2 Sums

2.1 NOTATION

$a_1 + ... + a_n$ could be presented as:

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} a_{k+1} = \sum_{1\le k\le n} a_k = \sum_{1\le k+1\le n} a_{k+1}. \tag{1}$$

Indicator is also useful.

$$\sum_{k=1}^{n} a_k = \sum_{k} a_k[1 \le k \le n]. \tag{2}$$

The indicator is **harder** than others.

$$\sum_{p} [p \le N]/p. \tag{3}$$

p could be 0 and the term $[0 \le N]/0$ is 0.

2.2 SUMS AND RECURRENCES

2.2.1 Simple Cases

$S_n = \sum_{k=0}^{n} a_k$ can be converted into a recurrence problem:

$$S_0 = a_0; \tag{4}$$
$$S_n = S_{n-1} + a_n. \qquad \{n > 0\} \tag{5}$$

Conversely, some recurrences can be reduced to sums.

$$T_0 = 0; \tag{6}$$
$$T_n = 2T_{n-1} + 1. \qquad \{n > 0\} \tag{7}$$

Let $S_n = T_n/(2n)$:

$$S_0 = 0; \tag{8}$$
$$S_n = S_{n-1} + 2^{-n}. \qquad \{n > 0\} \tag{9}$$

Then

$$S_n = \sum_{k=1}^{n} 2^{-k}. \tag{10}$$

2.2.2 A General Case

The general form is:

$$a_n T_n = b_n T_{n-1} + c_n. \tag{11}$$

Let $s_n b_n = s_{n-1} a_{n-1}$:

$$s_n a_n T_n = s_n b_n T_{n-1} + s_n c_n. \tag{12}$$

Then let $S_n = s_n a_n T_n$:

$$S_n = S_{n-1} + s_n c_n; \tag{13}$$
$$S_n = s_0 a_0 T_0 + \sum_{i=1}^{n} s_i c_i; \tag{14}$$
$$S_n = s_1 b_1 T_0 + \sum_{i=1}^{n} s_i c_i. \tag{15}$$

T_n is solved:

$$T_n = \frac{1}{s_n a_n}(s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k). \tag{16}$$

s_n is:

$$s_n = \frac{a_1 ... a_{n-1}}{b_2 ... b_n}. \tag{17}$$

2.2.3 A Quick Sort Case

$$C_0 = 0; \tag{18}$$
$$C_n = n + 1 + \frac{2}{n}\sum_{k=0}^{n-1} C_k. \qquad \{n > 0\} \tag{19}$$

Multiply n on both side:

$$nC_n = n^2 + n + 2\sum_{k=0}^{n-1} C_k. \qquad \{n > 0\} \tag{20}$$
$$(n-1)C_{n-1} = (n-1)^2 + (n-1) + 2\sum_{k=0}^{n-2} C_k. \qquad \{n - 1 > 0\} \tag{21}$$

Then

$$C_0 = 0; \tag{22}$$
$$nC_n = (n+1)C_{n-1} + 2n. \qquad \{n > 0\} \tag{23}$$

And

$$C_n = 2(n+1)\sum_{k=1}^{n} \frac{1}{k+1}. \tag{24}$$

Consider the harmonic number H_n.

$$H_n = \sum_{k=1}^{n} \frac{1}{k}. \tag{25}$$

So

$$C_n = 2(n+1)H_n - 2n. \tag{26}$$

2.3 MANIPULATION OF SUMS

2.3.1 Basic Rules

$$\sum_{k \in K} ca_k = c\sum_{k \in K} a_k; \qquad \text{(Distributive law)} \tag{27}$$
$$\sum_{k \in K} (a_k + b_k) = \sum_{k \in K} a_k + \sum_{k \in K} b_k; \qquad \text{(Associative law)} \tag{28}$$
$$\sum_{k \in K} a_k = \sum_{p(k) \in K} a_{p(k)}. \qquad \text{(Commutative law)} \tag{29}$$

where $p(k)$ is some permutation.

Rule one.

$$S_n = \sum_{0 \le k \le n} (a + bk) = \sum_{0 \le n-k \le n} (a + b(n-k)). \tag{30}$$
$$2S_n = \sum_{0 \le k \le n} (2a + bn) = (2a + bn)\sum_{0 \le k \le n} 1 = (2a + bn)(n+1). \tag{31}$$

Rule two.

$$\sum_{k\in K} a_k + \sum_{k\in K'} a_k = \sum_{k\in K\cap K'} a_k + \sum_{k\in K\cup K'} a_k. \tag{32}$$

Rule three.

$$S_n + a_{n+1} = a_0 + \sum_{0\le k\le n} a_{k+1}. \tag{33}$$

Example one.

$$S_n = \sum_{0\le k\le n} ax^k. \tag{34}$$

Use function 32.

$$S_n + ax^{n+1} = ax^0 + \sum_{0\le k\le n} ax^{k+1} = ax^0 + xS_n. \tag{35}$$

Solution is:

$$S_n = \frac{a - ax^{n+1}}{1-x}; \qquad \{1 \neq x\} \tag{36}$$

$$S_n = a(n+1). \qquad \{\text{else}\} \tag{37}$$

Example two.

$$S_n = \sum_{0\le k\le n} k2^k. \tag{38}$$

Use function 32.

$$S_n + (n+1)2^{n+1} = \sum_{0\le k\le n} (k+1)2^{k+1} \tag{39}$$

$$= \sum_{0\le k\le n} k2^{k+1} + \sum_{0\le k\le n} 2^{k+1} \tag{40}$$

$$= 2S_n + 2^{n+2} - 2. \tag{41}$$

Solution is:

$$S_n = (n-1)2^{n+1} + 2. \tag{42}$$

The general case.

$$\sum_{0\le k\le n} kx^k = \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2}. \qquad \{x \neq 1\} \tag{43}$$

2.4 MULTIPLE SUMS

Notation:

$$\sum_{1\le j,k\le 2} a_j b_k = a_1b_1 + a_1b_2 + a_2b_1 + a_2b_2. \tag{44}$$

Iverson's convention can also be applied in multiple sums.

$$\sum_{P(j,k)} a_{j,k} = \sum_{j,k} a_{j,k}[P(j,k)]. \tag{45}$$

A sum of sums.

$$\sum_j \sum_k a_{j,k}[P(j,k)] = \sum_j \Big(\sum_k a_{j,k}[P(j,k)]\Big). \tag{46}$$

A law called interchanging the order of summation.

$$\sum_j \sum_k a_{j,k}[P(j,k)] = \sum_{P(j,k)} a_{j,k} = \sum_k \sum_j a_{j,k}[P(j,k)]. \tag{47}$$

A general distributive law.

$$\sum_{\substack{j\in J\\ k\in K}} a_j b_k = \Big(\sum_{j\in J} a_j\Big)\Big(\sum_{k\in K} a_k\Big). \tag{48}$$

Another way of the interchanging the order of summation law.

$$\sum_{j\in J}\sum_{k\in K} a_{j,k} = \sum_{\substack{j\in J\\ k\in K}} a_j b_k = \sum_{k\in K}\sum_{j\in J} a_{j,k}. \tag{49}$$

When the range of an inner sum depends on the index variable of the outer sum, there is another way of the interchaning the order of summation law.

$$\sum_{j\in J}\sum_{k\in K(j)} a_{j,k} = \sum_{k\in K'}\sum_{j\in J'(k)} a_{j,k}. \tag{50}$$

where

$$[j \in J][k \in K(j)] = [k \in K'][j \in J'(k)]. \tag{51}$$

Example one.

$$[1 \le j \le n][j \le k \le n] = [1 \le j \le k \le n] = [1 \le k \le n][1 \le j \le k]. \tag{52}$$

Furthermore:

$$[1 \le j \le k \le n] + [1 \le k \le j \le n] = [1 \le k, j \le n] + [1 \le j = k \le n]. \tag{53}$$

Example two.

$$S = \sum_{1\le j<k\le n} (a_k - a_j)(b_k - b_j). \tag{54}$$

Use the identity:

$$[1 \le j < k \le n] + [1 \le k < j \le n] = [1 \le j, k \le n] - [1 \le j = k \le n] \tag{55}$$

Then

$$\begin{aligned}
2S &= \sum_{1\le j,k\le n} (a_k - a_j)(b_k - b_j) - 0 && (56)\\
&= \sum_{1\le j,k\le n} \big(a_k b_k + a_j b_j - a_k b_j - a_j b_k\big) && (57)\\
&= 2\sum_{1\le j,k\le n} a_j b_j - 2\sum_{1\le j,k\le n} a_j b_k && (58)\\
&= 2n\sum_{1\le j\le n} a_j b_j - 2\big(\sum_{1\le j\le n} a_j\big)\big(\sum_{1\le j\le n} b_j\big). && (59)\\
& && (60)
\end{aligned}$$

Solution is:

$$\sum_{1\le j<k\le n} (a_k - a_j)(b_k - b_j) = n\sum_{1\le j\le n} a_j b_j - \sum_{1\le j\le n} a_j \sum_{1\le j\le n} b_j. \tag{61}$$

This solution shows Chebyshev's monotonic inequalities:

$$\Big(\sum_{1\le j\le n} a_j\Big)\Big(\sum_{1\le j\le n} b_j\Big) \le n \sum_{1\le j\le n} a_j b_j; \qquad \{\text{if } a_1 \le ... \le a_n \text{ and } b_1 \le ... \le b_n\} \tag{62}$$

$$\Big(\sum_{1\le j\le n} a_j\Big)\Big(\sum_{1\le j\le n} b_j\Big) \ge n \sum_{1\le j\le n} a_j b_j. \qquad \{\text{if } a_1 \le ... \le a_n \text{ and } b_1 \ge ... \ge b_n\} \tag{63}$$

One interesting formula.

$$\sum_{0\le k<n} H_k = nH_n - n. \tag{64}$$

2.5 GENERAL METHODS

Different methods can be used to solve:

$$\Box_n = \sum_{0\le k\le n} k^2. \tag{65}$$

Method 0: look it up.
Method 1: Guess a solution, prove it by induction.
Method 2: Perturb the sum.

$$\sum_{0\le k\le n} k^3 + (n+1)^3 = \sum_{0\le k\le n+1} k^3 = \sum_{0\le k\le n} (k+1)^3 \tag{66}$$

$$= \sum_{0\le k\le n} (k^3 + 3k^2 + 3k + 1) \tag{67}$$

$$= \sum_{0\le k\le n} k^3 + \sum_{0\le k\le n} (3k^2 + 3k + 1); \tag{68}$$

$$(n+1)^3 = \sum_{0\le k\le n} 3k^2 + 3k + 1; \tag{69}$$

$$3\Box_n = n(n+1)(n+\frac{1}{2}). \tag{70}$$

Method 3: Build a repertoire.

$$R_0 = \alpha; \tag{71}$$

$$R_n = R_{n-1} + \beta + \gamma n + \sigma n^2; \tag{72}$$

$$R_n = A(n)\alpha + B(n)\beta + C(n)\gamma + D(n)\sigma. \tag{73}$$

Let $R_n = n^3$ there is $\alpha = 0$, $\beta = 1$, $\gamma = -3$ and $\sigma = 3$.

$$n^3 = 3D(n) - 3C(n) + B(n). \tag{74}$$

Let $R_n = \Box_n$ there is $\alpha = 0$, $\beta = 0$, $\gamma = 0$ and $\sigma = 1$.

$$D(n) = \Box_n. \tag{75}$$

Let $R_n = n$ there is $\alpha = 0$, $\beta = 1$, $\gamma = 0$ and $\sigma = 0$.

$$B(n) = n. \tag{76}$$

Let $R_n = n^2$ there is $\alpha = 0$, $\beta = -1$, $\gamma = 2$ and $\sigma = 0$.

$$C(n) = \frac{n^2+n}{2}. \tag{77}$$

Then

$$\Box_n = \frac{n^3 + 3C(n) - B(n)}{3}. \tag{78}$$

Method 4: Replace sums by integrals.

$$E_n = \Box_n - \int_0^n x^2 dx = \Box_n - \frac{1}{3}x^3 = E_{n-1} + n - \frac{1}{3}; \tag{79}$$

$$E_n = \sum_{1 \le k \le n} (k - \frac{1}{3}). \tag{80}$$

Method 5: Expand and contract.

$$\Box_n = \sum_{1 \le k \le n} k^2 \tag{81}$$

$$= \sum_{1 \le k \le n} \sum_{1 \le j \le k} k = \sum_{1 \le j \le n} \sum_{j \le k \le n} k \tag{82}$$

$$= \sum_{1 \le j \le n} \frac{n+j}{2}(n-j+1) \tag{83}$$

$$= \frac{1}{2}n(n+1)(n+\frac{1}{2}) - \frac{1}{2}\Box_n. \tag{84}$$

Method 6: Use finite calculus.
Method 7: Use generating functions.

2.6 FINITE AND INFINITE CALCULUS

Define $\triangle f(x) = f(x+1) - f(x)$, and

$$x^{\underline{m}} = x(x-1)...(x-m+1); \qquad \{m \ge 0\} \tag{85}$$

$$x^{\overline{m}} = x(x+1)...(x+m-1). \qquad \{m \ge 0\} \tag{86}$$

when m is 0:

$$x^{\underline{0}} = x^{\overline{0}} = 1. \tag{87}$$

This presentation is related to the factorial function.

$$n! = n^{\underline{n}} = 1^{\overline{n}}. \tag{88}$$

Then

$$\triangle(x^{\underline{m}}) = mx^{\underline{m-1}}. \tag{89}$$

The fundamental theorem of sum:

$$g(x) = \triangle f(x). \qquad \{\text{if and only if } \sum g(x)\delta x = f(x) + C\} \tag{90}$$

The finite sum:

$$\sum\nolimits_a^b g(x)\delta x = f(x)\Big|_a^b = f(b) - f(a) \qquad \{\text{if } g(x) = \triangle f(x)\} \tag{91}$$

$$= \sum\nolimits_{a \le j < b} g(j). \tag{92}$$

Rule one.

$$\sum\nolimits_a^b g(x)\delta x = -\sum\nolimits_b^a g(x)\delta x. \tag{93}$$

Rule two.

$$\sum\nolimits_a^b g(x)\delta x + \sum\nolimits_b^c g(x)\delta x = \sum\nolimits_a^c g(x)\delta x. \tag{94}$$

Sums of falling powers.

$$\sum_{0 \le k < n} k^{\underline{m}} = \sum\nolimits_0^n k^{\underline{m}} = \frac{k^{\underline{m+1}}}{m+1}\Big|_0^n = \frac{n^{\underline{m+1}}}{m+1}. \qquad \{\text{for } m \ne -1\} \tag{95}$$

Some examples.

$$\sum_{0\le k<n} k = \sum_{0\le k<n} k^{\underline{1}} = \frac{n^{\underline{2}}}{2}; \tag{96}$$

$$\sum_{0\le k<n} k^2 = \sum_{0\le k<n} (k^{\underline{2}} + k^{\underline{1}}) = \frac{n^{\underline{3}}}{3} + \frac{n^{\underline{2}}}{2}; \tag{97}$$

$$\sum_{0\le k<n} k^3 = \sum_{0\le k<n} (k^{\underline{3}} + 3k^{\underline{2}} + k^{\underline{1}}) = \frac{n^{\underline{4}}}{4} + n^{\underline{3}} + \frac{n^{\underline{2}}}{2}. \tag{98}$$

A negative rule.

$$x^{\underline{-m}} = \frac{1}{(x+1)...(x+m)}. \qquad \{\text{for } m > 0\} \tag{99}$$

Another rule:

$$x^{\underline{m+n}} = x^{\underline{m}}(x-m)^{\underline{n}}. \tag{100}$$

A complete description of the sums of falling powers.

$$\sum\nolimits_a^b x^{\underline{m}}\delta x = \begin{cases} \frac{k^{\underline{m+1}}}{m+1}\Big|_a^b; & \{\text{for } m \neq -1\} \\ H_x\Big|_a^b. & \{\text{for } m = -1\} \end{cases} \tag{101}$$

Corresponding to $D(e^x) = e^x$:

$$\triangle 2^x = 2^{x+1} - 2^x = 2^x. \tag{102}$$

One summary:

$f = \sum g$	$\triangle f = g$	$f = \sum g$	$\triangle f = g$
$x^{\underline{0}} = 1$	0	2^x	2^x
$x^{\underline{1}} = x$	1	c^x	$(c-1)c^x$
$x^{\underline{2}} = x(x-1)$	$2x$	$c^x/(c-1)$	c^x
$x^{\underline{m}}$	$mx^{\underline{m-1}}$	cf	$c \triangle f$
$x^{\underline{m+1}}/(m+1)$	$x^{\underline{m}}$	$f+g$	$\triangle f + \triangle g$
H_x	$x^{\underline{-1}} = 1/(x+1)$	fg	$f \triangle g + g \triangle f$

$\triangle(u(x)v(x))$ does not have a nice form:

$$\begin{aligned} \triangle(u(x)v(x)) &= u(x+1)v(x+1) - u(x)v(x) \qquad (103)\\ &= u(x+1)v(x+1) - u(x)v(x+1) + u(x)v(x+1) - u(x)v(x) \qquad (104)\\ &= u(x) \triangle v(x) + v(x+1) \triangle u(x). \qquad (105) \end{aligned}$$

Define

$$Ef(x) = f(x+1). \tag{106}$$

There is

$$\triangle(uv) = u \triangle v + Ev \triangle u. \tag{107}$$

and

$$\sum u \triangle v = uv - \sum Ev \triangle u. \tag{108}$$

Example one.

$$\sum x2^x\delta x = x2^x - \sum 2^{x+1}\delta x = x2^x - 2^{x+1} + C. \tag{109}$$

Example two.

$$\sum xH_x\delta x = \sum H_x \delta \frac{1}{2}x^{\underline{2}} \tag{110}$$
$$= \frac{x^{\underline{2}}}{2}H_x - \sum \frac{1}{2}(x+1)^{\underline{2}}\delta H_x \tag{111}$$
$$= \frac{x^{\underline{2}}}{2}H_x - \sum \frac{1}{2}(x+1)^{\underline{2}}x^{\underline{-1}}\delta x \tag{112}$$
$$= \frac{x^{\underline{2}}}{2}H_x - \sum \frac{1}{2}x^{\underline{1}}\delta x \tag{113}$$
$$= \frac{x^{\underline{2}}}{2}H_x - \frac{1}{4}x^{\underline{2}} + C. \tag{114}$$

2.7 INFINITE SUMS

Let $x = x^+ - x^-$ where $x^+ = x[x > 0]$ and $x^- = -x[x < 0]$. The infiite sums can be presented as:

$$\sum_{k\in K} a_k = \sum_{k\in K} a_k^+ - \sum_{k\in K} a_k^-. \tag{115}$$

Let $A^+ = \sum_{k\in K} a_k^+$ and $A^- = \sum_{k\in K} a_k^-$, $\sum_{k\in K} a_k$ is converge absolutely if both A^+ and A^- is finite, $\sum_{k\in K} a_k$ is diverge to inf or $-$inf if A^+ is infinite or A^- is infinite, else $\sum_{k\in K} a_k$ is undefined.

Many rules can be proved in converge absolutely cases.

The distributive law: if $\sum_{k\in K} a_k$ converges absolutely to A, then $\sum_{k\in K} ca_k$ converges absolutely to cA.

The associative law: if $\sum_{k\in K} a_k$ and $\sum_{k\in K} b_k$ converge absolutely to A and B, then $\sum_{k\in K}(a_b + b_k)$ converges absolutely to $A + B$.

The commutative law: absolutely convergent sums over two or more indices can always be summed first with respect to any one of those indices.

2.8 Exercises

Warmups 2.1:

$$\sum_{k=4}^{0} q_k = \sum_k q_k[4 \le k <= 0] = 0. \tag{116}$$

Warmups 2.2:

$$x([x > 0] - [x < 0]) = |x|. \tag{117}$$

Warmups 2.3:

$$\sum_{0\le k\le 5} a_k = a_0 + a_1 + a_2 + a_3 + a_4 + a_5; \tag{118}$$
$$\sum_{0\le k^2\le 5} a_{k^2} = a_0 + a_1 + a_4. \tag{119}$$

Warmups 2.4:

$$\sum_{1\le i<j<k\le 4} a_{ijk} = ((a_{123} + a_{124}) + a_{134}) + a_{234}; \qquad \{\text{case a}\} \tag{120}$$
$$= a_{123} + (a_{124} + (a_{134} + a_{234})). \qquad \{\text{case b}\} \tag{121}$$

Warmups 2.5:

$$(\sum_{j=1}^{n} a_j)(\sum_{k=1}^{n} \frac{1}{a_k}) = \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{a_j}{a_k} \tag{122}$$

$$\neq \sum_{k=1}^{n}\sum_{k=1}^{n} \frac{a_k}{a_k}. \tag{123}$$

Warmups 2.6:

$$\sum_k [1 \le j \le k \le n] = 0; \qquad \{\text{if } [1 \le j \le n]\} \tag{124}$$

$$= n - j + 1. \qquad \{\text{else}\} \tag{125}$$

Warmups 2.7:

$$\nabla(x^{\overline{m}}) = x^{\overline{m}} - (x-1)^{\overline{m}} = mx^{\overline{m-1}}. \tag{126}$$

Warmups 2.8:

$$0^{\underline{m}} = 0; \qquad \{m > 0\} \tag{127}$$

$$= \frac{1}{|m|!}. \qquad \{m \le 0\} \tag{128}$$

Warmups 2.9: The definition is:

$$x^{\overline{m}} = x(x+1)...(x+m-1); \qquad \{m > 0\} \tag{129}$$

$$= 1; \qquad \{m = 0\} \tag{130}$$

$$= \frac{1}{(x-1)...(x-m)}. \qquad \{m < 0\} \tag{131}$$

Based on the definition:

$$x^{\overline{m+n}} = x^{\overline{m}}(x+m)^{\overline{n}}. \tag{132}$$

The solution is:

$$x^{\overline{-n+n}} = 1 = x^{\overline{-n}}(x-n)^{\overline{n}}; \tag{133}$$

$$x^{\overline{-n}} = \frac{1}{(x-n)^{\overline{n}}}. \tag{134}$$

Warmups 2.10:

$$\triangle(uv) = u \triangle v + Ev \triangle u = v \triangle u + Eu \triangle v. \tag{135}$$

Basics 2.11:

$$\sum_{0 \le k < n} (a_{k+1} - a_k)b_k = \sum_{0 \le k < n} a_{k+1}b_k - \sum_{0 \le k < n} a_k b_k \tag{136}$$

$$= \sum_{0 \le k < n} a_{k+1}b_k - (a_0b_0 + \sum_{1 \le k \le n} a_k b_k - a_n b_n) \tag{137}$$

$$= a_n b_n - a_0 b_0 - (\sum_{1 \le k \le n} a_k b_k - \sum_{0 \le k < n} a_{k+1} b_k) \tag{138}$$

$$= a_n b_n - a_0 b_0 - \sum_{0 \le k < n} a_{k+1}(b_{k+1} - b_k). \tag{139}$$

Basics 2.12:

$$p(k) = k - c; \qquad \{k \text{ is odd}\} \tag{140}$$
$$= k + c. \qquad \{k \text{ is even}\} \tag{141}$$

Prove $p(k) \neq p(j)$ if $k \neq j$.

$$k - c \neq j - c; \qquad \{\text{k and j is odd}\} \tag{142}$$
$$k + c \neq j + c; \qquad \{\text{k and j is even}\} \tag{143}$$
$$k - c \neq j + c. \qquad \{\text{k is odd and j is even}\} \tag{144}$$

Prove any integer number n can be presented by $p(k)$.

$$n = k - c; \qquad \{n + c \text{ is odd}\} \tag{145}$$
$$n = k + c. \qquad \{n - c \text{ is even}\} \tag{146}$$

Basics 2.13:

$$f(0) = \alpha; \tag{147}$$
$$f(n) = f(n-1) + (-1)^n(\beta + n\gamma + n^2\delta); \tag{148}$$
$$f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma + D(n)\delta. \tag{149}$$

Try

$$f(n) = 1; \tag{150}$$
$$f(n) = (-1)^n; \tag{151}$$
$$f(n) = n(-1)^n; \tag{152}$$
$$f(n) = n^2(-1)^n. \tag{153}$$

Then

$$f(n) = D(n) = \frac{1}{2}(-1)^n(n^2 + n). \tag{154}$$

Basics 2.14:

$$\sum_{k=1}^{n} k2^k = \sum_{1 \leq j \leq k \leq n} 2^k = \sum_{1 \leq j \leq n} \sum_{j \leq k \leq n} 2^k \tag{155}$$
$$= \sum_{1 \leq j \leq n} (2^{n+1} - 2^j) \tag{156}$$
$$= (n-1)2^{n+1} + 2. \tag{157}$$

Basics 2.15:

$$S3 = \sum_{k=1}^{n} k^3; \tag{158}$$
$$S2 = \sum_{k=1}^{n} k^2; \tag{159}$$
$$S1 = \sum_{k=1}^{n} k. \tag{160}$$

Then

$$S3 + S2 = 2 \sum_{1 \leq j \leq k \leq n} jk = (\sum_{k=1}^{n} k)^2 + \sum_{k=1}^{n} k^2 = (\sum_{k=1}^{n} k)^2 + S2. \tag{161}$$

Solution is:

$$S3 = (\sum_{k=1}^{n} k)^2 = (\frac{1}{2}n(n+1))^2. \tag{162}$$

Basics 2.16:

$$x^{\underline{m+n}} = x^{\underline{m}}(x-m)^{\underline{n}} = x^{\underline{n}}(x-n)^{\underline{m}}. \tag{163}$$

Basics 2.17: Following rules and basic definitions can be used to prove states.

$$(x+m-1)^{\underline{m}}(x-1)^{\underline{-m}} = (x-1)^{\underline{0}} = 1. \tag{164}$$
$$(x-m+1)^{\overline{m}}(x+1)^{\overline{-m}} = (x+1)^{\overline{0}} = 1. \tag{165}$$

Basics 2.18: Unknown.

Homework exercises 2.19:

$$T_0 = 5; \tag{166}$$
$$2T_n = nT_{n-1} + 3n! \qquad \{n > 0\} \tag{167}$$

Let $s_n = \frac{2^{n-1}}{n!}$ and mutiply to both sides:

$$\frac{2^n T_n}{n!} = \frac{2^{n-1}T_{n-1}}{(n-1)!} + 3 * 2^{n-1}. \tag{168}$$

Then

$$S_0 = 5; \tag{169}$$
$$S_n = S_{n-1} + 3 * 2^{n-1}. \tag{170}$$

And

$$S_n = S_0 + 3(2^0 + ... + 2^{n-1}) = 5 + 3(2^n - 1) = 3 * 2^n + 2. \tag{171}$$

The solution is:

$$T_0 = 5; \tag{172}$$
$$T_n = n!(3 + 2^{1-n}). \tag{173}$$

Homework exercises 2.20:

$$\sum_{k=0}^{n} kH_k + (n+1)H_{n+1} = \sum_{k=0}^{n}(k+1)H_{k+1} \tag{174}$$
$$= \sum_{k=0}^{n} kH_k + \sum_{k=0}^{n} H_k + n. \tag{175}$$
$$\tag{176}$$

Then

$$\sum_{k=0}^{n} H_k = (n+1)H_{n+1} - n - 1. \tag{177}$$

Homework exercises 2.21: Rewrite formulas.

$$S_n = \sum_{k=0}^{n} (-1)^{n-k} = \sum_{k=0}^{n} (-1)^k; \tag{178}$$

$$T_n = \sum_{k=0}^{n} (-1)^{n-k} k = \sum_{k=0}^{n} (-1)^k (n-k); \tag{179}$$

$$U_n = \sum_{k=0}^{n} (-1)^{n-k} k^2 = \sum_{k=0}^{n} (-1)^k (n-k)^2. \tag{180}$$

Problem 1.

$$S_n + (-1)^{n+1} = 1 + \sum_{k=0}^{n} (-1)^{k+1} = 1 - \sum_{k=0}^{n} (-1)^k = 1 - S_n. \tag{181}$$

Soluiton is

$$S_n = \frac{1+(-1)^n}{2} = [\text{n is even}]. \tag{182}$$

Problem 2.

$$T_{n+1} = \sum_{k=0}^{n+1} (-1)^k (n+1-k) = \sum_{k=0}^{n} (-1)^k (n+1-k) = T_n + S_n. \tag{183}$$

Then

$$T_{n+1} = \sum_{k=0}^{n} S_k = \sum_{k=0}^{n} \frac{1+(-1)^k}{2} = \frac{n+1}{2} + \frac{S_n}{2}. \tag{184}$$

Solution is

$$T_n = \frac{n + S_{n-1}}{2} = \frac{n + [\text{n is odd}]}{2}. \tag{185}$$

Problem 3.

$$\begin{align} U_{n+1} &= \sum_{k=0}^{n+1} (-1)^k (n+1-k)^2 \tag{186}\\ &= \sum_{k=0}^{n} (-1)^k ((n-k)^2 + 2(n-k) + 1) \tag{187}\\ &= U_n + S_n + 2T_n \tag{188}\\ &= U_n + n + ([\text{n is odd}] + [\text{n is even}]) \tag{189}\\ &= U_n + n + 1. \tag{190} \end{align}$$

Then

$$U_n = \frac{n(n+1)}{2}. \tag{191}$$

Homework exercises 2.22:

$$\begin{align} \sum_{1\le j<k\le n} (a_j b_k - a_k b_j)(A_j B_k - A_k B_j) &= \frac{1}{2} \sum_{1\le j,\ k\le n} (a_j b_k - a_k b_j)(A_j B_k - A_k B_j) \tag{192}\\ &= \sum_{1\le j\le n} a_j A_j \sum_{1\le j\le n} b_j B_j - \sum_{1\le j\le n} b_j A_j \sum_{1\le j\le n} a_j B_j. \tag{193} \end{align}$$

Homework exercises 2.23: Method a.

$$\sum_{k=1}^{n} \frac{2k+1}{k(k+1)} = \sum_{k=1}^{n} (2k+1)(\frac{1}{k} - \frac{1}{k+1}) \tag{194}$$

$$= \sum_{k=1}^{n} (\frac{1}{k} + \frac{1}{k+1}) \tag{195}$$

$$= 2H_n + \frac{1}{n+1} - 1. \tag{196}$$

Method b.

$$\sum_{k=1}^{n} \frac{2k+1}{k(k+1)} = \sum\nolimits_{1}^{n+1} \frac{2x+1}{x(x+1)} \delta x \tag{197}$$

$$= \sum\nolimits_{1}^{n+1} -(2x+1)\delta(x-1)^{\underline{-1}} \tag{198}$$

$$= -(2k+1)k^{\underline{-1}}\Big|_{1}^{n+1} + \sum\nolimits_{1}^{n+1} k^{\underline{-1}} \delta 2k \tag{199}$$

$$= -1 - \frac{1}{n+1} + 2H_{n+1}. \tag{200}$$

Homework exercises 2.24:

$$\sum\nolimits_{0} n \frac{H_x}{(x+1)(x+2)} \delta x = \sum\nolimits_{0} n H_x x^{\underline{-2}} \delta x \tag{201}$$

$$= \sum\nolimits_{0} n - H_x \delta x^{\underline{-1}} \tag{202}$$

$$= -H_x x^{\underline{-1}}\Big|_{0}^{n} - \sum\nolimits_{0}^{n} -x^{\underline{-1}} \delta H_x \tag{203}$$

$$= -H_x x^{\underline{-1}}\Big|_{0}^{n+1} + \sum\nolimits_{0}^{n} x^{\underline{-2}} \delta x \tag{204}$$

$$= x^{\underline{-1}}(-1 - H_x)\Big|_{0}^{n+1} \tag{205}$$

$$= \frac{n - H_n}{n+1}. \tag{206}$$

Homework exercises 2.25:

$$\prod_{k \in K} a_k^c = (\prod_{k \in K} a_k)^c; \tag{207}$$

$$\prod_{k \in K} a_k b_k = \prod_{k \in K} a_k \prod_{k \in K} b_k; \tag{208}$$

$$\prod_{k \in K} a_k = \prod_{p(k) \in K} a_{p(k)}; \tag{209}$$

$$\prod_{j \in J, k \in K} a_{j,k} = \prod_{j \in J} \prod_{k \in K} a_{j,k}; \tag{210}$$

$$\prod_{k \in K} a_k = \prod_{k} a_k^{[k \in K]}; \tag{211}$$

$$\prod_{k \in K} c = c^{\# K}. \tag{212}$$

Homework exercises 2.26:

$$\prod_{1\le j\le k\le n} = \sqrt{\prod_{1\le j,k\le n} a_j a_k \prod_{1\le j=k\le n} a_j a_k} \tag{213}$$

$$= \sqrt{\prod_{1\le k\le n} a_k^{2n+2}} \tag{214}$$

$$= \prod_{1\le k\le n} a_k^{n+1}. \tag{215}$$

Homework exercises 2.27:

$$\triangle(c^{\underline{x}}) = c^{\underline{x+1}} - c^{\underline{x}} = c^{\underline{x}}(c-x-1) = \frac{c^{\underline{x+2}}}{c-x}. \tag{216}$$

Then

$$\sum_{k=1}^{n} \frac{(-2)^{\underline{k}}}{k} = \sum\nolimits_{1}^{n+1} \frac{(-2)^{\underline{x}}}{x}\delta x \tag{217}$$

$$= \sum\nolimits_{1}^{n+1} \delta - (-2)^{\underline{x-2}} \tag{218}$$

$$= -(-2)^{\underline{x-2}}\Big|_{1}^{n+1} \tag{219}$$

$$= -(-2)^{\underline{n-1}} + (-2)^{\underline{-1}} \tag{220}$$

$$= -(-1)^{n-1} n! - 1. \tag{221}$$

Homework exercises 2.28:

$$\sum_{k\ge 1}\sum_{j\ge 1}(\frac{k}{j}[j=k+1] - \frac{j}{k}[j=k-1]) \neq \sum_{j\ge 1}\sum_{k\ge 1}(\frac{k}{j}[j=k+1] - \frac{j}{k}[j=k-1]). \tag{222}$$

Because the function is not converge absolutely, so the exchange of the two sum cannot be applied.

Exam problems 2.29:

$$\sum_{k=1}^{n} \frac{(-1)^k k}{4k^2-1} = \sum_{k=1}^{n} \frac{(-1)^k k}{(2k+1)(2k-1)} \tag{223}$$

$$= \frac{1}{2}\sum_{k=1}^{n} \frac{(-1)^k}{4k-2} + \frac{1}{2}\sum_{k=1}^{n} \frac{(-1)^k}{4k+2} \tag{224}$$

$$= \frac{1}{2}(-\frac{1}{2} + \frac{(-1)^n}{4n+2}). \tag{225}$$

Exam problems 2.30:

$$\sum_{x=a}^{b-1} x = \sum\nolimits_{a}^{b} x\delta x = \frac{1}{2}x^{\underline{2}}\Big|_{a}^{b} = \frac{1}{2}(b-a)(b+a-1). \tag{226}$$

Then

$$(b-a)(b+a-1) = 2100 = 2^2 * 3 * 5^2 * 7 = 2^{p_2} * 3^{p_3} * 5^{p_5} * 7^{p_7}. \tag{227}$$

Beacause $(b-a)$ and $b+a-1$ is a even number and a odd number. The number of possible odd number could be

$$\prod_{k>2}(p_k+1) = (1+1)(2+1)(1+1) = 12. \tag{228}$$

Exam problems 2.31: a.

$$\sum_{k\geq 2}(\zeta(k)-1) = \sum_{k\geq 2}\sum_{j\geq 2}\frac{1}{j^k} \tag{229}$$

$$= \sum_{j\geq 2}\sum_{k\geq 2}\frac{1}{j^k} \tag{230}$$

$$= \sum_{j\geq 2}\frac{\frac{1}{j^2}}{1-\frac{1}{j}} \tag{231}$$

$$= \sum_{j\geq 2}(\frac{1}{j-1}-\frac{1}{j}) \tag{232}$$

$$= 1. \tag{233}$$

b.

$$\sum_{k\geq 1}(\zeta(2k)-1) = \sum_{k\geq 1}\sum_{j\geq 2}\frac{1}{j^{2k}} \tag{234}$$

$$= \sum_{j\geq 2}\sum_{k\geq 2}\frac{1}{j^{2k}} \tag{235}$$

$$= \sum_{j\geq 2}\frac{1}{2}(\frac{1}{j-1}-\frac{1}{j+1}) \tag{236}$$

$$= \frac{3}{4}. \tag{237}$$

Exam problems 2.32: Let S_0 be the left function and S_1 be the right function. When $2n \leq x < 2n+1$

$$S_0 = 1+2+3+...+n+(x-n-1)+...+(x-2n) \tag{238}$$

$$S_1 = (x-1)+(x-3)+...+(x-2n+1) \tag{239}$$

When $2n-1 \leq x < 2n$

$$S_0 = 0+1+2+3+...+n-1+(x-n)+...+(x-2n+1) \tag{240}$$

$$S_1 = (x-1)+(x-3)+...+(x-2n+1) \tag{241}$$

Then solution is $n(x-n)$.

Bonus problems 2.33:

$$\wedge_{k\in K} ca_k = c \wedge_{k\in K} a_k; \tag{242}$$

$$\wedge_{k\in K}(a_k+b_k) = \wedge_{k\in K} a_k + \wedge_{k\in K} b_k; \tag{243}$$

$$\wedge_{k\in K} a_k = \wedge_{p(k)\in K} a_{p(k)}; \tag{244}$$

$$\wedge_{j\in J,k\in K} a_{j,k} = \wedge_{j\in J} \wedge_{k\in K} a_{j,k}; \tag{245}$$

$$\wedge_{k\in K} a_k = \wedge_k a_k \infty^{[k\notin K]}. \tag{246}$$

Bonus problems 2.34: This problem is not perfectly solved.

If the $\sum_{k\in K} a_k$ is undefined, $\sum_{k\in K} a_k^+$ and $\sum_{k\in K} a_k^-$ are all equal to ∞. This means $\sum_{k\in K} a_k$ can be larger or smaller to any value. Then there must exist a E_1:

$$\sum_{k\in F_1} a_k = \sum_{k\in E_1} a_k \leq A^-. \tag{247}$$

For the rest F_n, let

$$F_n = F_{n-1} \cup E_n. \tag{248}$$

When n is even:

$$\sum_{k\in F_n} a_k = \sum_{k\in F_{n-1}\cup E_n} a_k = \sum_{k\in F_{n-1}} a_k + \sum_{k\in E_n} a_k - \sum_{k\in F_{n-1}\cap E_n} a_k \geq A^+. \tag{249}$$

Then there must exist a E_n that $F_{n-1}\cap E_n \neq \phi$:

$$\sum_{k\in E_n} a_k \geq A^+ - \sum_{k\in F_{n-1}} a_k + \sum_{k\in F_{n-1}\cap E_n} a_k. \tag{250}$$

When n is odd:

$$\sum_{k\in F_n} a_k = \sum_{k\in F_{n-1}\cup E_n} a_k = \sum_{k\in F_{n-1}} a_k + \sum_{k\in E_n} a_k - \sum_{k\in F_{n-1}\cap E_n} a_k \leq A^-. \tag{251}$$

Then there must exist a E_n that $F_{n-1}\cap E_n \neq \phi$:

$$\sum_{k\in E_n} a_k \leq A^- - \sum_{k\in F_{n-1}} a_k + \sum_{k\in F_{n-1}\cap E_n} a_k. \tag{252}$$

Bonus problems 2.35: Perfect power: n is a perfect power if there exist natural numbers $m > 1$, and $k > 1$ such that $m^k = n$.

Goldbach Euler theorem:

$$\sum_{k\in P} \frac{1}{k-1} = 1 \tag{253}$$

Let P is the perfect power set and T is the nopower set.

$$\begin{aligned}\sum_{k\in P} \frac{1}{k-1} &= \sum_{k\in P}(k-1)^{-1} &(254)\\ &= \sum_{i\geq 2}\sum_{a\in T}(a^i-1)^{-1} &(255)\\ &= \sum_{i\geq 2}\sum_{a\in T}\sum_{j\geq 1} a^{-ij} &(256)\\ &= \sum_{n\geq 2}\sum_{k\geq 2} n^{-k} &(257)\\ &= \sum_{n\geq 2}(n(n-1))^{-1} = 1. &(258)\end{aligned}$$

Bonus problems 2.36: a. follow the definition:

$$g(1) = 1; \tag{259}$$

$$g(n) - g(n-1) = f(n). \qquad \{n > 1\} \tag{260}$$

b. according to the definition, $f(g(k)) = n$ when $k \in (g(n-1), g(n)]$.

$$\begin{aligned}g(g(n)) - g(g(n-1)) &= \sum_{i=1}^{g}(n)f(i) - \sum_{i=1}^{g(n-1)} f(i) &(261)\\ &= \sum_i f(i)[g(n-1) < i \leq g(n)] &(262)\\ &= nf(n). &(263)\end{aligned}$$

c.

$$\begin{aligned}g(g(g(n))) - g(g(g(n-1))) &= \sum_{k=1}^{g(n)} kf(k) - \sum_{k=1}^{g(n-1)} kf(k) &(264)\\ &= \sum_k kf(k)[g(n-1) < k \leq g(n)] &(265)\\ &= n\sum_{k=g(n-1)+1}^{g(n)} k. &(266)\end{aligned}$$

Research problem 2.37: It seems that a book named ”Research Problems in Discrete Geometry” discussed this problem in chapter 3. However I cannot got a copy of the book.