3 Integer Functions

3.1 FLOORS AND CEILINGS

Define: $\lceil x \rceil$ is the least integer greater than or equal to x, and $\lfloor x \rfloor$ is the greatest integer less than or equal to x. Basic rules:

$$\lfloor x \rfloor \leq x; \tag{1}$$
$$\lceil x \rceil \geq x. \tag{2}$$

The two functions are equal precisely at the integer points:

$$\lfloor x \rfloor = x \iff x \text{ is an integer} \iff \lceil x \rceil = x. \tag{3}$$

The two functions are inequal if not at the integer points:

$$\lceil x \rceil - \lfloor x \rfloor = [x \text{ is not an integer}]. \tag{4}$$

The two functinos can be converted:

$$\lceil -x \rceil = -\lfloor x \rfloor; \tag{5}$$
$$\lfloor -x \rfloor = -\lceil x \rceil. \tag{6}$$

Integers can be easily removed or added in the two functions:

$$\lfloor x + n \rfloor = \lfloor x \rfloor + n; \qquad \{\text{integer n}\} \tag{7}$$
$$\lceil x + n \rceil = \lceil x \rceil + n. \qquad \{\text{integer n}\} \tag{8}$$

For important rules:

$$\lfloor x \rfloor = n \iff x - 1 < n \leq x < n + 1; \tag{9}$$
$$\lceil x \rceil = n \iff n - 1 < x \leq n < x + 1. \tag{10}$$

There are many situations in which floor and ceiling brackets are redundant:

$$x < n \iff \lfloor x \rfloor < n; \tag{11}$$
$$n < x \iff n < \lceil x \rceil; \tag{12}$$
$$x \leq n \iff \lceil x \rceil \leq n; \tag{13}$$
$$n \leq x \iff n \leq \lfloor x \rfloor. \tag{14}$$

Define: $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x, then $\lfloor x \rfloor$ is the integer part of x. A simple notation is $x = n + \theta$.

$$\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor + \lfloor \{x\} + \{y\} \rfloor. \tag{15}$$

3.2 FLOOR/CEILING APPLICATIONS

Problem 1: what is the bit number to express n in binary?

$$2^{m-1} \leq x < 2^m \iff \text{the bit number is } m; \tag{16}$$
$$m - 1 \leq \lg x < m \tag{17}$$
$$m = \lfloor \lg x \rfloor + 1. \qquad \{x > 0\} \tag{18}$$

To support $x = 0$, another better solution is $\lceil \lg(x + 1) \rceil$.
Problem 2: what is $m = \lfloor \sqrt{\lfloor x \rfloor} \rfloor$ when $x \geq 0$?

$$m \leq \sqrt{\lfloor x \rfloor} < m + 1; \tag{19}$$
$$m^2 \leq \lfloor x \rfloor < (m + 1)^2; \tag{20}$$
$$m^2 \leq x < (m + 1)^2; \tag{21}$$
$$m \leq \sqrt{x} < m + 1; \tag{22}$$
$$m = \lfloor \sqrt{x} \rfloor. \tag{23}$$

Problem 3: what is $m = \lceil \sqrt{\lceil x \rceil} \rceil$ when $x \geq 0$?

$$m - 1 < \sqrt{\lceil x \rceil} \leq m; \tag{24}$$
$$(m-1)^2 < \lceil x \rceil \leq m^2; \tag{25}$$
$$(m-1)^2 < x \leq m^2; \tag{26}$$
$$m - 1 < \sqrt{x} \leq m; \tag{27}$$
$$m = \lceil \sqrt{x} \rceil. \tag{28}$$

A general theorem: let $f(x)$ be any continuous, monotonically increasing function with the property that

$$f(x) = \text{integer} \implies x = \text{integer}. \tag{29}$$

Then there is:

$$\lfloor f(x) \rfloor = \lfloor f(\lfloor x \rfloor) \rfloor; \tag{30}$$
$$\lceil f(x) \rceil = \lceil f(\lceil x \rceil) \rceil. \tag{31}$$

A special case of the theorem:

$$\left\lfloor \frac{x+m}{n} \right\rfloor = \left\lfloor \frac{\lfloor x \rfloor + m}{n} \right\rfloor; \tag{32}$$
$$\left\lceil \frac{x+m}{n} \right\rceil = \left\lceil \frac{\lceil x \rceil + m}{n} \right\rceil. \tag{33}$$

Problem levels: **level 1** prove a given statement for a number; **level 2** prove a given statement for a set of numbers; **level 3** prove or disprove a given statement for a set of numbers; **level 4** find a necessary and suffcient condition that a statement is true; **level 5** find an interesting property given a set of numbers.
Consider the integer inside a range:

$$\alpha \leq n < \beta \iff \lceil \alpha \rceil \leq n < \lceil \beta \rceil; \tag{34}$$
$$\alpha < n \leq \beta \iff \lfloor \alpha \rfloor < n \leq \lfloor \beta \rfloor. \tag{35}$$

Then

$$[\alpha, \beta) \text{ contains } \lceil \beta \rceil - \lceil \alpha \rceil \text{ elements;} \qquad \{\alpha \leq \beta\} \tag{36}$$
$$(\alpha, \beta] \text{ contains } \lfloor \beta \rfloor - \lfloor \alpha \rfloor \text{ elements;} \qquad \{\alpha \leq \beta\} \tag{37}$$
$$(\alpha, \beta) \text{ contains } \lceil \beta \rceil - \lfloor \alpha \rfloor - 1 \text{ elements;} \qquad \{\alpha < \beta\} \tag{38}$$
$$[\alpha, \beta] \text{ contains } \lfloor \beta \rfloor - \lceil \alpha \rceil + 1 \text{ elements.} \qquad \{\alpha \leq \beta\} \tag{39}$$

Example 1:

$$W = \sum_{1 \leq n \leq 1000} [\lfloor \sqrt[3]{n} \rfloor \setminus n] \tag{40}$$
$$= \sum_{k,n} [k = \lfloor \sqrt[3]{n} \rfloor][1 \leq n \leq 1000][k \setminus n] \tag{41}$$
$$= \sum_{k,n,m} [k^3 \leq n < (k+1)^3][n = km][1 \leq n \leq 1000] \tag{42}$$
$$= 1 + \sum_{k,m} [k^3 \leq km < (k+1)^3][1 \leq k < 10] \tag{43}$$
$$= 1 + \sum_{k,m} [k^2 \leq m < (k+1)^3/k][1 \leq k < 10] \tag{44}$$
$$= 1 + \sum_{1 \leq k < 10} (\lceil (k+1)^3/k \rceil - \lceil k^2 \rceil) \tag{45}$$
$$= 1 + \sum_{1 \leq k < 10} (3k+4) = 172. \tag{46}$$

General case:

$$
\begin{aligned}
W &= \sum_{1\le n\le N} [\lfloor \sqrt[3]{n} \rfloor \setminus n] && (47)\\
&= \sum_{k,n} [k = \lfloor \sqrt[3]{n} \rfloor][1 \le n \le N][k \setminus n] && (48)\\
&= \sum_{k,n,m} [k^3 \le n < (k+1)^3][n = km][1 \le n \le N] && (49)\\
&= \sum_{k,m} [k^3 \le km < (k+1)^3][1 \le k < K] + \sum_{k,m} [K^3 \le Km \le N] && (50)\\
&= \sum_{k,m} [k^2 \le m < (k+1)^3/k][1 \le k < K] + \sum_{k,m} [K^2 \le m \le N/K] && (51)\\
&= \sum_{1\le k<K} (3k+4) + \sum_{m} [m \in [K^2, N/K]] && (52)\\
&= (7 + 3K + 1)(K-1)/2 + \lfloor N/K \rfloor - \lceil K^2 \rceil + 1 && (53)\\
&= \frac{1}{2}K^2 + \frac{5}{2}K - 3 + \lfloor N/K \rfloor. && \{K = \lfloor \sqrt[3]{N} \rfloor\} \quad (54)
\end{aligned}
$$

Define $Spec(\alpha) = \{\lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, ...\}$ then $Spec(\sqrt{2})$ and $Spec(\sqrt{2}+2)$ forms a partition of positive integers. Define $N(\alpha, n)$ is the number of elements in $Spec(\alpha)$ that are $\le n$.

$$
\begin{aligned}
N(\alpha, n) &= \sum_{k>0} [\lfloor \alpha k \rfloor \le n] && (55)\\
&= \sum_{k>0} [\lfloor \alpha k \rfloor < n+1] && (56)\\
&= \sum_{k>0} [\alpha k < n+1] && (57)\\
&= \sum_{k>0} [0 < k < (n+1)/\alpha] && (58)\\
&= \lceil (n+1)/\alpha \rceil - 1. && (59)
\end{aligned}
$$

Then $N(\sqrt{2}, n) + N(\sqrt{2}+2, n) = n$. And it is easy to prove that if $\alpha \ne \beta$ then $Spec(\alpha) \ne Spec(\beta)$.

3.3 FLOOR/CEILING RECURRENCES

Knuth numbers:

$$
\begin{aligned}
K_0 &= 1; && (60)\\
K_{n+1} &= 1 + min(2K_{\lfloor n/2 \rfloor}, 3K_{\lfloor n/3 \rfloor}). && (61)
\end{aligned}
$$

The Josephus problem:

$$
\begin{aligned}
J(1) &= 1; && (62)\\
J(n) &= 2J(\lfloor n/2 \rfloor) - (-1)^n. && (63)
\end{aligned}
$$

Consider the more authentic Josephus problem in which every third person is eliminated:

$$
J_3(n) = \left\lceil \frac{3}{2} J_3(\left\lfloor \frac{2}{3} n \right\rfloor) + a_n \right\rceil \mod (n+1). \qquad (64)
$$

where $a_n = -2, 1, -\frac{1}{2}$ when $n \mod 3 = 0, 1, 2$.
Another method: Whenever a person is passed over, we can assign a new number.

1	2	3	4	5	6	7	8	9	10
11	12		13	14		15	16		17
18			19	20			21		22
			23	24					25
			26						27
			28						
			29						
			30						

Any number $3k+1$ has a next value $10+3k+1-k$, and $3k+2$ has a next value $10+3k+2-k$. More general, there are n people at first and some person has a current number N. For this person his last number should be $3k+1$ or $3k+2$, and this current number is $N = n+2k+1$ or $N = n+2k+2$. This means

$$k = \left\lfloor \frac{N-n-1}{2} \right\rfloor. \tag{65}$$

And his last number can be converted into

$$3k + (N-n-2k) = k + N - n = \left\lfloor \frac{N-n-1}{2} \right\rfloor + N - n. \tag{66}$$

For the last one to be terminated, his number should be $3n$. Use the method we can always find his last number until the number is smaller than n which is his initial number.

```python
def J3(n):
    N = 3 * n
    while N > n:
        N = int((N - n - 1)/2) + N - n
    return N
```

Listing 1: Method 0

Let $D = 3n+1-N$, then $D = 1$ when $N = 3n$ and $D > 2n+1$ when $N < n$. D can also be updated as N:

$$\begin{aligned}
D &= 3n+1-N && (67)\\
&= 3n+1-(\left\lfloor \frac{(3n+1-D)-n-1}{2} \right\rfloor + (3n+1-D)-n) && (68)\\
&= n + D - \left\lfloor \frac{2n-D}{2} \right\rfloor && (69)\\
&= D - \left\lfloor \frac{-D}{2} \right\rfloor && (70)\\
&= D + \left\lceil \frac{D}{2} \right\rceil && (71)\\
&= \left\lceil \frac{3D}{2} \right\rceil. && (72)
\end{aligned}$$

```python
import math
def J3(n):
    D = 1
    while D <= 2*n:
        D = math.ceil(D*3/2)
    return 3*n + 1 - D
```

Listing 2: Method 1

More general:

```python
import math
def J(n,q):
    D = 1
    while D <= (q-1)*n:
        D = math.ceil(D*q/(q-1))
    return q*n + 1 - D
```

Listing 3: Method 2

Write it into a recurrence:

$$D_0^{(q)} = 1; \tag{73}$$
$$D_n^{(q)} = \left\lceil \frac{q}{q-1} D_{n-1}^{(q)} \right\rceil. \tag{74}$$

3.4 'MOD': THE BINARY OPERATION

Define operator 'mod':

$$x \bmod y = x - y\lfloor x/y \rfloor. \qquad \{y \neq 0\} \tag{75}$$

Based on the defination, there are some attributes:

$$0 \leq x \bmod y < y; \qquad \{y > 0\} \tag{76}$$
$$0 \geq x \bmod y > y. \qquad \{y < 0\} \tag{77}$$

To complete the defination, we can let $x \bmod y = x$ when $y = 0$.
The 'mod' operator can be used to show the fractional part of a number:

$$x = \lfloor x \rfloor + x \bmod 1. \tag{78}$$

A similar 'mumble' operator can be defined:

$$x \text{ mumble } y = y\lceil x/y \rceil - x. \qquad \{y \neq 0\} \tag{79}$$

The 'mod' operator follows the distributive law:

$$c(x \bmod y) = (cx) \bmod (cy). \tag{80}$$

Problem: how to partition n things into m groups as equally as possible?
There will be $n \bmod m$ groups contains $\lceil n/m \rceil$ things and the rest contains $\lfloor n/m \rfloor$ things. It also can be converted into:

$$n = \left\lceil \frac{n}{m} \right\rceil + \left\lceil \frac{n-1}{m} \right\rceil + ... + \left\lceil \frac{n-m+1}{m} \right\rceil. \tag{81}$$

and if change n to $km + n \bmod m$, the equation can be converted into:

$$n = \left\lfloor \frac{n}{m} \right\rfloor + \left\lfloor \frac{n+1}{m} \right\rfloor + ... + \left\lfloor \frac{n+m-1}{m} \right\rfloor. \tag{82}$$

If $n = \lfloor mx \rfloor$:

$$\lfloor mx \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{m} \right\rfloor + ... + \left\lfloor x + \frac{m-1}{m} \right\rfloor. \tag{83}$$

3.5 FLOOR/CEILING SUMS

Example 1, let $a = \lfloor\sqrt{n}\rfloor$:

$$\begin{aligned}
\sum_{0\le k<n} \lfloor\sqrt{k}\rfloor &= \sum_{k,m\ge 0} m[m = \lfloor\sqrt{k}\rfloor][k<n] && (84)\\
&= \sum_{k,m\ge 0} m[k<n][m \le \sqrt{k} < m+1] && (85)\\
&= \sum_{k,m\ge 0} m[k<n][m^2 \le k < (m+1)^2 && (86)\\
&= \sum_{k,m\ge 0} m[m^2 \le k < (m+1)^2 \le n] + \sum_{k,m\ge 0} m[m^2 \le k < n < (m+1)^2] && (87)\\
&= \sum_{m\ge 0} m((m+1)^2 - m^2)[m+1 \le a] + \sum_{m\ge 0} m(a^2 \le k < n) && (88)\\
&= \sum_{m\ge 0} (2m^2+m)[m+1\le a] + a(n-a^2) && (89)\\
&= \sum_{m\ge 0} (2m^{\underline{2}} + 3m^{\underline{1}})[m<a] - a^3 + an && (90)\\
&= \sum\nolimits_0^a (2m^{\underline{2}} + 3m^{\underline{1}})\delta m - a^3 + an && (91)\\
&= \frac{2}{3}m^{\underline{3}} + \frac{3}{2}m^{\underline{2}} - a^3 + an && (92)\\
&= na - \frac{1}{3}a^3 - \frac{1}{2}a^2 - \frac{1}{6}a. && (93)
\end{aligned}$$

Anothe method is le $\lfloor x\rfloor = \sum_j [1 \le j \le x]$:

$$\begin{aligned}
\sum_{0\le k<n} \lfloor\sqrt{k}\rfloor &= \sum_{j,k} [1 \le j \le \sqrt{k}][0 \le k \le a^2] && (94)\\
&= \sum_{1\le j<a} \sum_{k} [j^2 \le k < a^2] && (95)\\
&= \sum_{1\le j<a} (a^2 - j^2) && (96)\\
&= na - \frac{1}{3}a^3 - \frac{1}{2}a^2 - \frac{1}{6}a. && (97)
\end{aligned}$$

Equidistribution theorem:

$$\lim_{x\to\infty} \frac{1}{n} \sum_{0\le k<n} f(\{k\alpha\}) = \int_0^1 f(x)dx. \tag{98}$$

for all irrational α and all functions f that are continuous almost everywhere.
Example 2, let $d = gcd(m,n)$:

$$\begin{aligned}
\sum_{0\le k<m} \left\lfloor \frac{nk+x}{m} \right\rfloor &= d\left\lfloor\frac{x}{d}\right\rfloor + \frac{m-1}{2}n + \frac{d-m}{2} && (99)\\
&= d\left\lfloor\frac{x}{d}\right\rfloor + \frac{mn}{2} - \frac{n}{2} - \frac{m}{2} + \frac{d}{2} && (100)\\
&= \sum_{0\le k<n} \left\lfloor \frac{mk+x}{n} \right\rfloor. && (101)
\end{aligned}$$

3.6 Exercises

Warmups 3.1: Let

$$n = 2^m + l. \qquad \{0 \le l < 2^m\} \tag{102}$$

Then:

$$m = \lceil \lg (n+1) \rceil; \tag{103}$$
$$l = n - 2^{\lceil \lg (n+1) \rceil}. \tag{104}$$

Warmups 3.2:

$$round_{down}(x) = \lfloor x + 0.5 \rfloor; \tag{105}$$
$$round_{up}(x) = \lceil x - 0.5 \rceil. \tag{106}$$

Warmups 3.3:

$$\left\lfloor \frac{\lfloor m\alpha \rfloor n}{\alpha} \right\rfloor = \left\lfloor \frac{m\alpha n - \{m\alpha\} n}{\alpha} \right\rfloor = mn - \left\lceil \frac{\{m\alpha\} n}{\alpha} \right\rceil = mn - 1. \tag{107}$$

Warmups 3.4: Pass.

Warmups 3.5:

$$\lfloor n\lfloor x \rfloor + n\{x\} \rfloor = n\lfloor x \rfloor \iff \lfloor n\{x\} \rfloor = 0 \iff 0 \le n\{x\} < 1 \iff \{x\} < \frac{1}{n}. \tag{108}$$

Warmups 3.6:

$$\begin{aligned} \lfloor f(\lceil x \rceil) \rfloor = A &\iff A \le f(\lceil x \rceil) < A + 1 & (109)\\ &\iff f^{-1}(A) \ge \lceil x \rceil > f^{-1}(A+1) & (110)\\ &\iff f^{-1}(A) \ge x > f^{-1}(A+1) & (111)\\ &\iff \lfloor f(\lceil x \rceil) \rfloor = \lfloor f(x) \rfloor. & (112) \end{aligned}$$

Warmups 3.7:

$$X_n = n; \qquad \{0 \le n < m\} \tag{113}$$
$$X_n = X_{n-m} + 1. \qquad \{n \ge m\} \tag{114}$$

Solution:

$$X_n = \left\lfloor \frac{n}{m} \right\rfloor + n \bmod m. \tag{115}$$

Warmups 3.8: If m boxes contains $< \lceil n/m \rceil$ elements:

$$n \le m(\lceil n/m \rceil - 1) \iff n/m + 1 \le \lceil n/m \rceil. \tag{116}$$

If m boxes contains $> \lfloor n/m \rfloor$ elements:

$$n \ge m(\lfloor n/m \rfloor + 1) \iff n/m - 1 \ge \lfloor n/m \rfloor. \tag{117}$$

These two statements contradict function 9 and 11.

Warmups 3.9: Because

$$\frac{m}{n} - \frac{1}{q} = \frac{m\lceil \frac{n}{m} \rceil - n}{nq} = \frac{n \text{ mumble } m}{nq}. \tag{118}$$

Then n mumble m is smaller than m and nq is larger than q. This means 1) it is possible to splite a fractional number into a number series; 2) The number series $1/x_1, 1/x_2, ...$ has distinct numbers; 3) This is a finite number series.

Basics 3.10:

$$\left\lceil \frac{2x+1}{2} \right\rceil - \left\lceil \frac{2x+1}{4} \right\rceil + \left\lfloor \frac{2x+1}{4} \right\rfloor = \lceil x+0.5 \rceil - [x \neq 2k-0.5]. \qquad \{k \text{ is a integer}\} \tag{119}$$

This means:

$$\left\lceil \frac{2x+1}{2} \right\rceil - \left\lceil \frac{2x+1}{4} \right\rceil + \left\lfloor \frac{2x+1}{4} \right\rfloor = \lceil x \rceil; \qquad \{x = 2k-0.5 \text{ or } \{x\} > 0.5\} \tag{120}$$

$$= \lfloor x \rfloor. \qquad \{\text{else}\} \tag{121}$$

Basics 3.11:

$$\alpha < n < \beta \iff \lfloor \alpha \rfloor < n < \lceil \beta \rceil. \tag{122}$$

The number of possible n is

$$(\lceil \beta \rceil - \lfloor \alpha \rfloor - 1)[\lceil \beta \rceil > \lfloor \alpha \rfloor]. \tag{123}$$

If $\alpha = \beta =$ integer then $\lceil \beta \rceil = \lfloor \alpha \rfloor$, this reults a wrong answer.

Basics 3.12: Prove

$$\left\lceil \frac{n}{m} \right\rceil = \left\lfloor \frac{n+m-1}{m} \right\rfloor \iff \left\lceil \frac{km + n \bmod m}{m} \right\rceil = \left\lfloor \frac{km + n \bmod m + m - 1}{m} \right\rfloor \tag{124}$$

$$\iff k + \left\lceil \frac{n \bmod m}{m} \right\rceil = k + 1 + \left\lfloor \frac{n \bmod m - 1}{m} \right\rfloor \tag{125}$$

$$\tag{126}$$

If $n \bmod m = 0$

$$\left\lceil \frac{n}{m} \right\rceil = \left\lfloor \frac{n+m-1}{m} \right\rfloor \iff k = k + 1 - 1. \tag{127}$$

Else

$$\left\lceil \frac{n}{m} \right\rceil = \left\lfloor \frac{n+m-1}{m} \right\rfloor \iff k + 1 = k + 1 - 0. \tag{128}$$

Basics 3.13: Because

$$N(\alpha, n) + N(\beta, n) = N(\frac{\beta}{\beta - 1}, n) + N(\beta, n) \tag{129}$$

$$= n + 1 + \left\lceil \frac{n+1}{\beta} \right\rceil - \left\lfloor \frac{n+1}{\beta} \right\rfloor - 2 \tag{130}$$

$$= n. \tag{131}$$

And it is easy to prove that if $\alpha \neq \beta$ then $Spec(\alpha) \neq Spec(\beta)$.

Basics 3.14:

$$(x \bmod ny) \bmod y = x - \left\lfloor \frac{x}{ny} \right\rfloor ny - \left\lfloor \frac{x - \left\lfloor \frac{x}{ny} \right\rfloor ny}{y} \right\rfloor y \tag{132}$$

$$= x - y(\left\lfloor \frac{x}{ny} \right\rfloor n - \left\lfloor \frac{x}{y} + \left\lfloor \frac{x}{ny} \right\rfloor n \right\rfloor) \tag{133}$$

$$= x - \left\lfloor \frac{x}{y} \right\rfloor y = x \bmod y. \tag{134}$$

Basics 3.15:

$$\lceil mx \rceil = \lceil x \rceil + \left\lceil x - \frac{1}{m} \right\rceil + ... + \left\lceil x - \frac{m-1}{m} \right\rceil. \tag{135}$$

Basics 3.16: Prove

$$n \bmod 2 = (1 - (-1)^n)/2. \tag{136}$$

It is true in both even and odd cases.
Solve

$$n \bmod 3 = a + bw^n + cw^{2n}. \qquad \{w = (-1 + i\sqrt{3})/2\} \tag{137}$$

Try $n = 0, 1, 2$:

$$a + b + c = 0; \tag{138}$$

$$a + bw + cw^2 = a + bw + c(-1 - w) = 1; \tag{139}$$

$$a + bw^2 + cw^4 = a + b(-1 - w) + cw = 2. \tag{140}$$

Solution is $a = 1$, $b = (-1 - w)/(1 + 2w) = (w - 1)/3$, $c = -1 - b = -(w + 2)/3$.
To prove this is easy, because $w^3 = 1$.

Basics 3.17:

$$\sum_j \sum_k [0 \le k < m][1 \le j \le x + k/m] \tag{141}$$

$$= \sum_j \sum_k [0 \le k < m][1 \le j \le \lceil x \rceil][j \le x + k/m] \tag{142}$$

$$= \sum_j \sum_k [0 \le k < m][1 \le j \le \lceil x \rceil][k \ge m(j - x)] \tag{143}$$

$$= \sum_{1 \le j \le \lceil x \rceil} \sum_k [0 \le k < m] - \sum_{j = \lceil x \rceil} \sum_k [0 \le k < m(j - x)] \tag{144}$$

$$= m\lceil x \rceil - \lceil m(\lceil x \rceil - x) \rceil \tag{145}$$

$$= \lfloor mx \rfloor. \tag{146}$$

Basics 3.18: pass.

Homework exercises 3.19: To let $f(x) = \lfloor log_b(x) \rfloor = \lfloor log_b(\lfloor x \rfloor) \rfloor$:

$$f(x) = \text{integer} \implies x = \text{integer}. \tag{147}$$

Then b should be an integer.

Homework exercises 3.20:

$$\sum_k [\alpha \le kx \le \beta] xk = x \sum_k [\alpha/x \le k \le \beta/x] k \tag{148}$$

$$= x \sum_k [\lceil \alpha/x \rceil \le k \le \lfloor \beta/x \rfloor] k \tag{149}$$

$$= \frac{1}{2} x (\lfloor \beta/x \rfloor \lfloor \beta/x \rfloor + \lfloor \beta/x \rfloor - \lceil \alpha/x \rceil \lceil \alpha/x \rceil + \lceil \alpha/x \rceil). \tag{150}$$

Homework exercises 3.21: Small cases $1, 16, 128, 1024, ...$ show that for a number which has k digits in decimal notation, there always exactly one number 2^m has leading 1. This is true because for any number n:

$$log_{10}(n) < \lfloor log_{10}(n) + 1 \rfloor \iff 2n < n + 10^{\lfloor log_{10}(n) + 1 \rfloor}. \tag{151}$$

Then solution is $\lfloor Mlog_{10}2 \rfloor + 1$.

Homework exercises 3.22:

$$S_n = \sum_{k \ge 1} \left\lfloor \frac{n}{2^k} + \frac{1}{2} \right\rfloor. \tag{152}$$

Small cases show that $S_n = n$. Let $n = m_k 2^k + d$

$$\left\lfloor \frac{n}{2^k} + \frac{1}{2} \right\rfloor \neq \left\lfloor \frac{n-1}{2^k} + \frac{1}{2} \right\rfloor \iff \tag{153}$$

$$m_k + \left\lfloor \frac{d}{2^k} + \frac{1}{2} \right\rfloor \neq m_k + \left\lfloor \frac{d-1}{2^k} + \frac{1}{2} \right\rfloor \iff \tag{154}$$

$$d = 2^{k-1}. \tag{155}$$

Consider S_n and S_{n-1}, there always one and only one k to make $n \bmod (2^k) = 2^{k-1}$. Let $n = 2^{k'} q$ where q is a odd, here k' is the one. Then $S_n = S_{n-1} + 1 = n$.

$$T_n = \sum_{k \ge 1} 2^k \left\lfloor \frac{n}{2^k} + \frac{1}{2} \right\rfloor^2. \tag{156}$$

From the observation above, let $n = 2^{k'} q$ where q is a odd. The different terms between T_n and T_{n-1} is $\left\lfloor \frac{2^{k'} q}{2^{k'+1}} + \frac{1}{2} \right\rfloor$ and $\left\lfloor \frac{2^{k'} q - 1}{2^{k'+1}} + \frac{1}{2} \right\rfloor$. Then:

$$T_n - T_{n-1} = 2^{k'+1}\left(\left\lfloor \frac{2^{k'} q}{2^{k'+1}} + \frac{1}{2} \right\rfloor^2 - \left\lfloor \frac{2^{k'} q - 1}{2^{k'+1}} + \frac{1}{2} \right\rfloor^2 \right) \tag{157}$$

$$= 2^{k'+1}\left(\left\lfloor \frac{q}{2} + \frac{1}{2} \right\rfloor^2 - \left\lfloor \frac{q}{2} - \frac{1}{2^{k'+1}} + \frac{1}{2} \right\rfloor^2 \right) \tag{158}$$

$$= 2^{k'+1}\left(\left\lfloor \frac{q}{2} + \frac{1}{2} \right\rfloor^2 - \left\lfloor \frac{q}{2} \right\rfloor^2 \right) \tag{159}$$

$$= 2^{k'+1}\left(\left\lfloor \frac{q}{2} + \frac{1}{2} \right\rfloor - \left\lfloor \frac{q}{2} \right\rfloor \right)\left(\left\lfloor \frac{q}{2} + \frac{1}{2} \right\rfloor + \left\lfloor \frac{q}{2} \right\rfloor \right) \tag{160}$$

$$= 2^{k'+1} q = 2n. \tag{161}$$

Then $T_n = n(n+1)$.

Homework exercises 3.23: The nth elements should be $X_n = m$

$$\frac{1}{2}m(m-1) < n \le \frac{1}{2}m(m+1) \iff \tag{162}$$

$$m(m-1) < 2n \le m(m+1) \iff \tag{163}$$

$$m(m-1) + \frac{1}{4} < 2n < m(m+1) + \frac{1}{4} \iff \tag{164}$$

$$m - \frac{1}{2} < \sqrt{2n} < m + \frac{1}{2} \tag{165}$$

$$m < \sqrt{2n} + \frac{1}{2} \tag{166}$$

$$m = \left\lfloor \sqrt{2n} + \frac{1}{2} \right\rfloor. \tag{167}$$

Homework exercises 3.24: $Spec(\alpha/(\alpha+1)) = Spec(\alpha) + n + 1$.

Homework exercises 3.25: Can prove $K_n > n$ with induction.

The initial cases are true:

$$K_0 = 1; \tag{168}$$
$$K_1 = 3. \tag{169}$$

Assume that $K_i > i$ when $i \le n$, then:

$$\begin{aligned} K_{n+1} &= 1 + min(2K_{\lfloor n/2 \rfloor}, 3K_{\lfloor n/3 \rfloor}) && (170)\\ &\ge 1 + min(2(\lfloor n/2 \rfloor + 1), 3(\lfloor n/3 \rfloor + 1)) && (171)\\ &\ge 1 + min(n+1, n+1) && (172)\\ &> n+1. && (173) \end{aligned}$$

Proved.

Homework exercises 3.26: For the left side:

$$\begin{aligned} D_n^{(q)} = \left\lceil \frac{q}{q-1} D_{n-1}^{(q)} \right\rceil &\iff && (174)\\ \frac{q}{q-1} D_{n-1}^{(q)} &\le D_n^{(q)} && (175)\\ (\frac{q}{q-1})^n D_0^{(q)} &\le D_n^{(q)} && (176)\\ (\frac{q}{q-1})^n &\le D_n^{(q)} && (177) \end{aligned}$$

For the right side, can prove with induction:

$$D_n^{(q)} \le q(\frac{q}{q-1})^n - q + 1. \quad \{\text{this is wired but can be guessed from the initial case.}\} \tag{178}$$

The initial case is true:

$$D_0^{(q)} = 1 \ge q - q + 1. \tag{179}$$

Assume that it is true when $i \le n$, then:

$$\begin{aligned} D_{n+1}^{(q)} &= \left\lceil \frac{q}{q-1} D_{n-1}^{(q)} \right\rceil && (180)\\ &\le \left\lceil q(\frac{q}{q-1})^{n+1} - q \right\rceil && (181)\\ &\le q(\frac{q}{q-1})^{n+1} - q + 1. && (182) \end{aligned}$$

Proved.

Homework exercises 3.27: Let $D_n^{(3)} = 2^m b - a$ where $a = 0$ or 1 and b is odd, then:

$$D_{n+1}^{(3)} = \left\lceil \frac{3}{2}(2^m b - a) \right\rceil = 3 * 2^{m-1} b - \left\lfloor \frac{3}{2} a \right\rfloor = 3 * 2^{m-1} b - a; \tag{183}$$
$$D_{m+n}^{(3)} = 3^m b - a. \tag{184}$$

This formula shows another way to generate the next number in the $D_n^{(3)}$. This way does not generate the next number one by one which means the formula only generates a subset.
Let the seed number is $D_0^{(3)} = 2^1 - 1 = 1$, then $a = 1$. The numbers generated from the seed contians infinite numbers and these numbers are arranged in the order: odd, even, odd, even, ...

Homework exercises 3.28: Not solved yet.
From an initial number $a_n = m^2$, all numbers can be generated:

$$a_{n+2k+1} = (m+k)^2 + m - k; \qquad \{0 \le k \le m\} \tag{185}$$
$$a_{n+2k+2} = (m+k)^2 + 2m; \qquad \{0 \le k \le m\} \tag{186}$$
$$a_{n+2m+1} = 4m^2. \tag{187}$$

Then a_{n+2m+1} is the next initial number.

Homework exercises 3.29: pass.

Homework exercises 3.30: Small cases show that

$$X_n = \alpha^{2^n} + \alpha^{-2^n}. \tag{188}$$

It is easy to prove this with the induction.
$\alpha^{-2^n} < 1$ because $\alpha > 1$ which means the integer X_n should be $\lceil \alpha^{2^n} \rceil$.

Homework exercises 3.31: A really smart solution:

$$\begin{aligned} \lfloor x \rfloor + \lfloor y \rfloor + \lfloor x+y \rfloor &= \lfloor x + \lfloor y \rfloor \rfloor + \lfloor x+y \rfloor & (189)\\ &\le \left\lfloor x + \frac{1}{2}\lfloor 2y \rfloor \right\rfloor + \left\lfloor x + \frac{1}{2}\lfloor 2y \rfloor + \frac{1}{2} \right\rfloor & (190)\\ &= \lfloor 2x + \lfloor 2y \rfloor \rfloor = \lfloor 2x \rfloor + \lfloor 2y \rfloor. & (191) \end{aligned}$$

Homework exercises 3.32:

$$f(x) = \sum_k 2^k || \frac{x}{2^k} ||^2. \qquad \{||x|| = \min x - \lfloor x \rfloor, \lceil x \rceil - x\} \tag{192}$$

Firstly, consider some basic rules.
$f(x) = f(-x)$ because $||x|| = ||-x||$. Also:

$$f(2x) = \sum_k 2^k || \frac{2x}{2^k} ||^2 = 2\sum_k 2^{k-1} || \frac{x}{2^{k-1}} ||^2 = 2f(x). \tag{193}$$

Secondly, consider x in $0 \le x < 1$.
Let $f(x) = l(x) + r(x)$, where $l(x)$ is the sum when $k \le 0$ and $r(x)$ is the sum when $k > 0$.

$$l(x+1) = \sum_{k\le 0} 2^k || \frac{x+1}{2^k} ||^2 = \sum_{k\le 0} 2^k || \frac{x}{2^k} + 2^{-k} ||^2 = \sum_{k\le 0} 2^k || \frac{x}{2^k} ||^2 = l(x). \tag{194}$$

Because $||x|| \le 0.5$ then:

$$l(x) \le \sum_{k\le 0} 2^k (\frac{1}{2})^2 = \frac{1}{2}. \tag{195}$$

Then $r(x+1) = r(x) + 1$ because:

$$\begin{aligned} r(x) &= \frac{x^2}{2} + \frac{x^2}{4} + \frac{x^2}{8} + ... = x^2. & (196)\\ r(x+1) &= 2|| \frac{x+1}{2} ||^2 + \frac{x^2}{4} + \frac{x^2}{8} + ... & (197)\\ &= \frac{(x-1)^2}{2} + \frac{(x+1)^2}{4} + \frac{(x+1)^2}{8} + ... & (198)\\ &= \frac{(x-1)^2}{2} + \frac{(x+1)^2}{2} = x^2 + 1. & (199) \end{aligned}$$

Then $f(x+1) = l(x+1) + r(x+1) = l(x) + r(x) + 1 = f(x) + 1$ when $0 \le x < 1$. It also shows that $f(x+n) = f(x) + n$ and $f(0) = 0$ means $f(n) = n$. Finally small cases show that $f(x) = |x|$, try to prove that when $0 \le x < 1$:

$$\begin{aligned} f(x) &= 2^{-m} f(2^m x) & (200)\\ &= 2^{-m}(f(\lfloor 2^m x \rfloor + \{2^m x\}) & (201)\\ &= 2^{-m}\lfloor 2^m x \rfloor + 2^{-m} f(\{2^m x\}). \qquad \{m \text{ is any integer}\} & (202) \end{aligned}$$

Because

$$f(\{2^m x\}) = l(\{2^m x\}) + r(\{2^m x\}) = l(\{2^m x\}) + (\{2^m x\})^2 \leq 0.5 + 1 = 1.5. \tag{203}$$

Then

$$\begin{align}
|f(x) - x| &= |2^{-m}\lfloor 2^m x\rfloor + 2^{-m} f(\{2^m x\}) - x| \tag{204}\\
&= |2^{-m}\lfloor 2^m x\rfloor - x| + 2^{-m} f(\{2^m x\}) \tag{205}\\
&\leq |2^{-m}\lfloor 2^m x\rfloor - x| + 2^{-m}\frac{3}{2} \tag{206}\\
&= |2^{-m}(\lfloor 2^m x\rfloor - 2^m x)| + 2^{-m}\frac{3}{2} \tag{207}\\
&\leq 2^{-m}\frac{5}{2}. \tag{208}
\end{align}$$

Because m could be any integer, then there is $|f(x) - x| = 0$ which means $f(x) = |x|$ when $0 \leq x < 1$. In summary, because $f(x) = f(-x)$ and $f(x) + n = f(x + n)$, then $f(x) = |x|$ for all real x.

Exam problems 3.33: a.Split the area into for parts, and consider the top right square. Because r is a fractional number, corners are not crossed. The circle edge could be treated as a path go from the top to the right. Because no left or up steps, then the step number is $r + r = 2r = 2n - 1$. Then the number of cells of the board contain a segment of the circle is $4(2n - 1) = 8n - 4$.
b.This is the Guass's Circle Problem, and I have no idea why:

$$f(n, k) = \lfloor\sqrt{r^2 - k^2}\rfloor. \tag{209}$$

Exam problems 3.34: a.Small cases $0, 1, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 4, 4, 4, 5, ...$ show that it is a good idea to estimate $f(n)$ when n reached the last number in its level. Let $m = \lceil\lg n\rceil$, and there will be 2^m numbers if the last level is full.

$$\begin{align}
f(n) + (2^m - n)m &= \sum_{k=1}^{2^m}\lceil\lg k\rceil \tag{210}\\
&= \sum_{j,k} j[j = \lceil\lg k\rceil][1 \leq k \leq 2^m] \tag{211}\\
&= \sum_{j,k} j[2^{j-1} < k \leq 2^j][1 \leq j \leq m] \tag{212}\\
&= \sum_{1\leq j\leq m} j2^{j-1} = m2^m - 2^m + 1. \tag{213}
\end{align}$$

Then $f(n) = mn + 2^m - 1$.
b.When $n = 2k$:

$$\begin{align}
f(k) &= \lceil\lg k\rceil k + 2^{\lceil\lg k\rceil} - 1; \tag{214}\\
f(2k) &= \lceil\lg 2k\rceil 2k + 2^{\lceil\lg 2k\rceil} - 1 \tag{215}\\
&= \lceil\lg k + 1\rceil 2k + 2^{\lceil\lg k+1\rceil} - 1 \tag{216}\\
&= 2\lceil\lg k\rceil k + 2k + 2 * 2^{\lceil\lg k\rceil} - 1; \tag{217}\\
f(2k) &= 2f(k) + 2k - 1; \tag{218}\\
f(n) &= n - 1 + f(\lceil n/2\rceil) + f(\lfloor n/2\rfloor). \tag{219}
\end{align}$$

When $k = 2^m + 1$ and $n = 2k - 1$. Then $m = \lceil\lg k - 1\rceil$ and $m + 1 = \lceil\lg k\rceil$:

$$\begin{align}
f(\left\lfloor\frac{2k-1}{2}\right\rfloor) &= f(k - 1) = m(k - 1) - 2^m + 1; \tag{220}\\
f(\left\lceil\frac{2k-1}{2}\right\rceil) &= f(k) = (m + 1)k - 2 * 2^m + 1; \tag{221}\\
f(2k - 1) &= 2mk - m + 4k - 4 * 2^m - 1. \tag{222}
\end{align}$$

Then

$$f(2k-1) = f(\left\lfloor \frac{2k-1}{2} \right\rfloor) + f(\left\lceil \frac{2k-1}{2} \right\rceil) + 2k - 1 - 1. \tag{223}$$

When $k \neq 2^m + 1$ and $n = 2k - 1$. Then $m = \lceil \lg k - 1 \rceil = \lceil \lg k \rceil$:

$$f(\left\lfloor \frac{2k-1}{2} \right\rfloor) = m(k-1) - 2^m + 1; \tag{224}$$

$$f(\left\lceil \frac{2k-1}{2} \right\rceil) = mk - 2^m + 1; \tag{225}$$

$$f(2k-1) = 2mk - m + 2k - 2 * 2^m. \tag{226}$$

Then

$$f(2k-1) = f(\left\lfloor \frac{2k-1}{2} \right\rfloor) + f(\left\lceil \frac{2k-1}{2} \right\rceil) + 2k - 1 - 1. \tag{227}$$

Proved.

Exam problems 3.35:

$$\begin{aligned}
\lfloor (n+1)^2 n! e \rfloor &= \left\lfloor (n+1)^2 n! \sum_{k=0}^{\infty} \frac{1}{k!} \right\rfloor && (228)\\
&= \left\lfloor (\frac{n!}{0!} + \frac{n!}{1!} + ... + \frac{n!}{n!} + \frac{n!}{(n+1)!} + ... + \frac{n!}{\infty!})(n+1)^2 \right\rfloor && (229)\\
&= \left\lfloor (n+1)^2 n((n-1)! + ... + 1) + (n+1)^2 + (n+1) + \frac{n+1}{n+2} + ... + \frac{n!(n+1)^2}{\infty!} \right\rfloor && (230)\\
& && (231)
\end{aligned}$$

Add the mod operator:

$$\begin{aligned}
\lfloor (n+1)^2 n! e \rfloor \bmod n &= \left\lfloor (n+1)^2 + (n+1) + \frac{n+1}{n+2} + ... + \frac{n!(n+1)^2}{\infty!} \right\rfloor \bmod n && (232)\\
&= \left\lfloor 2 + \frac{n+1}{n+2} + ... + \frac{n!(n+1)^2}{\infty!} \right\rfloor \bmod n && (233)
\end{aligned}$$

Because

$$\begin{aligned}
\frac{n+1}{n+2} + ... + \frac{n!(n+1)^2}{\infty!} &= \frac{n+1}{n+2}(1 + \frac{1}{n+3} + \frac{1}{(n+3)(n+4)} + ...) && (234)\\
&< \frac{n+1}{n+2}(1 + \frac{1}{n+3} + \frac{1}{(n+3)(n+3)} + ...) && (235)\\
&= \frac{(n+1)(n+3)}{(n+2)^2} < 1 && (236)
\end{aligned}$$

Then result is 2 mod n.

Exam problems 3.36:

$$\begin{aligned}
\sum_{1<k<2^{2^n}} \frac{1}{2^{\lfloor \lg k \rfloor} 4^{\lfloor \lg \lg k \rfloor}} &= \sum_{k,l,m} 2^{-l} 4^{-m} [m = \lfloor \lg l \rfloor][l = \lfloor \lg k \rfloor][1 < k < 2^{2^n}] && (237)\\
&= \sum_{k,l,m} 2^{-l} 4^{-m} [2^m \leq l < 2^{m+1}][2^l \leq k < 2^{l+1}][0 \leq m < n] && (238)\\
&= \sum_{l,m} 4^{-m} [2^m \leq l < 2^{m+1}][0 \leq m < n] && (239)\\
&= \sum_{m} 2^{-m} [0 \leq m < n] && (240)\\
&= 1 + \frac{1}{2} + ... + \frac{1}{2^{n-1}} = 2(1 - 2^{-n}). && (241)
\end{aligned}$$

Exam problems 3.37:

Exam problems 3.38:

Exam problems 3.39:

Exam problems 3.40: