LINEAR ALGEBRA. LECTURE 3

Transposes, permutations, symmetric matrices

In this lecture we introduce vector spaces and their subspaces.

Row exchanges

What if there are row exchanges? In other words, what happens if there's a zero in a factor position. To swap two rows, we multiply on the left by a permutation matrix. For example,

$$P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \updownarrow$$

swaps the first and second rows of a *3*x*3* matrix. You can get it by changing first and second rows in the identity matrix. You also can easy check out that the inverse of any permutation matrix P is $P^{-1} = P^T$.

There are $n!$ different ways to permute the rows of an nxn matrix (including the permutation that leaves all rows fixed) so there are $n!$ permutation matrices. These matrices form a *multiplicative group*.

Permutations

Multiplication by a permutation matrix P swaps the rows of a matrix; when applying the method of elimination we use permutation matrices to move zeros out of pivot positions. Our factorization $A = LU$ then becomes $PA = LU$, where P is a permutation matrix which reorders any number of rows of A. Recall that $P^{-1} = P^T$, i.e. that $P^T P = I$.

If P swaps the rows of a matrix how to swap the columns of a matrix. Let's take the matrix $(PA)^T = A^T P^T$. We know that the matrix $(PA)^T$ swaps the columns of matrix A. So, to swap the columns of a matrix we need to multiply it on the Identity matrix with the same swapped columns at the right.

Transposes

When we take the transpose of a matrix, its rows become columns and its columns become rows. If we denote the entry in row i column j of matrix A by A_{ij}, then we can describe A^T by: $A^T_{ij} = A_{ji}$. For example:

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix}$$

Symmetric Matrices

A matrix A is *symmetric* if $A^T = A$. Given any matrix R (not necessarily square) the product $R^T R$ is always symmetric, because $\left(R^T R\right)^T = R^T \left(R^T\right)^T = R^T R$. (Note that $\left(R^T\right)^T = R$.). RR^T is also symmetric, but it is different from $R^T R$. Most scientific problems that start with a rectangular matrix R end up with $R^T R$ or RR^T or both.

Vector spaces

Elimination can simplify, one entry at a time, the linear system $Ax = b$. Fortunately it also simplifies the theory. The basic questions of *existence* and *uniqueness* — Is there one solution, or no solution, or an infinity of solutions — are much easier to answer after elimination, we need to devote one more section to those questions, to find every solution for an m by n system. Then that circle of ideas will be complete.

We can add vectors and multiply them by numbers, which means we can discuss *linear combinations* of vectors. These combinations follow the rules of a *vector space*. One such vector space is $\mathbf{R}^2$, the set of all vectors with exactly two real number components. We depict the vector $\{a,b\}$ by drawing an arrow from the origin to the point (a, b) which is a units to the right of the origin and b units above it, and we call $\mathbf{R}^2$ the "x - y plane". Another example of a space is $\mathbf{R}^n$, the set of vectors (columns) with n real number components.

Closure

The collection of vectors with exactly two *positive* real valued components is *not* a vector space. The sum of any two vectors in that collection is again in the collection, but multiplying any vector by, say, -5, gives a vector that's not in the collection. We say that this collection of positive vectors is *closed* under addition but not under multiplication.

If a collection of vectors is closed under linear combinations (i.e. under addition and multiplication by any real numbers), and if multiplication and addition behave in a reasonable way, then we call that collection a *vector space*.

Subspaces

A vector space that is contained inside of another vector space is called a *subspace* of that space. For example, take any non-zero vector V in $\mathbf{R}^2$. Then the set of all vectors cV, where c is a real number, forms a subspace of $\mathbf{R}^2$. This collection of vectors describes a line through $\{0,0\}$ in $\mathbf{R}^2$ and is closed under addition.

Definition. A *subspace* of a vector space is a nonempty subset that satisfies the requirements for a vector space: *Linear combinations stay in the subspace*.

(i) If we add any vectors x and y in the subspace, $x+y$ is *in the subspace*.

(ii) If we multiply any vector x in the subspace by any scalar c, cx is *in the subspace*.

A line in $\mathbf{R}^2$ that does not pass through the origin is *not* a subspace of $\mathbf{R}^2$. Multiplying any vector on that line by 0 gives the zero vector, which does not lie on the line. Every subspace must contain the zero vector because vector spaces are closed under multiplication.

The subspaces of $\mathbf{R}^2$ are:

1. all of $\mathbf{R}^2$,
2. any line through {0,0} and
3. the zero vector alone (Z).

The subspaces of $\mathbf{R}^3$ are:

1. all of $\mathbf{R}^3$,
2. any plane through the origin,
3. any line through the origin, and
4. the zero vector alone (Z).

Column space and nullspace

A vector space is a collection of vectors which is closed under linear combinations. In other words, for any two vectors *V* and *W* in the space and any two real numbers *c* and *d*, the vector *cV*+*dW* is also in the vector space. A subspace is a vector space contained inside a vector space. A plane *P* containing {0,0,0} and a line *L* containing {0,0,0} are both subspaces of $\mathbf{R}^3$. The union $P \bigcup L$ of those two subspaces is generally not a subspace, because the sum of a vector in *P* and a vector in *L* is probably not contained in $P \bigcup L$. The intersection $S \bigcap T$ of two subspaces *S* and *T* is a subspace. To prove this, use the fact that both *S* and *T* are closed under linear combinations to show that their intersection is closed under linear combinations.

Column space

Given a matrix *A* with columns in $\mathbf{R}^3$, these columns and all their linear combinations form a subspace of $\mathbf{R}^3$. This is the *column space C*(*A*). If $A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$ the column space of *A* is the plane through the origin in $\mathbf{R}^3$ containing vectors $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$.

Our next task will be to understand the equation *Ax* = *b* in terms of subspaces and the column space of *A*.

Column space of *A*

The *column space* of a matrix *A* (*mxn*) *contains all linear combinations of the columns of A*. It is a subspace of R^m. With *m* > *n* we have more equations than unknowns in *Ax*=*b* and *usually there will be no solution*. The system will be solvable only for a very "thin" subset of all possible *b*'s. One way of describing this thin subset is so simple that it is easy to overlook. The system *Ax* = *b* is solvable if and only if the vector *b* can be expressed as a *combination of the columns* of *A*. Then *b* is in the *column space*.

Solving *Ax* = *b*

Given a matrix *A*, for what vectors *b* does *Ax* = *b* has a solution *x*?

$$\text{Let } A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$

Then *Ax* = *b* does not have a solution for every choice of *b* because solving *Ax* = *b* is equivalent to solving four linear equations in three unknowns. If there is a solution *x* to *Ax* = *b*, then *b* must be a linear combination of the columns of *A*. Only three columns cannot fill the entire four dimensional vector space – any vectors *b* cannot be expressed as linear combinations of columns of *A*.

Big question: what *b*'s allow $Ax = b$ to be solved?

A useful approach is to choose *x* and find the vector $b = Ax$ corresponding to that solution. The components of *x* are just the coefficients in a linear combination of columns of *A*. The system of linear equations $Ax = b$ is *solvable* exactly when *b* is a vector in the *column space* of *A*.

For our example matrix *A*, what can we say about the column space of *A*? Are the columns of *A independent*? In other words, does each column contribute something new to the subspace? The third column of *A* is the sum of the first two columns, so does not add anything to the subspace. The column space of our matrix *A* is a two dimensional subspace $\mathbf{R}^2$ of $\mathbf{R}^4$.

Nullspace of *A*

The *nullspace* of a matrix *A* is the collection of all solutions $x = \{x_1, x_2, x_3\}$ to the equation $Ax = 0$. The column space of the matrix in our example was a $\mathbf{R}^2$ subspace of $\mathbf{R}^4$. The nullspace of *A* is a subspace of $\mathbf{R}^3$. To see that it's a vector space, check that any sum or multiple of solutions to $Ax = 0$ is also a solution: $A(x_1+x_2) = Ax_1 + Ax_2 = 0+0 = 0$ and $A(cx) = cAx = 0$.

In the example:

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

the nullspace $N(A)$ consists of all multiples of $c\{1,1,-1\} = \{c,c,-c\}$; column 1 plus column 2 minus column 3 equals the zero vector. The nullspace of *A* is the line in in $\mathbf{R}^3$ of all points $x_1 = c$, $x_2 = c$, $x_3 = -c$. (The line goes through the origin, as any subspace must.) We want to be able, for any system $Ax = b$, to find $C(A)$ and $N(A)$: all attainable right-hand sides *b* and all solutions to $Ax = 0$.

The vectors *b* are in the column space and the vectors *x* are in the nullspace. We shall compute the dimensions of those subspaces and a convenient set of vectors to generate them.

Other values of *b*

The solutions to the equation:

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

do not form a subspace. The zero vector is not a solution to this equation. The set of solutions forms a line in $\mathbf{R}^3$ that passes through the points {1,0,0} and {0,-1,1} but not {0,0,0}.