Data Structures & Algorithms

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Fundamental Techniques In Handling People – Dale Carnegie

1. Don’t Criticize, Condemn or Complain.
2. Give honest and sincere appreciation.
3. Arouse in the other person an eager want.

Recap

- MAP ADT
- Hashmap
- Time complexity of a hashmap

Objectives

- What is an algorithmic strategy?

- Learn about commonly used Algorithmic Strategies
  - ❖ Brute-force
  - ❖ Divide-and-conquer
  - ❖ Dynamic programming
  - ❖ Greedy algorithms

- You will also see an example of how classical algorithmic problems can appear in daily life

Algorithmic Strategies

- Approach to solving a problem
- Algorithms that use a similar problem solving approach can be grouped together
- Classification scheme for algorithms
- Purpose is not to learn how to classify an algorithm, but to highlight the various ways a problem can be attacked

Brute Force

Brute Force

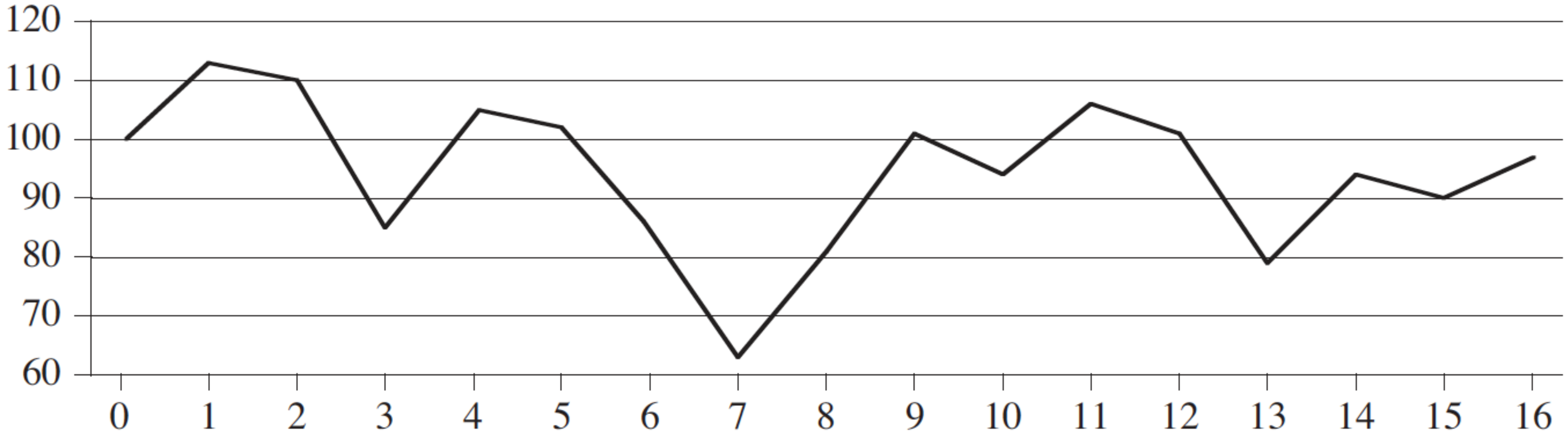
- Straightforward approach to solve a problem based on the simple formulation of the problem
- Often, does not required deep analysis of the problem
- Perhaps the easiest approach to apply and is useful for solving problems of small-size
- May results in naïve solutions with poor performance
- Examples
  - Computing $a^n (a > 0, n$ a non negative integer) by repetitive multiplication $a * a * \ \ldots * a$
  - Computing $n!$
  - Sequential search
  - Selection sort

Brute Force

- Maximum subarray problem
  - Given a sequence of integers $i_1, i_1, \dots, i_n$ find the sub-sequence with the maximum sum
    - If all the numbers are negative, the result is 0
  - Examples:
    - -2, 11, -4, 13, -4, 2   gives the solution ?
    - 1, -3, 4, -2, -1, 6   gives the solution ?

Max Subarray in Real-life

- Information about the price of stock in a Chemical manufacturing company after the close of trading over a period of 17 days

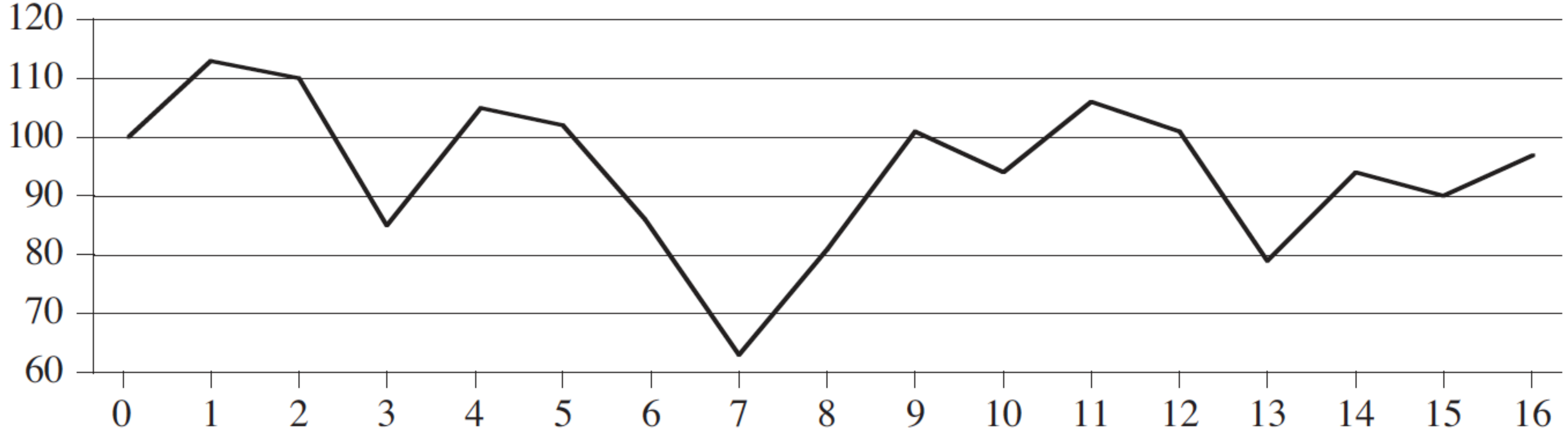


Day	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Price	100	113	110	85	105	102	86	63	81	101	94	106	101	79	94	90	97

- When to buy the stock and when to sell it to maximize the profit?

Max Subarray in Real-life

- Transformation to convert this problem into the max-subarry problem



Day	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Price	100	113	110	85	105	102	86	63	81	101	94	106	101	79	94	90	97
Change		13	−3	−25	20	−3	−16	−23	18	20	−7	12	−5	−22	15	−4	7

- When to buy the stock and when to sell it to maximize the profit? *Now we can answer this by finding the sequence of days over which the net change is maximum*

Formal Definition

Input: An array of reals $A[1 \ldots N]$.

The *value* of subarray $A[i \ldots j]$ is

$$V(i,j) = \sum_{x=i}^{j} A(x).$$

The **Maximum Contiguous subarray problem** is to find $i \le j$ such that

$$\forall (i', j'),\ V(i', j') \le V(i,j).$$

Output: $V(i,j)$ s.t. $\forall (i', j'), \quad V(i', j') \le V(i,j)$.

Note: Can modify the problem so it returns indices (i,j).

Brute Force

- We can easily devise a brute-force solution to this problem – O(?)

```
int grenanderBF(int a[], int n) {
    int maxSum = 0;
    for (int i = 0; i < n; i++) {
        for (int j = i; j < n; j++) {
            int thisSum = 0;
            for (int k = i; k <= j; k++) {
                thisSum += a[ k ];
            }
            if (thisSum > maxSum) {
                maxSum = thisSum;
            }
        }
    }
    return maxSum;
}
```

Brute Force

- Thus, it is the most straightforward and the easiest of all approach
- Often, does not required deep analysis of the problem
- May results in naïve solutions with poor performance, but easy to implement

Divide-and-Conquer

Divide-and-Conquer

- Solving a problem *recursively*, applying three steps at each level of *recursion*
  - **Divide** the problems into a number a sub-problems that are similar instances of the same problem
  - **Conquer** the sub-problems by solving them recursively. If the sub-problems size is small enough, just solve it in a straightforward manner
  - **Combine** the solutions to the sub-problems into the solution for the original problem

Recursion and Recurrence Relations

- Recursion
  - A wonderful programming tool
  - A function is said to be recursive if it calls itself – usually with "*smaller or simpler*" inputs
  - Two properties:
    a) A problem should be solvable by utilizing the solutions to the smaller versions of the same problem,
    b) The smaller versions should reduce to easily solvable cases

Recursion and Recurrence Relations

- Recursion

```
long power(long x, long n)
  if (n == 0)
    return 1;
  else
    return x * power(x, n-1);
```

Recursion and Recurrence Relations

- Recursion

```
long power(long x, long n)
  if (n == 0)
    return 1;
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```

Base case!

Recursive case!

Recursion and Recurrence Relations

- Recurrence Relations
  - Are used to determine the running time of recursive algorithms
  - Recurrence relations are themselves recursive
  - Let $T(n)$ = Time required to solve the problem of size n

$$
\begin{aligned}
T(0) = \quad & \text{time to solve problem of size } 0 \\
& \text{– Base Case} \\
T(n) = \quad & \text{time to solve problem of size } n \\
& \text{– Recursive Case}
\end{aligned}
$$

Recursion and Recurrence Relations

- Recurrence Relations

```
long power(long x, long n)
  if (n == 0)
    return 1;
  else
    return x * power(x, n-1);
```

$$\boxed{\begin{array}{ll} T(0) = c_1 & \text{for some constant } c_1 \\ T(n) = c_2 + T(n-1) & \text{for some constant } c_2 \end{array}}$$

Recursion and Recurrence Relations

- Recurrence Relations

$$T(0) = c_1$$
$$T(n) = T(n-1) + c_2$$

If we knew $T(n-1)$, we could solve $T(n)$.

$$\begin{aligned} T(n) &= T(n-1) + c_2 & T(n-1) = T(n-2) + c_2 \\ &= T(n-2) + c_2 + c_2 \\ &= T(n-2) + 2c_2 & T(n-2) = T(n-3) + c_2 \\ &= T(n-3) + c_2 + 2c_2 \\ &= T(n-3) + 3c_2 & T(n-3) = T(n-4) + c_2 \\ &= T(n-4) + 4c_2 \\ &= \ldots \\ &= \boxed{T(n-k) + kc_2} \end{aligned}$$

Recursion and Recurrence Relations

- Recurrence Relations

$$T(0) = c_1$$
$$T(n) = T(n-k) + k * c_2 \quad \text{for all } k$$

If we set $k = n$, we have:

$$\begin{aligned} T(n) &= T(n-n) + nc_2 \\ &= T(0) + nc_2 \\ &= c_1 + nc_2 \\ &\in \Theta(n) \end{aligned}$$

Back to Divide-and-Conquer

- Solving a problem *recursively*, applying three steps at each level of *recursion*
  - **Divide** the problems into a number a sub-problems that are similar instances of the same problem
  - **Conquer** the sub-problems by solving them recursively. If the sub-problems size is small enough, just solve it in a straightforward manner
  - **Combine** the solutions to the sub-problems into the solution for the original problem
- Examples: Quicksort, Mergesort, etc.

Divide-and-Conquer

Mergesort

UNSORTEDSEQUENCE

UNSORTED SEQUENCE

UNSO RTED SEQU ENCE

UN SO RT ED SE QU EN CE

NU OS RT DE ES QU EN CE

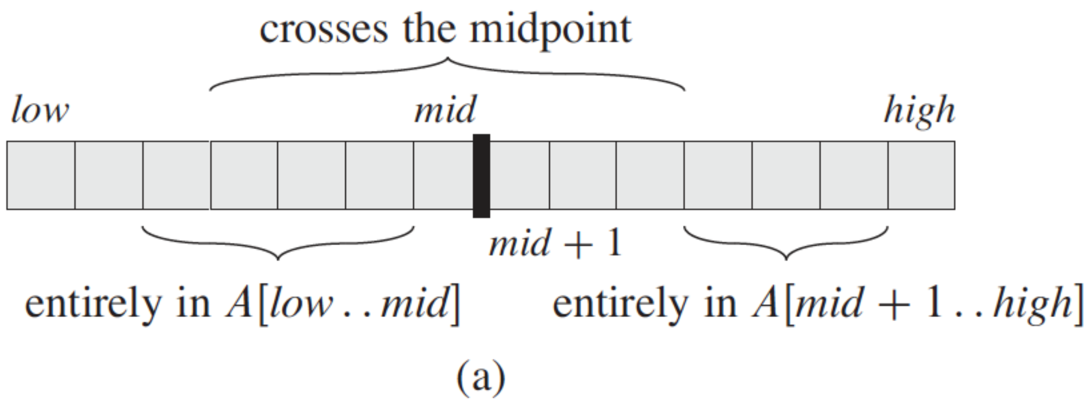
NOSU DERT EQSU CEEN

DENORSTU CEEENQSU

CDEEEENNOQRSSTUU

Divide-and-Conquer

- Max Subarray Problem



Divide-and-Conquer

- Max Subarray Problem

```
FIND-MAXIMUM-SUBARRAY(A, low, high)
if high == low
    return (low, high, A[low])              // base case: only one element
else mid = ⌊(low + high)/2⌋
    (left-low, left-high, left-sum) =
        FIND-MAXIMUM-SUBARRAY(A, low, mid)
    (right-low, right-high, right-sum) =
        FIND-MAXIMUM-SUBARRAY(A, mid + 1, high)
    (cross-low, cross-high, cross-sum) =
        FIND-MAX-CROSSING-SUBARRAY(A, low, mid, high)
    if left-sum ≥ right-sum and left-sum ≥ cross-sum
        return (left-low, left-high, left-sum)
    elseif right-sum ≥ left-sum and right-sum ≥ cross-sum
        return (right-low, right-high, right-sum)
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Divide-and-Conquer

- Max Subarray Problem

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    else return (cross-low, cross-high, cross-sum)
```

Divide

- Max Subarray Problem

```
FIND-MAX-CROSSING-SUBARRAY(A, low, mid, high)
left-sum = −∞
sum = 0
for i = mid downto low
    sum = sum + A[i]
    if sum > left-sum
        left-sum = sum
        max-left = i
right-sum = −∞
sum = 0
for j = mid + 1 to high
    sum = sum + A[j]
    if sum > right-sum
        right-sum = sum
        max-right = j
return (max-left, max-right, left-sum + right-sum)
```

```
FIND-MAXIMUM-SUBA
if high == low
    return (low, hi
else mid = ⌊(low -
    (left-low, left-h
        FIN
    (right-l
        FIND-MA
    (cross-low, cross-high, cross-sum) =
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Divide-and-Conquer

- Max Subarray Problem

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```

Divide-and-Conquer

- Max Subarray Problem – Time Complexity

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1\,, \\ 2T(n/2) + \Theta(n) & \text{if } n > 1\,. \end{cases}$$

- This type of recurrence is called "*Divide-and-Conquer*" recurrence

We can solve this recurrence using the **"Master Theorem"**

Divide-and-Conquer

- Master Theorem

Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the nonnegative integers by the recurrence

$$T(n) = aT(n/b) + f(n) ,$$

where we interpret n/b to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then $T(n)$ has the following asymptotic bounds:

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \lg n)$.
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$. ■

Divide-and-Conquer

- Max Subarray Problem – Time Complexity

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1\,, \\ 2T(n/2) + \Theta(n) & \text{if } n > 1\,. \end{cases}$$

- Case 2 applies, thus we have to solution

$$T(n) = \Theta(n \lg n).$$

Dynamic Programming

Dynamic Programming

- Like divide-and-conquer solves the problem by combining solutions to the sub-problems
- But it applies when sub-problems overlap
- That is, sub-problems **share sub-sub-problems**!
- To avoid solving the same sub-problems more than once, the results are stored in a data structure that is updated dynamically

Dynamic Programming

- Optimization problems
  - Such problems have many candidate solutions
  - Each solution has a value and we wish to find a solution with an optimal value
  - ***An*** optimal solution in contrast to ***the*** optimal solution

Dynamic Programming

- Fibonacci Numbers

```
Fibonacchi(N) = 0                                         for n=0
              = 1                                         for n=1
              = Fibonacchi(N-1)+Finacchi(N-2)              for n>1
```

Dynamic Programming

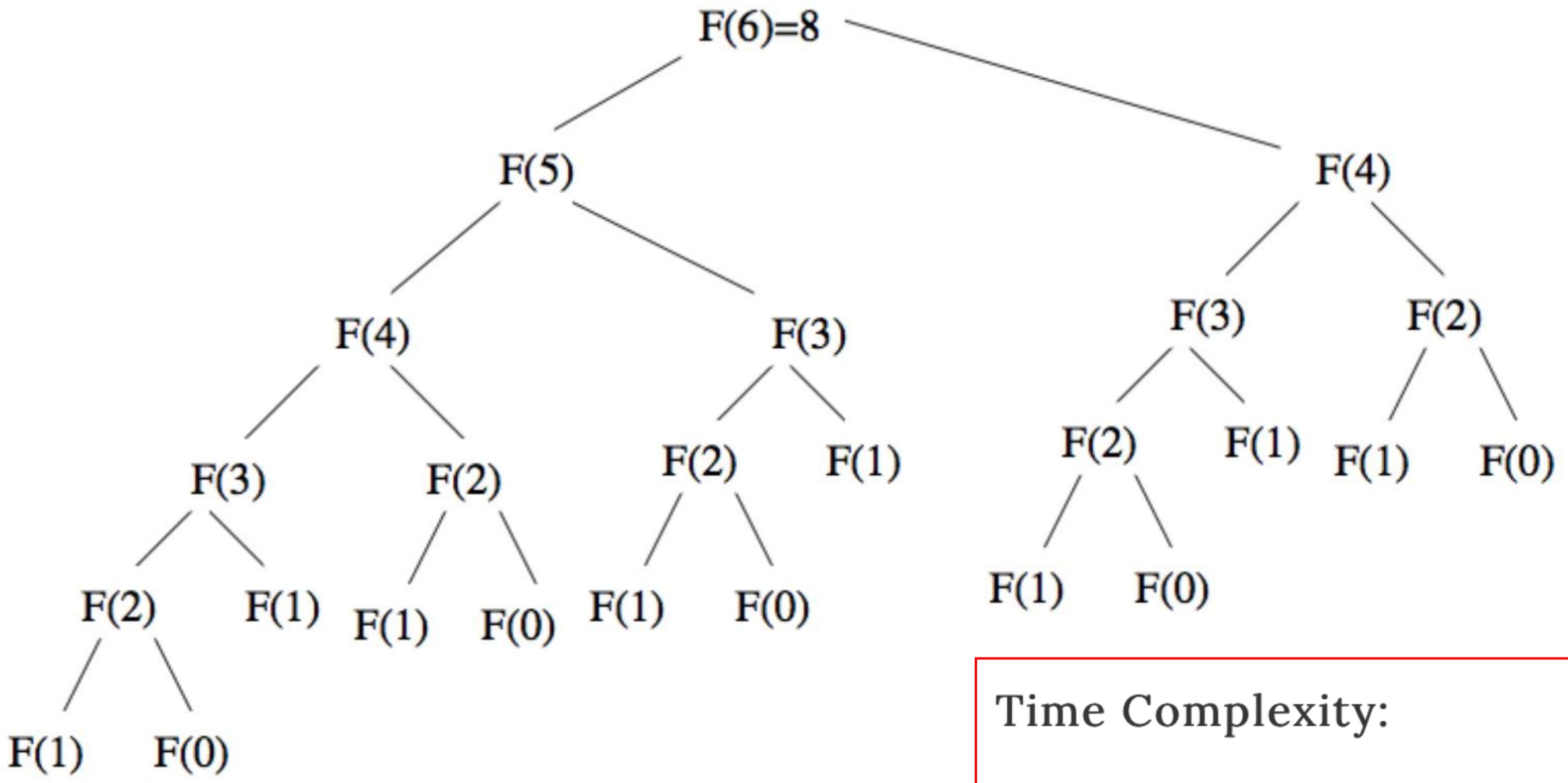
- Fibonacci Numbers

```
public int fibRecur(int x) {
        if (x == 0)
            return 0;
        if (x == 1)
            return 1;
        else {
            int f = fibRecur(x - 1) + fibRecur(x - 2);
            return f;
        }

    }
```

Dynamic Programming

- $n-th$ Fibonacci Numbers

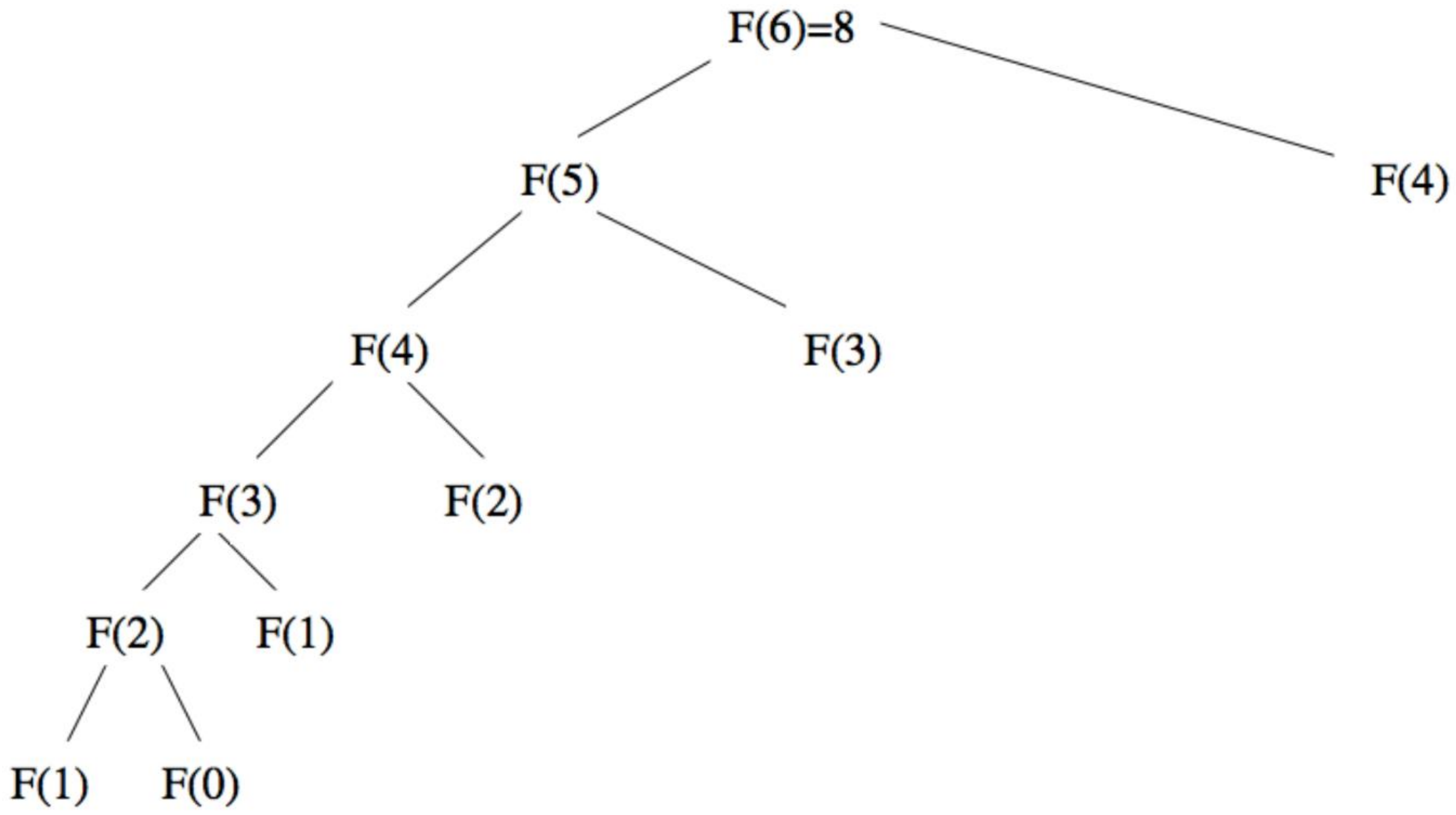


Time Complexity:

$T(n) = T(n-1) + T(n-2) + 1 = 2^n = O(2^n)$

Dynamic Programming

- $n - th$ Fibonacci Numbers



Dynamic Programming

- Fibonacci Numbers - Memoization

```
public int fibDP(int x) {
        int fib[] = new int[x + 1];
        fib[0] = 0;
        fib[1] = 1;
        for (int i = 2; i < x + 1; i++) {
                fib[i] = fib[i - 1] + fib[i - 2];
        }
        return fib[x];
    }
```

Time Complexity: O(n) , Space Complexity : O(n)

Dynamic Programming

- Key is to relate the solution of the whole problem and the solutions of subproblems.

  ❖ Same is true of divide & conquer, but here the subproblems need not be disjoint. – they need not divide the input (i.e., they can "overlap") – divide & conquer is a special case of dynamic programming

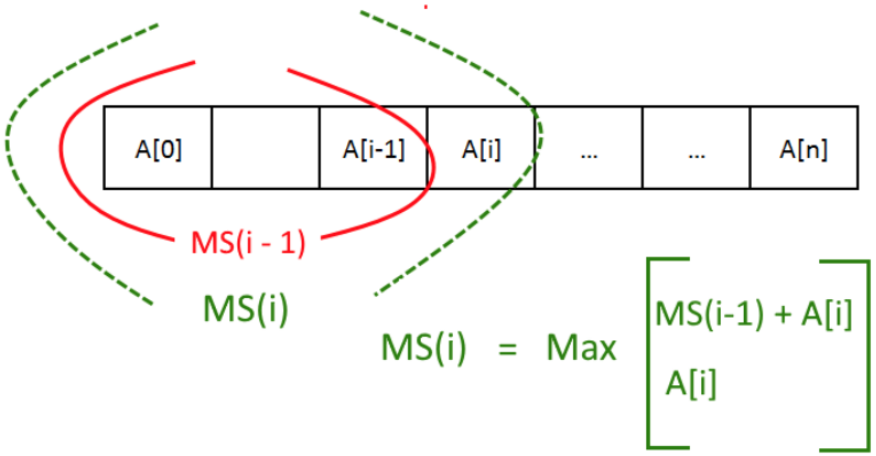
- A dynamic programming algorithm computes the solution of every subproblem needed to build up the solution for the whole problem.
  - compute each solution using the above relation
  - store all the solutions in an array (or matrix)
  - algorithm simply fills in the array entries in some order

Dynamic Programming

- Max Subarray Problem

- Let $S(i)$ be the sum at ith-index

A[0]		A[i-1]	A[i]	...		A[n]



- Then it can be recursively defined as

$$S(i) = \max((S(i-1) + A[i]), A[i])$$

Dynamic Programming

- Max Subarray Problem

$$S(i) = \max((S(i-1) + A[i]), A[i])$$

- Apply this definition to solve the problem for the following sequence

-2, -3, 4, -1, -2, 1, 5, -3

Greedy Algorithms

Greedy Algorithms

- Finding solutions to problem **step-by-step**
- A partial solution is incrementally expanded towards a complete solution
- In each step, there are several ways to expand the partial solution
- The best alternative for the moment is chosen, the others are discarded.
- Thus, at each step the choice must be **locally optimal** – this is the central point of this technique

Greedy Algorithms

- For example, counting to a desired value using the least number of coins

- Let’s say, we are given coins of value 1, 2, 5 and 10 of some currency. And the target value is 16 in that currency

- How will you proceed?

Greedy Algorithms

- Not always gives the optimal solution

- Let’s say, a monetary system consists of only coins of worth 1, 7 and 10.

- How would a greedy approach count out the value of 15?

Greedy Algorithms

- Examples
  - Finding the minimum spanning tree of a graph (Prim’s algorithm)
  - Finding the shortest distance in a graph (Dijkstra’s algorithm)
  - Using Huffman trees for optimal encoding of information
  - The Knapsack problem
- We will go through the first two algorithms in detail when we learn about Graphs; therefore, I will end today’s lecture here.
- You are strongly advised to read about the discussed topics, as well as other algorithmic strategies such as
  - “Combinatorial search & Backtracking”
  - “Branch and Bound”

Did we achieve today’s objectives?

- What is an algorithmic strategy?

- Learn about commonly used Algorithmic Strategies
  - ❖Brute-force
  - ❖Divide-and-conquer
  - ❖Dynamic programming
  - ❖Greedy algorithms

- You will also see an example of how classical algorithmic problems can appear in daily life