Probability Theory & Statistics

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Part I

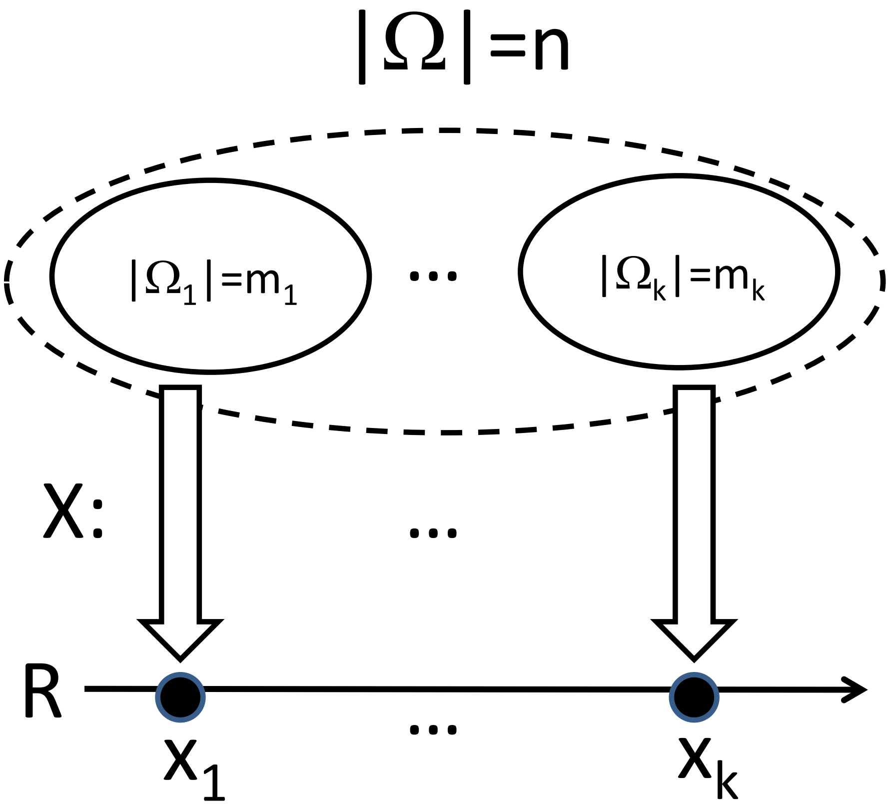
ANATOMY OF INDEPENDENCE

Back to a problem from Mid-Term

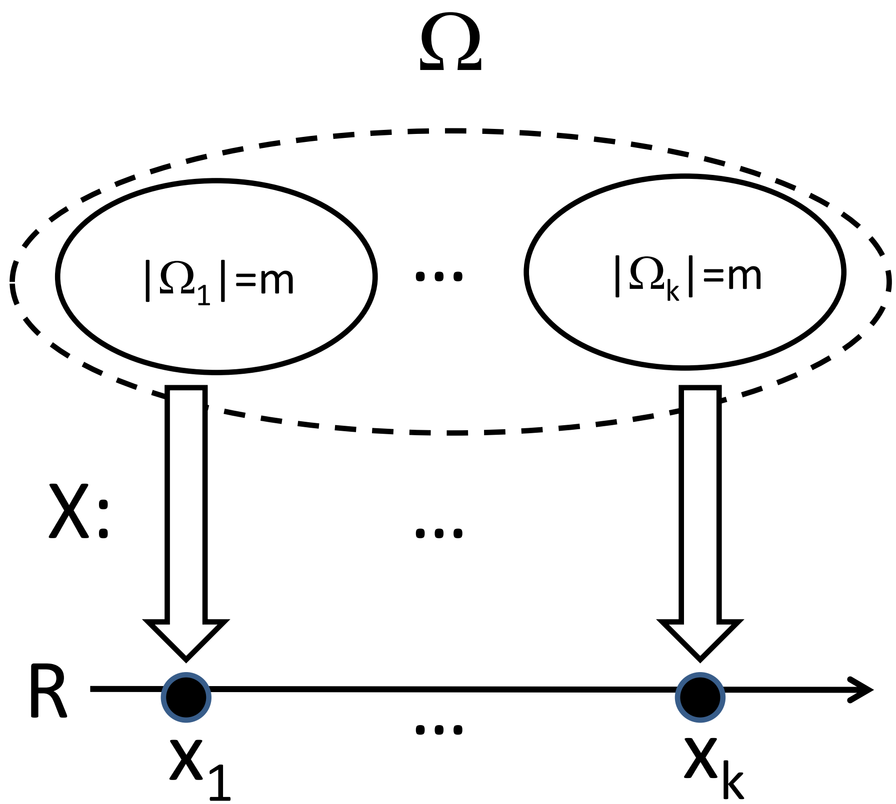
- Assume that Ω is a finite set of outcomes of a random variable X with uniform probability distribution and $|\Omega|= n>0$. What can be size of the image $X(\Omega)$ (i.e. how many element this set may have)?

How to solve it

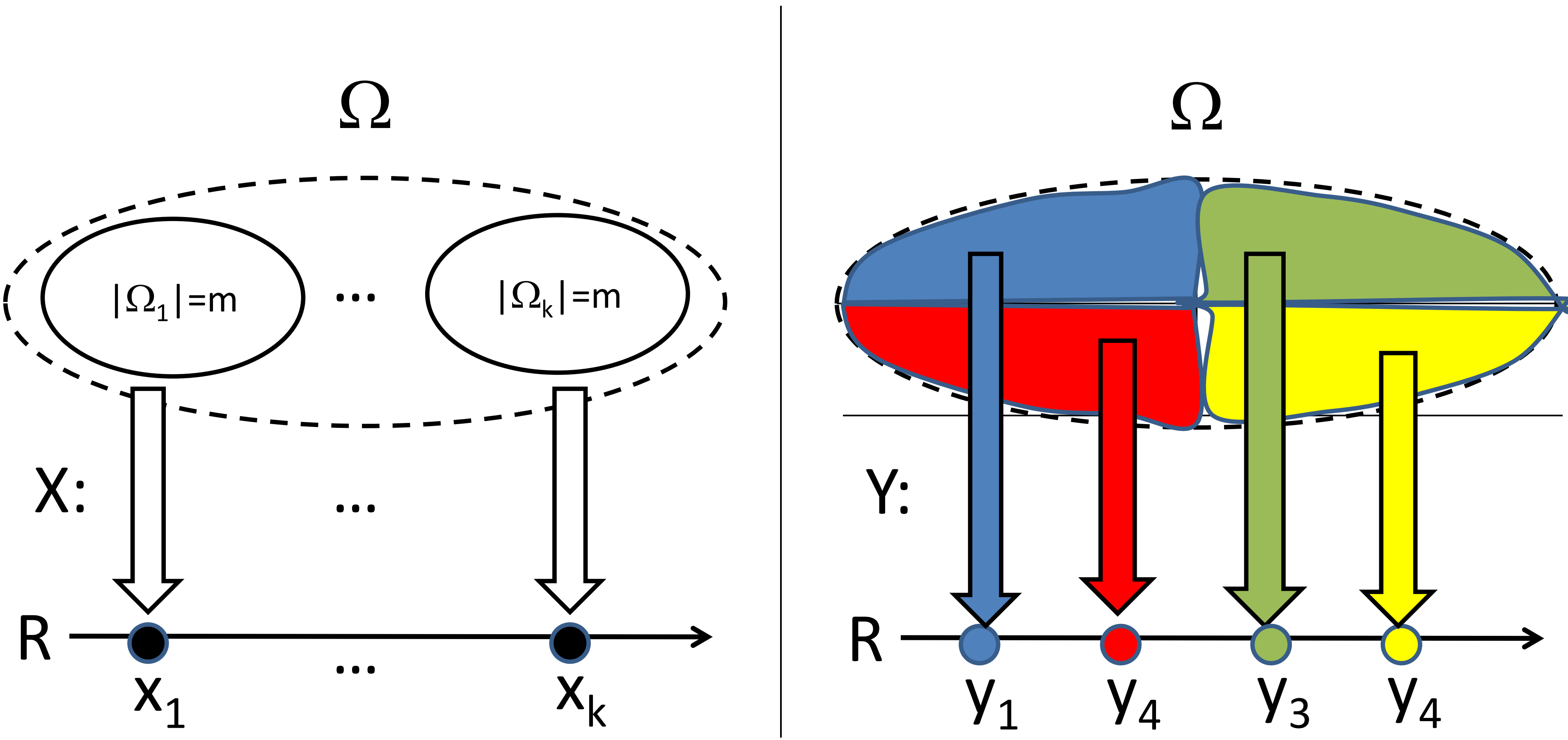
- $m_1 = \ldots = m_k$ since X is uniform (flat);
- hence $m = m_1 = \ldots = m_k$ divides n (notation: $m|n$),
- and k, the size of $X(\Omega)$, also divides n ($k|n$).



Anatomy of Discrete Uniform Distribution



What if X and Y are independent?



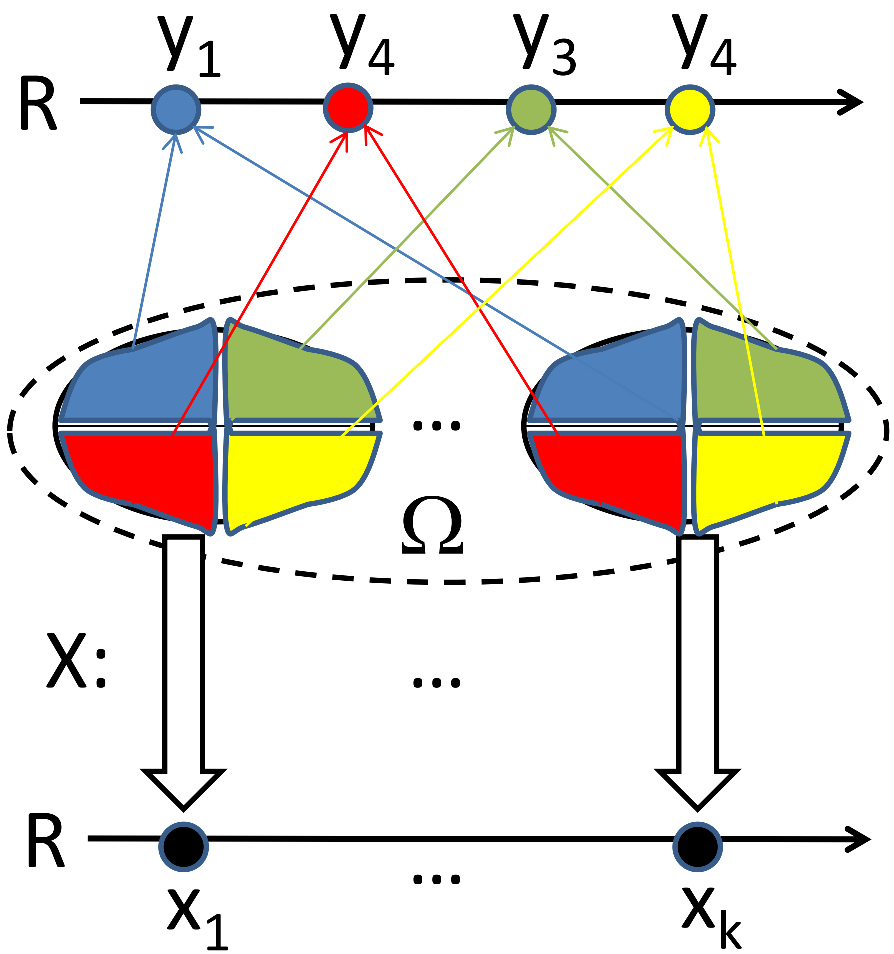
Anatomy of two Independent Uniform Distributions

For each Ω_j

the ratio of

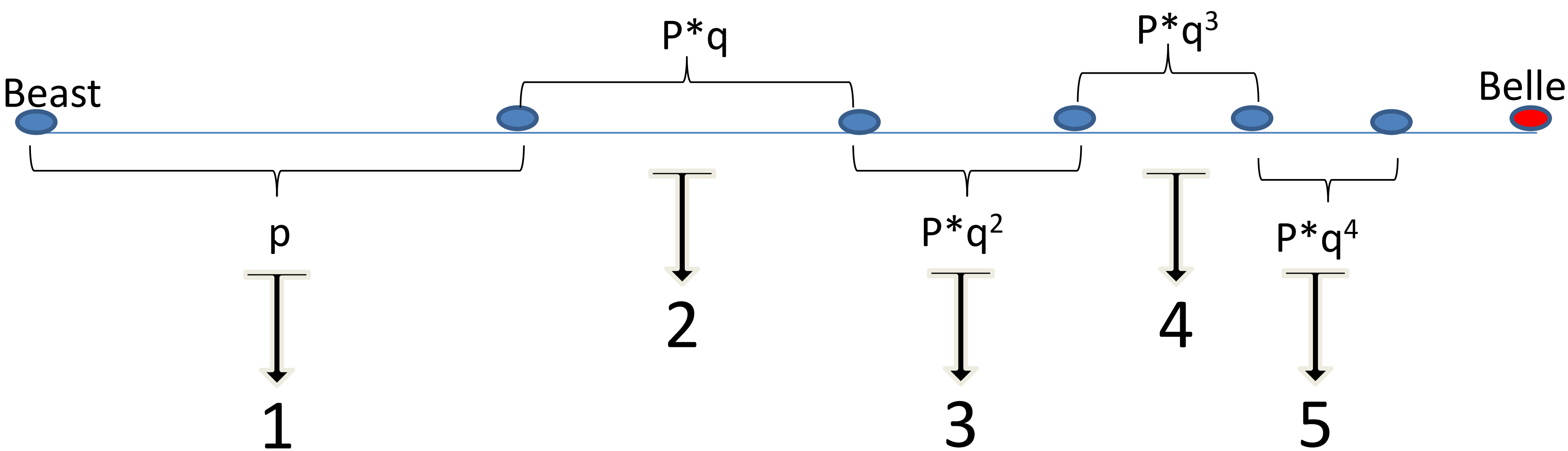
- blue within Ω_j to m,
- green within Ω_j to m,
- yellow within Ω_j to m,
- red within Ω_j to m

is 1/4.



Back to Beauty and the Beast example (from lecture for week 7)

- A staircase function $X(t)=k$ on $[(1-q^{(k-1)}), (1-q^k))$ is a random variable with outcomes $[0,1)$ and geometric distribution:



Back to Beauty and the Beast example (cont.)

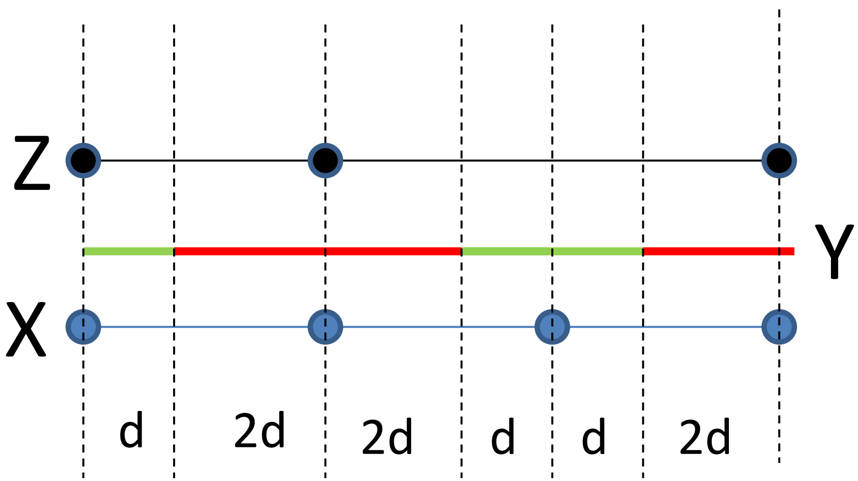
- Question: Build any random variable Y variable with the same outcomes [0,1) and the geometric distribution that is independent with the above random variable X.

Part II

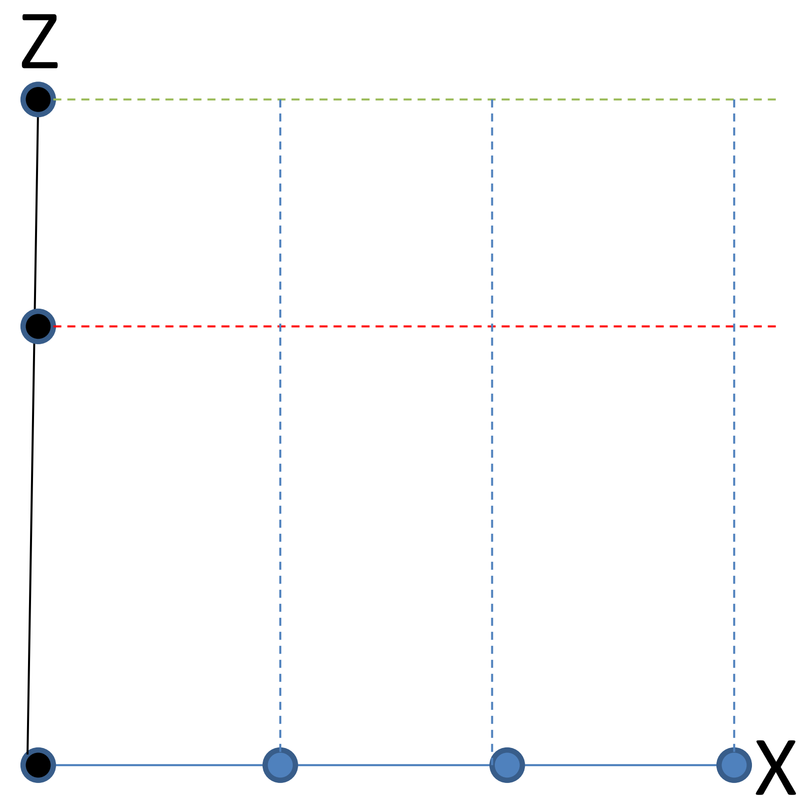
MULTIVARIATE DISCRETE DISTRIBUTIONS

Distinguish please…

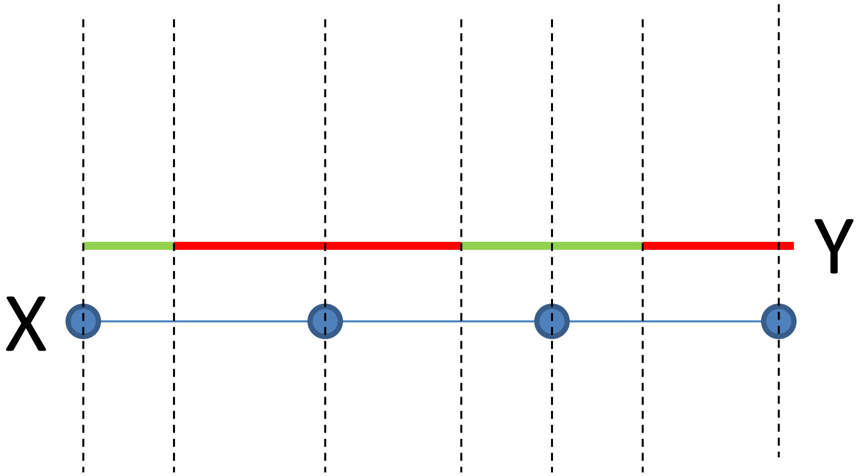
Equally distributed random variables X, Y, Z over Ω:



A random variable g(X,Z) over $\Omega\times\Omega$:



Two independent variables over Ω

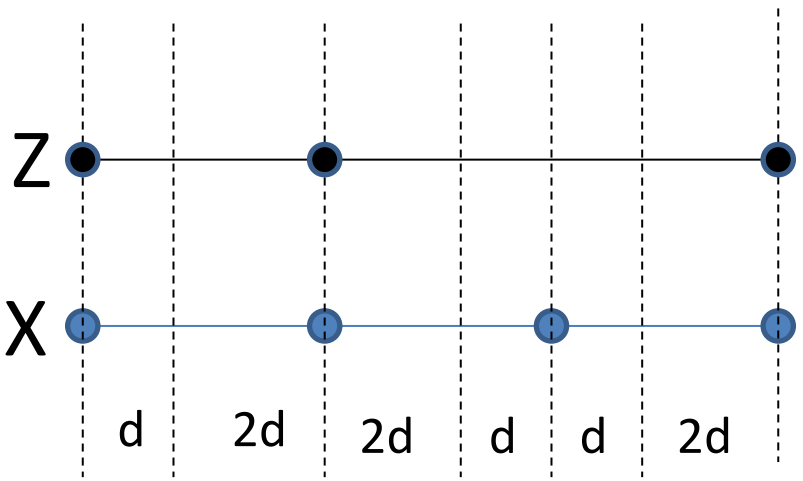


Sample distributions:

- $P_X(0) = P_X(1) =$
  $= P_X(2) = 1/3$
- $P_Y(0) = 2/3$,
  $P_Y(1) = 1/3$.

Two exercises

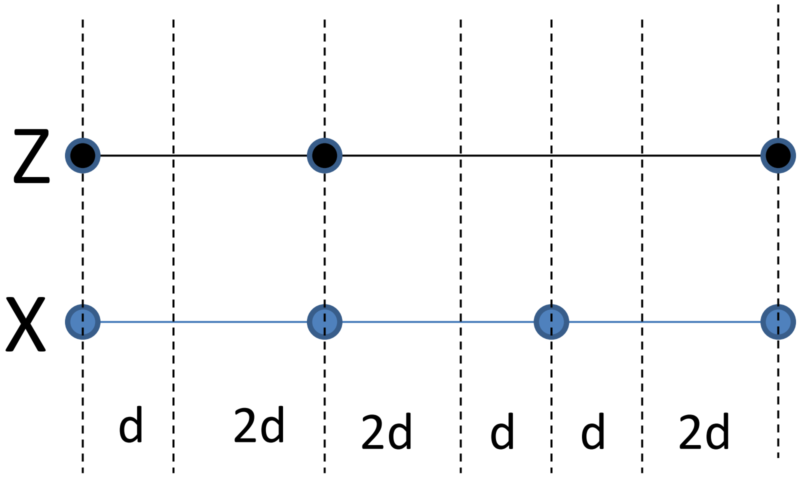
We have not proved expectation additivity. Prove that for any two random variables with depicted distributions.



Two exercises (cont)

"Design" values for dependent random variables X and Z such that

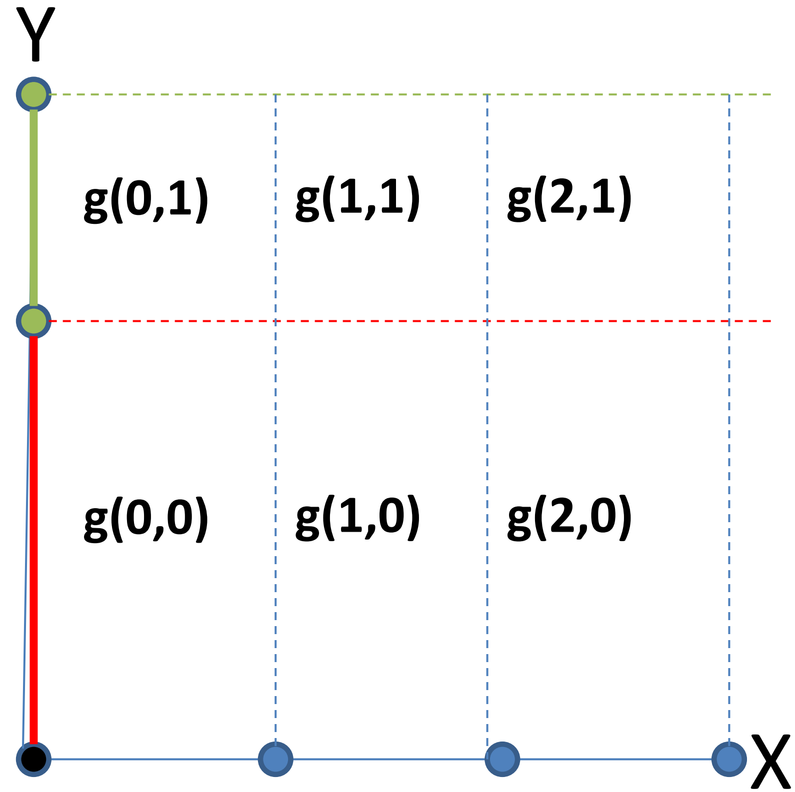
$$M(X*Z) = M(X)*M(Z)$$



Random variable g(X,Z) over $\Omega\times\Omega$

Sample *multivariate* distribution:

Y\X	0	1	2
0	2/9	2/9	2/9
1	1/9	1/9	1/9



Tuples of Random Variables (example)

- Experiment: flipping an ideal coin 3 times.
- Random variables:
  - X – number of tails (T);
  - Y – number of heads (H) before the first head.

Ω	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
P	1/8	1/8	1/8	1/8	1/8	1/8	1/8	1/8
X	0	1	1	2	1	2	2	3
Y	3	2	1	1	0	0	0	0

Multivariate Discrete Distribution (example)

Random variables:

Ω	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
P	1/8	1/8	1/8	1/8	1/8	1/8	1/8	1/8
X	0	1	1	2	1	2	2	3
Y	3	2	1	1	0	0	0	0

Joint Probability Mass Function:

X \ Y	**0**	**1**	**2**	**3**
0	**0**	**0**	**0**	**1/8**
1	**1/8**	**1/8**	**1/8**	**0**
2	**2/8**	**1/8**	**0**	**0**
3	**1/8**	**0**	**0**	**0**

Marginal Distributions

X \ Y	0	1	2	3	$P_X = \Sigma_{y \in [0..3]}$
0	0	0	0	1/8	**1/8**
1	1/8	1/8	1/8	0	**3/8**
2	2/8	1/8	0	0	**3/8**
3	1/8	0	0	0	**1/8**
$P_Y = \Sigma_{x \in [0..3]}$	**4/8**	**2/8**	**1/8**	**1/8**	

Exercises for slides 14-16

- Are random variables X and Y independent?
- Explain why the marginal distribution for X is the same as the distribution of X. (Same for Y).
- Build conditional distribution distributions $P_{X|Y}$ and $P_{Y|X}$ (i.e. distributions for each random variable given an admissible value for another one).

Part III

VARIANCE, COVARIANCE AND CORRELATION

Moments of a random variable

- Let k>0 be an integer, and X be a discrete random variable.
- k-th (*initial*) *moment* of X is expectation of X^k:

$$M(X^k) = \Sigma_{x \in R} x^k * P_X(x);$$

- k-th *central moment* of X is expectation of the random variable $[X - E(X)]^k$:

$$M([X - E(X)]^k) = \Sigma_{x \in R}(x-E(X))^k * P_X(x).$$

(Recall: <u>M</u>ean is <u>E</u>xpectation!)

Variance (дисперсия;-)

- *Variance* of a (discrete) random variable X is its 2^{nd} central moment:

$$D(X) = var(X) = M([X - E(X)]^2).$$

- $D(X) = M[X^2 - 2X*E(X) + E^2(X)] =$

$$= M(X^2) - 2*E(X)*M(X) + E^2(X) =$$

$$= M(X^2) - M^2(X).$$

Example and Exercise

- Example: X is "rolling dice" random variable:
  - $M(X) = 7/2$ (lecture for week 7);
  - $M(X^2) = \Sigma_{1\leq k\leq 6} k^2 * P_X(k) = 91/6$;
  - $D(X) = M(X^2) - M^2(X) = 91/6 - 49/4 = 35/12$.
- Exercise: compute expectation , 2nd moment and variance for a "dice" with n different values (i.e. a random variable X with values [1..n] and uniform distribution).

Some Properties of Variance

- Prove: if $a \in R$ is a constant then
  - $D(a) = 0$;
  - $D(X + a) = D(X)$;
  - $D(a*X) = a^2 * D(X)$.
- Prove: if X and Y have the same outcomes then

$$D(X + Y) = D(X) + D(Y) + \\ + 2[M(X*Y) - M(X)*M(Y)].$$

Prove yourself:

- If X and Y are independent random variables than $D(X + Y) = D(X) + D(Y)$;
- if $a,b \in R$ are constants and X and Y have the same outcomes then

$$D(a*X + b*Y) = a^2*D(X) + b^2*D(Y) + \\ + 2ab*[M(X*Y) - M(X)*M(Y)]$$

Variance of selected discrete distributions

- If X= Bernoulli(p) then

$$D(X) = M(X^2) - M^2(X) = p - p^2 = p*q,$$

where q=(1-p).

- If X= Binomial(n,p) then D(X)= n*D(X) = n*p*q where q=(1-p).
- Problem: prove that

$$D(X)= q/p^2,$$

where q=(1-p) and X=geom(p).

Deviation and Covariance

- For any discrete random variable X its (*standard*) *deviation* is $\sigma(X) = D^{1/2}(X)$.
- For any discrete random variable X and Y (with same set of outcomes) their *covariance* is

$$\text{cov}(X,Y) =$$

$$= M[(X\text{-}E(X))*(Y\text{-}E(X))] =$$

$$= M(X*Y) – M(X)*M(Y)];$$

  – if X and Y are independent then cov(X,Y)=0.

Correlation Coefficient

- For any discrete random variable X and Y (with same set of outcomes) their *correlation coefficient* is
$$r(X,Y) = \rho(X,Y) = corr(X,Y) = \frac{cov(X,Y)}{\sigma(X)*\sigma(Y)} .$$
- Some properties:
  - Prove: $-1 \leq cor(X,Y) \leq 1$ for all random variables X and Y.
  - If X and Y are independent then cor(X,Y)=0 (they are *uncorrelated*).

Correlation vs. Linear Expressibility

- Prove: if Y= a*X + b then

$$\mathrm{corr}(X,Y) = \begin{cases} 1, \text{ if } a>0; \\ -1, \text{ if } a<0. \end{cases}$$

- Question: is the opposite implication valid? Prove or refute by an example.