# Discrete Mathematics

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“It is not Certain that Everything is Uncertain!”

- Blaise Pascal -

# Recap

- Logic
- Prepositions
- Negation, Conjunction, Disjunction
- Condition Statement
- Contrapositive, Inverse, Converse
- Bi-conditional
- Logical Equivalence
- Rules of Inference
- Application (System Specification)

# Recap cont.

- Bi-conditionals are not always explicit in natural language.

  "if you finish your meal, **then** you can have dessert"

  "John will brake the worlds record for mile run **only if** he runs the mile in under four minutes".

Bi-Conditionals are often expressed using "if, then" or "only if" construction.

# Recap cont.

Determine whether these system specifications are consistent

- "The diagnostic message is stored in the buffer or it is retransmitted"
- "The diagnostic message is not stored in the buffer"
- "If the diagnostic message is stored in the buffer, then it is transmitted"

# Predicate Logic

- Can be considered as a model to investigate certain arguments that cannot be expressed in propositional logic.
  - Given the following propositions as premises
    - All human beings are mammals
    - Peter is a human being

  We should be able to conclude that
    - Peter is a mammal.

- This raises two questions
    - How to express propositions like the first premise?
    - How to provide rules to help judge the validity of the argument?

# Predicate Logic

- In propositional logic we may state.
  - "Peter is a human being"

    and

  - "Ann is a human being"
- But we have no means to express that the two propositions are about the same property: "**Is a human being**"

# Predicate Logic

- A predicate **p(x)** describes a property, say *"x" is a human being*. – also called propositional function

  Where **x** is a free variable that may be substituted by values in the **Universe of discourse** (UOD, also called domain) of the predicate.

- For example, we can write the two propositions on the previous slide as: **p(*Peter*)** and **p(*Ann*)**.

# Predicates

- Statements involving variables, such that

“**X > 3**”, “**X = Y+3**” , “**X + Y = Z**”

and

“Computer **X** is under attack by an intruder”

and

“Computer **X** is functioning properly”

# Predicates

- These statements are neither **true** nor **false**

- When variables are substituted by values (elements) in the domain, the resulting statement is either true or false.

- The set of all such elements that make the predicate true is called the **truth set** of the predicate.

# Predicates

- For example
  - Let $p(X)$ denote $X>3$,
    - What are the truth values of $p(4)$ and $p(2)$?
    - $P(4)$, set $X = 4$,
    - $\Rightarrow 4 > 3$, **<u>which is true</u>**.
    - $P(2)$, set $X = 2$,
    - $\Rightarrow 2 > 3$, **<u>which is false</u>**.

# Predicates

- For example
  - Let **Q(n)** be the predicate

    *“**n** is a factor of 8”*.

Find the **truth set** of **Q(n)** if,

- The **domain** of **n** is $Z^+$

  Truth set is {1 , 2 , 4 , 8}

- The **domain** of **n** is Z

  Truth set is {1 , 2 , 4 , 8 , -1 , -2 , -4 , -8}

# The Universal Quantifier

- One sure way to change propositional function into propositions is to assign specific values to all their variables.
- Another way is to add **quantifiers**.
- Words that refer to quantities such as “some” or “all”.
- The symbol $\forall$ denotes “for all” and is called the **universal quantifier**.
- Other expressions
  - *For every, for arbitrary, for any, for each, given any.*

# The Universal Quantifier

- Let **Q(x)** be a predicate and **D** the domain of **x**
- Universal statement is a statement of the form

$$\forall x \in D, Q(x)$$

- **True:** *iff* Q(x) is true **for every** x in D.
- **False:** *iff* Q(x) is false **for at least one** x in D
- Value of **x** for which Q(x) is false is called **counterexample.**

# The Universal Quantifier

- **Example:** truth and falsity of universal statement.
  - Let D = {1, 2, 3, 4, 5}, and consider the statement

$$\forall x \in D, x^2 \geq x$$

Show that the statement is true.

- **Solution:** Check that "$x^2 \geq x$" is true for each individual x in D.

$$1^2 \geq 1,\ 2^2 \geq 2,\ 3^2 \geq 3,\ 4^2 \geq 4,\ 5^2 \geq 5$$

Hence "$\forall x \in D, x^2 \geq x$" is true.

# The Universal Quantifier

- **Example:** truth and falsity of universal statement.

- Consider the statement

$$\forall x \in \boldsymbol{R}, x^2 \geq x$$

Find a counterexample to show that this statement is false.

- **Counterexample:** Take $x = \frac{1}{2}$, then x is in $\boldsymbol{R}$ (since $\frac{1}{2}$ is a real number) and

$$\left(\frac{1}{2}\right)^2 = \frac{1}{4} \ngeq \frac{1}{2}$$

Hence "$\forall x \in \boldsymbol{R}, x^2 \geq x$" is false.

# The Universal Quantifier and Empty Domain

- Generally D is considered to be non-empty
- However, if we consider an ***Empty Domain***, then the universal statement

$$\forall x \in D, Q(x)$$

- Is True, for any Q(x)

# The Existential Quantifier

- The symbol ∃ denotes "*there exists*".

**For Example:** The sentence "*There is a student in math 140 course*" can be written as

> "∃ a person p such that p is a student in math 140 course"

or more formally

> "∃ p ∈ $P$ such that p is student in math 140 course"

Other expressions:

> *"There is a", "we can find a", "there is at least one", "for some", "for at least one".*

# The Existential Quantifier

Let **Q(x)** be the predicate, and **D** the domain of **x**.

An existential statement is a statement of the form,

"$\exists\ p \in P$ such that Q(x)"

**True** *iff* Q(x) **is true for at least one x in D.**

**False** *iff* Q(x) **is false for all** x in D.

# The Existential Quantifier

- **Example:** Truth and falsity of Existential Statement.
  - Consider the statement

$$\exists\, m \in Z^+, \text{such that } m^2 = m$$

Show that the statement is true.

- **Solution:**
  - Observe that $1^2 = 1$. Thus $m^2 = m$ is true for at least on integer m.

  - hence "$\exists\, m \in Z$ such that $m^2 = m$ is true.

# The Existential Quantifier

- **Example:** Truth and falsity of Existential Statement.

  - Let E= [5, 6, 7, 8] and consider the statement

$$\exists\, m \in E\text{, such that } m^2 = m$$

Show that this statement is false.

- **Solution:**
  - Note that $m^2 = m$ is not true for any integers m form 5 through 8:

$$5^2 = 25 \neq 5,\ 6^2 = 36 \neq 6,\ 7^2 = 49 \neq 7\ ,\ 8^2 = 64 \neq 8$$

Hence "$\exists\, m \in E$, such that $m^2 = m$" is false.

# The Existential Quantifier and Empty Domain

- Generally D is considered to be non-empty
- However, if we consider an ***Empty Domain***, then the universal statement

$$\exists x \in D, Q(x)$$

- **Is false,** for any Q(x)

# Other Quantifiers

Are there just two kinds of quantifiers?

- Of other quantifiers, one most often see is the *Uniqueness Quantifier*

  - $\exists! x \in \boldsymbol{D}, P(x)$ or $\exists_1 x \in \boldsymbol{D}, P(x)$

  - Remember, we can use quantifiers and propositional logic to express uniqueness

  - Therefore, it is better to stick to universal and existential quantifiers so that rules of inference for these quantifiers can be used

# Quantifier Precedence

First the precedence of logical operators

| *Operator* | *Precedence* |
|---|---|
| $\neg$ | 1 |
| $\wedge$ | 2 |
| $\vee$ | 3 |
| $\rightarrow$ | 4 |
| $\leftrightarrow$ | 5 |

- $\forall$ and $\exists$ have higher precedence than all other logical operators

# Universal Conditional Statement

Probably, the most important form of statement in mathematics is

$$\forall x, \text{ if } P(x) \text{ then } Q(x).$$

- **Example:**
  - $\forall x \in \boldsymbol{R}$, if $x > 2$ then $x^2 > 4$
  - Whenever a real number is greater than 2, its square is greater than 4.

OR

  - The square of any real number greater than 2 is greater than 4.

- Translating sentences in English into logical expression is crucial task in mathematics, logic programming artificial intelligence and many other disciplines.

# Equivalent forms of Universal Statement

$\forall x \in U$, if $P(x)$ then $Q(x)$.

Can always be written as

$\forall x \in D, Q(x),$ By narrowing U to D consisting of all values of x that make P(x) true

- **Example:**
  - ∀ real numbers **x**, if **x** is an integer then **x** is rational. Means the same as
  - ∀ integers **x**, **x** is rational.

$\forall x \in \boldsymbol{R}$ if $P(x)$ then $Q(x)$

$\forall x \in Z, Q(x)$

P(x) = x is an integer

Q(x) = x is rational

# Equivalent forms of Existential Statement

$\exists$ x, such that P(x) and Q(x)

Can always be written as

$\exists$ x $\in$ D, such that Q(x)

Where D is the set of all x for which P(x) is true.

# Equivalent forms of Existential Statement

- **Example:** Express the proposition: "All human beings are intelligent" in predicate logic.

*Let I(x) = x, is intelligent*

*H(x) = x is human being*

Then

$$\forall x, H(x) \rightarrow I(x)$$

Or Equivalently

$$\forall x \in D, \; I(x)$$

Where D is a set of all **x** for which H(x) is true.

# Logical Equivalence with Quantifiers

- Statements involving predicates and quantifiers are logically equivalent if and only if they have the same truth values no matter what which predicates are substituted into those statements.

$$\forall x(P(x) \land Q(x)) \equiv \forall x P(x) \land \forall x Q(x).$$

# Negating Quantified Statements

# Negation of a Universal Statement

$$\forall x \in D, Q(x).$$

Its negation is logically equivalent to

$$\exists x \in D, \sim Q(x)$$

That is

$$\sim(\forall x \in D, Q(x)) \equiv \exists x \in D, \sim Q(x)$$

$\sim$ (all are) $\equiv$ "*Some are not*" or "*There is at least one that is not*"

# Negation of a Existential Statement

$$\exists x \in D, Q(x).$$

Its negation is logically equivalent to

$$\forall x \in D, \sim Q(x)$$

That is

$$\sim(\exists x \in D, Q(x)) \equiv \forall x \in D, \sim Q(x)$$

$\sim$ (some are) $\equiv$ "*None are*"

# Negation of Universal Conditional Statement

Negations of universal conditional statements are of special importance in mathematics.

The form of such negations can be derived from facts that have already been established.

By definition of the negation of a *for all* statement,

$$\sim(\forall x,\ P(x) \rightarrow Q(x)) \equiv \exists x \text{ such that } \sim(P(x) \rightarrow Q(x)). \qquad 3.2.1$$

But the negation of an if-then statement is logically equivalent to an *and* statement. More precisely,

$$\sim(P(x) \rightarrow Q(x)) \equiv P(x) \wedge \sim Q(x). \qquad 3.2.2$$

# Negation of Universal Conditional Statement

Substituting (3.2.2) into (3.2.1) gives

$$\sim(\forall x,\ P(x) \rightarrow Q(x)) \equiv \exists x \text{ such that } (P(x) \wedge \sim Q(x)).$$

Written less symbolically, this becomes

**Negation of a Universal Conditional Statement**

$$\sim(\forall x, \text{ if } P(x) \text{ then } Q(x)) \equiv \exists x \text{ such that } P(x) \text{ and } \sim Q(x).$$

Write a formal negation of the following statement.

If a computer program has more than 100,000 lines, then it contains a bug.

# Variants of Universal Conditional Statement

We have known that a conditional statement has a contrapositive, a converse, and an inverse.

The definitions of these terms can be extended to universal conditional statements.

- **Definition**

Consider a statement of the form: $\forall x \in D$, if $P(x)$ then $Q(x)$.

1. Its **contrapositive** is the statement: $\forall x \in D$, if $\sim Q(x)$ then $\sim P(x)$.
2. Its **converse** is the statement: $\forall x \in D$, if $Q(x)$ then $P(x)$.
3. Its **inverse** is the statement: $\forall x \in D$, if $\sim P(x)$ then $\sim Q(x)$.

# Nested Quantifiers

- Quantifier within the scope of another quantifier

$\forall x \exists y (x + y = 0)$, where the domain of x and y consists of all real numbers.

- For every real number x there is a real number y such that, $x + y = 0$.

OR

- Every real number has an additive inverse.

❑ Commonly occur in mathematics and computer science.

# Nested Quantifiers

- **Example:**

- Translate into English the following

$$\forall x, \forall y, \big((x > 0) \wedge (y < 0) \rightarrow (xy < 0)\big)$$

Where the domain of x and y consists of all real numbers.

# Nested Quantification as Loops

- It is sometimes easier to think of nested quantification in terms of nested loops.

  - $\forall x, \forall y, P(x,y)$ (1)

- To see whether it is true.
  - Loop through the values of **x**
  - For each **x**, loop through the values of **y**
  - If **P(x,y)** is true for all values of **x** and **y**, (1) is true.
  - If we ever hit a value **x**, for which we hit a value y for which **P(x,y)** is false, (1) is false.

# Nested Quantification as Loops

- Similarly,

  - $\forall x, \exists y, P(x, y)$ (2)

- To see whether it is true.
  - Loop through the values of **x**.
  - For each **x**, loop through the values of **y**, until we find a **y** for which P(x,y) is true.
  - If for every **x**, we hit such a **y**, (2) is true.
  - If for some **x**, we never hit such a **y**, (2) is false.

# Nested Quantification as Loops

- Furthermore,

  - $\exists x, \forall y, P(x, y)$

  - $\exists x, \exists y, P(x, y)$

Try it yourself

# Negating Nested Quantification

- Statements involving nested quantifiers can be negated by successively applying rules of negating single quantifier statements.

- For Example
  - Express the negation of $\forall x, \exists y\ (xy = 1)$
  - $\sim\left(\forall x, \exists y\ (xy = 1)\right) \equiv \exists x \sim \exists y\ (xy = 1)$

    $\equiv \exists x\ \forall y \sim(xy = 1)$

    $\equiv \exists x\ \forall y\ (xy \neq 1)$

# Arguments with Quantified Statements

- Universal modus ponens.

$$\begin{array}{c} \forall x, \text{ if } P(x) \text{ then } Q(x) \\ P(a) \text{ for a particular } a \\ \hdashline \therefore Q(a) \end{array}$$

$$\begin{array}{c} \forall x\,\big(p(x) \rightarrow Q(x)\big) \\ P(a) \text{ for a particular } a \\ \hdashline \therefore Q(a) \end{array}$$

Different Representations of the same concept (argument form)

# Arguments with Quantified Statements

- **Example**: Is this argument valid? Why?
  - If an integer is even, then its square is even
  - **k** is particular integer that is even

    $\therefore k^2$ is even

# Arguments with Quantified Statements

- **Example:** Is this argument valid? Why?
  - If an integer is even, then its square is even
  - k is particular integer that is even

    $\therefore k^2$ is even
- **Solution:** Let E(x) be "an integer is even"
  - S(x) be "its square is even"

  Then

  $\forall$ x , if E(x) then S(x)

  E(k) for a particular k

  ---

  $\therefore$ S(k)

- This argument has the form of "universal modus ponens". Therefore it is a valid argument.

# Arguments with Quantified Statements

- Universal modus Tollens:

$$\begin{array}{c} \forall\, x \text{, if } P(x) \text{ then } Q(x) \\ \\ \sim Q(a) \text{ for a particular } a \\ \hdashline \therefore \sim P(a) \end{array}$$

$$\begin{array}{c} \forall\, x \text{, } \big(p(x) \rightarrow Q(x)\big) \\ \\ \sim Q(x) \text{for a particular } a \\ \hdashline \therefore \sim P(a) \end{array}$$

# Arguments with Quantified Statements

- **Example**: Universal Modus Tollens.
  - Rewrite the following argument using quantifiers, variables, and predicate symbols. Write the major premise in conditional form. In this argument valid? why?

All human beings are mortal

Zeus is not mortal.

---

∴ Zeus is not human

# Arguments with Quantified Statements

- ❑ **Example: Recognizing the form of Universal Modus Tollens.**
  - **Rewrite the following argument using quantifiers, variables, and predicate symbols. Write the major premise in conditional form. In this argument valid? why?**

All human beings are mortal

Zeus is not mortal.

---

∴ Zeus is not human

❑ **Solution: The major premise can be rewritten as**

**∀x, if x is human then x is mortal.**

**Let H(x) be "x is human", let M(x) be "x is mortal", and let Z stand for Zeus. The argument becomes.**

$$\forall x, \text{ if } H(x), \text{ then } M(x)$$

$$\sim M(Z)$$

---

$$\therefore \sim H(Z)$$

**This argument has the form of universal modus Tollens and is therefore valid.**

# Creating Additional Forms of Arguments

Universal modus ponens and modus tollens were obtained by combining universal instantiation with modus ponens and modus tollens.

In the same way, additional forms of arguments involving universally quantified statements can be obtained by combining universal instantiation with other of the valid argument forms discussed earlier.

# Creating Additional Forms of Arguments

Consider the following argument:

$$p \to q$$
$$q \to r$$
$$\bullet\ p \to r$$

This argument form can be combined with universal instantiation to obtain the following valid argument form.

**Universal Transitivity**

| *Formal Version* | *Informal Version* |
|---|---|
| $\forall x\, P(x) \to Q(x)$. | Any $x$ that makes $P(x)$ true makes $Q(x)$ true. |
| $\forall x\, Q(x) \to R(x)$. | Any $x$ that makes $Q(x)$ true makes $R(x)$ true. |
| $\bullet\ \forall x\, P(x) \to R(x)$. | $\bullet$ Any $x$ that makes $P(x)$ true makes $R(x)$ true. |