1 Sums

1.1 NOTATION

$a_1 + ... + a_n$ could be presented as:

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} a_{k+1} = \sum_{1 \le k \le n} a_k = \sum_{1 \le k+1 \le n} a_{k+1}. \tag{1}$$

Indicator is also useful.

$$\sum_{k=1}^{n} a_k = \sum_{k} a_k [1 \le k \le n]. \tag{2}$$

The indicator is **harder** than others.

$$\sum_{p} [p \le N]/p. \tag{3}$$

p could be 0 and the term $[0 \le N]/0$ is 0.

1.2 SUMS AND RECURRENCES

1.2.1 Simple Cases

$S_n = \sum_{k=0}^{n} a_k$ can be converted into a recurrence problem:

$$S_0 = a_0; \tag{4}$$
$$S_n = S_{n-1} + a_n. \qquad \{n > 0\} \tag{5}$$

Conversely, some recurrences can be reduced to sums.

$$T_0 = 0; \tag{6}$$
$$T_n = 2T_{n-1} + 1. \qquad \{n > 0\} \tag{7}$$

Let $S_n = T_n/(2n)$:

$$S_0 = 0; \tag{8}$$
$$S_n = S_{n-1} + 2^{-n}. \qquad \{n > 0\} \tag{9}$$

Then

$$S_n = \sum_{k=1}^{n} 2^{-k}. \tag{10}$$

1.2.2 A General Case

The general form is:

$$a_n T_n = b_n T_{n-1} + c_n. \tag{11}$$

Let $s_n b_n = s_{n-1} a_{n-1}$:

$$s_n a_n T_n = s_n b_n T_{n-1} + s_n c_n. \tag{12}$$

Then let $S_n = s_n a_n T_n$:

$$S_n = S_{n-1} + s_n c_n; \tag{13}$$
$$S_n = s_0 a_0 T_0 + \sum_{i=1}^{n} s_i c_i; \tag{14}$$
$$S_n = s_1 b_1 T_0 + \sum_{i=1}^{n} s_i c_i. \tag{15}$$

T_n is solved:

$$T_n = \frac{1}{s_n a_n}(s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k). \tag{16}$$

s_n is:

$$s_n = \frac{a_1 ... a_{n-1}}{b_2 ... b_n}. \tag{17}$$

1.2.3 A Quick Sort Case

$$C_0 = 0; \tag{18}$$

$$C_n = n + 1 + \frac{2}{n}\sum_{k=0}^{n-1} C_k. \qquad \{n > 0\} \tag{19}$$

Multiply n on both side:

$$nC_n = n^2 + n + 2\sum_{k=0}^{n-1} C_k. \qquad \{n > 0\} \tag{20}$$

$$(n-1)C_{n-1} = (n-1)^2 + (n-1) + 2\sum_{k=0}^{n-2} C_k. \qquad \{n - 1 > 0\} \tag{21}$$

Then

$$C_0 = 0; \tag{22}$$

$$nC_n = (n+1)C_{n-1} + 2n. \qquad \{n > 0\} \tag{23}$$

And

$$C_n = 2(n+1)\sum_{k=1}^{n} \frac{1}{k+1}. \tag{24}$$

Consider the harmonic number H_n.

$$H_n = \sum_{k=1}^{n} \frac{1}{k}. \tag{25}$$

So

$$C_n = 2(n+1)H_n - 2n. \tag{26}$$

1.3 MANIPULATION OF SUMS

1.3.1 Basic Rules

$$\sum_{k \in K} ca_k = c\sum_{k \in K} a_k; \qquad \text{(Distributive law)} \tag{27}$$

$$\sum_{k \in K} (a_k + b_k) = \sum_{k \in K} a_k + \sum_{k \in K} b_k; \qquad \text{(Associative law)} \tag{28}$$

$$\sum_{k \in K} a_k = \sum_{p(k) \in K} a_{p(k)}. \qquad \text{(Commutative law)} \tag{29}$$

where $p(k)$ is some permutation.

Rule one.

$$S_n = \sum_{0 \le k \le n} (a + bk) = \sum_{0 \le n-k \le n} (a + b(n-k)). \tag{30}$$

$$2S_n = \sum_{0 \le k \le n} (2a + bn) = (2a + bn)\sum_{0 \le k \le n} 1 = (2a + bn)(n+1). \tag{31}$$

Rule two.

$$\sum_{k\in K} a_k + \sum_{k\in K'} a_k = \sum_{k\in K\cap K'} a_k + \sum_{k\in K\cup K'} a_k. \tag{32}$$

Rule three.

$$S_n + a_{n+1} = a_0 + \sum_{0\le k\le n} a_{k+1}. \tag{33}$$

Example one.

$$S_n = \sum_{0\le k\le n} ax^k. \tag{34}$$

Use function 32.

$$S_n + ax^{n+1} = ax^0 + \sum_{0\le k\le n} ax^{k+1} = ax^0 + xS_n. \tag{35}$$

Solution is:

$$S_n = \frac{a - ax^{n+1}}{1-x}; \qquad \{1 \neq x\} \tag{36}$$

$$S_n = a(n+1). \qquad \{\text{else}\} \tag{37}$$

Example two.

$$S_n = \sum_{0\le k\le n} k2^k. \tag{38}$$

Use function 32.

$$S_n + (n+1)2^{n+1} = \sum_{0\le k\le n} (k+1)2^{k+1} \tag{39}$$

$$= \sum_{0\le k\le n} k2^{k+1} + \sum_{0\le k\le n} 2^{k+1} \tag{40}$$

$$= 2S_n + 2^{n+2} - 2. \tag{41}$$

Solution is:

$$S_n = (n-1)2^{n+1} + 2. \tag{42}$$

The general case.

$$\sum_{0\le k\le n} kx^k = \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2}. \qquad \{x \neq 1\} \tag{43}$$

1.4 MULTIPLE SUMS

Notation:

$$\sum_{1\le j,k\le 2} a_j b_k = a_1b_1 + a_1b_2 + a_2b_1 + a_2b_2. \tag{44}$$

Iverson's convention can also be applied in multiple sums.

$$\sum_{P(j,k)} a_{j,k} = \sum_{j,k} a_{j,k}[P(j,k)]. \tag{45}$$

A sum of sums.

$$\sum_j \sum_k a_{j,k}[P(j,k)] = \sum_j \Big(\sum_k a_{j,k}[P(j,k)] \Big). \tag{46}$$

A law called interchanging the order of summation.

$$\sum_j \sum_k a_{j,k}[P(j,k)] = \sum_{P(j,k)} a_{j,k} = \sum_k \sum_j a_{j,k}[P(j,k)]. \tag{47}$$

A general distributive law.

$$\sum_{\substack{j\in J\\ k\in K}} a_j b_k = \Big(\sum_{j\in J} a_j\Big)\Big(\sum_{k\in K} a_k\Big). \tag{48}$$

Another way of the interchanging the order of summation law.

$$\sum_{j\in J}\sum_{k\in K} a_{j,k} = \sum_{\substack{j\in J\\ k\in K}} a_j b_k = \sum_{k\in K}\sum_{j\in J} a_{j,k}. \tag{49}$$

When the range of an inner sum depends on the index variable of the outer sum, there is another way of the interchaning the order of summation law.

$$\sum_{j\in J}\sum_{k\in K(j)} a_{j,k} = \sum_{k\in K'}\sum_{j\in J'(k)} a_{j,k}. \tag{50}$$

where

$$[j \in J][k \in K(j)] = [k \in K'][j \in J'(k)]. \tag{51}$$

Example one.

$$[1 \le j \le n][j \le k \le n] = [1 \le j \le k \le n] = [1 \le k \le n][1 \le j \le k]. \tag{52}$$

Furthermore:

$$[1 \le j \le k \le n] + [1 \le k \le j \le n] = [1 \le k, j \le n] + [1 \le j = k \le n]. \tag{53}$$

Example two.

$$S = \sum_{1\le j<k\le n} (a_k - a_j)(b_k - b_j). \tag{54}$$

Use the identity:

$$[1 \le j < k \le n] + [1 \le k < j \le n] = [1 \le j, k \le n] - [1 \le j = k \le n] \tag{55}$$

Then

$$\begin{aligned}
2S &= \sum_{1\le j,k\le n} (a_k - a_j)(b_k - b_j) - 0 && (56)\\
&= \sum_{1\le j,k\le n} \big(a_k b_k + a_j b_j - a_k b_j - a_j b_k\big) && (57)\\
&= 2\sum_{1\le j,k\le n} a_j b_j - 2\sum_{1\le j,k\le n} a_j b_k && (58)\\
&= 2n\sum_{1\le j\le n} a_j b_j - 2\Big(\sum_{1\le j\le n} a_j\Big)\Big(\sum_{1\le j\le n} b_j\Big). && (59)\\
& && (60)
\end{aligned}$$

Solution is:

$$\sum_{1\le j<k\le n} (a_k - a_j)(b_k - b_j) = n\sum_{1\le j\le n} a_j b_j - \sum_{1\le j\le n} a_j \sum_{1\le j\le n} b_j. \tag{61}$$

This solution shows Chebyshev's monotonic inequalities:

$$\Big(\sum_{1\le j\le n} a_j\Big)\Big(\sum_{1\le j\le n} b_j\Big) \le n \sum_{1\le j\le n} a_j b_j; \qquad \{\text{if } a_1 \le ... \le a_n \text{ and } b_1 \le ... \le b_n\} \tag{62}$$

$$\Big(\sum_{1\le j\le n} a_j\Big)\Big(\sum_{1\le j\le n} b_j\Big) \ge n \sum_{1\le j\le n} a_j b_j. \qquad \{\text{if } a_1 \le ... \le a_n \text{ and } b_1 \ge ... \ge b_n\} \tag{63}$$

One interesting formula.

$$\sum_{0\le k<n} H_k = nH_n - n. \tag{64}$$

1.5 GENERAL METHODS

Different methods can be used to solve:

$$\square_n = \sum_{0\le k\le n} k^2. \tag{65}$$

Method 0: look it up.
Method 1: Guess a solution, prove it by induction.
Method 2: Perturb the sum.

$$\sum_{0\le k\le n} k^3 + (n+1)^3 = \sum_{0\le k\le n+1} k^3 = \sum_{0\le k\le n} (k+1)^3 \tag{66}$$

$$= \sum_{0\le k\le n} (k^3 + 3k^2 + 3k + 1) \tag{67}$$

$$= \sum_{0\le k\le n} k^3 + \sum_{0\le k\le n} (3k^2 + 3k + 1); \tag{68}$$

$$(n+1)^3 = \sum_{0\le k\le n} 3k^2 + 3k + 1; \tag{69}$$

$$3\square_n = n(n+1)(n+\frac{1}{2}). \tag{70}$$

Method 3: Build a repertoire.

$$R_0 = \alpha; \tag{71}$$

$$R_n = R_{n-1} + \beta + \gamma n + \sigma n^2; \tag{72}$$

$$R_n = A(n)\alpha + B(n)\beta + C(n)\gamma + D(n)\sigma. \tag{73}$$

Let $R_n = n^3$ there is $\alpha = 0$, $\beta = 1$, $\gamma = -3$ and $\sigma = 3$.

$$n^3 = 3D(n) - 3C(n) + B(n). \tag{74}$$

Let $R_n = \square_n$ there is $\alpha = 0$, $\beta = 0$, $\gamma = 0$ and $\sigma = 1$.

$$D(n) = \square_n. \tag{75}$$

Let $R_n = n$ there is $\alpha = 0$, $\beta = 1$, $\gamma = 0$ and $\sigma = 0$.

$$B(n) = n. \tag{76}$$

Let $R_n = n^2$ there is $\alpha = 0$, $\beta = -1$, $\gamma = 2$ and $\sigma = 0$.

$$C(n) = \frac{n^2+n}{2}. \tag{77}$$

Then

$$\square_n = \frac{n^3 + 3C(n) - B(n)}{3}. \tag{78}$$

Method 4: Replace sums by integrals.

$$E_n = \square_n - \int_0^n x^2 dx = \square_n - \frac{1}{3}x^3 = E_{n-1} + n - \frac{1}{3}; \tag{79}$$

$$E_n = \sum_{1\le k\le n} (k - \frac{1}{3}). \tag{80}$$

Method 5: Expand and contract.

$$\square_n = \sum_{1\le k\le n} k^2 \tag{81}$$

$$= \sum_{1\le k\le n} \sum_{1\le j\le k} k = \sum_{1\le j\le n} \sum_{j\le k\le n} k \tag{82}$$

$$= \sum_{1\le j\le n} \frac{n+j}{2}(n-j+1) \tag{83}$$

$$= \frac{1}{2}n(n+1)(n+\frac{1}{2}) - \frac{1}{2}\square_n. \tag{84}$$

Method 6: Use finite calculus.
Method 7: Use generating functions.

1.6 FINITE AND INFINITE CALCULUS

Define $\triangle f(x) = f(x+1) - f(x)$, and

$$x^{\underline{m}} = x(x-1)...(x-m+1); \qquad \{m \ge 0\} \tag{85}$$

$$x^{\overline{m}} = x(x+1)...(x+m-1). \qquad \{m \ge 0\} \tag{86}$$

when m is 0:

$$x^{\underline{0}} = x^{\overline{0}} = 1. \tag{87}$$

This presentation is related to the factorial function.

$$n! = n^{\underline{n}} = 1^{\overline{n}}. \tag{88}$$

Then

$$\triangle(x^{\underline{m}}) = mx^{\underline{m-1}}. \tag{89}$$

The fundamental theorem of sum:

$$g(x) = \triangle f(x). \qquad \{\text{if and only if } \sum g(x)\delta x = f(x) + C\} \tag{90}$$

The finite sum:

$$\sum\nolimits_a^b g(x)\delta x = f(x)\Big|_a^b = f(b) - f(a) \qquad \{\text{if } g(x) = \triangle f(x)\} \tag{91}$$

$$= \sum\nolimits_{a\le j<b} g(j). \tag{92}$$

Rule one.

$$\sum\nolimits_a^b g(x)\delta x = -\sum\nolimits_b^a g(x)\delta x. \tag{93}$$

Rule two.

$$\sum\nolimits_a^b g(x)\delta x + \sum\nolimits_b^c g(x)\delta x = \sum\nolimits_a^c g(x)\delta x. \tag{94}$$

Sums of falling powers.

$$\sum_{0\le k<n} k^{\underline{m}} = \sum\nolimits_0^n k^{\underline{m}} = \frac{k^{\underline{m+1}}}{m+1}\Big|_0^n = \frac{n^{\underline{m+1}}}{m+1}. \qquad \{\text{for } m \ne -1\} \tag{95}$$

Some examples.

$$\sum_{0\le k<n} k = \sum_{0\le k<n} k^{\underline{1}} = \frac{n^{\underline{2}}}{2}; \tag{96}$$

$$\sum_{0\le k<n} k^2 = \sum_{0\le k<n} (k^{\underline{2}} + k^{\underline{1}}) = \frac{n^{\underline{3}}}{3} + \frac{n^{\underline{2}}}{2}; \tag{97}$$

$$\sum_{0\le k<n} k^3 = \sum_{0\le k<n} (k^{\underline{3}} + 3k^{\underline{2}} + k^{\underline{1}}) = \frac{n^{\underline{4}}}{4} + n^{\underline{3}} + \frac{n^{\underline{2}}}{2}. \tag{98}$$

A negative rule.

$$x^{\underline{-m}} = \frac{1}{(x+1)...(x+m)}. \qquad \{\text{for } m > 0\} \tag{99}$$

Another rule:

$$x^{\underline{m+n}} = x^{\underline{m}}(x-m)^{\underline{n}}. \tag{100}$$

A complete description of the sums of falling powers.

$$\sum\nolimits_a^b x^{\underline{m}}\delta x = \begin{cases} \frac{k^{\underline{m+1}}}{m+1}\Big|_a^b; & \{\text{for } m \neq -1\} \\ H_x\Big|_a^b. & \{\text{for } m = -1\} \end{cases} \tag{101}$$

Corresponding to $D(e^x) = e^x$:

$$\triangle 2^x = 2^{x+1} - 2^x = 2^x. \tag{102}$$

One summary:

$f = \sum g$	$\triangle f = g$	$f = \sum g$	$\triangle f = g$
$x^{\underline{0}} = 1$	0	2^x	2^x
$x^{\underline{1}} = x$	1	c^x	$(c-1)c^x$
$x^{\underline{2}} = x(x-1)$	$2x$	$c^x/(c-1)$	c^x
$x^{\underline{m}}$	$mx^{\underline{m-1}}$	cf	$c \triangle f$
$x^{\underline{m+1}}/(m+1)$	$x^{\underline{m}}$	$f+g$	$\triangle f + \triangle g$
H_x	$x^{\underline{-1}} = 1/(x+1)$	fg	$f \triangle g + g \triangle f$

$\triangle(u(x)v(x))$ does not have a nice form:

$$\begin{aligned} \triangle(u(x)v(x)) &= u(x+1)v(x+1) - u(x)v(x) && (103)\\ &= u(x+1)v(x+1) - u(x)v(x+1) + u(x)v(x+1) - u(x)v(x) && (104)\\ &= u(x) \triangle v(x) + v(x+1) \triangle u(x). && (105) \end{aligned}$$

Define

$$Ef(x) = f(x+1) \tag{106}$$

There is

$$\triangle(uv) = u \triangle v + Ev \triangle u. \tag{107}$$

and

$$\sum u \triangle v = uv - \sum Ev \triangle u. \tag{108}$$

Example one.

$$\sum x2^x\delta x = x2^x - \sum 2^{x+1}\delta x = x2^x - 2^{x+1} + C. \tag{109}$$

Example two.

$$\sum xH_x\delta x = \sum H_x \delta \frac{1}{2}x^{\underline{2}} \tag{110}$$

$$= \frac{x^{\underline{2}}}{2}H_x - \sum \frac{1}{2}(x+1)^{\underline{2}}\delta H_x \tag{111}$$

$$= \frac{x^{\underline{2}}}{2}H_x - \sum \frac{1}{2}(x+1)^{\underline{2}}x^{\underline{-1}}\delta x \tag{112}$$

$$= \frac{x^{\underline{2}}}{2}H_x - \sum \frac{1}{2}x^{\underline{1}}\delta x \tag{113}$$

$$= \frac{x^{\underline{2}}}{2}H_x - \frac{1}{4}x^{\underline{2}} + C. \tag{114}$$

1.7 INFINITE SUMS