3 Integer Functions

3.1 FLOORS AND CEILINGS

Define: $\lceil x \rceil$ is the least integer greater than or equal to x, and $\lfloor x \rfloor$ is the greatest integer less than or equal to x. Basic rules:

$$\lfloor x \rfloor \leq x; \tag{1}$$
$$\lceil x \rceil \geq x. \tag{2}$$

The two functions are equal precisely at the integer points:

$$\lfloor x \rfloor = x \iff x \text{ is an integer} \iff \lceil x \rceil = x. \tag{3}$$

The two functions are inequal if not at the integer points:

$$\lceil x \rceil - \lfloor x \rfloor = [x \text{ is not an integer}]. \tag{4}$$

The two functinos can be converted:

$$\lceil -x \rceil = -\lfloor x \rfloor; \tag{5}$$
$$\lfloor -x \rfloor = -\lceil x \rceil. \tag{6}$$

Integers can be easily removed or added in the two functions:

$$\lfloor x + n \rfloor = \lfloor x \rfloor + n; \qquad \{\text{integer n}\} \tag{7}$$
$$\lceil x + n \rceil = \lceil x \rceil + n. \qquad \{\text{integer n}\} \tag{8}$$

For important rules:

$$\lfloor x \rfloor = n \iff n \leq x < n + 1; \tag{9}$$
$$\lfloor x \rfloor = n \iff x - 1 < n \leq x; \tag{10}$$
$$\lceil x \rceil = n \iff n - 1 < x \leq n; \tag{11}$$
$$\lceil x \rceil = n \iff x \leq n < x + 1. \tag{12}$$

There are many situations in which floor and ceiling brackets are redundant:

$$x < n \iff \lfloor x \rfloor < n; \tag{13}$$
$$n < x \iff n < \lceil x \rceil; \tag{14}$$
$$x \leq n \iff \lceil x \rceil \leq n; \tag{15}$$
$$n \leq x \iff n \leq \lfloor x \rfloor. \tag{16}$$

Define: $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x, then $\lfloor x \rfloor$ is the integer part of x. A simple notation is $x = n + \theta$.

$$\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor + \lfloor \{x\} + \{y\} \rfloor. \tag{17}$$

3.2 FLOOR/CEILING APPLICATIONS

Problem 1: what is the bit number to express n in binary?

$$2^{m-1} \leq x < 2^m \iff \text{the bit number is } m; \tag{18}$$
$$m - 1 \leq \lg x < m \tag{19}$$
$$m = \lfloor \lg x \rfloor + 1. \qquad \{x > 0\} \tag{20}$$

To support $x = 0$, another better solution is $\lceil \lg(x + 1) \rceil$.
Problem 2: what is $m = \lfloor \sqrt{\lfloor x \rfloor} \rfloor$ when $x \geq 0$?

$$m \leq \sqrt{\lfloor x \rfloor} < m + 1; \tag{21}$$
$$m^2 \leq \lfloor x \rfloor < (m + 1)^2; \tag{22}$$
$$m^2 \leq x < (m + 1)^2; \tag{23}$$
$$m \leq \sqrt{x} < m + 1; \tag{24}$$
$$m = \lfloor \sqrt{x} \rfloor. \tag{25}$$

Problem 3: what is $m = \lceil \sqrt{\lceil x \rceil} \rceil$ when $x \geq 0$?

$$
\begin{aligned}
& m - 1 < \sqrt{\lceil x \rceil} \leq m; && (26)\\
& (m-1)^2 < \lceil x \rceil \leq m^2; && (27)\\
& (m-1)^2 < x \leq m^2; && (28)\\
& m - 1 < \sqrt{x} \leq m; && (29)\\
& m = \lceil \sqrt{x} \rceil. && (30)
\end{aligned}
$$

A general theorem: let $f(x)$ be any continuous, monotonically increasing function with the property that

$$f(x) = \text{integer} \implies x = \text{integer}. \tag{31}$$

Then there is:

$$
\begin{aligned}
& \lfloor f(x) \rfloor = \lfloor f(\lfloor x \rfloor) \rfloor; && (32)\\
& \lceil f(x) \rceil = \lceil f(\lceil x \rceil) \rceil. && (33)
\end{aligned}
$$

A special case of the theorem:

$$
\begin{aligned}
& \left\lfloor \frac{x+m}{n} \right\rfloor = \left\lfloor \frac{\lfloor x \rfloor + m}{n} \right\rfloor; && (34)\\
& \left\lceil \frac{x+m}{n} \right\rceil = \left\lceil \frac{\lceil x \rceil + m}{n} \right\rceil. && (35)
\end{aligned}
$$

Problem levels: **level 1** prove a given statement for a number; **level 2** prove a given statement for a set of numbers; **level 3** prove or disprove a given statement for a set of numbers; **level 4** find a necessary and suffcient condition that a statement is true; **level 5** find an interesting property given a set of numbers.
Consider the integer inside a range:

$$
\begin{aligned}
& \alpha \leq n < \beta \iff \lceil \alpha \rceil \leq n < \lceil \beta \rceil; && (36)\\
& \alpha < n \leq \beta \iff \lfloor \alpha \rfloor < n \leq \lfloor \beta \rfloor. && (37)
\end{aligned}
$$

Then

$$
\begin{aligned}
& [\alpha, \beta) \text{ contains } \lceil \beta \rceil - \lceil \alpha \rceil \text{ elements;} && \{\alpha \leq \beta\} && (38)\\
& (\alpha, \beta] \text{ contains } \lfloor \beta \rfloor - \lfloor \alpha \rfloor \text{ elements;} && \{\alpha \leq \beta\} && (39)\\
& (\alpha, \beta) \text{ contains } \lceil \beta \rceil - \lfloor \alpha \rfloor - 1 \text{ elements;} && \{\alpha < \beta\} && (40)\\
& [\alpha, \beta] \text{ contains } \lfloor \beta \rfloor - \lceil \alpha \rceil + 1 \text{ elements.} && \{\alpha \leq \beta\} && (41)
\end{aligned}
$$

Example 1:

$$
\begin{aligned}
W &= \sum_{1 \leq n \leq 1000} [\lfloor \sqrt[3]{n} \rfloor \setminus n] && (42)\\
&= \sum_{k,n} [k = \lfloor \sqrt[3]{n} \rfloor][1 \leq n \leq 1000][k \setminus n] && (43)\\
&= \sum_{k,n,m} [k^3 \leq n < (k+1)^3][n = km][1 \leq n \leq 1000] && (44)\\
&= 1 + \sum_{k,m} [k^3 \leq km < (k+1)^3][1 \leq k < 10] && (45)\\
&= 1 + \sum_{k,m} [k^2 \leq m < (k+1)^3/k][1 \leq k < 10] && (46)\\
&= 1 + \sum_{1 \leq k < 10} (\lceil (k+1)^3/k \rceil - \lceil k^2 \rceil) && (47)\\
&= 1 + \sum_{1 \leq k < 10} (3k + 4) = 172. && (48)
\end{aligned}
$$

General case:

$$
\begin{aligned}
W &= \sum_{1\le n\le N} [\lfloor \sqrt[3]{n} \rfloor \setminus n] && (49)\\
&= \sum_{k,n} [k = \lfloor \sqrt[3]{n} \rfloor][1 \le n \le N][k \setminus n] && (50)\\
&= \sum_{k,n,m} [k^3 \le n < (k+1)^3][n = km][1 \le n \le N] && (51)\\
&= \sum_{k,m} [k^3 \le km < (k+1)^3][1 \le k < K] + \sum_{k,m} [K^3 \le Km \le N] && (52)\\
&= \sum_{k,m} [k^2 \le m < (k+1)^3/k][1 \le k < K] + \sum_{k,m} [K^2 \le m \le N/K] && (53)\\
&= \sum_{1\le k<K} (3k+4) + \sum_{m} [m \in [K^2, N/K]] && (54)\\
&= (7 + 3K + 1)(K-1)/2 + \lfloor N/K \rfloor - \lceil K^2 \rceil + 1 && (55)\\
&= \frac{1}{2}K^2 + \frac{5}{2}K - 3 + \lfloor N/K \rfloor. && \{K = \lfloor \sqrt[3]{N} \rfloor\} \quad (56)
\end{aligned}
$$

Define $Spec(\alpha) = \{\lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, ...\}$ then $Spec(\sqrt{2})$ and $Spec(\sqrt{2}+2)$ forms a partition of positive integers. Define $N(\alpha, n)$ is the number of elements in $Spec(\alpha)$ that are $\le n$.

$$
\begin{aligned}
N(\alpha, n) &= \sum_{k>0} [\lfloor \alpha k \rfloor \le n] && (57)\\
&= \sum_{k>0} [\lfloor \alpha k \rfloor < n+1] && (58)\\
&= \sum_{k>0} [\alpha k < n+1] && (59)\\
&= \sum_{k>0} [0 < k < (n+1)/\alpha] && (60)\\
&= \lceil (n+1)/\alpha \rceil - 1. && (61)
\end{aligned}
$$

Then $N(\sqrt{2}, n) + N(\sqrt{2}+2, n) = n$. And it is easy to prove that if $\alpha \ne \beta$ then $Spec(\alpha) \ne Spec(\beta)$.

3.3 FLOOR/CEILING RECURRENCES

Knuth numbers:

$$
\begin{aligned}
K_0 &= 1; && (62)\\
K_{n+1} &= 1 + min(2K_{\lfloor n/2 \rfloor}, 3K_{\lfloor n/3 \rfloor}). && (63)
\end{aligned}
$$

The Josephus problem:

$$
\begin{aligned}
J(1) &= 1; && (64)\\
J(n) &= 2J(\lfloor n/2 \rfloor) - (-1)^n. && (65)
\end{aligned}
$$

Consider the more authentic Josephus problem in which every third person is eliminated:

$$
J_3(n) = \left\lceil \frac{3}{2} J_3(\left\lfloor \frac{2}{3} n \right\rfloor) + a_n \right\rceil \mod (n+1). \qquad (66)
$$

where $a_n = -2, 1, -\frac{1}{2}$ when $n \mod 3 = 0, 1, 2$.
Another method: Whenever a person is passed over, we can assign a new number.

1	2	3	4	5	6	7	8	9	10
11	12		13	14		15	16		17
18			19	20			21		22
			23	24					25
			26						27
			28						
			29						
			30						

Any number $3k+1$ has a next value $10+3k+1-k$, and $3k+2$ has a next value $10+3k+2-k$. More general, there are n people at first and some person has a current number N. For this person his last number should be $3k+1$ or $3k+2$, and this current number is $N = n+2k+1$ or $N = n+2k+2$. This means

$$k = \left\lfloor \frac{N-n-1}{2} \right\rfloor. \tag{67}$$

And his last number can be converted into

$$3k + (N-n-2k) = k + N - n = \left\lfloor \frac{N-n-1}{2} \right\rfloor + N - n. \tag{68}$$

For the last one to be terminated, his number should be $3n$. Use the method we can always find his last number until the number is smaller than n which is his initial number.

```
def J3(n):
    N = 3 * n
    while N > n:
        N = int((N - n - 1)/2) + N - n
    return N
```

Listing 1: Method 0

Let $D = 3n+1-N$, then $D = 1$ when $N = 3n$ and $D > 2n+1$ when $N < n$. D can also be updated as N:

$$\begin{aligned} D &= 3n+1-N && (69)\\ &= 3n+1-(\left\lfloor \frac{(3n+1-D)-n-1}{2} \right\rfloor + (3n+1-D)-n) && (70)\\ &= n+D-\left\lfloor \frac{2n-D}{2} \right\rfloor && (71)\\ &= D-\left\lfloor \frac{-D}{2} \right\rfloor && (72)\\ &= D+\left\lceil \frac{D}{2} \right\rceil && (73)\\ &= \left\lceil \frac{3D}{2} \right\rceil. && (74) \end{aligned}$$

```
import math
def J3(n):
    D = 1
    while D <= 2*n:
        D = math.ceil(D*3/2)
    return 3*n + 1 - D
```

Listing 2: Method 1

More general:

```python
import math
def J(n,q):
    D = 1
    while D <= (q-1)*n:
        D = math.ceil(D*q/(q-1))
    return q*n + 1 - D
```

Listing 3: Method 2

Write it into a recurrence:

$$D_0^{(q)} = 1; \tag{75}$$

$$D_n^{(q)} = \left\lceil \frac{q}{q-1} D_{n-1}^{(q)} \right\rceil. \tag{76}$$

3.4 'MOD': THE BINARY OPERATION

Define operator 'mod':

$$x \bmod y = x - y\lfloor x/y \rfloor. \qquad \{y \neq 0\} \tag{77}$$

Based on the defination, there are some attributes:

$$0 \leq x \bmod y < y; \qquad \{y > 0\} \tag{78}$$

$$0 \geq x \bmod y > y. \qquad \{y < 0\} \tag{79}$$

To complete the defination, we can let $x \bmod y = x$ when $y = 0$.
The 'mod' operator can be used to show the fractional part of a number:

$$x = \lfloor x \rfloor + x \bmod 1. \tag{80}$$

A similar 'mumble' operator can be defined:

$$x \text{ mumble } y = y\lceil x/y \rceil - x. \qquad \{y \neq 0\} \tag{81}$$

The 'mod' operator follows the distributive law:

$$c(x \bmod y) = (cx) \bmod (cy). \tag{82}$$

Problem: how to partition n things into m groups as equally as possible?
There will be $n \bmod m$ groups contains $\lceil n/m \rceil$ things and the rest contains $\lfloor n/m \rfloor$ things. It also can be converted into:

$$n = \left\lceil \frac{n}{m} \right\rceil + \left\lceil \frac{n-1}{m} \right\rceil + ... + \left\lceil \frac{n-m+1}{m} \right\rceil. \tag{83}$$

and if change n to $km + n \bmod m$, the equation can be converted into:

$$n = \left\lfloor \frac{n}{m} \right\rfloor + \left\lfloor \frac{n+1}{m} \right\rfloor + ... + \left\lfloor \frac{n+m-1}{m} \right\rfloor. \tag{84}$$

If $n = \lfloor mx \rfloor$:

$$\lfloor mx \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{m} \right\rfloor + ... + \left\lfloor x + \frac{m-1}{m} \right\rfloor. \tag{85}$$

3.5 FLOOR/CEILING SUMS

Example 1, let $a = \lfloor \sqrt{n} \rfloor$:

$$\sum_{0 \le k < n} \lfloor \sqrt{k} \rfloor = \sum_{k,m \ge 0} m[m = \lfloor \sqrt{k} \rfloor][k < n] \tag{86}$$
$$= \sum_{k,m \ge 0} m[k < n][m \le \sqrt{k} < m+1] \tag{87}$$
$$= \sum_{k,m \ge 0} m[k < n][m^2 \le k < (m+1)^2 \tag{88}$$
$$= \sum_{k,m \ge 0} m[m^2 \le k < (m+1)^2 \le n] + \sum_{k,m \ge 0} m[m^2 \le k < n < (m+1)^2] \tag{89}$$
$$= \sum_{m \ge 0} m((m+1)^2 - m^2)[m+1 \le a] + \sum_{m \ge 0} m(a^2 \le k < n) \tag{90}$$
$$= \sum_{m \ge 0} (2m^2 + m)[m+1 \le a] + a(n - a^2) \tag{91}$$
$$= \sum_{m \ge 0} (2m^{\underline{2}} + 3m^{\underline{1}})[m < a] - a^3 + an \tag{92}$$
$$= \sum\nolimits_0^a (2m^{\underline{2}} + 3m^{\underline{1}})\delta m - a^3 + an \tag{93}$$
$$= \frac{2}{3}m^{\underline{3}} + \frac{3}{2}m^{\underline{2}} - a^3 + an \tag{94}$$
$$= na - \frac{1}{3}a^3 - \frac{1}{2}a^2 - \frac{1}{6}a. \tag{95}$$

Anothe method is le $\lfloor x \rfloor = \sum_j [1 \le j \le x]$:

$$\sum_{0 \le k < n} \lfloor \sqrt{k} \rfloor = \sum_{j,k} [1 \le j \le \sqrt{k}][0 \le k \le a^2] \tag{96}$$
$$= \sum_{1 \le j < a} \sum_{k} [j^2 \le k < a^2] \tag{97}$$
$$= \sum_{1 \le j < a} (a^2 - j^2) \tag{98}$$
$$= na - \frac{1}{3}a^3 - \frac{1}{2}a^2 - \frac{1}{6}a. \tag{99}$$

Equidistribution theorem:

$$\lim_{x \to \infty} \frac{1}{n} \sum_{0 \le k < n} f(\{k\alpha\}) = \int_0^1 f(x)dx. \tag{100}$$

for all irrational α and all functions f that are continuous almost everywhere.
Example 2, let $d = gcd(m, n)$:

$$\sum_{0 \le k < m} \left\lfloor \frac{nk + x}{m} \right\rfloor = d\left\lfloor \frac{x}{d} \right\rfloor + \frac{m-1}{2}n + \frac{d-m}{2} \tag{101}$$
$$= d\left\lfloor \frac{x}{d} \right\rfloor + \frac{mn}{2} - \frac{n}{2} - \frac{m}{2} + \frac{d}{2} \tag{102}$$
$$= \sum_{0 \le k < n} \left\lfloor \frac{mk + x}{n} \right\rfloor. \tag{103}$$

3.6 Exercises

Warmups 3.1:

Warmups 3.2:

Warmups 3.3:

Warmups 3.4:

Warmups 3.5:

Warmups 3.6:

Warmups 3.7:

Warmups 3.8:

Warmups 3.9: