

# 1 Recurrent Problems

## 1.1 THE TOWER OF HANOI

$T_n$  is the minimum number of steps that can move  $n$  disks from one peg to another.

### 1.1.1 Three Towers

There are three towers A, B and C. At the begining all disks are on A, and C is the target. Move top  $n-1$  from A to B ( $T_{n-1}$ ), move the last one from A to C, move  $n-1$  from B to C.

$$T_0 = 0; \quad (1)$$

$$T_n = 2T_{n-1}. \quad \{n \geq 1\} \quad (2)$$

Then combine these two:

$$T_n = 2^n - 1. \quad \{n \geq 0\} \quad (3)$$

## 1.2 Lines In The Plane

$L_n$  is the maximum region number defined by  $n$  lines in the plane. The  $n$ th line at most crosses  $n-1$  lines if not parallels to any other lines. The  $n$ th line at most splits  $n$  new spaces if not goes through any existing intersection point.

$$L_0 = 1; \quad (4)$$

$$L_n = L_{n-1} + n. \quad \{n \geq 1\} \quad (5)$$

Then combine these two:

$$L_n = \frac{1}{2}n(n+1) + 1. \quad \{n \geq 0\} \quad (6)$$

### 1.2.1 Zig Lines

$Z_n$  is the maximum region number defined by  $n$  zig lines in the plane. The  $n$ th zig line corresponds to the  $2n$ th line in the last problems. Each zig line generates 2 less spaces than 2 lines.

$$Z_n = L_{2n} - 2n. \quad \{n \geq 0\} \quad (7)$$

Which is:

$$Z_n = 2n^2 - n + 1. \quad \{n \geq 0\} \quad (8)$$

## 1.3 The Josephus Problem

$n$  people numbered 1 to  $n$  stand around a circle. Every second remaining person are eliminated until only one survives.  $J(n)$  is the survivor's number. Case  $2n$ : after the first round (1, 2, 3, ...,  $2n$ ) becomes (1, 3, 5, ...,  $2n-1$ ); Case  $2n+1$ : after the first round (1, 2, 3, ...,  $2n+1$ ) becomes (3, 5, ...,  $2n+1$ ); In the case  $2n$ , rename the left  $n$  people using the map  $(n+1)/2$  to (1, 2, ...,  $n$ ) and continue play this game. In the case  $2n+1$ , rename the left  $n$  people using the map  $(n-1)/2$  to (1, 2, ...,  $n$ ) and continue play this game.

$$J(1) = 1; \quad (9)$$

$$J(2n) = 2J(n) - 1; \quad \{n \geq 1\} \quad (10)$$

$$J(2n+1) = 2J(n) + 1. \quad \{n \geq 1\} \quad (11)$$

Solution:

$$J(2^m + l) = 2l + 1. \quad \{m \geq 0 \text{ and } 0 \leq l < 2^m\} \quad (12)$$

### 1.3.1 Binary Solution

Let  $n = (1b_{m-1}...b_1b_0)_2$ :

$$l = (0b_{m-1}b_{m-2}...b_1b_0)_2; \quad (13)$$

$$J(n) = (b_{m-1}b_{m-2}...b_1b_01)_2. \quad (14)$$

Can get  $J(n)$  from  $n$  with a one-bit cyclic shift left!

### 1.3.2 The Repertoire Method

For example:

$$f(1) = \alpha; \quad (15)$$

$$f(2n) = 2f(n) + \beta; \quad \{n \geq 1\} \quad (16)$$

$$f(2n+1) = 2f(n) + \gamma. \quad \{n \geq 1\} \quad (17)$$

Solution should be

$$f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma \quad (18)$$

Some pairs of  $(f(n), \alpha, \beta\gamma)$  are need to solve  $A(n)$ ,  $B(n)$  and  $C(n)$ .

$(1, 1, -1, -1)$ ,  $(n, 1, 0, 1)$  and  $(2^m, 2^m, 0, 0)$ :

$$1 = A(n) - B(n) - C(n); \quad \{n \geq 1\} \quad (19)$$

$$n = A(n) + C(n); \quad \{n \geq 1\} \quad (20)$$

$$A(n) = 2^m. \quad \{n \geq 1 \text{ and } 2^m + l = n \text{ and } 0 \leq l < 2^m\} \quad (21)$$

I guess this method (the repertoire method) is not used for solving  $A(n)$  in this example because the  $2^m$  is hard to guess. The  $A(n)$  is solved by intuition. However  $B(n)$  and  $C(n)$  can be easily solved by the repertoire method.

### 1.3.3 Generalized Josephus Recurrence

$$f(1) = \alpha; \quad (22)$$

$$f(2n+j) = 2f(n) + \beta_j. \quad \{j = 0, 1 \text{ and } n \geq 1\} \quad (23)$$

Solution is  $f((b_mb_{m-1}...b_1b_0)_2) = (\alpha\beta_{b_{m-1}}\beta_{b_{m-2}}...\beta_{b_1}\beta_{b_0})_2$ .

$$f(j) = \alpha_j; \quad \{1 \leq j < d\} \quad (24)$$

$$f(dn+j) = cf(n) + \beta_j. \quad \{0 \leq j < d \text{ and } n \geq 1\} \quad (25)$$

Solution is  $f((b_mb_{m-1}...b_1b_0)_d) = (\alpha_{b_m}\beta_{b_{m-1}}\beta_{b_{m-2}}...\beta_{b_1}\beta_{b_0})_c$ .

## 1.4 Exercises

**Warmups 1.1:** cannot prove the 1st horse and the 2nd one has one same color when  $n = 2$ .

**Warmups 1.2:** move  $n-1$  from A via C to B, move the last one from A to C. Move  $n-1$  from B via C to A, move the last one from C to B. Move  $n-1$  from A via C to B.

$$f(1) = 2; \quad (26)$$

$$f(n) = 3f(n-1) + 2. \quad \{n \geq 2\} \quad (27)$$

Solution is  $f(n) = 3^n - 1$  where  $n \geq 1$ .

**Warmups 1.3:** the result of **warmups 1.2** is  $3^n - 1$ , which is the minimal step number. The minimal step number means no two arrangements generated by any two different steps are same. Then  $3^n$  different arrangements have been encountered (plus the beginning one). There are at

most  $3^n$  different arrangements in this case, because for each disk there are 3 possible needles. Then all arrangements have been encountered.

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**Warmups 1.4:** prove by induction that from any disk arrangement the minimal step number of moving all disks to one needle (B) is  $2^n - 1$  when  $n \geq 1$ .

This state is true when  $n = 1$ .

If this state is true when  $n = k$  where  $k \geq 1$ , there will be two cases when  $n = k + 1$ . Case 1 the max disk is on B. Case 2 the max disk is not on B (but on A).

In the case 1, the minimal step is  $2^{n-1} - 1$  when move the rest  $n-1$  disks to B (use the assumption). In the case 2, the minimal step is also  $2^{n-1} - 1 + 1 + 2^{n-1} - 1 = 2^n - 1$  when move rest  $n-1$  disks to C, move the last one to B, move the rest  $n-1$  disk from C to B.

Then the state is when  $n \geq 1$ .

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**Warmups 1.5:** the  $n$ th circle generates  $2(n - 1)$  new intersection points. All points are on the  $n$ th circle, and every two connected points create a new area. Then  $n - 1$  new areas are generated.

$$f(1) = 2; \quad (28)$$

$$f(n) = f(n - 1) + 2(n - 1). \quad \{n \geq 2\} \quad (29)$$

Solution is  $f(n) = n^2 - n + 2$  where  $n \geq 1$ .

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**Warmups 1.6:** the  $n$ th line generates  $n-1$  new intersection points. These  $n-1$  points creates  $n-2$  bounded areas.

$$f(3) = 1; \quad (30)$$

$$f(n) = f(n - 1) + n - 2. \quad \{n \geq 4\} \quad (31)$$

Solution is  $f(n) = \frac{1}{2}(n^2 - 3n + 2)$  where  $n \geq 3$ .

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**Warmups 1.7:** the state is false when  $n = 1$ :

$$H(1) = J(2) - J(1) = 0. \quad (32)$$

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**Homework exercises 1.8:** some small cases shows a loop:

$$Q_0 = \alpha; Q_1 = \beta; \quad (33)$$

$$Q_2 = \frac{1 + \beta}{\alpha}; \quad (34)$$

$$Q_3 = \frac{1 + \alpha + \beta}{\alpha\beta}; \quad (35)$$

$$Q_4 = \frac{1 + \alpha}{\beta}; \quad (36)$$

$$Q_5 = \alpha; Q_6 = \beta. \quad (37)$$

Solution is:

$$Q_{5n} = \alpha; Q_{5n+1} = \beta; \quad (38)$$

$$Q_{5n+2} = \frac{1 + \beta}{\alpha}; \quad (39)$$

$$Q_{5n+3} = \frac{1 + \alpha + \beta}{\alpha\beta}; \quad (40)$$

$$Q_{5n+4} = \frac{1 + \alpha}{\beta}. \quad (41)$$

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**Homework exercises 1.9:** (a) Rewrite  $x_n$  in the right:

$$x_1 \dots x_{n-1} x_n \leq \left( \frac{x_1 + \dots + x_n}{n} \right)^n = \left( \frac{x_1 + \dots + x_{n-1}}{n-1} \right)^n. \quad (42)$$

Then rewrite  $x_n$  in the left proves the state  $P(n-1)$ :

$$x_1 \dots x_{n-1} \leq \left( \frac{x_1 + \dots + x_{n-1}}{n-1} \right)^{n-1}. \quad (43)$$

(b) Combine following two inequations:

$$x_1 x_2 \leq \left( \frac{x_1 + x_2}{2} \right)^2; \quad (44)$$

$$x_1 \dots x_n \leq \left( \frac{x_1 + \dots + x_n}{n} \right)^n. \quad (45)$$

implies  $P(2n)$ :

$$(x_1 \dots x_n)(x_{n+1} \dots x_{2n}) \leq \left( \frac{x_1 + \dots + x_n}{n} \right)^n \left( \frac{x_{n+1} + \dots + x_{2n}}{n} \right)^n \leq \left( \frac{x_1 + \dots + x_{2n}}{2n} \right)^{2n}. \quad (46)$$

**Homework exercises 1.10:**  $Q_n$ : move  $n-1$  from A to C, move the last one from A to B, move  $n-1$  from C to B.  $R_n$ : move  $n-1$  from B to A, move the last one from B to C, move  $n-1$  from A to B, move the last one from C to A, move the  $n-1$  from B to A.

**Homework exercises 1.11:** (a) This is similar to the single tower of hanoi, and every step in the single tower because two steps. so minimal step number is  $2T_n$ . (b) Consider the last two disks  $\alpha$  covers  $\beta$  at the begining. Move top  $2n-2$  from A to B, move  $\alpha$  from A to C, move  $2n-2$  from B to C, move  $\beta$  from A to B, move  $2n-n$  from C to A, move  $\alpha$  from C to B, move  $2n-2$  from A to B.

$$A_0 = 0; \quad (47)$$

$$A_{2n} = 4A_{2n-2} + 3. \quad \{n > 0\} \quad (48)$$

Solution is  $A_{2n} = 4^n - 1$  where  $n \geq 0$ .

**Homework exercises 1.12:**

$$A(m_1) = m_1; \quad (49)$$

$$A(m_1, \dots, m_n) = 2A(m_1, \dots, m_{n-1}) + m_n. \quad \{n > 1\} \quad (50)$$

Solution is  $A(m_1, \dots, m_n) = (m_1, \dots, m_n)_2$  where  $n \geq 1$ .

**Homework exercises 1.13:** every three lines can generate 7 planes at most, however one zig-zag line will only generate 2 planes at most. Each zig-zag line generates 5 less planes than three lines:

$$ZZ_n = L_{3n} - 5n. \quad \{n \geq 0\} \quad (51)$$

Solution is  $ZZ_n = \frac{1}{2}(9n^2 - 7n) + 1$  where  $n \geq 0$ .

**Homework exercises 1.14:** put the last plane added plane in a table, to achieve maximum space number, each other plane must have a intersection line with it. New spaces below the table correspond to the planes segmented by the intersection lines on the table.

$$P_0 = 1; \quad (52)$$

$$P_n = P_{n-1} + L_{n-1}. \quad \{n \geq 1\} \quad (53)$$

Solution is  $P_n = 1 + \sum_{i=0}^{n-1} L_i$  where  $n \geq 0$ .

**Homework exercises 1.15:** the process is un-changed, so the function is unchanged. But the initial value has changed.

$$I(2) = 2; \quad (54)$$

$$I(3) = 1; \quad (55)$$

$$I(2n) = 2I(n) - 1; \quad \{n > 2\} \quad (56)$$

$$I(2n+1) = 2I(n) + 1. \quad \{n > 2\} \quad (57)$$

Solution is  $I((b_m b_{m-1} \dots b_0)_2) = (\alpha_{(b_m b_{m-1})_2} \beta_{b_{m-2}} \beta_{b_{m-3}} \dots \beta_{b_0})$  where  $\alpha_{(10)_2} = 2$ ,  $\alpha_{(11)_2} = 1$ ,  $\beta_0 = -1$  and  $\beta_1 = 1$ .

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**Homework exercises 1.16:** given

$$g(1) = \alpha; \quad (58)$$

$$g(2n + j) = 3g(n) + \gamma n + \beta_j. \quad \{\text{for } j = 0, 1 \text{ and } n \geq 1\} \quad (59)$$

Solution should be

$$g(n) = A(n)\alpha + B(n)\beta_0 + C(n)\beta_1 + D(n)\gamma. \quad \{n \geq 1\} \quad (60)$$

When  $\gamma = 0$  there is

$$g((1b_{m-1} \dots b_0)_2) = A(n)\alpha + B(n)\beta_0 + C(n)\beta_1 = (\alpha\beta_{b_{m-1}} \dots \beta_{b_0})_3. \quad (61)$$

Use  $g(n) = n$

$$g(1) = \alpha = 1; \quad (62)$$

$$0 = n + \gamma n + \beta_0; \quad \{n \geq 1\} \quad (63)$$

$$1 = n + \gamma n + \beta_1. \quad \{n \geq 1\} \quad (64)$$

We have  $\gamma = -1$ ,  $\beta_0 = 0$  and  $\beta_1 = 1$ , which means

$$n = A(n) + C(n) - D(n). \quad \{n \geq 1\} \quad (65)$$

$A(n)$  and  $C(n)$  is required to solve  $D(n)$ .

$\alpha = 1$  and  $\beta_0 = \beta_1 = \gamma = 0$  can solve  $A(n) = g(1b_{m-1} \dots b_0)_2) = 3^m$ .

$\beta_1 = 1$  and  $\alpha = \beta_0 = \gamma = 0$  can solve  $D(n) = g(1b_{m-1} \dots b_0)_2) = (b_{m-1} \dots b_0)_3$ .

Use function 61, 65 with  $A(n)$  and  $B(n)$ :

$$g((1b_{m-1} \dots b_0)_2) = (\alpha\beta_{b_{m-1}} \dots \beta_{b_0})_3 + \gamma(3^m + (b_{m-1} \dots b_0)_3 - n) \quad (66)$$

$$= (\alpha\beta_{b_{m-1}} \dots \beta_{b_0})_3 + \gamma((1b_{m-1} \dots b_0)_3 - n). \quad (67)$$

where  $n = (1b_{m-1} \dots b_0)_2$ .

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**Exam problems 1.17:** Set  $g(n) = \frac{1}{2}n(n+1)$  where  $n \geq 0$ :

$$W_{g(n)} \leq T_n + 2W_{g(n-1)} \quad (68)$$

$$\leq T_n + 2(T_{n-1} + W_{g(n-2)}) \quad (69)$$

$$\dots \quad (70)$$

$$\leq T_n + 2^1 T_{n-1} + \dots + 2^{n-2} T_2 + 2^{n-1} W_{g(1)} \quad (71)$$

$$\leq (n-1)2^n - (2^{n-1} - 1) + 2^{n-1} \quad (72)$$

$$\leq (n-1)2^n + 1. \quad (73)$$

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**Exam problems 1.18:** If there are  $n$  zag lines, two rays of the  $i$ th ( $1 \leq i \leq n$ ) line is:

$$y = -\frac{1}{n^i}(x - n^{2i}); \quad \{x \leq n^{2i}\} \quad (74)$$

$$y = -\frac{1}{n^i + n^{-n}}(x - n^{2i}). \quad \{x \leq n^{2i}\} \quad (75)$$

To prove the state, following states should be proved: A. Each ray of the  $i$ th zag line should have  $2(n-1)$  intersection points with other rays. B. All intersection points are different.

In the case  $1 \leq i < j \leq n$ :

$$-\frac{1}{n^i} < -\frac{1}{n^i + n^{-n}} < -\frac{1}{n^j} < -\frac{1}{n^j + n^{-n}} < 0 \quad (76)$$

State A can be proved by the induction method.

There are four kinds of intersections, given  $(1 \leq i < j \leq n)$

$$y = -\frac{1}{n^i}(x - n^{2i}); \quad \{x \leq n^{2i}\} \quad (77)$$

$$y = -\frac{1}{n^j}(x - n^{2j}); \quad \{x \leq n^{2j}\} \quad (78)$$

Solution is  $x_0 = -n^{i+j}$  and  $y_0 = n^i + n^j$ .

$$y = -\frac{1}{n^i + n^{-n}}(x - n^{2i}). \quad \{x \leq n^{2i}\} \quad (79)$$

$$y = -\frac{1}{n^j + n^{-n}}(x - n^{2j}). \quad \{x \leq n^{2j}\} \quad (80)$$

Solution is  $x_1 = -n^{i+j} - n^{-n}(n^i + n^j)$  and  $y_1 = n^i + n^j$ .

$$y = -\frac{1}{n^i}(x - n^{2i}); \quad \{x \leq n^{2i}\} \quad (81)$$

$$y = -\frac{1}{n^j + n^{-n}}(x - n^{2j}). \quad \{x \leq n^{2j}\} \quad (82)$$

Solution is  $x_2 = \frac{n^{2i+j} + n^{2i-n} - n^{i+2j}}{n^j + n^{-n} - n^i}$  and  $y_2 = \frac{n^{2j} - n^{2i}}{n^j + n^{-n} - n^i}$ .

$$y = -\frac{1}{n^i + n^{-n}}(x - n^{2i}). \quad \{x \leq n^{2i}\} \quad (83)$$

$$y = -\frac{1}{n^j}(x - n^{2j}); \quad \{x \leq n^{2j}\} \quad (84)$$

Solution is  $x_3 = \frac{n^{i+2j} + n^{2j-n} - n^{2i+j}}{n^i + n^{-n} - n^j}$  and  $y_3 = \frac{n^{2i} - n^{2j}}{n^i + n^{-n} - n^j}$ .

$x_0 \neq x_1$  for any  $i$  and  $j$  because  $n^i + n^j \neq 0$ .

$y_0 \neq y_2$  for any  $i$  and  $j$  because  $y_0 = \frac{n^{2i} - n^{2j}}{n^i - n^j}$  and  $n^i - n^j \neq n^i - n^j - n^{-n}$ .

$y_0 \neq y_3$  for any  $i$  and  $j$  because  $y_0 = \frac{n^{2i} - n^{2j}}{n^i - n^j}$  and  $n^i - n^j \neq n^i - n^j + n^{-n}$ .

$y_2 \neq y_3$  for any  $i$  and  $j$  because  $n^i - n^j + n^n \neq n^i - n^j + n^{-n}$ . Then the state is proved.

**Exam problems 1.19:**  $Z_n$  means any two rays have a intersection points, and this means  $n \leq 11$ .

**Exam problems 1.20:** given

$$h(1) = \alpha; \quad (85)$$

$$h(2n + j) = 4g(n) + \gamma_j n + \beta_j. \quad \{\text{for } j = 0, 1 \text{ and } n \geq 1\} \quad (86)$$

Solution should be

$$h(n) = A(n)\alpha + B(n)\beta_0 + C(n)\beta_1 + D(n)\gamma_0 + E(n)\gamma_1. \quad \{n \geq 1\} \quad (87)$$

when  $\gamma_0 = \gamma_1 = 0$  there is

$$h((1b_{m-1}...b_0)_2) = A(n)\alpha + B(n)\beta_0 + C(n)\beta_1 = (\alpha\beta_{b_{m-1}}...\beta_{b_0})_4. \quad (88)$$

Use  $h(n) = n$

$$h(1) = \alpha = 1; \quad (89)$$

$$2n + 0 = 4n + \gamma_0 n + \beta_0; \quad (90)$$

$$2n + 1 = 4n + \gamma_1 n + \beta_1. \quad (91)$$

We have  $\alpha = 1$ ,  $\gamma_0 = \gamma_1 = -2$ ,  $\beta_0 = 0$  and  $\beta_1 = 1$  which means:

$$n = A(n) + C(n) - 2D(n) - 2E(n). \quad (92)$$

Use  $h(n) = n^2$

$$h(1) = \alpha = 1; \quad (93)$$

$$4n^2 = 4n^2 + \gamma_0 n + \beta_0; \quad (94)$$

$$4n^2 + 4n + 1 = 4n^2 + \gamma_1 n + \beta_1. \quad (95)$$

We have  $\alpha = 1$ ,  $\gamma_0 = \beta_0 = 0$ ,  $\gamma_1 = 4$  and  $\beta_1 = 1$  which means:

$$n^2 = A(n) + C(n) + 4E(n). \quad (96)$$

$A(n)$  and  $C(n)$  is required to solve  $D(n) = (3A(n) + 3C(n) - n^2 - 2n)/4$  and  $E(n) = (n^2 - A(n) - C(n))/4$ .

$\alpha = 1$  and  $\beta_0 = \beta_1 = \gamma_0 = \gamma_1 = 0$  can solve  $A((1b_{m-1}...b_0)_2) = 4^m$ .

$\beta_1 = 1$  and  $\beta_0 = \alpha = \gamma_0 = \gamma_1 = 0$  can solve  $C((1b_{m-1}...b_0)_2) = (1b_{m-1}...b_0)_4$ .

Use function  $A(n)$ ,  $C(n)$ , (96) and (92):

$$g((1b_{m-1}...b_0)_2) = (\alpha\beta_{b_{m-1}}...\beta_{b_0})_4 \quad (97)$$

$$+ \gamma_0(3 * (2b_{m-1}...b_0)_4 - n^2 - 2n)/4 \quad (98)$$

$$+ \gamma_1(n^2 - (2b_{m-1}...b_0)_4)/4. \quad (99)$$

**Exam problems 1.21:** excute last people every time can excute bad peole firstly. Then m could be any common multiple of  $n+1$ ,  $n+2$ , ...,  $n+n$ .

**Bonus problems 1.22:** can use a De Bruijn cycle which is a De Bruijn sequence of  $B(2, n)$ . The cycle is similar to a regular polygon and each edge is labeled as 0 or 1. Each edge labeled as 1 becomes a curve. Rotate and copy this shape  $n - 1$  times and consider these  $n$  shapes. There are  $2^n - 2$  small spaces between edges and curves, because there is a  $0...0$  edge and a  $1...1$  curve. Add the space inside and the space outside, there are  $2^n$  spaces.

The algorithm for generating Eulerian Path can help to generate a De Bruijn sequence.

**Bonus problems 1.23:** case 1: given  $p$  where  $1 \leq n-p < j \leq n/2$ , and one way to save himself is to remove people in the order of  $1, 2, \dots, n-p$  then  $j+1, j+2, \dots, n$  then  $n-p+1, n-p+2, \dots, j-1$ . This order means remove the first people in the first  $n-p$  moves. At this moment, the first people is  $n-p+1$ , and remove the  $j+1-(n-p)$ -th people whose id is  $j+1$ . At last remove the first people in rest moves.

For the first  $n-p$  people and the last  $p-1$  people,  $q \equiv 1 \pmod{\text{lcm}(n, n-1, \dots, 1)/p}$ . To jump from  $n-p$  to  $j+1-(n-p)$  when there are  $p$  people,  $q \equiv j+1-n \pmod{p}$ . According to the Chinese remainder theorem if  $p$  is a prime there is a solution for  $q$ . According to the Bertrand's postulate there always exists at least one prime between  $n/2$  and  $n$ .

case 2: given  $p$  where  $1 \leq j < n/2$ , and one way to save himself is to remove people in the order of  $n, n-1, \dots, p+1$  then  $j+1, j+2, \dots, p$  then  $1, 2, \dots, j-1$ . This order means remove the last people in the first  $n-p$  moves. At this moment, the first people is  $p$ , and remove the  $j+1$ -th people whose id is  $j+1$ . At last remove the first people in rest moves.

For the first  $n-p$  people and the last  $p-1$  people,  $q \equiv 0 \pmod{\text{lcm}(n, n-1, \dots, 1)/p}$ . To jump from  $p$  to  $j+1$  when there are  $p$  people,  $q \equiv j+1 \pmod{p}$ . According to the Chinese remainder theorem if  $p$  is a prime there is a solution for  $q$ . According to the Bertrand's postulate there always exists at least one prime between  $n/2$  and  $n$ .

So he can always save himself.