3. **Cool application I: Sums of odd perfect squares.** Can a sum of two perfect squares be another perfect square? Sure; for example, $3^2 + 4^2 = 5^2$, $5^2 + 12^2 = 13^2$, $6^2 + 8^2 = 10^2$, $7^2 + 24^2 = 25^2$. However, no matter how much you try, you won't find any examples in which the two perfect squares on the left are both odd. Your task is to prove this, i.e.:

   Prove that a sum of two odd perfect squares is never a perfect square.

   (An interesting consequence of this result is that in any right triangle in which all sides have integer length at least one of the two shorter sides must be of even length.)

   Proof: *The argument given below makes use of the results of Problems 1 and 2. Without using these results, the proof would become a bit longer. One would need to consider separately the cases when n, m, p below have a specified parity (even or odd), and show that in each of these cases a contradiction arises.*

   We first restate the result to be proved in a more explicitly form:

   $(*)$ *If s and t are odd perfect squares, then $s + t$ is not a perfect square.*

   Proof of $(*)$: *We use the method of contradiction.*

   - *Suppose s and t are odd perfect squares, and assume that $s + t$ is also a perfect square.*
   - *Then $s = n^2$, $t = m^2$, and $s + t = p^2$, for some $n, m, p \in \mathbb{Z}$, by the definition of a perfect square.*
   - *Since, by assumption, $s = n^2$ and $t = m^2$ are odd, the integers n and m must be odd as well (by Problem 2).*
   - *Hence $n = 2k + 1$ and $m = 2l + 1$ for some $k, l \in \mathbb{Z}$, by the definition of an odd integer.*
   - *Since the sum of two odd numbers is even, $s + t = p^2$ is even.*
   - *Hence p, must be even as well.*
   - *Therefore $p = 2h$ for some $h \in \mathbb{Z}$.*
   - *Thus we have*

   $$s = n^2 = (2k+1)^2, \quad t = m^2 = (2l+1)^2, \quad s + t = p^2 = (2h)^2.$$

- *Equating the two expressions for $s + t$ and simplifying, we get*

$$\begin{aligned}
(2k+1)^2 + (2l+1)^2 &= (2h)^2, \\
4k^2 + 4k + 1 + 4l^2 + 4l + 1 &= 4h^2, \\
4(k^2 + k + l^2 + l) + 2 &= 4h^2, \\
k^2 + k + l^2 + l - h^2 &= -\frac{1}{2}.
\end{aligned}$$

- *In the latter equation the left side is an integer, while the right-hand side is not an integer.*
- *Thus we have arrived at a contradiction.*
- *Therefore our assumption that $s + t$ is a perfect square is false, and we have shown that if s and t are odd perfect squares, then $s + t$ cannot be not a perfect square.* ■