LINEAR ALGEBRA. LECTURE 5

Solving $Ax = b$: Summary

When does $Ax = b$ has solutions x, and how can we describe those solutions? If R is in row reduced form with factor (pivot) columns first (RREF), the table below summarizes our results.

	$r = m = n$	$r = n < m$	$r = m < n$	$r < m, r < n$
R	I	$\begin{bmatrix} I \\ 0 \end{bmatrix}$	$\begin{bmatrix} I & F \end{bmatrix}$	$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$
Number solutions to $Ax = b$	1	0 or 1	infinitely many	0 or infinitely many

Matrix Rank

The rank of a matrix equals the number of factors(pivots) of that matrix. If A is an m by n matrix of rank r, we know $r \le m$ and $r \le n$.

Full row and column rank

If $r = m = n$ is the number of factors of A, then A is an invertible square matrix and R is the identity matrix. The nullspace has dimension zero, and $Ax = b$ has a unique solution $x = A^{-1}b$ for every b in $\mathbf{R}^{\mathbf{m}}$.

Full column rank

If $r = n$, then we know that the nullspace has dimension $n - r = 0$ and contains only the zero vector. There are no free variables or special solutions. If $Ax = b$ has a solution, it is unique. There is either 0 or 1 solution. One way to find out whether $Ax = b$ is solvable or not is to use elimination on the augmented matrix $\left[A | b \right]$. From an earlier lecture, we know that $Ax = b$ is solvable exactly when b is in the column space $C(A)$. If a combination of the rows of A gives the zero row, then the same combination of the entries of b must equal zero. In that case we can eliminate the zero rows of A and zero elements of b and solve the reduced system like in the previous case:

$$\begin{bmatrix} a_{11} & . & . & a_{1n} \\ . & . & . & . \\ . & . & . & . \\ a_{n1} & . & . & a_{nn} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ . \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ . \\ . \\ b_n \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & . & . & a_{1n} \\ . & . & . & . \\ . & . & . & . \\ a_{n1} & . & . & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ . \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ . \\ . \\ b_n \end{bmatrix} \Rightarrow \vec{x} = A^{-1}_{(nxn)} b_{(n)}$$

Full row rank

If $r = m$, then the reduced matrix $R=\begin{bmatrix} I & F\end{bmatrix}$ has no rows of zeros and so there are no requirements for the entries of b to satisfy. The equation $Ax = b$ is solvable for every b. There are $n-r = n-m$ free variables, which we can use to express factor variables:

$$\begin{bmatrix} a_{11} & . & . & . & . & a_{1n} \\ . & . & . & . & . & . \\ a_{m1} & . & . & . & . & a_{mn} \end{bmatrix}\begin{bmatrix} x_1 \\ . \\ . \\ . \\ . \\ x_n \end{bmatrix}=\begin{bmatrix} b_1 \\ . \\ b_m \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & . & 0 & f_{1\,m+1} & . & f_{1\,n} \\ . & . & . & . & . & . \\ 0 & . & 1 & f_{m\,m+1} & . & f_{m\,n} \end{bmatrix}\begin{bmatrix} x_1 \\ . \\ . \\ . \\ . \\ x_n \end{bmatrix}=\begin{bmatrix} d_1 \\ . \\ d_m \end{bmatrix} \Rightarrow$$

$$\Rightarrow \begin{bmatrix} x_1 \\ . \\ x_m \end{bmatrix}=\begin{bmatrix} d_1 \\ . \\ d_m \end{bmatrix}-x_{m+1}\vec{f}_{m+1}-...-x_n\vec{f}_n \Rightarrow \begin{bmatrix} x_1 \\ . \\ . \\ . \\ . \\ x_n \end{bmatrix}=\begin{bmatrix} d_1 \\ . \\ d_m \\ 0 \\ 0 \\ 0 \end{bmatrix}+c_{m+1}\begin{bmatrix} -f_{1\,m+1} \\ . \\ -f_{m\,m+1} \\ 1 \\ 0 \\ 0 \end{bmatrix}+c_{m+2}\begin{bmatrix} -f_{1\,m+2} \\ . \\ -f_{m\,m+2} \\ 0 \\ 1 \\ 0 \end{bmatrix}+...+c_n\begin{bmatrix} -f_{1\,n} \\ . \\ -f_{m\,n} \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Complete solution

If $r < m$, $r < n$, then the reduced matrix $R=\begin{bmatrix} I & F \\ 0 & 0\end{bmatrix}$. In order to find all solutions to $Ax = b$ we first check that the equation is solvable. If a combination of the rows of A gives the zero row, then the same combination of the entries of b must equal zero. In that case we can eliminate the zero rows of A and zero elements of b and solve the reduced system like in the previous case when $R=\begin{bmatrix} I & F\end{bmatrix}$:

$$\begin{bmatrix} a_{11} & . & . & a_{1n} \\ . & . & . & . \\ a_{r1} & . & . & a_{rn} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ . \\ . \\ x_n \end{bmatrix}=\begin{bmatrix} b_1 \\ . \\ b_r \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & . & . & a_{1n} \\ . & . & . & . \\ a_{r1} & . & . & a_{rn} \end{bmatrix}\begin{bmatrix} x_1 \\ . \\ . \\ x_n \end{bmatrix}=\begin{bmatrix} b_1 \\ . \\ b_r \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} a_{11} & . & . & . & . & a_{1n} \\ . & . & . & . & . & . \\ a_{r1} & . & . & . & . & a_{rn} \end{bmatrix}\begin{bmatrix} x_1 \\ . \\ . \\ . \\ . \\ x_n \end{bmatrix}=\begin{bmatrix} b_1 \\ . \\ b_r \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & . & 0 & f_{1\,m+1} & . & f_{1\,n} \\ . & . & . & . & . & . \\ 0 & . & 1 & f_{r\,m+1} & . & f_{r\,n} \end{bmatrix}\begin{bmatrix} x_1 \\ . \\ . \\ . \\ . \\ x_n \end{bmatrix}=\begin{bmatrix} d_1 \\ . \\ d_r \end{bmatrix} \Rightarrow$$

$$\Rightarrow \begin{bmatrix} x_1 \\ . \\ x_r \end{bmatrix}=\begin{bmatrix} d_1 \\ . \\ d_r \end{bmatrix}-x_{r+1}\vec{f}_{r+1}-...-x_n\vec{f}_n \Rightarrow \begin{bmatrix} x_1 \\ . \\ . \\ . \\ . \\ x_n \end{bmatrix}=\begin{bmatrix} d_1 \\ . \\ d_r \\ 0 \\ 0 \\ 0 \end{bmatrix}+c_{r+1}\begin{bmatrix} -f_{1\,m+1} \\ . \\ -f_{r\,m+1} \\ 1 \\ 0 \\ 0 \end{bmatrix}+c_{r+2}\begin{bmatrix} -f_{1\,m+2} \\ . \\ -f_{r\,m+2} \\ 0 \\ 1 \\ 0 \end{bmatrix}+...+c_n\begin{bmatrix} -f_{1\,n} \\ . \\ -f_{r\,n} \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Independence, basis, and dimension

What does it mean for vectors to be independent? How does the idea of independence help us describe subspaces like the nullspace?

Linear independence

Suppose A is an m by n matrix with $m < n$ (so $Ax = b$ has more unknowns than equations). A has at least one free variable, so there are nonzero solutions to $Ax = 0$. A combination of the columns is zero, so the columns of this A are *dependent*.

We say vectors $x_1, x_2, \ldots, x_n$ are *linearly independent* (or just *independent*) if $c_1x_1 + c_2x_2 +$ $+ \ldots + c_nx_n = 0$ only when $c_1, c_2, \ldots, c_n$ are all 0. When those vectors are the columns of A, the only solution to $Ax = 0$ is $x = 0$.

Two vectors are independent if they do not lie on the same line. Three vectors are independent if they do not lie in the same plane. Thinking of Ax as a linear combination of the column vectors of A, we see that the column vectors of A are independent exactly when the nullspace of A contains only the zero vector.

If the columns of A are independent then all columns are pivot columns, the rank of A is n, and there are no free variables. If the columns of A are dependent then the rank of A is less than n and there are free variables.

Spanning a space

Vectors $v_1, v_2, \ldots, v_k$ *span* a space when the space consists of all combinations of those vectors. For example, the column vectors of A span the column space of A. If vectors $v_1, v_2, \ldots, v_k$ span a space S, then S is the smallest space containing those vectors.

Basis and dimension

A *basis* for a vector space is a sequence of vectors $v_1, v_2, \ldots, v_d$ with two properties:

- $v_1, v_2, \ldots, v_d$ are independent
- $v_1, v_2, \ldots, v_d$ span the vector space.

The basis of a space tells us everything we need to know about that space.

Example: $\mathbf{R}^3$

One basis for $\mathbf{R}^3$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. These are independent because:

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

is only possible when $c_1 = c_2 = c_3 = 0$. These vectors span $\mathbf{R}^3$. The vectors $\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}$ and $\begin{bmatrix}3\\3\\8\end{bmatrix}$ do not form a basis for $\mathbf{R}^3$ because these are the column vectors of a matrix that has two identical rows. The three vectors are not linearly independent. In general, n vectors in $\mathbf{R}^n$ form a basis *if they are the column vectors of an invertible matrix*.

Basis for a subspace

The vectors $\begin{bmatrix}1\\1\\2\end{bmatrix}$ and $\begin{bmatrix}2\\2\\5\end{bmatrix}$ span a plane in $\mathbf{R}^3$ but they cannot form a basis for $\mathbf{R}^3$. Given a space, every basis for that space has the same number of vectors; that number is the *dimension* of the space. So there are exactly n vectors in every basis for $\mathbf{R}^n$.

Bases of a column space and nullspace

Suppose: $A = \begin{bmatrix}1 & 2 & 3 & 1\\1 & 1 & 2 & 1\\1 & 2 & 3 & 1\end{bmatrix}$.

By definition, the four column vectors of A span the column space of A. The third and fourth column vectors are dependent on the first and second, and the first two columns are independent. Therefore, the first two column vectors are the factor(pivot) columns. They form a basis for the column space $C(A)$. The matrix has rank 2. In fact, for any matrix A we can say:

Rank(A)= number of factor(pivot) columns of A = dimension of $C(A)$.

(Note that matrices have a rank but not a dimension. Subspaces have a dimension but not a rank.)

The column vectors of this A are not independent, so the nullspace $N(A)$ contains more than just the zero vector. Because the third column is the sum of the first two, we know that the vector $\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}$ is in the nullspace. Similarly $\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}$ is also in $N(A)$. These are the two special solutions to $A\mathbf{x} = \mathbf{0}$. We'll see that:

Dimension of $N(A)$= number of free variables = $n - r$,

so we know that the dimension of $N(A)$ is 4 - 2 = 2. These two special solutions form a basis for the nullspace.

The four fundamental subspaces

In this lecture we discuss the four fundamental spaces associated with a matrix and the relations between them.

Four subspaces

Any m by n matrix A determines four subspaces (possibly containing only the zero vector).

Column space, C(A)

$C(A)$ consists of all combinations of the columns of A and is a vector space in $\mathbf{R}^m$.

Nullspace, N(A)

This consists of all solutions x of the equation Ax = 0 and lies in Rn.

Row space, $C(A^T)$

The combinations of the row vectors of A form a subspace of R^n. We equate this with $C(A^T)$, the column space of the transpose of A.

Left nullspace, $N(A^T)$

We call the nullspace of A^T the *left nullspace* of A. This is a subspace of $\mathbf{R}^m$.

Basis and Dimension

Column space

The r factor (pivot) columns form a basis for $C(A)$:

$$\text{Dim } C(A) = r.$$

Nullspace

The special solutions to $Ax = 0$ correspond to free variables and form a basis for $N(A)$. An m by n matrix has $n - r$ free variables:

$$\text{Dim } N(A) = n - r.$$

Row space

We could perform row reduction on A^T, but instead we make use of R, the row reduced echelon form of A.

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \to \ldots \to \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} = R$$

Although the column spaces of A and R are different, the row space of R is the same as the row space of A. The rows of R are combinations of the rows of A, and because reduction is reversible the rows of A are combinations of the rows of R.

The first r rows of R are the "echelon" basis for the row space of A:

$$\text{Dim } C(A^T) = r.$$

Left nullspace

The matrix A^T has m columns. We just saw that r is the rank of A^T, so the number of free columns of A^T must be $m - r$:

$$\text{Dim } N(A^T) = m - r.$$

The left nullspace is the collection of vectors y for which $A^T y = 0$. Equivalently, $y^T A = 0$; here y and 0 are row vectors. We say "left nullspace" because y^T is on the left of A in this equation.

To find a basis for the left nullspace we reduce an augmented version of A:

$$\begin{bmatrix} A_{m\times n} & I_{m\times m} \end{bmatrix} \rightarrow \begin{bmatrix} R_{m\times n} & E_{m\times m} \end{bmatrix}$$

From this we get the matrix E for which $EA = R$. (If A is a square, invertible matrix then $E = A^{-1}$.) In our example,

$$EA = \begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

The bottom $m - r$ rows of E describe linear dependencies of rows of A, because the bottom $m - r$ rows of R are zero. Here $m - r = 1$ (one zero row in R). The bottom $m - r$ rows of E satisfy the equation $y^T A = \mathbf{0}$ and form a basis for the left nullspace of A.

New vector space

The collection of all 3x3 matrices forms a vector space; call it M. We can add matrices and multiply them by scalars and there's a zero matrix (additive identity). If we ignore the fact that we can multiply matrices by each other, they behave just like vectors.

Some subspaces of M include:

- all upper triangular matrices
- all symmetric matrices
- D, all diagonal matrices

D is the intersection of the first two spaces. Its dimension is 3; one basis for D is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$