

2 Sums

2.1 NOTATION

$a_1 + \dots + a_n$ could be presented as:

$$\sum_{k=1}^n a_k = \sum_{k=0}^{n-1} a_{k+1} = \sum_{1 \leq k \leq n} a_k = \sum_{1 \leq k+1 \leq n} a_{k+1}. \quad (1)$$

Indicator is also useful.

$$\sum_{k=1}^n a_k = \sum_k a_k [1 \leq k \leq n]. \quad (2)$$

The indicator is **harder** than others.

$$\sum_p [p \leq N]/p. \quad (3)$$

p could be 0 and the term $[0 \leq N]/0$ is 0.

2.2 SUMS AND RECURRENCES

2.2.1 Simple Cases

$S_n = \sum_{k=0}^n a_k$ can be converted into a recurrence problem:

$$S_0 = a_0; \quad (4)$$

$$S_n = S_{n-1} + a_n. \quad \{n > 0\} \quad (5)$$

Conversely, some recurrences can be reduced to sums.

$$T_0 = 0; \quad (6)$$

$$T_n = 2T_{n-1} + 1. \quad \{n > 0\} \quad (7)$$

Let $S_n = T_n/(2n)$:

$$S_0 = 0; \quad (8)$$

$$S_n = S_{n-1} + 2^{-n}. \quad \{n > 0\} \quad (9)$$

Then

$$S_n = \sum_{k=1}^n 2^{-k}. \quad (10)$$

2.2.2 A General Case

The general form is:

$$a_n T_n = b_n T_{n-1} + c_n. \quad (11)$$

Let $s_n b_n = s_{n-1} a_{n-1}$:

$$s_n a_n T_n = s_n b_n T_{n-1} + s_n c_n. \quad (12)$$

Then let $S_n = s_n a_n T_n$:

$$S_n = S_{n-1} + s_n c_n; \quad (13)$$

$$S_n = s_0 a_0 T_0 + \sum_{i=1}^n s_i c_i; \quad (14)$$

$$S_n = s_1 b_1 T_0 + \sum_{i=1}^n s_i c_i. \quad (15)$$

T_n is solved:

$$T_n = \frac{1}{s_n a_n} (s_1 b_1 T_0 + \sum_{k=1}^n s_k c_k). \quad (16)$$

s_n is:

$$s_n = \frac{a_1 \dots a_{n-1}}{b_2 \dots b_n}. \quad (17)$$

2.2.3 A Quick Sort Case

$$C_0 = 0; \quad (18)$$

$$C_n = n + 1 + \frac{2}{n} \sum_{k=0}^{n-1} C_k. \quad \{n > 0\} \quad (19)$$

Multiply n on both side:

$$nC_n = n^2 + n + 2 \sum_{k=0}^{n-1} C_k. \quad \{n > 0\} \quad (20)$$

$$(n-1)C_{n-1} = (n-1)^2 + (n-1) + 2 \sum_{k=0}^{n-2} C_k. \quad \{n-1 > 0\} \quad (21)$$

Then

$$C_0 = 0; \quad (22)$$

$$nC_n = (n+1)C_{n-1} + 2n. \quad \{n > 0\} \quad (23)$$

And

$$C_n = 2(n+1) \sum_{k=1}^n \frac{1}{k+1}. \quad (24)$$

Consider the harmonic number H_n .

$$H_n = \sum_{k=1}^n \frac{1}{k}. \quad (25)$$

So

$$C_n = 2(n+1)H_n - 2n. \quad (26)$$

2.3 MANIPULATION OF SUMS

2.3.1 Basic Rules

$$\sum_{k \in K} ca_k = c \sum_{k \in K} a_k; \quad (\text{Distributive law}) \quad (27)$$

$$\sum_{k \in K} (a_k + b_k) = \sum_{k \in K} a_k + \sum_{k \in K} b_k; \quad (\text{Associative law}) \quad (28)$$

$$\sum_{k \in K} a_k = \sum_{p(k) \in K} a_{p(k)}. \quad (\text{Commutative law}) \quad (29)$$

where $p(k)$ is some permutation.

Rule one.

$$S_n = \sum_{0 \leq k \leq n} (a + bk) = \sum_{0 \leq n-k \leq n} (a + b(n-k)). \quad (30)$$

$$2S_n = \sum_{0 \leq k \leq n} (2a + bn) = (2a + bn) \sum_{0 \leq k \leq n} 1 = (2a + bn)(n+1). \quad (31)$$

Rule two.

$$\sum_{k \in K} a_k + \sum_{k \in K'} a_k = \sum_{k \in K \cap K'} a_k + \sum_{k \in K \cup K'} a_k. \quad (32)$$

Rule three.

$$S_n + a_{n+1} = a_0 + \sum_{0 \leq k \leq n} a_{k+1}. \quad (33)$$

Example one.

$$S_n = \sum_{0 \leq k \leq n} ax^k. \quad (34)$$

Use function 32.

$$S_n + ax^{n+1} = ax^0 + \sum_{0 \leq k \leq n} ax^{k+1} = ax^0 + xS_n. \quad (35)$$

Solution is:

$$S_n = \frac{a - ax^{n+1}}{1 - x}; \quad \{1 \neq x\} \quad (36)$$

$$S_n = a(n+1). \quad \{\text{else}\} \quad (37)$$

Example two.

$$S_n = \sum_{0 \leq k \leq n} k2^k. \quad (38)$$

Use function 32.

$$S_n + (n+1)2^{n+1} = \sum_{0 \leq k \leq n} (k+1)2^{k+1} \quad (39)$$

$$= \sum_{0 \leq k \leq n} k2^{k+1} + \sum_{0 \leq k \leq n} 2^{k+1} \quad (40)$$

$$= 2S_n + 2^{n+2} - 2. \quad (41)$$

Solution is:

$$S_n = (n-1)2^{n+1} + 2. \quad (42)$$

The general case.

$$\sum_{0 \leq k \leq n} kx^k = \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2}. \quad \{x \neq 1\} \quad (43)$$

2.4 MULTIPLE SUMS

Notation:

$$\sum_{1 \leq j, k \leq 2} a_j b_k = a_1 b_1 + a_1 b_2 + a_2 b_1 + a_2 b_2. \quad (44)$$

Iverson's convention can also be applied in multiple sums.

$$\sum_{P(j,k)} a_{j,k} = \sum_{j,k} a_{j,k} [P(j,k)]. \quad (45)$$

A sum of sums.

$$\sum_j \sum_k a_{j,k} [P(j,k)] = \sum_j \left(\sum_k a_{j,k} [P(j,k)] \right). \quad (46)$$

A law called interchanging the order of summation.

$$\sum_j \sum_k a_{j,k} [P(j, k)] = \sum_{P(j, k)} a_{j,k} = \sum_k \sum_j a_{j,k} [P(j, k)]. \quad (47)$$

A general distributive law.

$$\sum_{\substack{j \in J \\ k \in K}} a_j b_k = \left(\sum_{j \in J} a_j \right) \left(\sum_{k \in K} a_k \right). \quad (48)$$

Another way of the interchanging the order of summation law.

$$\sum_{j \in J} \sum_{k \in K} a_{j,k} = \sum_{\substack{j \in J \\ k \in K}} a_j b_k = \sum_{k \in K} \sum_{j \in J} a_{j,k}. \quad (49)$$

When the range of an inner sum depends on the index variable of the outer sum, there is another way of the interchanging the order of summation law.

$$\sum_{j \in J} \sum_{k \in K(j)} a_{j,k} = \sum_{k \in K'} \sum_{j \in J'(k)} a_{j,k}. \quad (50)$$

where

$$[j \in J][k \in K(j)] = [k \in K'][j \in J'(k)]. \quad (51)$$

Example one.

$$[1 \leq j \leq n][j \leq k \leq n] = [1 \leq j \leq k \leq n] = [1 \leq k \leq n][1 \leq j \leq k]. \quad (52)$$

Furthermore:

$$[1 \leq j \leq k \leq n] + [1 \leq k \leq j \leq n] = [1 \leq k, j \leq n] + [1 \leq j = k \leq n]. \quad (53)$$

Example two.

$$S = \sum_{1 \leq j < k \leq n} (a_k - a_j)(b_k - b_j). \quad (54)$$

Use the identity:

$$[1 \leq j < k \leq n] + [1 \leq k < j \leq n] = [1 \leq j, k \leq n] - [1 \leq j = k \leq n] \quad (55)$$

Then

$$2S = \sum_{1 \leq j, k \leq n} (a_k - a_j)(b_k - b_j) - 0 \quad (56)$$

$$= \sum_{1 \leq j, k \leq n} (a_k b_k + a_j b_j - a_k b_j - a_j b_k) \quad (57)$$

$$= 2 \sum_{1 \leq j, k \leq n} a_j b_j - 2 \sum_{1 \leq j, k \leq n} a_j b_k \quad (58)$$

$$= 2n \sum_{1 \leq j \leq n} a_j b_j - 2 \left(\sum_{1 \leq j \leq n} a_j \right) \left(\sum_{1 \leq j \leq n} b_j \right). \quad (59)$$

$$(60)$$

Solution is:

$$\sum_{1 \leq j < k \leq n} (a_k - a_j)(b_k - b_j) = n \sum_{1 \leq j \leq n} a_j b_j - \sum_{1 \leq j \leq n} a_j \sum_{1 \leq j \leq n} b_j. \quad (61)$$

This solution shows Chebyshev's monotonic inequalities:

$$\left(\sum_{1 \leq j \leq n} a_j\right) \left(\sum_{1 \leq j \leq n} b_j\right) \leq n \sum_{1 \leq j \leq n} a_j b_j; \quad \{\text{if } a_1 \leq \dots \leq a_n \text{ and } b_1 \leq \dots \leq b_n\} \quad (62)$$

$$\left(\sum_{1 \leq j \leq n} a_j\right) \left(\sum_{1 \leq j \leq n} b_j\right) \geq n \sum_{1 \leq j \leq n} a_j b_j. \quad \{\text{if } a_1 \leq \dots \leq a_n \text{ and } b_1 \geq \dots \geq b_n\} \quad (63)$$

One interesting formula.

$$\sum_{0 \leq k < n} H_k = nH_n - n. \quad (64)$$

2.5 GENERAL METHODS

Different methods can be used to solve:

$$\square_n = \sum_{0 \leq k \leq n} k^2. \quad (65)$$

Method 0: look it up.

Method 1: Guess a solution, prove it by induction.

Method 2: Perturb the sum.

$$\sum_{0 \leq k \leq n} k^3 + (n+1)^3 = \sum_{0 \leq k \leq n+1} k^3 = \sum_{0 \leq k \leq n} (k+1)^3 \quad (66)$$

$$= \sum_{0 \leq k \leq n} (k^3 + 3k^2 + 3k + 1) \quad (67)$$

$$= \sum_{0 \leq k \leq n} k^3 + \sum_{0 \leq k \leq n} (3k^2 + 3k + 1); \quad (68)$$

$$(n+1)^3 = \sum_{0 \leq k \leq n} 3k^2 + 3k + 1; \quad (69)$$

$$3\square_n = n(n+1)\left(n + \frac{1}{2}\right). \quad (70)$$

Method 3: Build a repertoire.

$$R_0 = \alpha; \quad (71)$$

$$R_n = R_{n-1} + \beta + \gamma n + \sigma n^2; \quad (72)$$

$$R_n = A(n)\alpha + B(n)\beta + C(n)\gamma + D(n)\sigma. \quad (73)$$

Let $R_n = n^3$ there is $\alpha = 0$, $\beta = 1$, $\gamma = -3$ and $\sigma = 3$.

$$n^3 = 3D(n) - 3C(n) + B(n). \quad (74)$$

Let $R_n = \square_n$ there is $\alpha = 0$, $\beta = 0$, $\gamma = 0$ and $\sigma = 1$.

$$D(n) = \square_n. \quad (75)$$

Let $R_n = n$ there is $\alpha = 0$, $\beta = 1$, $\gamma = 0$ and $\sigma = 0$.

$$B(n) = n. \quad (76)$$

Let $R_n = n^2$ there is $\alpha = 0$, $\beta = -1$, $\gamma = 2$ and $\sigma = 0$.

$$C(n) = \frac{n^2 + n}{2}. \quad (77)$$

Then

$$\square_n = \frac{n^3 + 3C(n) - B(n)}{3}. \quad (78)$$

Method 4: Replace sums by integrals.

$$E_n = \square_n - \int_0^n x^2 dx = \square_n - \frac{1}{3}x^3 = E_{n-1} + n - \frac{1}{3}; \quad (79)$$

$$E_n = \sum_{1 \leq k \leq n} (k - \frac{1}{3}). \quad (80)$$

Method 5: Expand and contract.

$$\square_n = \sum_{1 \leq k \leq n} k^2 \quad (81)$$

$$= \sum_{1 \leq k \leq n} \sum_{1 \leq j \leq k} k = \sum_{1 \leq j \leq n} \sum_{j \leq k \leq n} k \quad (82)$$

$$= \sum_{1 \leq j \leq n} \frac{n+j}{2} (n-j+1) \quad (83)$$

$$= \frac{1}{2}n(n+1)(n+\frac{1}{2}) - \frac{1}{2}\square_n. \quad (84)$$

Method 6: Use finite calculus.

Method 7: Use generating functions.

2.6 FINITE AND INFINITE CALCULUS

Define $\triangle f(x) = f(x+1) - f(x)$, and

$$x^{\overline{m}} = x(x-1)\dots(x-m+1); \quad \{m \geq 0\} \quad (85)$$

$$x^{\overline{m}} = x(x+1)\dots(x+m-1). \quad \{m \geq 0\} \quad (86)$$

when m is 0:

$$x^{\overline{0}} = x^{\overline{0}} = 1. \quad (87)$$

This presentation is related to the factorial function.

$$n! = n^{\overline{n}} = 1^{\overline{n}}. \quad (88)$$

Then

$$\triangle(x^{\overline{m}}) = mx^{\overline{m-1}}. \quad (89)$$

The fundamental theorem of sum:

$$g(x) = \triangle f(x). \quad \{\text{if and only if } \sum g(x)\delta x = f(x) + C\} \quad (90)$$

The finite sum:

$$\sum_a^b g(x)\delta x = f(x)|_a^b = f(b) - f(a) \quad \{\text{if } g(x) = \triangle f(x)\} \quad (91)$$

$$= \sum_{a \leq j < b} g(j). \quad (92)$$

Rule one.

$$\sum_a^b g(x)\delta x = -\sum_b^a g(x)\delta x. \quad (93)$$

Rule two.

$$\sum_a^b g(x)\delta x + \sum_b^c g(x)\delta x = \sum_a^c g(x)\delta x. \quad (94)$$

Sums of falling powers.

$$\sum_{0 \leq k < n} k^{\overline{m}} = \sum_0^n k^{\overline{m}} = \frac{k^{\overline{m+1}}}{m+1} \Big|_0^n = \frac{n^{\overline{m+1}}}{m+1}. \quad \{\text{for } m \neq -1\} \quad (95)$$

Some examples.

$$\sum_{0 \leq k < n} k = \sum_{0 \leq k < n} k^{\underline{1}} = \frac{n^{\underline{2}}}{2}; \quad (96)$$

$$\sum_{0 \leq k < n} k^2 = \sum_{0 \leq k < n} (k^{\underline{2}} + k^{\underline{1}}) = \frac{n^{\underline{3}}}{3} + \frac{n^{\underline{2}}}{2}; \quad (97)$$

$$\sum_{0 \leq k < n} k^3 = \sum_{0 \leq k < n} (k^{\underline{3}} + 3k^{\underline{2}} + k^{\underline{1}}) = \frac{n^{\underline{4}}}{4} + n^{\underline{3}} + \frac{n^{\underline{2}}}{2}. \quad (98)$$

A negative rule.

$$x^{-m} = \frac{1}{(x+1)\dots(x+m)}. \quad \{\text{for } m > 0\} \quad (99)$$

Another rule:

$$x^{m+n} = x^m(x-m)^{\underline{n}}. \quad (100)$$

A complete description of the sums of falling powers.

$$\sum_a^b x^m \delta x = \begin{cases} \left. \frac{k^{\underline{m+1}}}{m+1} \right|_a^b; & \{\text{for } m \neq -1\} \\ H_x \Big|_a^b. & \{\text{for } m = -1\} \end{cases} \quad (101)$$

Corresponding to $D(e^x) = e^x$:

$$\Delta 2^x = 2^{x+1} - 2^x = 2^x. \quad (102)$$

One summary:

$f = \sum g$	$\Delta f = g$	$f = \sum g$	$\Delta f = g$
$x^0 = 1$	0	2^x	2^x
$x^{\underline{1}} = x$	1	c^x	$(c-1)c^x$
$x^{\underline{2}} = x(x-1)$	$2x$	$c^x/(c-1)$	c^x
$x^{\underline{m}}$	$mx^{\underline{m-1}}$	cf	$c \Delta f$
$x^{\underline{m+1}}/(m+1)$	$x^{\underline{m}}$	$f+g$	$\Delta f + \Delta g$
H_x	$x^{\underline{-1}} = 1/(x+1)$	fg	$f \Delta g + g \Delta f$

$\Delta(u(x)v(x))$ does not have a nice form:

$$\Delta(u(x)v(x)) = u(x+1)v(x+1) - u(x)v(x) \quad (103)$$

$$= u(x+1)v(x+1) - u(x)v(x+1) + u(x)v(x+1) - u(x)v(x) \quad (104)$$

$$= u(x) \Delta v(x) + v(x+1) \Delta u(x). \quad (105)$$

Define

$$Ef(x) = f(x+1). \quad (106)$$

There is

$$\Delta(uv) = u \Delta v + Ev \Delta u. \quad (107)$$

and

$$\sum u \Delta v = uv - \sum Ev \Delta u. \quad (108)$$

Example one.

$$\sum x 2^x \delta x = x 2^x - \sum 2^{x+1} \delta x = x 2^x - 2^{x+1} + C. \quad (109)$$

Example two.

$$\sum x H_x \delta x = \sum H_x \delta \frac{1}{2} x^2 \quad (110)$$

$$= \frac{x^2}{2} H_x - \sum \frac{1}{2} (x+1)^2 \delta H_x \quad (111)$$

$$= \frac{x^2}{2} H_x - \sum \frac{1}{2} (x+1)^2 x^{-1} \delta x \quad (112)$$

$$= \frac{x^2}{2} H_x - \sum \frac{1}{2} x^1 \delta x \quad (113)$$

$$= \frac{x^2}{2} H_x - \frac{1}{4} x^2 + C. \quad (114)$$

2.7 INFINITE SUMS

Let $x = x^+ - x^-$ where $x^+ = x[x > 0]$ and $x^- = -x[x < 0]$. The infinite sums can be presented as:

$$\sum_{k \in K} a_k = \sum_{k \in K} a_k^+ - \sum_{k \in K} a_k^-. \quad (115)$$

Let $A^+ = \sum_{k \in K} a_k^+$ and $A^- = \sum_{k \in K} a_k^-$, $\sum_{k \in K} a_k$ is convergent absolutely if both A^+ and A^- are finite, $\sum_{k \in K} a_k$ is divergent to ∞ or $-\infty$ if A^+ is infinite or A^- is infinite, else $\sum_{k \in K} a_k$ is undefined.

Many rules can be proved in convergent absolute cases.

The distributive law: if $\sum_{k \in K} a_k$ converges absolutely to A , then $\sum_{k \in K} c a_k$ converges absolutely to cA .

The associative law: if $\sum_{k \in K} a_k$ and $\sum_{k \in K} b_k$ converge absolutely to A and B , then $\sum_{k \in K} (a_k + b_k)$ converges absolutely to $A + B$.

The commutative law: absolutely convergent sums over two or more indices can always be summed first with respect to any one of those indices.

2.8 Exercises

Warmups 2.1:

$$\sum_{k=4}^0 q_k = \sum_k q_k [4 \leq k \leq 0] = 0. \quad (116)$$

Warmups 2.2:

$$x([x > 0] - [x < 0]) = |x|. \quad (117)$$

Warmups 2.3:

$$\sum_{0 \leq k \leq 5} a_k = a_0 + a_1 + a_2 + a_3 + a_4 + a_5; \quad (118)$$

$$\sum_{0 \leq k^2 \leq 5} a_{k^2} = a_0 + a_1 + a_4. \quad (119)$$

Warmups 2.4:

$$\sum_{1 \leq i < j < k \leq 4} a_{ijk} = ((a_{123} + a_{124}) + a_{134}) + a_{234}; \quad \{\text{case a}\} \quad (120)$$

$$= a_{123} + (a_{124} + (a_{134} + a_{234})). \quad \{\text{case b}\} \quad (121)$$

Warmups 2.5:

$$\left(\sum_{j=1}^n a_j\right)\left(\sum_{k=1}^n \frac{1}{a_k}\right) = \sum_{j=1}^n \sum_{k=1}^n \frac{a_j}{a_k} \quad (122)$$

$$\neq \sum_{k=1}^n \sum_{k=1}^n \frac{a_k}{a_k}. \quad (123)$$

Warmups 2.6:

$$\sum_k [1 \leq j \leq k \leq n] = 0; \quad \{\text{if } [1 \leq j \leq n]\} \quad (124)$$

$$= n - j + 1. \quad \{\text{else}\} \quad (125)$$

Warmups 2.7:

$$\nabla(x^{\overline{m}}) = x^{\overline{m}} - (x-1)^{\overline{m}} = mx^{\overline{m-1}}. \quad (126)$$

Warmups 2.8:

$$0^{\overline{m}} = 0; \quad \{m > 0\} \quad (127)$$

$$= \frac{1}{|m|!}. \quad \{m \leq 0\} \quad (128)$$

Warmups 2.9: The definition is:

$$x^{\overline{m}} = x(x+1)\dots(x+m-1); \quad \{m > 0\} \quad (129)$$

$$= 1; \quad \{m = 0\} \quad (130)$$

$$= \frac{1}{(x-1)\dots(x-m)}. \quad \{m < 0\} \quad (131)$$

Based on the definition:

$$x^{\overline{m+n}} = x^{\overline{m}}(x+m)^{\overline{n}}. \quad (132)$$

The solution is:

$$x^{\overline{-n+n}} = 1 = x^{\overline{-n}}(x-n)^{\overline{n}}; \quad (133)$$

$$x^{\overline{-n}} = \frac{1}{(x-n)^{\overline{n}}}. \quad (134)$$

Warmups 2.10:

$$\Delta(uv) = u \Delta v + Ev \Delta u = v \Delta u + Eu \Delta v. \quad (135)$$

Basics 2.11:

$$\sum_{0 \leq k < n} (a_{k+1} - a_k)b_k = \sum_{0 \leq k < n} a_{k+1}b_k - \sum_{0 \leq k < n} a_k b_k \quad (136)$$

$$= \sum_{0 \leq k < n} a_{k+1}b_k - (a_0b_0 + \sum_{1 \leq k \leq n} a_k b_k - a_n b_n) \quad (137)$$

$$= a_n b_n - a_0 b_0 - \left(\sum_{1 \leq k \leq n} a_k b_k - \sum_{0 \leq k < n} a_{k+1} b_k \right) \quad (138)$$

$$= a_n b_n - a_0 b_0 - \sum_{0 \leq k < n} a_{k+1} (b_{k+1} - b_k). \quad (139)$$

Basics 2.12:

$$p(k) = k - c; \quad \{k \text{ is odd}\} \quad (140)$$

$$= k + c. \quad \{k \text{ is even}\} \quad (141)$$

Prove $p(k) \neq p(j)$ if $k \neq j$.

$$k - c \neq j - c; \quad \{k \text{ and } j \text{ is odd}\} \quad (142)$$

$$k + c \neq j + c; \quad \{k \text{ and } j \text{ is even}\} \quad (143)$$

$$k - c \neq j + c. \quad \{k \text{ is odd and } j \text{ is even}\} \quad (144)$$

Prove any integer number n can be presented by $p(k)$.

$$n = k - c; \quad \{n + c \text{ is odd}\} \quad (145)$$

$$n = k + c. \quad \{n - c \text{ is even}\} \quad (146)$$

Basics 2.13:

$$f(0) = \alpha; \quad (147)$$

$$f(n) = f(n-1) + (-1)^n(\beta + n\gamma + n^2\delta); \quad (148)$$

$$f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma + D(n)\delta. \quad (149)$$

Try

$$f(n) = 1; \quad (150)$$

$$f(n) = (-1)^n; \quad (151)$$

$$f(n) = n(-1)^n; \quad (152)$$

$$f(n) = n^2(-1)^n. \quad (153)$$

Then

$$f(n) = D(n) = \frac{1}{2}(-1)^n(n^2 + n). \quad (154)$$

Basics 2.14:

$$\sum_{k=1}^n k2^k = \sum_{1 \leq j \leq k \leq n} 2^k = \sum_{1 \leq j \leq n} \sum_{j \leq k \leq n} 2^k \quad (155)$$

$$= \sum_{1 \leq j \leq n} (2^{n+1} - 2^j) \quad (156)$$

$$= (n-1)2^{n+1} + 2. \quad (157)$$

Basics 2.15:

$$S3 = \sum_{k=1}^n k^3; \quad (158)$$

$$S2 = \sum_{k=1}^n k^2; \quad (159)$$

$$S1 = \sum_{k=1}^n k. \quad (160)$$

Then

$$S3 + S2 = 2 \sum_{1 \leq j \leq k \leq n} jk = \left(\sum_{k=1}^n k\right)^2 + \sum_{k=1}^n k^2 = \left(\sum_{k=1}^n k\right)^2 + S2. \quad (161)$$

Solution is:

$$S3 = \left(\sum_{k=1}^n k\right)^2 = \left(\frac{1}{2}n(n+1)\right)^2. \quad (162)$$

Basics 2.16:

$$x^{\overline{m+n}} = x^{\overline{m}}(x-m)^{\underline{n}} = x^{\underline{n}}(x-n)^{\overline{m}}. \quad (163)$$

Basics 2.17: Following rules and basic definitions can be used to prove states.

$$(x+m-1)^{\overline{m}}(x-1)^{-\overline{m}} = (x-1)^{\underline{0}} = 1. \quad (164)$$

$$(x-m+1)^{\overline{m}}(x+1)^{-\overline{m}} = (x+1)^{\underline{0}} = 1. \quad (165)$$

Basics 2.18: Unknown.

Homework exercises 2.19:

$$T_0 = 5; \quad (166)$$

$$2T_n = nT_{n-1} + 3n! \quad \{n > 0\} \quad (167)$$

Let $s_n = \frac{2^{n-1}}{n!}$ and mutiply to both sides:

$$\frac{2^n T_n}{n!} = \frac{2^{n-1} T_{n-1}}{(n-1)!} + 3 * 2^{n-1}. \quad (168)$$

Then

$$S_0 = 5; \quad (169)$$

$$S_n = S_{n-1} + 3 * 2^{n-1}. \quad (170)$$

And

$$S_n = S_0 + 3(2^0 + \dots + 2^{n-1}) = 5 + 3(2^n - 1) = 3 * 2^n + 2. \quad (171)$$

The solution is:

$$T_0 = 5; \quad (172)$$

$$T_n = n!(3 + 2^{1-n}). \quad (173)$$

Homework exercises 2.20:

$$\sum_{k=0}^n kH_k + (n+1)H_{n+1} = \sum_{k=0}^n (k+1)H_{k+1} \quad (174)$$

$$= \sum_{k=0}^n kH_k + \sum_{k=0}^n H_k + n. \quad (175)$$

$$(176)$$

Then

$$\sum_{k=0}^n H_k = (n+1)H_{n+1} - n - 1. \quad (177)$$

Homework exercises 2.21: Rewrite formulas.

$$S_n = \sum_{k=0}^n (-1)^{n-k} = \sum_{k=0}^n (-1)^k; \quad (178)$$

$$T_n = \sum_{k=0}^n (-1)^{n-k} k = \sum_{k=0}^n (-1)^k (n-k); \quad (179)$$

$$U_n = \sum_{k=0}^n (-1)^{n-k} k^2 = \sum_{k=0}^n (-1)^k (n-k)^2. \quad (180)$$

Problem 1.

$$S_n + (-1)^{n+1} = 1 + \sum_{k=0}^n (-1)^{k+1} = 1 - \sum_{k=0}^n (-1)^k = 1 - S_n. \quad (181)$$

Solution is

$$S_n = \frac{1 + (-1)^n}{2} = [\text{n is even}]. \quad (182)$$

Problem 2.

$$T_{n+1} = \sum_{k=0}^{n+1} (-1)^k (n+1-k) = \sum_{k=0}^n (-1)^k (n+1-k) = T_n + S_n. \quad (183)$$

Then

$$T_{n+1} = \sum_{k=0}^n S_k = \sum_{k=0}^n \frac{1 + (-1)^k}{2} = \frac{n+1}{2} + \frac{S_n}{2}. \quad (184)$$

Solution is

$$T_n = \frac{n + S_{n-1}}{2} = \frac{n + [\text{n is odd}]}{2}. \quad (185)$$

Problem 3.

$$U_{n+1} = \sum_{k=0}^{n+1} (-1)^k (n+1-k)^2 \quad (186)$$

$$= \sum_{k=0}^n (-1)^k ((n-k)^2 + 2(n-k) + 1) \quad (187)$$

$$= U_n + S_n + 2T_n \quad (188)$$

$$= U_n + n + ([\text{n is odd}] + [\text{n is even}]) \quad (189)$$

$$= U_n + n + 1. \quad (190)$$

Then

$$U_n = \frac{n(n+1)}{2}. \quad (191)$$

Homework exercises 2.22:

$$\sum_{1 \leq j < k \leq n} (a_j b_k - a_k b_j)(A_j B_k - A_k B_j) = \frac{1}{2} \sum_{1 \leq j, k \leq n} (a_j b_k - a_k b_j)(A_j B_k - A_k B_j) \quad (192)$$

$$= \sum_{1 \leq j \leq n} a_j A_j \sum_{1 \leq k \leq n} b_k B_k - \sum_{1 \leq j \leq n} b_j A_j \sum_{1 \leq k \leq n} a_k B_k. \quad (193)$$

Homework exercises 2.23: Method a.

$$\sum_{k=1}^n \frac{2k+1}{k(k+1)} = \sum_{k=1}^n (2k+1) \left(\frac{1}{k} - \frac{1}{k+1} \right) \quad (194)$$

$$= \sum_{k=1}^n \left(\frac{1}{k} + \frac{1}{k+1} \right) \quad (195)$$

$$= 2H_n + \frac{1}{n+1} - 1. \quad (196)$$

Method b.

$$\sum_{k=1}^n \frac{2k+1}{k(k+1)} = \sum_1^{n+1} \frac{2x+1}{x(x+1)} \delta x \quad (197)$$

$$= \sum_1^{n+1} -(2x+1) \delta(x-1)^{-1} \quad (198)$$

$$= -(2k+1)k^{-1} \Big|_1^{n+1} + \sum_1^{n+1} k^{-1} \delta 2k \quad (199)$$

$$= -1 - \frac{1}{n+1} + 2H_{n+1}. \quad (200)$$

Homework exercises 2.24:

$$\sum_0^n n \frac{H_x}{(x+1)(x+2)} \delta x = \sum_0^n n H_x x^{-2} \delta x \quad (201)$$

$$= \sum_0^n n - H_x \delta x^{-1} \quad (202)$$

$$= -H_x x^{-1} \Big|_0^n - \sum_0^n -x^{-1} \delta H_x \quad (203)$$

$$= -H_x x^{-1} \Big|_0^{n+1} + \sum_0^n x^{-2} \delta x \quad (204)$$

$$= x^{-1}(-1 - H_x) \Big|_0^{n+1} \quad (205)$$

$$= \frac{n - H_n}{n+1}. \quad (206)$$

Homework exercises 2.25:

$$\prod_{k \in K} a_k^c = \left(\prod_{k \in K} a_k \right)^c; \quad (207)$$

$$\prod_{k \in K} a_k b_k = \prod_{k \in K} a_k \prod_{k \in K} b_k; \quad (208)$$

$$\prod_{k \in K} a_k = \prod_{p(k) \in K} a_{p(k)}; \quad (209)$$

$$\prod_{j \in J, k \in K} a_{j,k} = \prod_{j \in J} \prod_{k \in K} a_{j,k}; \quad (210)$$

$$\prod_{k \in K} a_k = \prod_k a_k^{[k \in K]}; \quad (211)$$

$$\prod_{k \in K} c = c^{\#K}. \quad (212)$$

Homework exercises 2.26:

$$\prod_{1 \leq j \leq k \leq n} = \sqrt{\prod_{1 \leq j, k \leq n} a_j a_k \prod_{1 \leq j=k \leq n} a_j a_k} \quad (213)$$

$$= \sqrt{\prod_{1 \leq k \leq n} a_k^{2n+2}} \quad (214)$$

$$= \prod_{1 \leq k \leq n} a_k^{n+1}. \quad (215)$$

Homework exercises 2.27:

$$\triangle(c^x) = c^{x+1} - c^x = c^x(c - x - 1) = \frac{c^{x+2}}{c - x}. \quad (216)$$

Then

$$\sum_{k=1}^n \frac{(-2)^k}{k} = \sum_1^{n+1} \frac{(-2)^x}{x} \delta x \quad (217)$$

$$= \sum_1^{n+1} \delta - (-2)^{x-2} \quad (218)$$

$$= -(-2)^{x-2} \Big|_1^{n+1} \quad (219)$$

$$= -(-2)^{n-1} + (-2)^{-1} \quad (220)$$

$$= -(-1)^{n-1} n! - 1. \quad (221)$$

Homework exercises 2.28:

$$\sum_{k \geq 1} \sum_{j \geq 1} \left(\frac{k}{j} [j = k+1] - \frac{j}{k} [j = k-1] \right) \neq \sum_{j \geq 1} \sum_{k \geq 1} \left(\frac{k}{j} [j = k+1] - \frac{j}{k} [j = k-1] \right). \quad (222)$$

Because the function is not converge absolutely, so the exchange of the two sum cannot be applied.

Exam problems 2.29:

$$\sum_{k=1}^n \frac{(-1)^k k}{4k^2 - 1} = \sum_{k=1}^n \frac{(-1)^k k}{(2k+1)(2k-1)} \quad (223)$$

$$= \frac{1}{2} \sum_{k=1}^n \frac{(-1)^k}{4k-2} + \frac{1}{2} \sum_{k=1}^n \frac{(-1)^k}{4k+2} \quad (224)$$

$$= \frac{1}{2} \left(-\frac{1}{2} + \frac{(-1)^n}{4n+2} \right). \quad (225)$$

Exam problems 2.30:

$$\sum_{x=a}^{b-1} x = \sum_a^b x \delta x = \frac{1}{2} x^2 \Big|_a^b = \frac{1}{2} (b-a)(b+a-1). \quad (226)$$

Then

$$(b-a)(b+a-1) = 2100 = 2^2 * 3 * 5^2 * 7 = 2^{p_2} * 3^{p_3} * 5^{p_5} * 7^{p_7}. \quad (227)$$

Beacause $(b-a)$ and $b+a-1$ is a even number and a odd number. The number of possible odd number could be

$$\prod_{k>2} (p_k + 1) = (1+1)(2+1)(1+1) = 12. \quad (228)$$

Exam problems 2.31: a.

$$\sum_{k \geq 2} (\zeta(k) - 1) = \sum_{k \geq 2} \sum_{j \geq 2} \frac{1}{j^k} \quad (229)$$

$$= \sum_{j \geq 2} \sum_{k \geq 2} \frac{1}{j^k} \quad (230)$$

$$= \sum_{j \geq 2} \frac{\frac{1}{j^2}}{1 - \frac{1}{j}} \quad (231)$$

$$= \sum_{j \geq 2} \left(\frac{1}{j-1} - \frac{1}{j} \right) \quad (232)$$

$$= 1. \quad (233)$$

b.

$$\sum_{k \geq 1} (\zeta(2k) - 1) = \sum_{k \geq 1} \sum_{j \geq 2} \frac{1}{j^{2k}} \quad (234)$$

$$= \sum_{j \geq 2} \sum_{k \geq 1} \frac{1}{j^{2k}} \quad (235)$$

$$= \sum_{j \geq 2} \frac{1}{2} \left(\frac{1}{j-1} - \frac{1}{j+1} \right) \quad (236)$$

$$= \frac{3}{4}. \quad (237)$$

Exam problems 2.32: Let S_0 be the left function and S_1 be the right function. When $2n \leq x < 2n+1$

$$S_0 = 1 + 2 + 3 + \dots + n + (x - n - 1) + \dots + (x - 2n) \quad (238)$$

$$S_1 = (x - 1) + (x - 3) + \dots + (x - 2n + 1) \quad (239)$$

When $2n - 1 \leq x < 2n$

$$S_0 = 0 + 1 + 2 + 3 + \dots + n - 1 + (x - n) + \dots + (x - 2n + 1) \quad (240)$$

$$S_1 = (x - 1) + (x - 3) + \dots + (x - 2n + 1) \quad (241)$$

Then solution is $n(x - n)$.

Bonus problems 2.33:

Bonus problems 2.34:

Bonus problems 2.35:

Bonus problems 2.36: