a De Bruijn cycle which is a De Bruign sequence of $B(2, n)$. The cycle is similar to a regular polygon and each edge is labeled as 0 or 1. Each edge labeled as 1 becomes a curve. Rotate and copy this shape $n-1$ times and consider these n shapes. There are $2^n - 2$ small spaces between edges and curves, because there is a 0...0 edge and a 1...1 curve. Add the space inside and the space outide, there are 2^n spaces.

The algorithm for generating Eulerian Path can help to generate a De Bruijn sequence.

Bonus problems 1.23: case 1: given p where $1 \leq n-p < j \leq n/2$, and one way to save himself is to remove people in the order of $1, 2, ..., n-p$ then $j+1, j+2, ..., n$ then $n-p+1, n-p+2, .., .j-1$. This order means remove the first people in the first $n-p$ moves. At this moment, the first people is n-p+1, and remove the j+1-(n-p)-th people whose id is $j+1$. At last remove the first people in rest moves.

For the first $n-p$ people and the last $p-1$ people, $q \equiv 1 \pmod{lcm(n, n-1, ..., 1)/p}$. To jump from $n-p$ to $j+1-(n-p)$ when there are p people, $q \equiv j+1-n \pmod{p}$. According to the Chinese remainder theorem if p is a prime there is a solution for q. According to the Bertrand's postulate there always exists at least one prime between n/2 and n.

case 2: given p where $1 \leq j < n/2$, and one way to save himself is to remove people in the order of $n, n-1, ..., p+1$ then $j+1, j+2, ..., p$ then $1, 2, ..., j-1$. This order means remove the last people in the first $n-p$ moves. At this moment, the first people is p, and remove the j+1-th people whose id is $j+1$. At last remove the first people in rest moves.

For the first $n-p$ people and the last $p-1$ people, $q \equiv 0 \pmod{lcm(n, n-1, ..., 1)/p}$. To jump from p to $j+1$ when there are p people, $q \equiv j+1 \pmod{p}$. According to the Chinese remainder theorem if p is a prime there is a solution for q. According to the Bertrand's postulate there always exists at least one prime between n/2 and n.

So he can always save himself.