1 Sums

1.1 NOTATION

$a_1, ..., a_n$ could be presented as:

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} a_{k+1} = \sum_{1 \le k \le n} a_k = \sum_{1 \le k+1 \le n} a_{k+1}. \tag{1}$$

Indicator is also can be used:

$$\sum_{k=1}^{n} a_k = \sum_{k} a_k [i \le k \le n]. \tag{2}$$

The indicator could be more powerful than others:

$$\sum_{p} [p \le N]/p. \tag{3}$$

p could be 0 and the term $[0 \le N]/0$ is 0.

1.2 SUMS AND RECURRENCES

1.2.1 Simple Cases

One way to solve $S_n = \sum_{k=0}^{n} a_k$ is to convert the problem into a recurrence problem:

$$S_0 = a_0; \tag{4}$$

$$S_n = S_{n-1} + a_n. \qquad \{n > 0\} \tag{5}$$

Conversely, some recurrences can be reduced to sums:

$$T_0 = 0; \tag{6}$$

$$T_n = 2T_{n-1} + 1. \qquad \{n > 0\} \tag{7}$$

Set $S_n = \frac{T_n}{2n}$ it turns:

$$S_0 = 0; \tag{8}$$

$$S_n = S_{n-1} + 2^{-n}. \qquad \{n > 0\} \tag{9}$$

which means $S_n = \sum_{k=1}^{n} 2^{-k}$.

1.2.2 A General Case

A more general case is

$$a_n T_n = b_n T_{n-1} + c_n. \tag{10}$$

and use a s_n where $s_n b_n = s_{n-1} a_{n-1}$:

$$s_n a_n T_n = s_n b_n T_{n-1} + s_n c_n. \tag{11}$$

Let $S_n = s_n a_n T_n$ there is

$$S_n = S_{n-1} + s_n c_n. \tag{12}$$

and

$$S_n = s_0 a_0 T_0 + \sum_{i=1}^{n} s_i c_i = s_1 b_1 T_0 + \sum_{i=1}^{n} s_i c_i. \tag{13}$$

Hence T_n is solved:

$$T_n = \frac{1}{s_n a_n}(s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k). \tag{14}$$

The choice of s_n is:

$$s_n = \frac{a_1 ... a_{n-1}}{b_2 ... b_n}. \tag{15}$$

1.2.3 A Quick Sort Case

$$C_0 = 0; \tag{16}$$

$$C_n = n + 1 + \frac{2}{n} \sum_{k=0}^{n-1} C_k. \qquad \{n > 0\} \tag{17}$$

The solution is to multiply n on both side:

$$nC_n = n^2 + n + 2 \sum_{k=0}^{n-1} C_k. \qquad \{n > 0\} \tag{18}$$

$$(n-1)C_{n-1} = (n-1)^2 + (n-1) + 2 \sum_{k=0}^{n-2} C_k. \qquad \{n - 1 > 0\} \tag{19}$$

then

$$C_0 = 0; \tag{20}$$

$$nC_n = (n+1)C_{n-1} + 2n. \qquad \{n > 0\} \tag{21}$$

So the solution is

$$C_n = 2(n+1) \sum_{k=1}^{n} \frac{1}{k+1}. \tag{22}$$

Harmonic H_n is

$$H_n = \sum_{k=1}^{n} \frac{1}{k}. \tag{23}$$

So C_n can be presented in a shorter way:

$$C_n = 2(n+1)H_n - 2n. \tag{24}$$

1.3 MANIPULATION OF SUMS

1.3.1 Basic Rules

$$\sum_{k \in K} ca_k = c \sum_{k \in K} a_k; \qquad \text{Distributive law} \tag{25}$$

$$\sum_{k \in K} (a_k + b_k) = \sum_{k \in K} a_k + \sum_{k \in K} b_k; \qquad \text{Associative law} \tag{26}$$

$$\sum_{k \in K} a_k = \sum_{p(k) \in K} a_{p(k)}. \qquad \text{Commutative law} \tag{27}$$