3 Integer Functions

3.1 FLOORS AND CEILINGS

Define: $\lceil x \rceil$ is the least integer greater than or equal to x, and $\lfloor x \rfloor$ is the greatest integer less than or equal to x. Basic rules:

$$\lfloor x \rfloor \le x; \tag{1}$$
$$\lceil x \rceil \ge x. \tag{2}$$

The two functions are equal precisely at the integer points:

$$\lfloor x \rfloor = x \iff x \text{ is an integer} \iff \lceil x \rceil = x. \tag{3}$$

The two functions are inequal if not at the integer points:

$$\lceil x \rceil - \lfloor x \rfloor = [x \text{ is not an integer}]. \tag{4}$$

The two functinos can be converted:

$$\lceil -x \rceil = -\lfloor x \rfloor; \tag{5}$$
$$\lfloor -x \rfloor = -\lceil x \rceil. \tag{6}$$

Integers can be easily removed or added in the two functions:

$$\lfloor x + n \rfloor = \lfloor x \rfloor + n; \qquad \{\text{integer n}\} \tag{7}$$
$$\lceil x + n \rceil = \lceil x \rceil + n. \qquad \{\text{integer n}\} \tag{8}$$

For important rules:

$$\lfloor x \rfloor = n \iff n \le x < n + 1; \tag{9}$$
$$\lfloor x \rfloor = n \iff x - 1 < n \le x; \tag{10}$$
$$\lceil x \rceil = n \iff n - 1 < x \le n; \tag{11}$$
$$\lceil x \rceil = n \iff x \le n < x + 1. \tag{12}$$

There are many situations in which floor and ceiling brackets are redundant:

$$x < n \iff \lfloor x \rfloor < n; \tag{13}$$
$$n < x \iff n < \lceil x \rceil; \tag{14}$$
$$x \le n \iff \lceil x \rceil \le n; \tag{15}$$
$$n \le x \iff n \le \lfloor x \rfloor. \tag{16}$$

Define: $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x, then $\lfloor x \rfloor$ is the integer part of x. A simple notation is $x = n + \theta$.

$$\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor + \lfloor \{x\} + \{y\} \rfloor. \tag{17}$$

3.2 FLOOR/CEILING APPLICATIONS

Problem 1: what is the bit number to express n in binary?

$$2^{m-1} \le x < 2^m \iff \text{the bit number is } m; \tag{18}$$
$$m - 1 \le \lg x < m \tag{19}$$
$$m = \lfloor \lg x \rfloor + 1. \qquad \{x > 0\} \tag{20}$$

To support $x = 0$, another better solution is $\lceil \lg(x+1) \rceil$.
Problem 2: what is $m = \lfloor \sqrt{\lfloor x \rfloor} \rfloor$ when $x \ge 0$?

$$m \le \sqrt{\lfloor x \rfloor} < m + 1; \tag{21}$$
$$m^2 \le \lfloor x \rfloor < (m+1)^2; \tag{22}$$
$$m^2 \le x < (m+1)^2; \tag{23}$$
$$m \le \sqrt{x} < m + 1; \tag{24}$$
$$m = \lfloor \sqrt{x} \rfloor. \tag{25}$$

Problem 3: what is $m = \lceil \sqrt{\lceil x \rceil} \rceil$ when $x \geq 0$?

$$\begin{aligned}
& m - 1 < \sqrt{\lceil x \rceil} \leq m; && (26)\\
& (m-1)^2 < \lceil x \rceil \leq m^2; && (27)\\
& (m-1)^2 < x \leq m^2; && (28)\\
& m - 1 < \sqrt{x} \leq m; && (29)\\
& m = \lceil \sqrt{x} \rceil. && (30)
\end{aligned}$$

A general theorem: let $f(x)$ be any continuous, monotonically increasing function with the property that

$$f(x) = \text{integer} \implies x = \text{integer}. \tag{31}$$

Then there is:

$$\begin{aligned}
& \lfloor f(x) \rfloor = \lfloor f(\lfloor x \rfloor) \rfloor; && (32)\\
& \lceil f(x) \rceil = \lceil f(\lceil x \rceil) \rceil. && (33)
\end{aligned}$$

A special case of the theorem:

$$\begin{aligned}
& \left\lfloor \frac{x+m}{n} \right\rfloor = \left\lfloor \frac{\lfloor x \rfloor + m}{n} \right\rfloor; && (34)\\
& \left\lceil \frac{x+m}{n} \right\rceil = \left\lceil \frac{\lceil x \rceil + m}{n} \right\rceil. && (35)
\end{aligned}$$

Problem levels: **level 1** prove a given statement for a number; **level 2** prove a given statement for a set of numbers; **level 3** prove or disprove a given statement for a set of numbers; **level 4** find a necessary and suffcient condition that a statement is true; **level 5** find an interesting property given a set of numbers.
Consider the integer inside a range:

$$\begin{aligned}
& \alpha \leq n < \beta \iff \lceil \alpha \rceil \leq n < \lceil \beta \rceil; && (36)\\
& \alpha < n \leq \beta \iff \lfloor \alpha \rfloor < n \leq \lfloor \beta \rfloor. && (37)
\end{aligned}$$

Then

$$\begin{aligned}
& [\alpha, \beta) \text{ contains } \lceil \beta \rceil - \lceil \alpha \rceil \text{ elements;} && \{\alpha \leq \beta\} && (38)\\
& (\alpha, \beta] \text{ contains } \lfloor \beta \rfloor - \lfloor \alpha \rfloor \text{ elements;} && \{\alpha \leq \beta\} && (39)\\
& (\alpha, \beta) \text{ contains } \lceil \beta \rceil - \lfloor \alpha \rfloor - 1 \text{ elements;} && \{\alpha < \beta\} && (40)\\
& [\alpha, \beta] \text{ contains } \lfloor \beta \rfloor - \lceil \alpha \rceil + 1 \text{ elements.} && \{\alpha \leq \beta\} && (41)
\end{aligned}$$

Example 1:

$$\begin{aligned}
W &= \sum_{1 \leq n \leq 1000} [\lfloor \sqrt[3]{n} \rfloor \setminus n] && (42)\\
&= \sum_{k,n} [k = \lfloor \sqrt[3]{n} \rfloor][1 \leq n \leq 1000][k \setminus n] && (43)\\
&= \sum_{k,n,m} [k^3 \leq n < (k+1)^3][n = km][1 \leq n \leq 1000] && (44)\\
&= 1 + \sum_{k,m} [k^3 \leq km < (k+1)^3][1 \leq k < 10] && (45)\\
&= 1 + \sum_{k,m} [k^2 \leq m < (k+1)^3/k][1 \leq k < 10] && (46)\\
&= 1 + \sum_{1 \leq k < 10} (\lceil (k+1)^3/k \rceil - \lceil k^2 \rceil) && (47)\\
&= 1 + \sum_{1 \leq k < 10} (3k+4) = 172. && (48)
\end{aligned}$$

General case:

$$\begin{aligned}
W &= \sum_{1\le n\le N} [\lfloor \sqrt[3]{n} \rfloor \setminus n] && (49)\\
&= \sum_{k,n} [k = \lfloor \sqrt[3]{n} \rfloor][1\le n\le N][k \setminus n] && (50)\\
&= \sum_{k,n,m} [k^3 \le n < (k+1)^3][n = km][1\le n\le N] && (51)\\
&= \sum_{k,m} [k^3 \le km < (k+1)^3][1\le k< K] + \sum_{k,m} [K^3 \le Km \le N] && (52)\\
&= \sum_{k,m} [k^2 \le m < (k+1)^3/k][1\le k< K] + \sum_{k,m} [K^2 \le m \le N/K] && (53)\\
&= \sum_{1\le k<K} (3k+4) + \sum_{m} [m \in [K^2, N/K]] && (54)\\
&= (7+3K+1)(K-1)/2 + \lfloor N/K \rfloor - \lceil K^2 \rceil + 1 && (55)\\
&= \frac{1}{2}K^2 + \frac{5}{2}K - 3 + \lfloor N/K \rfloor. \qquad \{K = \lfloor \sqrt[3]{N} \rfloor\} && (56)
\end{aligned}$$

Define $Spec(\alpha) = \{\lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, ...\}$ then $Spec(\sqrt{2})$ and $Spec(\sqrt{2}+2)$ forms a partition of positive integers. Define $N(\alpha, n)$ is the number of elements in $Spec(\alpha)$ that are $\le n$.

$$\begin{aligned}
N(\alpha, n) &= \sum_{k>0} [\lfloor \alpha k \rfloor \le n] && (57)\\
&= \sum_{k>0} [\lfloor \alpha k \rfloor < n+1] && (58)\\
&= \sum_{k>0} [\alpha k < n+1] && (59)\\
&= \sum_{k>0} [0 < k < (n+1)/\alpha] && (60)\\
&= \lceil (n+1)/\alpha \rceil - 1. && (61)
\end{aligned}$$

Then $N(\sqrt{2}, n) + N(\sqrt{2}+2, n) = n$. And it is easy to prove that if $\alpha \ne \beta$ then $Spec(\alpha) \ne Spec(\beta)$.

3.3 FLOOR/CEILING RECURRENCES

Knuth numbers:

$$\begin{aligned}
K_0 &= 1; && (62)\\
K_{n+1} &= 1 + min(2K_{\lfloor n/2 \rfloor}, 3K_{\lfloor n/3 \rfloor}). && (63)
\end{aligned}$$

The Josephus problem:

$$\begin{aligned}
J(1) &= 1; && (64)\\
J(n) &= 2J(\lfloor n/2 \rfloor) - (-1)^n. && (65)
\end{aligned}$$

Consider the more authentic Josephus problem in which every third person is eliminated:

$$J_3(n) = \left\lceil \frac{3}{2} J_3(\left\lfloor \frac{2}{3} n \right\rfloor) + a_n \right\rceil \mod (n+1). \tag{66}$$

where $a_n = -2, 1, -\frac{1}{2}$ when $n \mod 3 = 0, 1, 2$.
Another method: Whenever a person is passed over, we can assign a new number.

1	2	3	4	5	6	7	8	9	10
11	12		13	14		15	16		17
18			19	20			21		22
			23	24					25
			26						27
			28						
			29						
			30						

Any number $3k+1$ has a next value $10+3k+1-k$, and $3k+2$ has a next value $10+3k+2-k$. More general, there are n people at first and some person has a current number N. For this person his last number should be $3k+1$ or $3k+2$, and this current number is $N = n+2k+1$ or $N = n+2k+2$. This means

$$k = \left\lfloor \frac{N-n-1}{2} \right\rfloor. \tag{67}$$

And his last number can be converted into

$$3k + (N-n-2k) = k + N - n = \left\lfloor \frac{N-n-1}{2} \right\rfloor + N - n. \tag{68}$$

For the last one to be terminated, his number should be $3n$. Use the method we can always find his last number until the number is smaller than n which is his initial number.

```python
def J3(n):
    N = 3 * n
    while N > n:
        N = int((N - n - 1)/2) + N - n
    return N
```

Listing 1: Method 0

Let $D = 3n+1-N$, then $D = 1$ when $N = 3n$ and $D > 2n+1$ when $N < n$. D can also be updated as N:

$$
\begin{aligned}
D &= 3n+1-N && (69)\\
&= 3n+1-(\left\lfloor \frac{(3n+1-D)-n-1}{2} \right\rfloor + (3n+1-D)-n) && (70)\\
&= n+D-\left\lfloor \frac{2n-D}{2} \right\rfloor && (71)\\
&= D-\left\lfloor \frac{-D}{2} \right\rfloor && (72)\\
&= D+\left\lceil \frac{D}{2} \right\rceil && (73)\\
&= \left\lceil \frac{3D}{2} \right\rceil. && (74)
\end{aligned}
$$

```python
import math
def J3(n):
    D = 1
    while D <= 2*n:
        D = math.ceil(D*3/2)
    return 3*n + 1 - D
```

Listing 2: Method 1

More general:

```python
import math
def J(n,q):
    D = 1
    while D <= (q-1)*n:
        D = math.ceil(D*q/(q-1))
    return q*n + 1 - D
```

Listing 3: Method 2

Write it into a recurrence:

$$D_0^{(q)} = 1; \tag{75}$$

$$D_n^{(q)} = \left\lceil \frac{q}{q-1} D_{n-1}^{(q)} \right\rceil. \tag{76}$$

3.4 'MOD': THE BINARY OPERATION

Define operator 'mod':

$$x \text{ mod } y = x - y\lfloor x/y \rfloor. \qquad \{y \neq 0\} \tag{77}$$

Based on the defination, there are some attributes:

$$0 \leq x \text{ mod } y < y; \qquad \{y > 0\} \tag{78}$$

$$0 \geq x \text{ mod } y > y. \qquad \{y < 0\} \tag{79}$$

To complete the defination, we can let x mod $y = x$ when $y = 0$.
The 'mod' operator can be used to show the fractional part of a number:

$$x = \lfloor x \rfloor + x \text{ mod } 1. \tag{80}$$

A similar 'mumble' operator can be defined:

$$x \text{ mumble } y = y\lceil x/y \rceil - x. \qquad \{y \neq 0\} \tag{81}$$

The 'mod' operator follows the distributive law:

$$c(x \text{ mod } y) = (cx) \text{ mod } (cy). \tag{82}$$

Problem: how to partition n things into m groups as equally as possible?
There will be n mod m groups contains $\lceil n/m \rceil$ things and the rest contains $\lfloor n/m \rfloor$ things. It also can be converted into:

$$n = \left\lceil \frac{n}{m} \right\rceil + \left\lceil \frac{n-1}{m} \right\rceil + ... + \left\lceil \frac{n-m+1}{m} \right\rceil. \tag{83}$$

and if change n to $km + n$ mod m, the equation can be converted into:

$$n = \left\lfloor \frac{n}{m} \right\rfloor + \left\lfloor \frac{n+1}{m} \right\rfloor + ... + \left\lfloor \frac{n+m-1}{m} \right\rfloor. \tag{84}$$

If $n = \lfloor mx \rfloor$:

$$\lfloor mx \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{m} \right\rfloor + ... + \left\lfloor x + \frac{m-1}{m} \right\rfloor. \tag{85}$$

3.5 FLOOR/CEILING SUMS

Example 1, let $a = \lfloor\sqrt{n}\rfloor$:

$$\sum_{0\le k<n} \lfloor\sqrt{k}\rfloor = \sum_{k,m\ge 0} m[m = \lfloor\sqrt{k}\rfloor][k<n] \tag{86}$$
$$= \sum_{k,m\ge 0} m[k<n][m \le \sqrt{k} < m+1] \tag{87}$$
$$= \sum_{k,m\ge 0} m[k<n][m^2 \le k < (m+1)^2 \tag{88}$$
$$= \sum_{k,m\ge 0} m[m^2 \le k < (m+1)^2 \le n] + \sum_{k,m\ge 0} m[m^2 \le k < n < (m+1)^2] \tag{89}$$
$$= \sum_{m\ge 0} m((m+1)^2 - m^2)[m+1 \le a] + \sum_{m\ge 0} m(a^2 \le k < n) \tag{90}$$
$$= \sum_{m\ge 0} (2m^2+m)[m+1\le a] + a(n-a^2) \tag{91}$$
$$= \sum_{m\ge 0} (2m^{\underline{2}} + 3m^{\underline{1}})[m<a] - a^3 + an \tag{92}$$
$$= \sum\nolimits_0^a (2m^{\underline{2}} + 3m^{\underline{1}})\delta m - a^3 + an \tag{93}$$
$$= \frac{2}{3}m^{\underline{3}} + \frac{3}{2}m^{\underline{2}} - a^3 + an \tag{94}$$
$$= na - \frac{1}{3}a^3 - \frac{1}{2}a^2 - \frac{1}{6}a. \tag{95}$$

Anothe method is le $\lfloor x\rfloor = \sum_j [1 \le j \le x]$:

$$\sum_{0\le k<n} \lfloor\sqrt{k}\rfloor = \sum_{j,k} [1 \le j \le \sqrt{k}][0 \le k \le a^2] \tag{96}$$
$$= \sum_{1\le j<a} \sum_k [j^2 \le k < a^2] \tag{97}$$
$$= \sum_{1\le j<a} (a^2 - j^2) \tag{98}$$
$$= na - \frac{1}{3}a^3 - \frac{1}{2}a^2 - \frac{1}{6}a. \tag{99}$$

Equidistribution theorem:

$$\lim_{x\to\infty} \frac{1}{n} \sum_{0\le k<n} f(\{k\alpha\}) = \int_0^1 f(x)dx. \tag{100}$$

for all irrational α and all functions f that are continuous almost everywhere.
Example 2, let $d = gcd(m,n)$:

$$\sum_{0\le k<m} \left\lfloor \frac{nk+x}{m} \right\rfloor = d\left\lfloor \frac{x}{d} \right\rfloor + \frac{m-1}{2}n + \frac{d-m}{2} \tag{101}$$
$$= d\left\lfloor \frac{x}{d} \right\rfloor + \frac{mn}{2} - \frac{n}{2} - \frac{m}{2} + \frac{d}{2} \tag{102}$$
$$= \sum_{0\le k<n} \left\lfloor \frac{mk+x}{n} \right\rfloor. \tag{103}$$

3.6 Exercises

Warmups 3.1: Let

$$n = 2^m + l. \qquad \{0 \le l < 2^m\} \tag{104}$$

Then:

$$m = \lceil \lg{(n+1)} \rceil; \tag{105}$$
$$l = n - 2^{\lceil \lg{(n+1)} \rceil}. \tag{106}$$

Warmups 3.2:

$$round_{down}(x) = \lfloor x + 0.5 \rfloor; \tag{107}$$
$$round_{up}(x) = \lceil x - 0.5 \rceil. \tag{108}$$

Warmups 3.3:

$$\left\lfloor \frac{\lfloor m\alpha \rfloor n}{\alpha} \right\rfloor = \left\lfloor \frac{m\alpha n - \{m\alpha\} n}{\alpha} \right\rfloor = mn - \left\lceil \frac{\{m\alpha\} n}{\alpha} \right\rceil = mn - 1. \tag{109}$$

Warmups 3.4: Pass.

Warmups 3.5:

$$\lfloor n\lfloor x \rfloor + n\{x\} \rfloor = n\lfloor x \rfloor \iff \lfloor n\{x\} \rfloor = 0 \iff 0 \le n\{x\} < 1 \iff \{x\} < \frac{1}{n}. \tag{110}$$

Warmups 3.6:

$$\begin{aligned} \lfloor f(\lceil x \rceil) \rfloor = A &\iff A \le f(\lceil x \rceil) < A + 1 && (111) \\ &\iff f^{-1}(A) \ge \lceil x \rceil > f^{-1}(A+1) && (112) \\ &\iff f^{-1}(A) \ge x > f^{-1}(A+1) && (113) \\ &\iff \lfloor f(\lceil x \rceil) \rfloor = \lfloor f(x) \rfloor. && (114) \end{aligned}$$

Warmups 3.7:

$$X_n = n; \qquad \{0 \le n < m\} \tag{115}$$
$$X_n = X_{n-m} + 1. \qquad \{n \ge m\} \tag{116}$$

Solution:

$$X_n = \left\lfloor \frac{n}{m} \right\rfloor + n \bmod m. \tag{117}$$

Warmups 3.8: If m boxes contains $< \lceil n/m \rceil$ elements:

$$n \le m(\lceil n/m \rceil - 1) \iff n/m + 1 \le \lceil n/m \rceil. \tag{118}$$

If m boxes contains $> \lfloor n/m \rfloor$ elements:

$$n \ge m(\lfloor n/m \rfloor + 1) \iff n/m - 1 \ge \lfloor n/m \rfloor. \tag{119}$$

These two statements contradict function 9 and 11.

Warmups 3.9: Because

$$\frac{m}{n} - \frac{1}{q} = \frac{m\lceil \frac{n}{m} \rceil - n}{nq} = \frac{n \text{ mumble } m}{nq}. \tag{120}$$

Then n mumble m is smaller than m and nq is larger than q. This means 1) it is possible to splite a fractional number into a number series; 2) The number series $1/x_1, 1/x_2, ...$ has distinct numbers; 3) This is a finite number series.

Basics 3.10:

$$\left\lceil \frac{2x+1}{2} \right\rceil - \left\lceil \frac{2x+1}{4} \right\rceil + \left\lfloor \frac{2x+1}{4} \right\rfloor = \lceil x+0.5 \rceil - [x \neq 2k-0.5]. \qquad \{k \text{ is a integer}\} \tag{121}$$

This means:

$$\left\lceil \frac{2x+1}{2} \right\rceil - \left\lceil \frac{2x+1}{4} \right\rceil + \left\lfloor \frac{2x+1}{4} \right\rfloor = \lceil x \rceil; \qquad \{x = 2k-0.5 \text{ or } \{x\} > 0.5\} \tag{122}$$

$$= \lfloor x \rfloor. \qquad \{\text{else}\} \tag{123}$$

Basics 3.11:

$$\alpha < n < \beta \iff \lfloor \alpha \rfloor < n < \lceil \beta \rceil. \tag{124}$$

The number of possible n is

$$(\lceil \beta \rceil - \lfloor \alpha \rfloor - 1)[\lceil \beta \rceil > \lfloor \alpha \rfloor]. \tag{125}$$

If $\alpha = \beta =$ integer then $\lceil \beta \rceil = \lfloor \alpha \rfloor$, this reults a wrong answer.

Basics 3.12: Prove

$$\left\lceil \frac{n}{m} \right\rceil = \left\lfloor \frac{n+m-1}{m} \right\rfloor \iff \left\lceil \frac{km+n \bmod m}{m} \right\rceil = \left\lfloor \frac{km+n \bmod m+m-1}{m} \right\rfloor \tag{126}$$

$$\iff k + \left\lceil \frac{n \bmod m}{m} \right\rceil = k+1+ \left\lfloor \frac{n \bmod m - 1}{m} \right\rfloor \tag{127}$$

$$\tag{128}$$

If $n \bmod m = 0$

$$\left\lceil \frac{n}{m} \right\rceil = \left\lfloor \frac{n+m-1}{m} \right\rfloor \iff k = k+1-1. \tag{129}$$

Else

$$\left\lceil \frac{n}{m} \right\rceil = \left\lfloor \frac{n+m-1}{m} \right\rfloor \iff k+1 = k+1-0. \tag{130}$$

Basics 3.13: Because

$$N(\alpha, n) + N(\beta, n) = N(\frac{\beta}{\beta-1}, n) + N(\beta, n) \tag{131}$$

$$= n+1+ \left\lceil \frac{n+1}{\beta} \right\rceil - \left\lfloor \frac{n+1}{\beta} \right\rfloor - 2 \tag{132}$$

$$= n. \tag{133}$$

And it is easy to prove that if $\alpha \neq \beta$ then $Spec(\alpha) \neq Spec(\beta)$.

Basics 3.14:

$$(x \bmod ny) \bmod y = x - \left\lfloor \frac{x}{ny} \right\rfloor ny - \left\lfloor \frac{x - \left\lfloor \frac{x}{ny} \right\rfloor ny}{y} \right\rfloor y \tag{134}$$

$$= x - y(\left\lfloor \frac{x}{ny} \right\rfloor n - \left\lfloor \frac{x}{y} + \left\lfloor \frac{x}{ny} \right\rfloor n \right\rfloor) \tag{135}$$

$$= x - \left\lfloor \frac{x}{y} \right\rfloor y = x \bmod y. \tag{136}$$

Basics 3.15:

$$\lceil mx \rceil = \lceil x \rceil + \left\lceil x - \frac{1}{m} \right\rceil + ... + \left\lceil x - \frac{m-1}{m} \right\rceil. \tag{137}$$

Basics 3.16: Prove

$$n \bmod 2 = (1 - (-1)^n)/2. \tag{138}$$

It is true in both even and odd cases.
Solve

$$n \bmod 3 = a + bw^n + cw^{2n}. \qquad \{w = (-1 + i\sqrt{3})/2\} \tag{139}$$

Try $n = 0, 1, 2$:

$$a + b + c = 0; \tag{140}$$

$$a + bw + cw^2 = a + bw + c(-1 - w) = 1; \tag{141}$$

$$a + bw^2 + cw^4 = a + b(-1 - w) + cw = 2. \tag{142}$$

Solution is $a = 1$, $b = (-1 - w)/(1 + 2w) = (w - 1)/3$, $c = -1 - b = -(w + 2)/3$.
To prove this is easy, because $w^3 = 1$.

Basics 3.17:

$$\sum_j \sum_k [0 \le k < m][1 \le j \le x + k/m] \tag{143}$$

$$= \sum_j \sum_k [0 \le k < m][1 \le j \le \lceil x \rceil][j \le x + k/m] \tag{144}$$

$$= \sum_j \sum_k [0 \le k < m][1 \le j \le \lceil x \rceil][k \ge m(j - x)] \tag{145}$$

$$= \sum_{1 \le j \le \lceil x \rceil} \sum_k [0 \le k < m] - \sum_{j = \lceil x \rceil} \sum_k [0 \le k < m(j - x)] \tag{146}$$

$$= m\lceil x \rceil - \lceil m(\lceil x \rceil - x) \rceil \tag{147}$$

$$= \lfloor mx \rfloor. \tag{148}$$

Basics 3.18: pass.

Homework exercises 3.19: To let $f(x) = \lfloor log_b(x) \rfloor = \lfloor log_b(\lfloor x \rfloor) \rfloor$:

$$f(x) = \text{integer} \implies x = \text{integer}. \tag{149}$$

Then b should be an integer.

Homework exercises 3.20:

$$\sum_k [\alpha \le kx \le \beta] xk = x \sum_k [\alpha/x \le k \le \beta/x] k \tag{150}$$

$$= x \sum_k [\lceil \alpha/x \rceil \le k \le \lfloor \beta/x \rfloor] k \tag{151}$$

$$= \frac{1}{2} x (\lfloor \beta/x \rfloor \lfloor \beta/x \rfloor + \lfloor \beta/x \rfloor - \lceil \alpha/x \rceil \lceil \alpha/x \rceil + \lceil \alpha/x \rceil). \tag{152}$$

Homework exercises 3.21: Small cases $1, 16, 128, 1024, ...$ show that for a number which has k digits in decimal notation, there always exactly one number 2^m has leading 1. This is true because for any number n:

$$log_{10}(n) < \lfloor log_{10}(n) + 1 \rfloor \iff 2n < n + 10^{\lfloor log_{10}(n) + 1 \rfloor}. \tag{153}$$

Then solution is $\lfloor M log_{10} 2 \rfloor + 1$.

Homework exercises 3.22:

$$S_n = \sum_{k\geq 1} \left\lfloor \frac{n}{2^k} + \frac{1}{2} \right\rfloor. \tag{154}$$

Small cases show that $S_n = n$. Let $n = m_k 2^k + d$

$$\begin{aligned} \left\lfloor \frac{n}{2^k} + \frac{1}{2} \right\rfloor \neq \left\lfloor \frac{n-1}{2^k} + \frac{1}{2} \right\rfloor \iff \qquad & \qquad (155) \\ m_k + \left\lfloor \frac{d}{2^k} + \frac{1}{2} \right\rfloor \neq m_k + \left\lfloor \frac{d-1}{2^k} + \frac{1}{2} \right\rfloor \iff & \qquad (156) \\ d = 2^{k-1}. & \qquad (157) \end{aligned}$$

Consider S_n and S_{n-1}, there always one and only one k to make $n \bmod (2^k) = 2^{k-1}$. Let $n = 2^{k'} q$ where q is a odd, here k' is the one. Then $S_n = S_{n-1} + 1 = n$.

$$T_n = \sum_{k\geq 1} 2^k \left\lfloor \frac{n}{2^k} + \frac{1}{2} \right\rfloor^2. \tag{158}$$

From the observation above, let $n = 2^{k'} q$ where q is a odd. The different terms between T_n and T_{n-1} is $\left\lfloor \frac{2^{k'} q}{2^{k'+1}} + \frac{1}{2} \right\rfloor$ and $\left\lfloor \frac{2^{k'} q - 1}{2^{k'+1}} + \frac{1}{2} \right\rfloor$. Then:

$$\begin{aligned} T_n - T_{n-1} &= 2^{k'+1}\left(\left\lfloor \frac{2^{k'} q}{2^{k'+1}} + \frac{1}{2} \right\rfloor^2 - \left\lfloor \frac{2^{k'} q - 1}{2^{k'+1}} + \frac{1}{2} \right\rfloor^2 \right) & (159) \\ &= 2^{k'+1}\left(\left\lfloor \frac{q}{2} + \frac{1}{2} \right\rfloor^2 - \left\lfloor \frac{q}{2} - \frac{1}{2^{k'+1}} + \frac{1}{2} \right\rfloor^2 \right) & (160) \\ &= 2^{k'+1}\left(\left\lfloor \frac{q}{2} + \frac{1}{2} \right\rfloor^2 - \left\lfloor \frac{q}{2} \right\rfloor^2 \right) & (161) \\ &= 2^{k'+1}\left(\left\lfloor \frac{q}{2} + \frac{1}{2} \right\rfloor - \left\lfloor \frac{q}{2} \right\rfloor \right)\left(\left\lfloor \frac{q}{2} + \frac{1}{2} \right\rfloor + \left\lfloor \frac{q}{2} \right\rfloor \right) & (162) \\ &= 2^{k'+1} q = 2n. & (163) \end{aligned}$$

Then $T_n = n(n+1)$.

Homework exercises 3.23: The nth elements should be $X_n = m$

$$\begin{aligned} \frac{1}{2} m(m-1) < n \leq \frac{1}{2} m(m+1) \iff \qquad & \qquad (164) \\ m(m-1) < 2n \leq m(m+1) \iff & \qquad (165) \\ m(m-1) + \frac{1}{4} < 2n < m(m+1) + \frac{1}{4} \iff & \qquad (166) \\ m - \frac{1}{2} < \sqrt{2n} < m + \frac{1}{2} & \qquad (167) \\ m < \sqrt{2n} + \frac{1}{2} & \qquad (168) \\ m = \left\lfloor \sqrt{2n} + \frac{1}{2} \right\rfloor. & \qquad (169) \end{aligned}$$

Homework exercises 3.24: $Spec(\alpha/(\alpha+1)) = Spec(\alpha) + n + 1$.

Homework exercises 3.25: Can prove $K_n > n$ with induction.

The initial cases are true:

$$K_0 = 1; \tag{170}$$
$$K_1 = 3. \tag{171}$$

Assume that $K_i > i$ when $i \leq n$, then:

$$\begin{aligned}
K_{n+1} &= 1 + min(2K_{\lfloor n/2 \rfloor}, 3K_{\lfloor n/3 \rfloor}) && (172)\\
&\geq 1 + min(2(\lfloor n/2 \rfloor + 1), 3(\lfloor n/3 \rfloor + 1)) && (173)\\
&\geq 1 + min(n+1, n+1) && (174)\\
&> n+1. && (175)
\end{aligned}$$

Proved.

Homework exercises 3.26: For the left side:

$$\begin{aligned}
D_n^{(q)} = \left\lceil \frac{q}{q-1} D_{n-1}^{(q)} \right\rceil \iff && (176)\\
\frac{q}{q-1} D_{n-1}^{(q)} \leq D_n^{(q)} && (177)\\
(\frac{q}{q-1})^n D_0^{(q)} \leq D_n^{(q)} && (178)\\
(\frac{q}{q-1})^n \leq D_n^{(q)} && (179)
\end{aligned}$$

For the right side, can prove with induction:

$$D_n^{(q)} \leq q(\frac{q}{q-1})^n - q + 1. \quad \{\text{this is wired but can be guessed from the initial case.}\} \tag{180}$$

The initial case is true:

$$D_0^{(q)} = 1 \geq q - q + 1. \tag{181}$$

Assume that it is true when $i \leq n$, then:

$$\begin{aligned}
D_{n+1}^{(q)} &= \left\lceil \frac{q}{q-1} D_{n-1}^{(q)} \right\rceil && (182)\\
&\leq \left\lceil q(\frac{q}{q-1})^{n+1} - q \right\rceil && (183)\\
&\leq q(\frac{q}{q-1})^{n+1} - q + 1. && (184)
\end{aligned}$$

Proved.

Homework exercises 3.27: Let $D_n^{(3)} = 2^m b - a$ where $a = 0$ or 1 and b is odd, then:

$$D_{n+1}^{(3)} = \left\lceil \frac{3}{2}(2^m b - a) \right\rceil = 3 * 2^{m-1} b - \left\lfloor \frac{3}{2} a \right\rfloor = 3 * 2^{m-1} b - a; \tag{185}$$
$$D_{m+n}^{(3)} = 3^m b - a. \tag{186}$$

This formula shows another way to generate the next number in the $D_n^{(3)}$. This way does not generate the next number one by one which means the formula only generates a subset.
Let the seed number is $D_0^{(3)} = 2^1 - 1 = 1$, then $a = 1$. The numbers generated from the seed contians infinite numbers and these numbers are arranged in the order: odd, even, odd, even, ...

Homework exercises 3.28: Not solved yet.
From an initial number $a_n = m^2$, all numbers can be generated:

$$a_{n+2k+1} = (m+k)^2 + m - k; \qquad \{0 \leq k \leq m\} \tag{187}$$
$$a_{n+2k+2} = (m+k)^2 + 2m; \qquad \{0 \leq k \leq m\} \tag{188}$$
$$a_{n+2m+1} = 4m^2. \tag{189}$$

Homework exercises 3.29: pass.

Homework exercises 3.30: Small cases show that

$$X_n = \alpha^{2^n} + \alpha^{-2^n}. \tag{190}$$

It is easy to prove this with the induction.
$\alpha^{-2^n} < 1$ because $\alpha > 1$ which means the integer X_n should be $\lceil \alpha^{2^n} \rceil$.

Homework exercises 3.31: A really smart solution:

$$\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor = \lfloor x + \lfloor y \rfloor \rfloor + \lfloor x + y \rfloor \tag{191}$$

$$\le \left\lfloor x + \frac{1}{2}\lfloor 2y \rfloor \right\rfloor + \left\lfloor x + \frac{1}{2}\lfloor 2y \rfloor + \frac{1}{2} \right\rfloor \tag{192}$$

$$= \lfloor 2x + \lfloor 2y \rfloor \rfloor = \lfloor 2x \rfloor + \lfloor 2y \rfloor. \tag{193}$$

Homework exercises 3.32: