Linear Algebra. Final - Variant#2

Problem 1: Write down three equations for the line, which is go through points (1, 7), (-1, 7), (-2, 21), find least squares solution and draw the closest line. (**5** points)

$$y = 9 - 4x$$

Scoring: 1 point for writing down system of equations, 3 points for solving it with least squares, 1 point for the plot.

Problem 2: Is A a projection matrix? If so, find basis of the subspace of R^3 onto which A is projecting, and the basis of its orthogonal complement. (**6** points):

$$A = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Here we have matrix A already decomposed into its eigenvalues and eigenvectors. So, we see that vector $[1, 0, -1]^T$is not modified by A (it has eigenvalue 1), and vectors $[0, 1, 0]^T$ and $[1, 0, 1]^T$ are brought down to zero (both have $\lambda = 0$).

So, A projects onto a line, basis being $\{[1, 0, -1]^T\}$, and the orthogonal complement is a plane with basis $\{[1, 0, 1]^T, [0, 1, 0]^T\}$.

Scoring: 3 points for one basis, 3 points for the other.

Problem 3: What values of α produce stability $v_{n+1} = \alpha(v_n - w_n), w_{n+1} = \alpha(w_n - v_n)$. (**6** points).
Solution is **stable** if all eigenvalues have absolute value less than 1: $|\lambda_i| < 1$. If one or more $|\lambda_i| = 1$, than it is **neutrally stable**.
In our case, eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 2\alpha$. So, $|\alpha| < 0.5$ or $|\alpha| \le 0.5$ are both acceptable solutions.
Scoring: no strict criteria.

Problem 4: Matrix A has singular values 10, 9, and 1. Using SVD, we can find the best rank 2 approximation of A: the matrix $\tilde{A}$. What is upper bound of $|A\boldsymbol{x} - \tilde{A}\boldsymbol{x}|$ for all possible unit vectors $\boldsymbol{x}$? (**6** points).

The idea is straightforwardly taken from section 6.7 of "Introduction to Linear Algebra" by Strang, specifically "Image Compression" paragraph. To get $\tilde{A}$, we must perform SVD decomposition of A, and then replace all but two largest singular values with zero. Since our matrix has three eigenvalues, this means:

$$A = U\Sigma V^T = u_1\sigma_1 v_1^T + u_2\sigma_2 v_2^T + u_3\sigma_3 v_3^T$$

$$\tilde{A} = U\tilde{\Sigma} V^T = u_1\sigma_1 v_1^T + u_2\sigma_2 v_2^T$$

Therefore, $A - \tilde{A} = u_3\sigma_3 v_3^T = \sigma_3 u_3 v_3^T$, where u_3 and v_3 are two unit vectors, and $\sigma_3 = 1$.

Now, $Ax - \tilde{A}x = u_3 v_3^T x = u_3(v_3^T x)$, where $v_3^T x$ is dot-product of two unit vectors – so it has value in range $[-1,1]$, and u_3 is another unit vector. We have unit vector, multiplied by $\sigma_3 = 1$, and then multiplied by some number in range $[-1,1]$. The result is a vector with length between zero and 2, so upper bound of $|Ax - \tilde{A}x|$ is equal to 1.

Scoring: 4 points for writing down expression for $A - \tilde{A}$, 2 points for finishing the job.

Problem 5: Find A^k for the matrix $A = \begin{bmatrix} 6 & 9 \\ 1 & 6 \end{bmatrix}$. (**5** points)

$\lambda_1 = 3, \lambda_2 = 9$, and so on:

$$A = \begin{bmatrix} -3 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 9 \end{bmatrix}^k \begin{bmatrix} -1/3 & 1 \\ 1/3 & 1 \end{bmatrix} = \begin{bmatrix} 3^k + 9^k & 3^{2k+1} - 3^{k+1} \\ 3^{2k-1} - 3^{k-1} & 3^k + 9^k \end{bmatrix}$$

Scoring: 1 point for eigenvalues, 3 points for writing down A as product of three matrices, 1 point for actually computing A.

Problem 6: Transformation T first performs rotation by π radians clockwise, then translation by vector (-1,1), then another rotation by π radians (counterclockwise), another translation by (-1,1), and finally performs projection on axis OY, and takes the length of the resulting vector. Find matrix corresponding to T, or show why it can not be done. (**6** points)

All these translations and rotations do not matter – they cancel each other. So, only projection and length matter. Projection P is linear transformation. Taking the length of vector L usually is *not* a linear transformation, but in this case, length is equal to y coordinate, and taking the value of a particular vector coordinate *is* linear transformation:

$$T = LP = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Scoring: 3 points for P, 3 points for L.

Problem 7: Solve the differential equation, (**6** points):

$$5y'' - 2y = 0, y(0) = 1, y'(0) = 1.$$

What happens to y(t) as $t \to \infty$?

$$\frac{d}{dt}\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 0.4 & 0 \end{bmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}$$

$$u(t) = \frac{\sqrt{2} + \sqrt{5}}{2\sqrt{2}} e^{\sqrt{0.4}t} \begin{pmatrix} 1 \\ \sqrt{0.4} \end{pmatrix} + \frac{\sqrt{2} - \sqrt{5}}{2\sqrt{2}} e^{-\sqrt{0.4}t} \begin{pmatrix} 1 \\ -\sqrt{0.4} \end{pmatrix}$$

$$y(t) = \frac{(\sqrt{2} + \sqrt{5})e^{\sqrt{0.4}t} + (\sqrt{2} - \sqrt{5})e^{-\sqrt{0.4}t}}{2\sqrt{2}} \to \infty$$

Common mistake here: you can not write stuff like this:

$$\frac{d}{dt}\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0.4 \end{bmatrix} \begin{pmatrix} y' \\ y \end{pmatrix}$$

You either have $(y'\ y)^T$ or $(y\ y')^T$ – not both, because they are different variables!

Scoring: 4 points for solving differential equation, 2 points for finding $y(\infty)$ (only 1 point if that's $u(\infty)$).