3 Integer Functions

3.1 FLOORS AND CEILINGS

Define: $\lceil x \rceil$ is the least integer greater than or equal to x, and $\lfloor x \rfloor$ is the greatest integer less than or equal to x. Basic rules:

$$\lfloor x \rfloor \le x; \tag{1}$$
$$\lceil x \rceil \ge x. \tag{2}$$

The two functions are equal precisely at the integer points:

$$\lfloor x \rfloor = x \iff x \text{ is an integer} \iff \lceil x \rceil = x. \tag{3}$$

The two functions are inequal if not at the integer points:

$$\lceil x \rceil - \lfloor x \rfloor = [x \text{ is not an integer}]. \tag{4}$$

The two functinos can be converted:

$$\lceil -x \rceil = -\lfloor x \rfloor; \tag{5}$$
$$\lfloor -x \rfloor = -\lceil x \rceil. \tag{6}$$

Integers can be easily removed or added in the two functions:

$$\lfloor x + n \rfloor = \lfloor x \rfloor + n; \qquad \{\text{integer n}\} \tag{7}$$
$$\lceil x + n \rceil = \lceil x \rceil + n. \qquad \{\text{integer n}\} \tag{8}$$

For important rules:

$$\lfloor x \rfloor = n \iff n \le x < n + 1; \tag{9}$$
$$\lfloor x \rfloor = n \iff x - 1 < n \le x; \tag{10}$$
$$\lceil x \rceil = n \iff n - 1 < x \le n; \tag{11}$$
$$\lceil x \rceil = n \iff x \le n < x + 1. \tag{12}$$

There are many situations in which floor and ceiling brackets are redundant:

$$x < n \iff \lfloor x \rfloor < n; \tag{13}$$
$$n < x \iff n < \lceil x \rceil; \tag{14}$$
$$x \le n \iff \lceil x \rceil \le n; \tag{15}$$
$$n \le x \iff n \le \lfloor x \rfloor. \tag{16}$$

Define: $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x, then $\lfloor x \rfloor$ is the integer part of x. A simple notation is $x = n + \theta$.

$$\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor + \lfloor \{x\} + \{y\} \rfloor. \tag{17}$$

3.2 FLOOR/CEILING APPLICATIONS

Problem 1: what is the bit number to express n in binary?

$$2^{m-1} \le x < 2^m \iff \text{the bit number is } m; \tag{18}$$
$$m - 1 \le \lg x < m \tag{19}$$
$$m = \lfloor \lg x \rfloor + 1. \qquad \{x > 0\} \tag{20}$$

To support $x = 0$, another better solution is $\lceil \lg(x+1) \rceil$.
Problem 2: what is $m = \lfloor \sqrt{\lfloor x \rfloor} \rfloor$ when $x \ge 0$?

$$m \le \sqrt{\lfloor x \rfloor} < m + 1; \tag{21}$$
$$m^2 \le \lfloor x \rfloor < (m+1)^2; \tag{22}$$
$$m^2 \le x < (m+1)^2; \tag{23}$$
$$m \le \sqrt{x} < m + 1; \tag{24}$$
$$m = \lfloor \sqrt{x} \rfloor. \tag{25}$$

Problem 3: what is $m = \lceil \sqrt{\lceil x \rceil} \rceil$ when $x \geq 0$?

$$
\begin{aligned}
m - 1 &< \sqrt{\lceil x \rceil} \leq m; && (26) \\
(m-1)^2 &< \lceil x \rceil \leq m^2; && (27) \\
(m-1)^2 &< x \leq m^2; && (28) \\
m - 1 &< \sqrt{x} \leq m; && (29) \\
m &= \lceil \sqrt{x} \rceil. && (30)
\end{aligned}
$$

A general theorem: let $f(x)$ be any continuous, monotonically increasing function with the property that

$$f(x) = \text{integer} \implies x = \text{integer}. \tag{31}$$

Then there is:

$$
\begin{aligned}
\lfloor f(x) \rfloor &= \lfloor f(\lfloor x \rfloor) \rfloor; && (32) \\
\lceil f(x) \rceil &= \lceil f(\lceil x \rceil) \rceil. && (33)
\end{aligned}
$$

A special case of the theorem:

$$
\begin{aligned}
\left\lfloor \frac{x+m}{n} \right\rfloor &= \left\lfloor \frac{\lfloor x \rfloor + m}{n} \right\rfloor; && (34) \\
\left\lceil \frac{x+m}{n} \right\rceil &= \left\lceil \frac{\lceil x \rceil + m}{n} \right\rceil. && (35)
\end{aligned}
$$

Problem levels: **level 1** prove a given statement for a number; **level 2** prove a given statement for a set of numbers; **level 3** prove or disprove a given statement for a set of numbers; **level 4** find a necessary and suffcient condition that a statement is true; **level 5** find an interesting property given a set of numbers.
Consider the integer inside a range:

$$
\begin{aligned}
\alpha \leq n < \beta &\iff \lceil \alpha \rceil \leq n < \lceil \beta \rceil; && (36) \\
\alpha < n \leq \beta &\iff \lfloor \alpha \rfloor < n \leq \lfloor \beta \rfloor. && (37)
\end{aligned}
$$

Then

$$
\begin{aligned}
&[\alpha, \beta) \text{ contains } \lceil \beta \rceil - \lceil \alpha \rceil \text{ elements}; && \{\alpha \leq \beta\} && (38) \\
&(\alpha, \beta] \text{ contains } \lfloor \beta \rfloor - \lfloor \alpha \rfloor \text{ elements}; && \{\alpha \leq \beta\} && (39) \\
&(\alpha, \beta) \text{ contains } \lceil \beta \rceil - \lfloor \alpha \rfloor - 1 \text{ elements}; && \{\alpha < \beta\} && (40) \\
&[\alpha, \beta] \text{ contains } \lfloor \beta \rfloor - \lceil \alpha \rceil + 1 \text{ elements}. && \{\alpha \leq \beta\} && (41)
\end{aligned}
$$

Example 1:

$$
\begin{aligned}
W &= \sum_{1 \leq n \leq 1000} [\lfloor \sqrt[3]{n} \rfloor \backslash n] && (42) \\
&= \sum_{k,n} [k = \lfloor \sqrt[3]{n} \rfloor][1 \leq n \leq 1000][k \backslash n] && (43) \\
&= \sum_{k,n,m} [k^3 \leq n < (k+1)^3][n = km][1 \leq n \leq 1000] && (44) \\
&= 1 + \sum_{k,m} [k^3 \leq km < (k+1)^3][1 \leq k < 10] && (45) \\
&= 1 + \sum_{k,m} [k^2 \leq m < (k+1)^3/k][1 \leq k < 10] && (46) \\
&= 1 + \sum_{1 \leq k < 10} (\lceil (k+1)^3/k \rceil - \lceil k^2 \rceil) && (47) \\
&= 1 + \sum_{1 \leq k < 10} (3k + 4) = 172. && (48)
\end{aligned}
$$

General case:

$$
\begin{aligned}
W &= \sum_{1\le n\le N} [\lfloor \sqrt[3]{n} \rfloor \setminus n] && (49)\\
&= \sum_{k,n} [k = \lfloor \sqrt[3]{n} \rfloor][1 \le n \le N][k \setminus n] && (50)\\
&= \sum_{k,n,m} [k^3 \le n < (k+1)^3][n = km][1 \le n \le N] && (51)\\
&= \sum_{k,m} [k^3 \le km < (k+1)^3][1 \le k < K] + \sum_{k,m} [K^3 \le Km \le N] && (52)\\
&= \sum_{k,m} [k^2 \le m < (k+1)^3/k][1 \le k < K] + \sum_{k,m} [K^2 \le m \le N/K] && (53)\\
&= \sum_{1\le k<K} (3k+4) + \sum_{m} [m \in [K^2, N/K]] && (54)\\
&= (7 + 3K + 1)(K-1)/2 + \lfloor N/K \rfloor - \lceil K^2 \rceil + 1 && (55)\\
&= \frac{1}{2}K^2 + \frac{5}{2}K - 3 + \lfloor N/K \rfloor. \qquad \{K = \lfloor \sqrt[3]{N} \rfloor\} && (56)
\end{aligned}
$$

Define $Spec(\alpha) = \{\lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, ...\}$ then $Spec(\sqrt{2})$ and $Spec(\sqrt{2}+2)$ forms a partition of positive integers. Define $N(\alpha, n)$ is the number of elements in $Spec(\alpha)$ that are $\le n$.

$$
\begin{aligned}
N(\alpha, n) &= \sum_{k>0} [\lfloor \alpha k \rfloor \le n] && (57)\\
&= \sum_{k>0} [\lfloor \alpha k \rfloor < n+1] && (58)\\
&= \sum_{k>0} [\alpha k < n+1] && (59)\\
&= \sum_{k>0} [0 < k < (n+1)/\alpha] && (60)\\
&= \lceil (n+1)/\alpha \rceil - 1. && (61)
\end{aligned}
$$

Then $N(\sqrt{2}, n) + N(\sqrt{2}+2, n) = n$. And it is easy to prove that if $\alpha \neq \beta$ then $Spec(\alpha) \neq Spec(\beta)$.

3.3 FLOOR/CEILING RECURRENCES