LINEAR ALGEBRA. LECTURE 4

Solving $Ax = 0$: factor variables, special solutions

We have a definition for the column space and the nullspace of a matrix, but how do we compute these subspaces?

Computing the nullspace

The *Nullspace* of a matrix A is made up of the vectors x for which $Ax = 0$. Suppose:

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

(Note that the columns of this matrix A are not independent.) Our algorithm for computing the nullspace of this matrix uses the method of elimination, despite the fact that A is not invertible. We don't need to use an augmented matrix because the right side (the vector b) is 0 in this computation.

The row operations used in the method of elimination don't change the solution to $Ax = b$ so they don't change the nullspace. (They do affect the column space.)

The first step of elimination gives us:

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

We don't find a factor in the second column, so our next factor is the 2 in the third column of the second row:

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix} \rightarrow U = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix U is in *Echelon* (staircase) form. The third row is zero because row 3 was a linear combination of rows 1 and 2; it was eliminated. The *Rank* of a matrix A equals the number of factors it has. In this example, the rank of A (and of U) is 2.

Special solutions

Once we've found U we can use back-substitution to find the solutions x to the equation $Ux = 0$. In our example, columns 1 and 3 are *factor columns* containing factors, and columns 2 and 4 are *free columns*. We can assign any value to x_2 and x_4; we call these *free variables*. Suppose $x_2 = 1$ and $x_4 = 0$. Then:

$$2x_3 + 4x_4 = 0 \;\Rightarrow\; x_3 = 0 \;\text{ and }\; x_1 + 2x_2 + 2x_3 + 2x_4 = 0 \;\Rightarrow\; x_1 = -2$$

So, one solution is $\vec{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, because the second column is just twice the first column.

Any multiple of this vector is in the nullspace.

Letting a different free variable equal 1 and setting the other free variables equal to zero gives us other vectors in the nullspace. For example:

$$\vec{x} = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

has $x_4 = 1$ and $x_2 = 0$. The nullspace of A is the collection of all linear combinations of these "special solution" vectors.

The rank r of A equals the number of factor columns, so the number of free columns is $n - r$: the number of columns (variables) minus the number of factor columns. This equals the number of special solution vectors and the dimension of the nullspace.

Reduced row echelon form

By continuing to use the method of elimination we can convert U to a matrix R in *Reduced Row Echelon Form* (RREF form), with factors equal to 1 and zeros above and below the factors.

$$U = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow R = \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By exchanging some columns, R can be rewritten with a copy of the identity matrix in the upper left corner, possibly followed by some free columns on the right. If some rows of A are linearly dependent, the lower rows of the matrix R will be filled with zeros:

$$R = \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

Here I is an rxr square matrix and F is an rx$(n - r)$ free columns matrix.

If N is the *Nullspace matrix:* $N = \begin{bmatrix} -F \\ I \end{bmatrix}$ then $RN = 0$. Here I is an $(n - r)$x$(n - r)$ square matrix and N is an mx$(n - r)$ matrix. The columns of N are the special solution.

Example: Find the *Nullspace* of a matrix A^{T}:

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix}$$

First find the matrix U in *Echelon* (staircase) form.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} \rightarrow U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

In this example, the rank of A (and of U) is 2 again.

Once we've found U we can use back-substitution to find the solutions x to the equation $Ux = 0$. In this example, columns 1 and 2 are *factor columns* containing factors, and column 3 is *free column*. We can assign any value to x_3; we call these *free variable*. Suppose $x_3 = 1$. Then:

$$2x_2 + 2x_3 = 0 \Rightarrow x_2 = -1 \text{ and } x_1 + 2x_2 + 3x_3 = 0 \Rightarrow x_1 = -1$$

So one solution is $\vec{x} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$, because the third column is just the sum of the first and second columns. Any multiple of this vector is in the nullspace: $\vec{x}_N = c\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

By continuing to use the method of elimination we can convert U to a matrix R in *reduced row echelon form* (RREF form):

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

Here I is an rxr square matrix and F is an rx$(n - r)$ one free column matrix.

Here the *nullspace matrix* $N = \begin{bmatrix} -F \\ I \end{bmatrix} = \vec{x}_N = c\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ and $RN = 0$. Here I is an $(n - r)$x$(n - r)$ one element matrix and N is an mx$(n - r)$ matrix-vector. The column of N is the special solution.

Solving $Ax = b$: row reduced form R

When does $Ax = b$ has solutions x, and how can we describe those solutions?

Solvability conditions on b

We again use the example:

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

The third row of A is the sum of its first and second rows, so we know that if $Ax = b$ the third component of b equals the sum of its first and second components. If b does not satisfy $b_3 = b_1 + b_2$ the system has no solution. If a combination of the rows of A gives the zero row, then the same combination of the entries of b must equal zero.

One way to find out whether $Ax = b$ is solvable is to use elimination on the augmented matrix. If a row of A is completely eliminated, so is the corresponding entry in b. In our example, row 3 of A is completely eliminated:

$$\begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{bmatrix}$$

If $Ax=b$ has a solution, then $b_3 - b_2 - b_1 = 0$. For example, we could choose $\vec{b} = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$.

From an earlier lecture, we know that $Ax = b$ is solvable exactly when b is in the column space $C(A)$. We have these two conditions on b; in fact, they are equivalent.

Complete solution

In order to find all solutions to $Ax = b$ we first check that the equation is solvable, then find a particular solution. We get the complete solution of the equation by adding the particular solution to all the vectors in the nullspace.

A particular solution

One way to find a particular solution to the equation $Ax = b$ is to set all free variables to zero, then solve for the factor variables. For our example matrix A, we let $x_2 = x_4 = 0$ to get the system of equations:

$$\begin{cases} x_1 + 2x_3 = 1 \\ 2x_3 = 3 \end{cases}$$

which has the solution $x_3 = 3/2$, $x_1 = -2$. Our *Particular solution* is: $\vec{x}_p = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$.

The general solution to $Ax = b$ is given by $\vec{x}_{complete} = \vec{x}_N + \vec{x}_p$, where $\vec{x}_N$ is a generic vector in the nullspace. To see this, we add $Ax_p = b$ to $Ax_N = 0$ and get $A(x_p + x_N) = b$ for every vector x_N in the nullspace.

We know that the nullspace of A is the collection of all combinations of the special solutions:

$\vec{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$. So the complete solution to the equation $A\vec{x} = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$ is:

$$\vec{x}_{complete} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

The nullspace of A is a two-dimensional subspace of $\mathbf{R}^4$, and the solutions to the equation $Ax = b$ form a plane parallel to that through $\vec{x}_p = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$.

Rank

The rank of a matrix equals the number of factors(pivots) of that matrix. If A is an m by n matrix of rank r, we know $r \leq m$ and $r \leq n$.

Full column rank

If $r = n$, then we know that the nullspace has dimension $n - r = 0$ and contains only the zero vector. There are no free variables or special solutions.

If $Ax = b$ has a solution, it is unique; there is either 0 or 1 solution. Examples like this, in which the columns are independent, are common in applications.

We know $r \leq m$, so if $r = n$ the number of columns of the matrix is less than or equal to the number of rows. The row reduced echelon form of the matrix will look like $R = \begin{bmatrix} I \\ 0 \end{bmatrix}$.

For any vector b in $\mathbf{R}^m$ that's not a linear combination of the columns of A, there is no solution to $Ax = b$.

Full row rank

If $r = m$, then the reduced matrix $R = \begin{bmatrix} I & F \end{bmatrix}$ has no rows of zeros and so there are no requirements for the entries of b to satisfy. The equation $Ax = b$ is solvable for every b. There are $n - r = n - m$ free variables, so there are $n - m$ special solutions to $Ax = 0$.

Full row and column rank

If $r = m = n$ is the number of factors of A, then A is an invertible square matrix and R is the identity matrix. The nullspace has dimension zero, and $Ax = b$ has a unique solution for every b in $\mathbf{R}^{\mathbf{m}}$.

Summary

If R is in row reduced form with pivot columns first (RREF), the table below summarizes our results.

	$r = m = n$	$r = n < m$	$r = m < n$	$r < m, r < n$
R	I	$\begin{bmatrix} I \\ 0 \end{bmatrix}$	$\begin{bmatrix} I & F \end{bmatrix}$	$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$
Number solutions to $Ax = b$	1	0 or 1	infinitely many	0 or infinitely many