Discrete Mathematics

Inductions-II

Adil M. Khan

Professor of Computer Science

Innopolis University

"Induction is a process of inference it proceeds from the known to the unknown!"

- John Stuart Mill -

Validity of Mathematical Induction Revisited!

Why is mathematical induction a valid proof technique?

The reason comes from the well-ordering property --

states that every nonempty subset of the set of positive integers has a least element.

Validity of Mathematical Induction Revisited!

Why is mathematical induction a valid proof technique?

The reason comes from the well-ordering property --

states that every nonempty subset of the set of positive integers has a least element.

So, suppose

we know that P(1) is true and that the proposition P(k) → P(k + 1) is true for all positive

integers k.

Validity of Mathematical Induction Revisited!

Why is mathematical induction a valid proof technique?

The reason comes from the well-ordering property --

states that every nonempty subset of the set of positive integers has a least element.

So, suppose

we know that P(1) is true and that the proposition P(k) →P(k + 1) is true for all positive

integers k.

We say that P(n) must be true for all positive integers n.

Validity of Mathematical Induction Revisited!

Why is mathematical induction a valid proof technique?

The reason comes from the well-ordering property --

states that every nonempty subset of the set of positive integers has a least element.

So, suppose

we know that P(1) is true and that the proposition P(k) →P(k + 1) is true for all positive

integers k.

We say that P(n) must be true for all positive integers n.

We will prove that this is valid by "Proof by Contradiction"

Validity of Mathematical Induction Revisited!

Assume that there is at

least one positive integer for which P(n) is false.

Validity of Mathematical Induction Revisited!

Assume that there is at

least one positive integer for which P(n) is false.

Then the set S of positive integers for which

P(n) is false is nonempty.

Validity of Mathematical Induction Revisited!

Assume that there is at

least one positive integer for which P(n) is false.

Then the set S of positive integers for which

P(n) is false is nonempty

Thus, by the well-ordering property, S has a least element, which will be denoted by m.

Validity of Mathematical Induction Revisited!

We know that m cannot be 1, because P(1) is true.

Validity of Mathematical Induction Revisited!

We know that m cannot be 1, because P(1) is true.

Because m is positive

and greater than 1, m − 1 is a positive integer.

Validity of Mathematical Induction Revisited!

We know that m cannot be 1, because P(1) is true.

Because m is positive

and greater than 1, m − 1 is a positive integer.

Furthermore, because m − 1 is less than m, it is not in S,

Validity of Mathematical Induction Revisited!

We know that m cannot be 1, because P(1) is true.

Because m is positive

and greater than 1, m − 1 is a positive integer.

Furthermore, because m − 1 is less than m, it is

not in S,

so P(m − 1) must be true.

Validity of Mathematical Induction Revisited!

We know that m cannot be 1, because P(1) is true.

Because m is positive

and greater than 1, m − 1 is a positive integer.

Furthermore, because m − 1 is less than m, it is

not in S,

so P(m− 1) must be true.

Because the conditional statement P(m − 1) → P(m) is also true, it must be the case that P(m) is true.

Validity of Mathematical Induction Revisited!

We know that m cannot be 1, because P(1) is true.

Because m is positive

and greater than 1, m − 1 is a positive integer.

Furthermore, because m − 1 is less than m, it is

not in S,

so P(m− 1) must be true.

Because the conditional statement P(m − 1) → P(m) is also true, it must be the case that P(m) is true.

This contradicts the choice of m. Hence, P(n) must be true for every positive integer n.

Variants of Induction

Induction starting at *0 [Remember that it could be 1 as well, all depends on the definition of natural numbers]*

To prove that *P(n)* is true for all natural numbers greater than or equal to *0*

- Show that *P(0)* is true.
- Show that for any $k \geq 0$, that if *P(k)* is true, then *P(k+1)* is true.
- Conclude *P(n)* holds for all natural numbers greater than or equal to *0*.

Variants of Induction

Induction starting at *m* (Starting Induction Later)

To prove that *P(n)* is true for all natural numbers **greater than or equal to *m***:

- Show that *P(m)* is true.

- Show that for any $k \geq m$, that if *P(k)* is true, then *P(k+1)* is true.

- Conclude *P(n)* holds for all natural numbers greater than or equal to *m*.

Starting Induction Later

Theorem: For all $n \in N$, where $n \geq 5$, $n^2 < 2^n$.

Starting Induction Later

Theorem: For all $n \in N$, where $n \geq 5$, $n^2 < 2^n$.

Proof: By Induction:

$$\text{Let } P(n) \equiv (n+5)^2 < 2^{(n+5)}.$$

We will prove that $P(n)$ holds for all $n \in N$.

Starting Induction Later

Theorem: For all $n \in N$, where $n \geq 5$, $n^2 < 2^n$.

Proof: By Induction:

$$\text{Let } P(n) \equiv (n+5)^2 < 2^{(n+5)}.$$

We will prove that $P(n)$ holds for all $n \in N$.

From this we can conclude that for any $n \in N$ with $n \geq 5$ that $(n+5)^2 < 2^{(n+5)}$.

This holds because for any natural number $n \geq 5$ we have that n-5 $\in N$. $P(n-5)$ then implies that $n^2 < 2^n$.

Starting Induction Later

Theorem: For all $n \in N$, where $n \geq 5$, $n^2 < 2^n$.

Proof: By Induction: Let $P(n) \equiv (n+5)^2 < 2^{(n+5)}$. We will prove that $P(n)$ holds for all $n \in N$. From this we can conclude that for any $n \in N$ with $n \geq 5$ that $(n+5)^2 < 2^{(n+5)}$. This holds because for any natural number $n \geq 5$ we have that $n-5 \in N$. $P(n-5)$ then implies that $n^2 < 2^n$.

For our base case,

We prove $P(5)$, that $5^2 < 2^5$.

Starting Induction Later

Theorem: For all $n \in N$, where $n \geq 5$, $n^2 < 2^n$.

Proof: By Induction: Let $P(n) \equiv (n+5)^2 < 2^{(n+5)}$. We will prove that $P(n)$ holds for all $n \in N$. From this we can conclude that for any $n \in N$ with $n \geq 5$ that $(n+5)^2 < 2^{(n+5)}$. This holds because for any natural number $n \geq 5$ we have that $n-5 \in N$. $P(n-5)$ then implies that $n^2 < 2^n$.

For our base case,

We prove $P(5)$, that $5^2 < 2^5$.

Since $5^2 = 25$ and $2^5 = 32$, this claim is true.

Starting Induction Later

Theorem: For all $n \in N$, where $n \geq 5$, $n^2 < 2^n$.

Proof: By Induction: Let $P(n) \equiv (n+5)^2 < 2^{(n+5)}$. We will prove that $P(n)$ holds for all $n \in N$. From this we can conclude that for any $n \in N$ with $n \geq 5$ that $(n+5)^2 < 2^{(n+5)}$. This holds because for any natural number $n \geq 5$ we have that n-5 $\in N$. $P(n-5)$ then implies that $n^2 < 2^n$.

For our base case,

We prove $P(5)$, that $5^2 < 2^5$.

Since $5^2 = 25$ and $2^5 = 32$, this claim is true.

For the inductive step assume for some $k \in N$ that $P(k)$ holds and $(k+5)^2 < 2^{(k+5)}$.

We will prove that P(k+1) holds, meaning that $((k+5)+1)^2 < 2^{((k+5)+1)}$. To see this, note that

Starting Induction Later

Theorem: For all $n \in N$, where $n \geq 5$, $n^2 < 2^n$.

Proof: By Induction: Let $P(n) \equiv (n+5)^2 < 2^{(n+5)}$. We will prove that $P(n)$ holds for all $n \in N$. From this we can conclude that for any $n \in N$ with $n \geq 5$ that $(n+5)^2 < 2^{(n+5)}$. This holds because for any natural number $n \geq 5$ we have that n-5 $\in N$. $P(n\text{-}5)$ then implies that $n^2 < 2^n$.

For our base case, we prove P(5), that 52<25. Since 52=25 and 25=32, this claim is true. For the inductive step assume for some k $\in$ N that P(k) holds and $(k+5)^2 < 2^{(k+5)}$. We will prove that P(k+1) holds, meaning that $((k+5)+1)^2 < 2^{((k+5)+1)}$. To see this, note that

$$((k+5)+1)^2$$
$$= (k+5)^2 + 2(k+5) + 1 \qquad \text{(Distributing)}$$

Starting Induction Later

Theorem: For all $n \in N$, where $n \geq 5$, $n^2 < 2^n$.

Proof: By Induction: Let $P(n) \equiv (n+5)^2 < 2^{(n+5)}$. We will prove that $P(n)$ holds for all $n \in N$. From this we can conclude that for any $n \in N$ with $n \geq 5$ that $(n+5)^2 < 2^{(n+5)}$. This holds because for any natural number $n \geq 5$ we have that n-5 $\in N$. $P(n-5)$ then implies that $n^2 < 2^n$.

For our base case, we prove P(5), that 52<25. Since 52=25 and 25=32, this claim is true. For the inductive step assume for some k $\in$ N that P(k) holds and $(k+5)^2 < 2^{(k+5)}$. We will prove that P(k+1) holds, meaning that $((k+5)+1)^2 < 2^{((k+5)+1)}$. To see this, note that

$$((k+5)+1)^2$$
$$= (k+5)^2 + 2(k+5) + 1 \qquad \text{(Distributing)}$$

Starting Induction Later

Theorem: For all $n \in N$, where $n \geq 5$, $n^2 < 2^n$.

Proof: By Induction: Let $P(n) \equiv (n+5)^2 < 2^{(n+5)}$. We will prove that $P(n)$ holds for all $n \in N$. From this we can conclude that for any $n \in N$ with $n \geq 5$ that $(n+5)^2 < 2^{(n+5)}$. This holds because for any natural number $n \geq 5$ we have that $n-5 \in N$. $P(n-5)$ then implies that $n^2 < 2^n$.

For our base case, we prove P(5), that $52 < 25$. Since $52 = 25$ and $25 = 32$, this claim is true. For the inductive step assume for some $k \in N$ that P(k) holds and $(k+5)^2 < 2^{(k+5)}$. We will prove that P(k+1) holds, meaning that $((k+5)+1)^2 < 2^{((k+5)+1)}$. To see this, note that

$$((k+5)+1)^2$$
$$= (k+5)^2 + 2(k+5) + 1 \quad \text{(Distributing)}$$
$$< 2^{k+5} + 2(k+5) + 1 \quad \text{(By the Inductive Hypothesis)}$$

Starting Induction Later

Theorem: For all $n \in N$, where $n \geq 5$, $n^2 < 2^n$.

Proof: By Induction: Let $P(n) \equiv (n+5)^2 < 2^{(n+5)}$. We will prove that $P(n)$ holds for all $n \in N$. From this we can conclude that for any $n \in N$ with $n \geq 5$ that $(n+5)^2 < 2^{(n+5)}$. This holds because for any natural number $n \geq 5$ we have that n-5 $\in N$. $P(n-5)$ then implies that $n^2 < 2^n$.

For our base case, we prove P(5), that 52<25. Since 52=25 and 25=32, this claim is true. For the inductive step assume for some k $\in$ N that P(k) holds and $(k+5)^2 < 2^{(k+5)}$. We will prove that P(k+1) holds, meaning that $((k+5)+1)^2 < 2^{((k+5)+1)}$. To see this, note that

$$((k+5)+1)^2$$
$$= (k+5)^2 + 2(k+5) + 1 \quad \text{(Distributing)}$$
$$< 2^{k+5} + 2(k+5) + 1 \quad \text{(By the Inductive Hypothesis)}$$
$$< 2^{k+5} + 2(k+5) + 5 \quad \text{(Since 1<5)}$$

{Remember that it might have taken the creator of this proof quite some time and creativity to come up with this. Same technique is used in the rest of the proof}

Starting Induction Later

Theorem: For all $n \in N$, where $n \geq 5$, $n^2 < 2^n$.

Proof: By Induction: Let $P(n) \equiv (n+5)^2 < 2^{(n+5)}$. We will prove that $P(n)$ holds for all $n \in N$. From this we can conclude that for any $n \in N$ with $n \geq 5$ that $(n+5)^2 < 2^{(n+5)}$. This holds because for any natural number $n \geq 5$ we have that n-5 $\in N$. $P(n-5)$ then implies that $n^2 < 2^n$.

For our base case, we prove P(5), that 52<25. Since 52=25 and 25=32, this claim is true. For the inductive step assume for some k $\in$ N that P(k) holds and $(k+5)^2 < 2^{(k+5)}$. We will prove that P(k+1) holds, meaning that $((k+5)+1)^2 < 2^{((k+5)+1)}$. To see this, note that

$$((k+5)+1)^2$$
$$= (k+5)^2 + 2(k+5) + 1 \qquad \text{(Distributing)}$$
$$< 2^{k+5} + 2(k+5) + 1 \qquad \text{(By the Inductive Hypothesis)}$$
$$< 2^{k+5} + 2(k+5) + 5 \qquad \text{(Since 1<5)}$$
$$\leq 2^{k+5} + 2(k+5) + k + 5 \qquad \text{(Since 0}\leq\text{k)}$$

Starting Induction Later

Theorem: For all $n \in N$, where $n \geq 5$, $n^2 < 2^n$.

Proof: By Induction: Let $P(n) \equiv (n+5)^2 < 2^{(n+5)}$. We will prove that $P(n)$ holds for all $n \in N$. From this we can conclude that for any $n \in N$ with $n \geq 5$ that $(n+5)^2 < 2^{(n+5)}$. This holds because for any natural number $n \geq 5$ we have that $n-5 \in N$. $P(n-5)$ then implies that $n^2 < 2^n$.

For our base case, we prove P(5), that $52 < 25$. Since $52 = 25$ and $25 = 32$, this claim is true. For the inductive step assume for some $k \in N$ that P(k) holds and $(k+5)^2 < 2^{(k+5)}$. We will prove that P(k+1) holds, meaning that $((k+5)+1)^2 < 2^{((k+5)+1)}$. To see this, note that

$$((k+5)+1)^2$$
$$= (k+5)^2 + 2(k+5) + 1 \qquad \text{(Distributing)}$$
$$< 2^{k+5} + 2(k+5) + 1 \qquad \text{(By the Inductive Hypothesis)}$$
$$< 2^{k+5} + 2(k+5) + 5 \qquad \text{(Since 1<5)}$$
$$\leq 2^{k+5} + 2(k+5) + k + 5 \qquad \text{(Since 0≤k)}$$
$$= 2^{k+5} + 3(k+5) \qquad \text{(Simplifying)}$$

Starting Induction Later

Theorem: For all $n \in N$, where $n \geq 5$, $n^2 < 2^n$.

Proof: By Induction: Let $P(n) \equiv (n+5)^2 < 2^{(n+5)}$. We will prove that $P(n)$ holds for all $n \in N$. From this we can conclude that for any $n \in N$ with $n \geq 5$ that $(n+5)^2 < 2^{(n+5)}$. This holds because for any natural number $n \geq 5$ we have that n-5 $\in N$. $P(n-5)$ then implies that $n^2 < 2^n$.

For our base case, we prove P(5), that 52<25. Since 52=25 and 25=32, this claim is true. For the inductive step assume for some k $\in$ N that P(k) holds and $(k+5)^2 < 2^{(k+5)}$. We will prove that P(k+1) holds, meaning that $((k+5)+1)^2 < 2^{((k+5)+1)}$. To see this, note that

$$((k+5)+1)^2$$
$$= (k+5)^2 + 2(k+5) + 1 \qquad \text{(Distributing)}$$
$$< 2^{k+5} + 2(k+5) + 1 \qquad \text{(By the Inductive Hypothesis)}$$
$$< 2^{k+5} + 2(k+5) + 5 \qquad \text{(Since 1<5)}$$
$$\leq 2^{k+5} + 2(k+5) + k + 5 \qquad \text{(Since 0$\leq$k)}$$
$$= 2^{k+5} + 3(k+5) \qquad \text{(Simplifying)}$$
$$< 2^{k+5} + 5(k+5) \qquad \text{(Since 3<5)}$$

Starting Induction Later

Theorem: For all $n \in N$, where $n \geq 5$, $n^2 < 2^n$.

Proof: By Induction: Let $P(n) \equiv (n+5)^2 < 2^{(n+5)}$. We will prove that $P(n)$ holds for all $n \in N$. From this we can conclude that for any $n \in N$ with $n \geq 5$ that $(n+5)^2 < 2^{(n+5)}$. This holds because for any natural number $n \geq 5$ we have that n-5 $\in N$. $P(n-5)$ then implies that $n^2 < 2^n$.

For our base case, we prove P(5), that $52 < 25$. Since $52 = 25$ and $25 = 32$, this claim is true. For the inductive step assume for some k $\in$ N that P(k) holds and $(k+5)^2 < 2^{(k+5)}$. We will prove that P(k+1) holds, meaning that $((k+5)+1)^2 < 2^{((k+5)+1)}$. To see this, note that

$$((k+5)+1)^2$$
$$= (k+5)^2 + 2(k+5) + 1 \qquad \text{(Distributing)}$$
$$< 2^{k+5} + 2(k+5) + 1 \qquad \text{(By the Inductive Hypothesis)}$$
$$< 2^{k+5} + 2(k+5) + 5 \qquad \text{(Since } 1 < 5\text{)}$$
$$\leq 2^{k+5} + 2(k+5) + k + 5 \qquad \text{(Since } 0 \leq k\text{)}$$
$$= 2^{k+5} + 3(k+5) \qquad \text{(Simplifying)}$$
$$< 2^{k+5} + 5(k+5) \qquad \text{(Since } 3 < 5\text{)}$$
$$\leq 2^{k+5} + (k+5)(k+5) \qquad \text{(Since } 0 \leq k\text{)}$$

Starting Induction Later

Theorem: For all $n \in N$, where $n \geq 5$, $n^2 < 2^n$.

Proof: By Induction: Let $P(n) \equiv (n+5)^2 < 2^{(n+5)}$. We will prove that $P(n)$ holds for all $n \in N$. From this we can conclude that for any $n \in N$ with $n \geq 5$ that $(n+5)^2 < 2^{(n+5)}$. This holds because for any natural number $n \geq 5$ we have that n-5 $\in N$. $P(n-5)$ then implies that $n^2 < 2^n$.

For our base case, we prove P(5), that $52 < 25$. Since $52 = 25$ and $25 = 32$, this claim is true. For the inductive step assume for some k $\in$ N that P(k) holds and $(k+5)^2 < 2^{(k+5)}$. We will prove that P(k+1) holds, meaning that $((k+5)+1)^2 < 2^{((k+5)+1)}$. To see this, note that

$$((k+5)+1)^2$$
$$= (k+5)^2 + 2(k+5) + 1 \qquad \text{(Distributing)}$$
$$< 2^{k+5} + 2(k+5) + 1 \qquad \text{(By the Inductive Hypothesis)}$$
$$< 2^{k+5} + 2(k+5) + 5 \qquad \text{(Since } 1 < 5\text{)}$$
$$\leq 2^{k+5} + 2(k+5) + k + 5 \qquad \text{(Since } 0 \leq k\text{)}$$
$$= 2^{k+5} + 3(k+5) \qquad \text{(Simplifying)}$$
$$< 2^{k+5} + 5(k+5) \qquad \text{(Since } 3 < 5\text{)}$$
$$\leq 2^{k+5} + (k+5)(k+5) \qquad \text{(Since } 0 \leq k\text{)}$$
$$= 2^{k+5} + (k+5)^2 \qquad \text{(Simplifying)}$$

Starting Induction Later

Theorem: For all $n \in N$, where $n \geq 5$, $n^2 < 2^n$.

Proof: By Induction: Let $P(n) \equiv (n+5)^2 < 2^{(n+5)}$. We will prove that $P(n)$ holds for all $n \in N$. From this we can conclude that for any $n \in N$ with $n \geq 5$ that $(n+5)^2 < 2^{(n+5)}$. This holds because for any natural number $n \geq 5$ we have that n-5 $\in N$. $P(n-5)$ then implies that $n^2 < 2^n$.

For our base case, we prove P(5), that 52<25. Since 52=25 and 25=32, this claim is true. For the inductive step assume for some k $\in$ N that P(k) holds and $(k+5)^2 < 2^{(k+5)}$. We will prove that P(k+1) holds, meaning that $((k+5)+1)^2 < 2^{((k+5)+1)}$. To see this, note that

$$((k+5)+1)^2$$
$$= (k+5)^2 + 2(k+5) + 1 \qquad \text{(Distributing)}$$
$$< 2^{k+5} + 2(k+5) + 1 \qquad \text{(By the Inductive Hypothesis)}$$
$$< 2^{k+5} + 2(k+5) + 5 \qquad \text{(Since 1<5)}$$
$$\leq 2^{k+5} + 2(k+5) + k + 5 \qquad \text{(Since 0≤k)}$$
$$= 2^{k+5} + 3(k+5) \qquad \text{(Simplifying)}$$
$$< 2^{k+5} + 5(k+5) \qquad \text{(Since 3<5)}$$
$$\leq 2^{k+5} + (k+5)(k+5) \qquad \text{(Since 0≤k)}$$
$$= 2^{k+5} + (k+5)^2 \qquad \text{(Simplifying)}$$
$$< 2^{k+5} + 2^{k+5} \qquad \text{(By the Inductive Hypothesis)}$$

Starting Induction Later

Theorem: For all $n \in N$, where $n \geq 5$, $n^2 < 2^n$.

Proof: By Induction: Let $P(n) \equiv (n+5)^2 < 2^{(n+5)}$. We will prove that $P(n)$ holds for all $n \in N$. From this we can conclude that for any $n \in N$ with $n \geq 5$ that $(n+5)^2 < 2^{(n+5)}$. This holds because for any natural number $n \geq 5$ we have that n-5 $\in N$. $P(n-5)$ then implies that $n^2 < 2^n$.

For our base case, we prove P(5), that 52<25. Since 52=25 and 25=32, this claim is true. For the inductive step assume for some k $\in$ N that P(k) holds and $(k+5)^2 < 2^{(k+5)}$. We will prove that P(k+1) holds, meaning that $((k+5)+1)^2 < 2^{((k+5)+1)}$. To see this, note that

$$((k+5)+1)^2$$
$= (k+5)^2 + 2(k+5) + 1$ (Distributing)
$< 2^{k+5} + 2(k+5) + 1$ (By the Inductive Hypothesis)
$< 2^{k+5} + 2(k+5) + 5$ (Since 1<5)
$\leq 2^{k+5} + 2(k+5) + k + 5$ (Since 0≤k)
$= 2^{k+5} + 3(k+5)$ (Simplifying)
$< 2^{k+5} + 5(k+5)$ (Since 3<5)
$\leq 2^{k+5} + (k+5)(k+5)$ (Since 0≤k)
$= 2^{k+5} + (k+5)^2$ (Simplifying)
$< 2^{k+5} + 2^{k+5}$ (By the Inductive Hypothesis)
$= 2(2^{k+5})$ (Simplifying)

Starting Induction Later

Theorem: For all $n \in N$, where $n \geq 5$, $n^2 < 2^n$.

Proof: By Induction: Let $P(n) \equiv (n+5)^2 < 2^{(n+5)}$. We will prove that $P(n)$ holds for all $n \in N$. From this we can conclude that for any $n \in N$ with $n \geq 5$ that $(n+5)^2 < 2^{(n+5)}$. This holds because for any natural number $n \geq 5$ we have that n-5 $\in N$. $P(n-5)$ then implies that $n^2 < 2^n$.

For our base case, we prove P(5), that 52<25. Since 52=25 and 25=32, this claim is true. For the inductive step assume for some k $\in$ N that P(k) holds and $(k+5)^2 < 2^{(k+5)}$. We will prove that P(k+1) holds, meaning that $((k+5)+1)^2 < 2^{((k+5)+1)}$. To see this, note that

$$((k+5)+1)^2$$
$$= (k+5)^2+2(k+5)+1 \qquad \text{(Distributing)}$$
$$< 2^{k+5} +2(k+5)+1 \qquad \text{(By the Inductive Hypothesis)}$$
$$< 2^{k+5} +2(k+5)+5 \qquad \text{(Since 1<5)}$$
$$\leq 2^{k+5} +2(k+5)+k+5 \qquad \text{(Since 0≤k)}$$
$$= 2^{k+5} +3(k+5) \qquad \text{(Simplifying)}$$
$$< 2^{k+5} +5(k+5) \qquad \text{(Since 3<5)}$$
$$\leq 2^{k+5} +(k+5)(k+5) \qquad \text{(Since 0≤k)}$$
$$= 2^{k+5} +(k+5)^2 \qquad \text{(Simplifying)}$$
$$< 2^{k+5} +2^{k+5} \qquad \text{(By the Inductive Hypothesis)}$$
$$= 2(2^{k+5}) \qquad \text{(Simplifying)}$$
$$= 2^{((k+5)+1)} \qquad \text{(By Powers of Exponents)}$$

Thus $((k+5)+1)^2 < 2^{((k+5)+1)}$, so $P(k+1)$ holds.

Important Bits

This proof is interesting for a few reasons.

- **First**, it shows that we can use induction to reason about properties of numbers larger than a certain size.

- **Second**, it shows a style of proof that we have not seen before.
  - To prove an inequality holds between two quantities, we can often expand out the inequality across multiple steps, at each point showing one smaller piece of the inequality.
  - Since inequalities are transitive, the net result of these inequalities gives us the result that we want.

Practice

Prove the following theorem using Induction.

Theorem: For all $n \in N$, where $n \geq 4$, $2^n < n!$.

Strong Induction

Normal induction, and **induction starting at m**, enable us to prove a variety of useful results.

We will now turn to an even more powerful form of induction called **strong induction**

- It has numerous applications within computer science, from the analysis of algorithms to the understanding of the structure of numbers themselves.

Strong Induction – Cont.

Similar to **ordinary mathematical induction** in that it is a technique for establishing the truth of a sequence of statements about natural numbers.

Also, a proof by **strong mathematical induction** consists of a basis step and an inductive step.

Strong Induction – Cont.

However, the basis step may contain proofs for several initial values.

And, in the inductive step the truth of the predicate $P(n)$ is assumed not just for one value of n but for all values through k, and then the truth of $P(k + 1)$ is proved.

Strong Induction – Cont.

Theorem: Let *P(n)* be a property that applies to natural numbers. If the following are true:

- *P(0)* is true.

- For any *k ε N*, if *P(0)*, *P(1)*, … , *P(k)* are true, then *P(k+1)* is true.

Then for any *n ε N*, *P(n)* is true.

Why Strong Induction

The following example will illustrate how strong induction can help us prove a result that cannot easily be proved using the principle of mathematical induction.

Example

Suppose:

- We can reach the first and second steps of an infinite ladder, and

- We know that if we can reach a step, then we can reach two steps higher.

Can we prove that we can reach every step of this ladder?

Example

Attempt through Normal Induction:

Basis Step *P(1)*:

The basis step of such a proof holds; here it simply verifies that we can reach the first step.

Example

Attempt through Normal Induction:

Basis Step *P(1)*:

The basis step of such a proof holds; here it simply verifies that we can reach the first step.

Inductive Step:

The inductive hypothesis is the statement that we can reach the kth step of the ladder.

Example

Attempt through Normal Induction:

Basis Step *P(1)*:

The basis step of such a proof holds; here it simply verifies that we can reach the first step.

Inductive Step:

The inductive hypothesis is the statement that we can reach the kth step of the ladder.

To complete the inductive step, we need to show that if we assume that we can reach the kth step of the ladder, then we can show that we can reach the $(k+1)^{st}$ step of the ladder.

Example

Attempt through Normal Induction:

Basis Step *P(1)*:

The basis step of such a proof holds; here it simply verifies that we can reach the first step.

Inductive Step:

The inductive hypothesis is the statement that we can reach the kth step of the ladder.

To complete the inductive step, we need to show that if we assume the inductive hypothesis for the positive integer k, if we assume that we can reach the kth step of the ladder, then we can show that we can reach the $(k+1)^{st}$ step of the ladder.

However, there is no obvious way to complete this inductive step because we do not know from the given information that we can reach the $(k+1)^{st}$ step from the kth step. After all, **we only know that if we can reach a step we can reach the step two higher**.

Example

Attempt through Strong Induction:

Basis Step: The basis step is the same as before; it simply verifies that we can reach the first step.

Example

Attempt through Strong Induction:

Basis Step: The basis step is the same as before; it simply verifies that we can reach the first step.

Inductive Step: The inductive hypothesis states that we can reach each of the first k steps.

Example

Attempt through Strong Induction:

Basis Step: The basis step is the same as before; it simply verifies that we can reach the first step.

Inductive Step: The inductive hypothesis states that we can reach each of the first k steps.

To complete the inductive step, we need to show that if we can reach each of the first k steps, then we can reach the $(k+1)^{st}$ step.

Example

We already know that we can reach the second step of the ladder.

Example

We can complete the inductive step by noting that as long as $k \geq 2$, **we can reach the *(k+1)ˢᵗ* step of the ladder from the *(k-1)ˢᵗ* step** because we know we can climb two steps from a step we can already reach,

Example

We can complete the inductive step by noting that as long as $k \geq 2$, **we can reach the *(k+1)st* step of the ladder from the *(k-1)st* step** because we know we can climb two steps from a step we can already reach, and because *k-1 $\leq$ k*, by inductive hypothesis we can reach the *(k-1)st* step.

Example

We can complete the inductive step by noting that as long as $k \geq 2$, **we can reach the *(k+1)ˢᵗ* step of the ladder from the *(k-1)ˢᵗ* step** because we know we can climb two steps from a step we can already reach, and because $k\text{-}1 \leq k$, by inductive hypothesis we can reach the *(k-1)ˢᵗ* step.

This completes the inductive step and finishes the proof by strong induction.

Example

Attempt through Strong Induction:

Basis Step: The basis step is the same as before; it simply verifies that we can reach the first step.

Inductive Step: The inductive hypothesis states that we can reach each of the first k steps.

To complete the inductive step, we need to show that if we assume that the inductive hypothesis is true, that is, if we can reach each of the first k steps, then we can reach the $(k+1)^{st}$ step.

We already know that we can reach the second step of the ladder. We can complete the inductive step by noting that as long as $k \geq 2$, we can reach the $(k+1)^{st}$ step of the ladder from the $(k+1)^{st}$ step because we know we can climb two steps from a step we can already reach, and because $k-1 \leq k$, by inductive hypothesis we can reach the $(k-1)^{st}$ step.

This completes the inductive step and finishes the proof by strong induction.

Example

Prove: Any integer greater than 1 is divisible by a prime number.

Example

Proof (by strong mathematical induction):

Let the property $P(n)$ be the sentence

n is divisible by a prime number $n > 1$

Example

Proof (by strong mathematical induction):

Let the property $P(n)$ be the sentence

n is divisible by a prime number.

Basis Step: Show that $P(2)$ is true:

Example

Proof (by strong mathematical induction):

Let the property $P(n)$ be the sentence

n is divisible by a prime number.

Basis Step: Show that $P(2)$ is true:

This is true because 2 is divisible by 2 and 2 is a prime number.

Example

Inductive Step: Show that for all integers $k > 2$, if $P(i)$ is true for all integers *i from 2 through k*, then $P(k + 1)$ is also true:

Example

Case 1 ($k + 1$ is prime): In this case $k + 1$ is divisible by a prime number, namely itself.

Example

Case 1 ($k + 1$ is prime): In this case $k + 1$ is divisible by a prime number, namely itself.

Case 2 ($k + 1$ is not prime): In this case $k + 1 = ab$ where a and b are integers with $1 < a < k + 1$ and $1 < b < k + 1$.

Thus, in particular, $2 \le a \le k$, and so by <u>inductive hypothesis</u>, a is divisible by a prime number p.

Example

Case 1 ($k + 1$ is prime): In this case $k + 1$ is divisible by a prime number, namely itself.

Case 2 ($k + 1$ is not prime): In this case $k + 1 = ab$ where a and b are integers with $1 < a < k + 1$ and $1 < b < k + 1$.

Thus, in particular, $2 \geq a \leq k$, and so by inductive hypothesis, a is divisible by a prime number p.

In addition because $k + 1 = ab$, we have that $k + 1$ is divisible by a.

Example

Hence, $k + 1$ is divisible by a and a is divisible by p,

Example

Hence, $k + 1$ is divisible by a and a is divisible by p, by transitivity of divisibility, $k + 1$ is divisible by the prime number p.

Example

Hence, $k + 1$ is divisible by a and a is divisible by p, by transitivity of divisibility, $k + 1$ is divisible by the prime number p.

Therefore, regardless of whether $k + 1$ is prime or not, it is divisible by a prime number *[as was to be shown]*.

[Since we have proved both the basis and the inductive step of the strong mathematical induction, we conclude that the given statement is true.]

Example

Which amounts of money can be formed using just two dollar bills and five dollar bills? Justify your answer by formulating a proof!

How would you solve this problem?

Recurrence Relations

We started our journey of "mathematical induction" by introducing sequences. Let us end by telling you that sequences can be represented in many ways.

One of such ways is called a "**recurrence relation**".

Example

n people are invited to a party. If every person shakes hand of every other person, how many handshakes will be there?

Example

n people are invited to a party. If every person speaks to every other person on the phone, how many calls will be there?

NO. of People	1	2	3	4	5
NO. of handshakes	0	1	3	6	10

Example

n people are invited to a party. If every person speaks to every other person on the phone, how many calls will be there?

NO. of People	1	2	3	4	5
NO. of handshakes	0	1	3	6	10

So, we have 0, 1, 2, 3, 6, 10, . . .

$$a_k = \begin{cases} 0, & k = 1 \\ a_{k-1} + k - 1, & k \geq 2 \end{cases}$$

Which is the recurrence relation for the above sequence.