Probability Theory & Statistics

Innopolis University, BS-I,II

Spring Semester 2016-17

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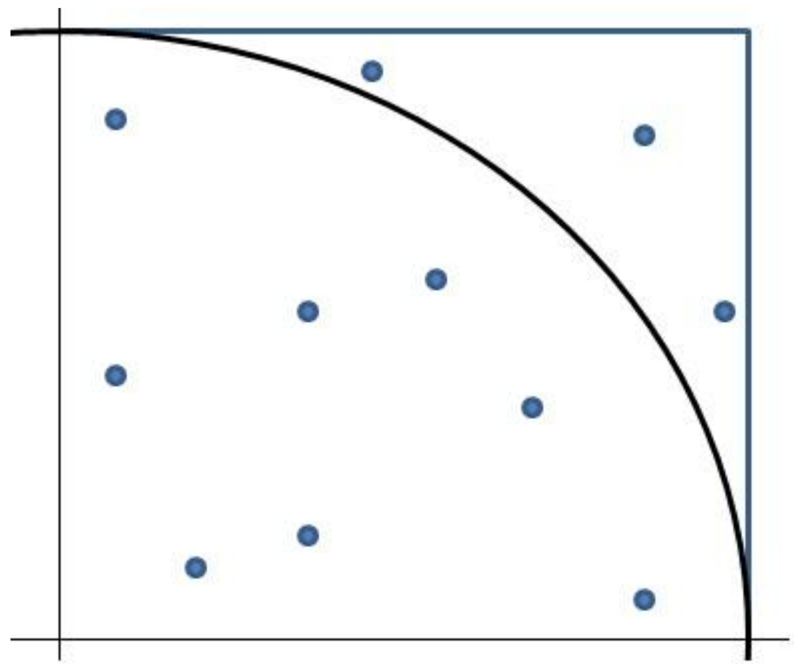
Part I

CONDITIONAL PROBABILITY

Towards Conditional Probability

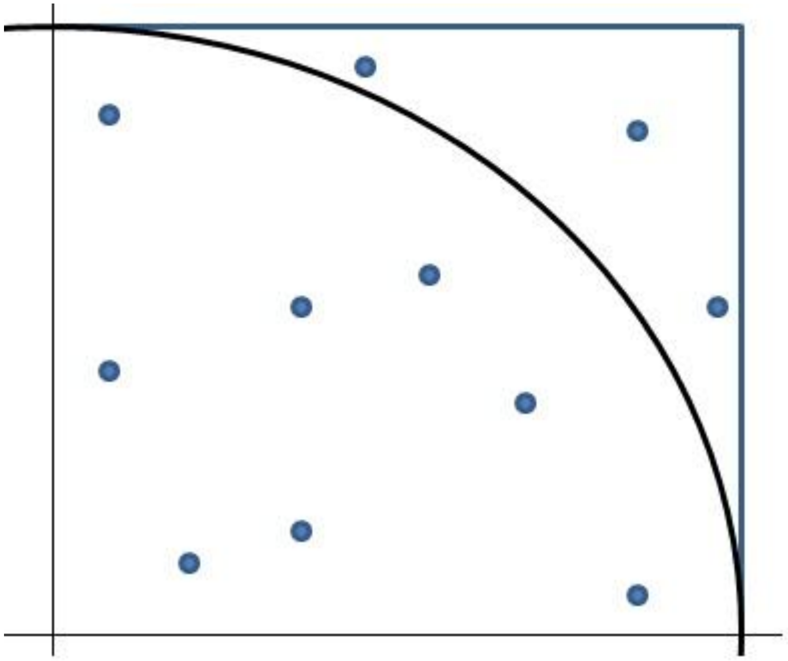
Recall the first lecture and the figure illustrating an idea behind Monte Carlo method to approximate π.



Towards Conditional Probability (cont.)

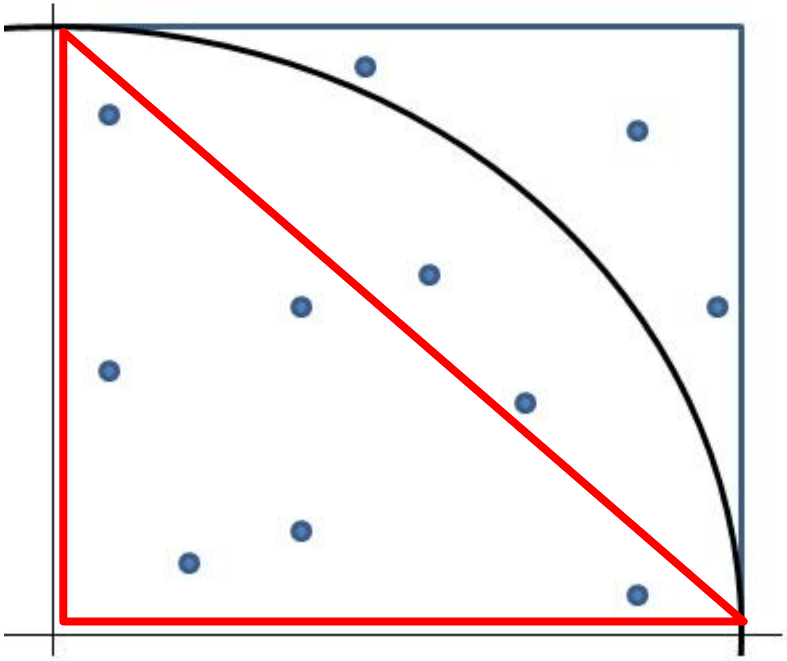
If the dots are random then

- the probability of the sector S is …
- and π is approximately …



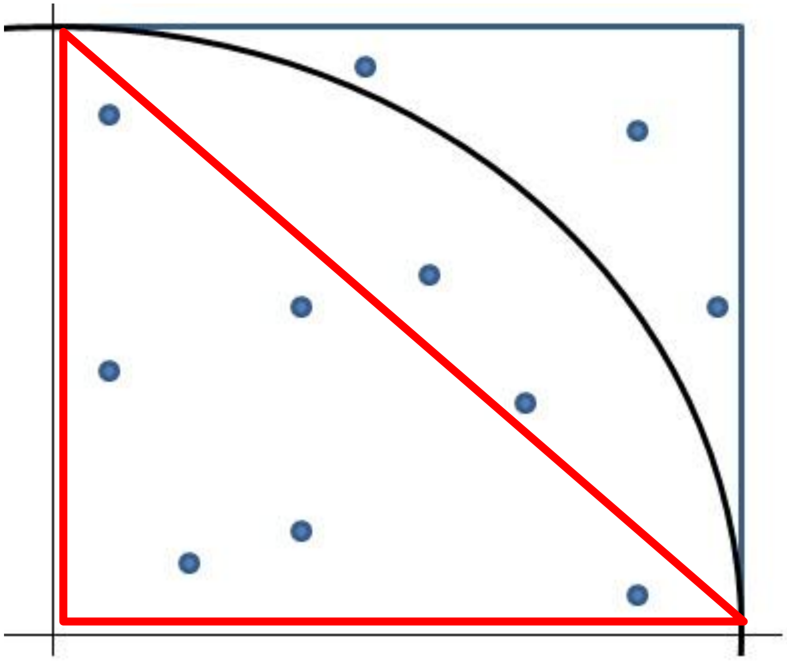
Towards Conditional Probability (cont.)

- the probability of the triangle T is …
- and the probability of its complement T^c is …



Conditional Probability: Diving into a Subspace

- the probability of the triangle T *within the sector S* is …
- and the probability of its complement T^c *within the sector S* is …



Conditional Probability Definition

- Given two events A and B of a probability space with P(B)>0, the *conditional probability* of A given/assuming B is defined as the quotient of the probability of the product $A \cap B$, and the probability of B:

$$P(A|B) = P(A \cap B)/P(B).$$

Conditional Probability Definition

- Thus
  - *the probability of the triangle T within the sector S*

  and
  - *the probability of its complement T^c within the sector S*

  both are conditional probabilities of

Part II

EXAMPLE:
HOW TO SURVIVE NEXT 10 YEARS

Example

(http://www.cut-the-knot.org/Probability/ConditionalProbability.shtml)

- In describing the survival rate and life expectancy in a certain population, let A_N denote the event of reaching the age of n years and P(n) = (A_n) be the corresponding probability. In other words, P(n) stands for the probability of a new-born to reach the age of n years.

Example (cont.)

(http://www.cut-the-knot.org/Probability/ConditionalProbability.shtml)

- We are given that

  $P(50) = 0.913$, $P(55) = 0.881$, $P(65) = 0.746$.

  This information suggests several questions.

Example (cont.)

(http://www.cut-the-knot.org/Probability/ConditionalProbability.shtml)

- For example, what is the probability of a 50 years old man to reach the age of 55, i.e. what is $P(55|50) = P(A_{55}|A_{50})$?
- Since obviously $A_{55} \cap A_{50} = A_{55}$, we have by definition,

$$P(55|50) = P(A_{55} \cap A_{50})/P(A_{50}) =$$

$$= P(A_{55})/P(50) \approx 0.965.$$

Example (cont.)

(http://www.cut-the-knot.org/Probability/ConditionalProbability.shtml)

- A probability that a 50 years old will die within 5 years is then a rather comforting 1 - 0 .965 = 0.035.
- However, as it should, the probability of dying within the next 5 years grows with age. So if, for example, the probability that a man who just turned 65 will die within 5 years is 0.16, what is the probability for a man to survive till his 70^{th} birthday, i.e., what is P(70)?

Example (cont.)

(http://www.cut-the-knot.org/Probability/ConditionalProbability.shtml)

- As before, $P(70|65) = P(70)/P(65)$ so that $P(70) = P(65)*P(70|65)$, but

$$P(70|65) = 1 -- 0.16 = 0.84.$$

- Therefore,

$$P(70) = P(65)*P(70|65) = 0.746*0.84 \approx 0.627.$$

To be, or not to be – that is the question

- As today, February 6-11, 2017, I am 55. What is probability that I will survive next 10 years? (Please be mercy…)

Part III

ONE MORE RULE FOR PROBABILITY CALCULUS

Just Other Way Around?

- Conditional Probability Definition : $P(A|B) = P(A \cap B)/P(B)$.
- Other way around: $P(A|B) * P(B) = P(A \cap B)$ – valid even in the case of impossible B!
- Corollary: $P(B|A) = P(A|B)*P(B) /P(A)$ assuming that P(A) is not an impossible event.

Multiplication Rule

- $P(A \cap B) = P(A) * P(B|A)$
- $P(A \cap B \cap C) = P(C|A \cap B) * P(A \cap B) =$

$$= P(C|A \cap B) * (P(B|A) * P(A)) =$$

$$= P(A) * P(B|A) * P(C|A \cap B)$$

- $P(A_1 \cap A_2 \cap \ldots \cap A_n) =$

$= P(A_1) * P(A_2|A_1) * P(A_3|A_1 \cap A_2) *$

$P(A_4|A_1 \cap A_2 \cap A_3) * \ldots * P(A_n|A_1 \cap A_2 \cap \ldots \cap A_{(n-1)})$

Example: same day Birth-day

- Assuming that university students have random birth-days. What is the probability that some people in a class of n students have same birth-day?

Example: same day Birth-day (cont.)

- Let us enumerate students in the class and let A_k (k in [1..n]) be the following event: student k has birth-day other than students in [1..(k-1)].
- Question: What is the probability space?
- Then for the event SBD (same birth-day) we have

$$P(SBD) = 1 - P(SBD^C),$$

$$P(SBD^C) = P(A_1 \cap A_2 \cap \ldots \cap A_n).$$

Example: same day Birth-day (cont.)

- For simplicity let n=4. Then
- $P(A_1 \cap A_2 \cap \ldots \cap A_n) =$

$= P(A_1) * P(A_2 | A_1) * P(A_3 | A_1 \cap A_2) *$

$* P(A_4 | A_1 \cap A_2 \cap A_3) =$

$= 1 * (364/365) * (363/365) * (362/365) =$

$= 365!/(365^n * (365-n)!)$