Discrete Mathematics

Functions

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"Go Down Deep Enough into Anything and You Will Find Mathematics!"

-Dean Schlicter-

Functions

- A way of transforming objects of one type into objects of another type.

- Imagine

{Set of Strings}

Length(w) → produces its length

Where w is a string.

e.g. length(*smile*)=*5*.

Functions

- Question that we might want to discuss when studying functions.

  - Means by which a function is computed?
  - Are there different ways?
  - Are some worse than others?
  - Are there different ways to represent a function?

Functions

- But we are mainly interested in the following.

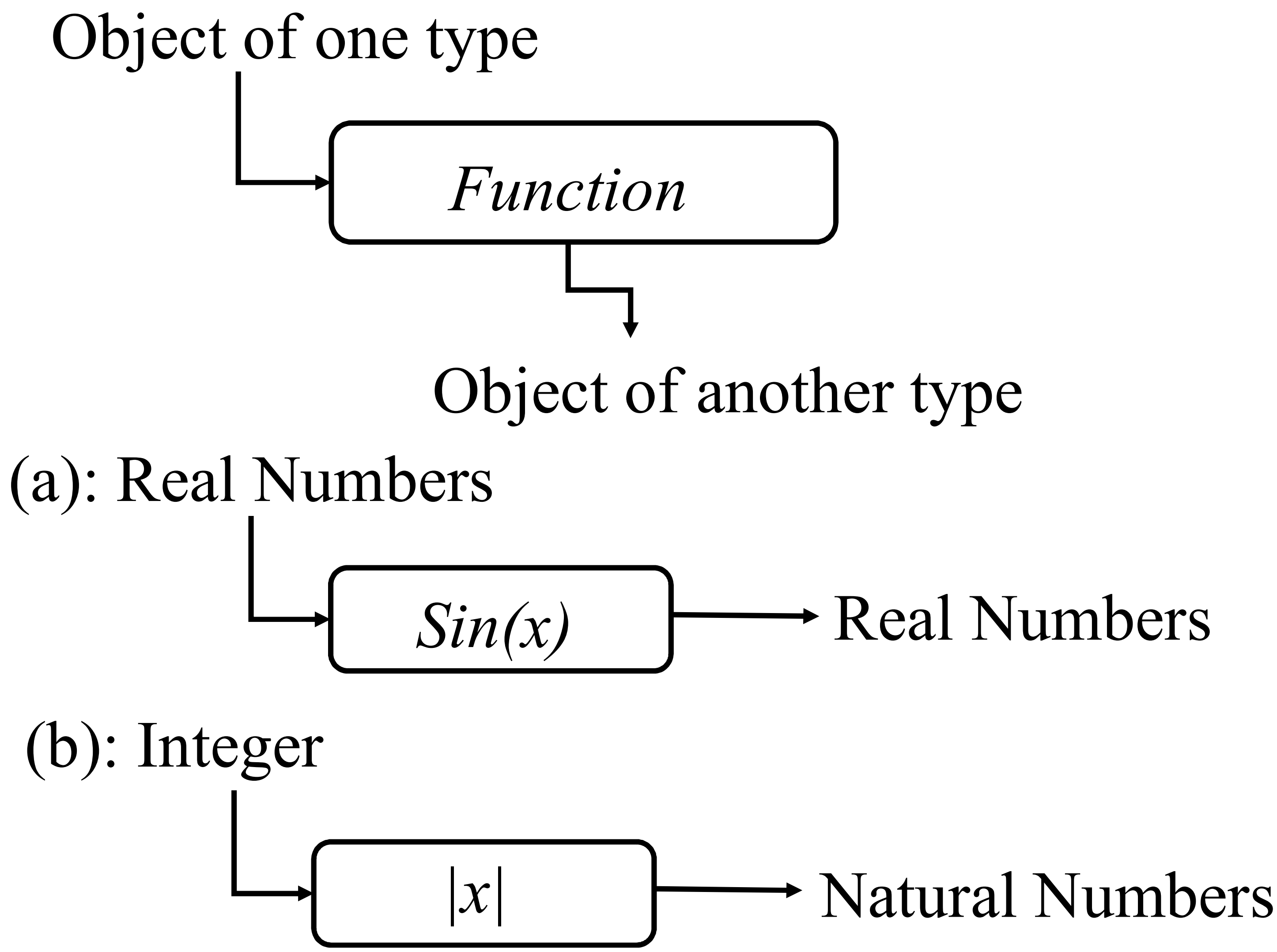
  - ❑ What exactly is a function?

  - ❑ How can we classify function into different kinds?

  - ❑ How can we build new functions from the existing ones?

- What exactly is a function?

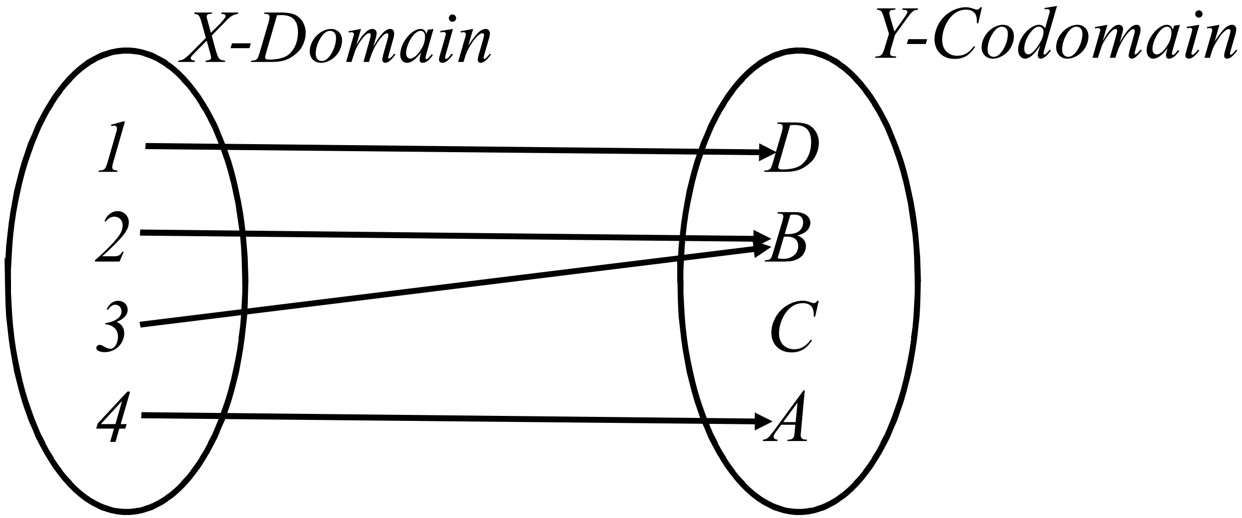
Functions – Basic Definitions



Functions – Basic Definitions—Cont.

- Let ***A* and *B*** be arbitrary sets
- f: a function from ***A* to *B***.
- Associates $\boxed{\text{every}}$ element of *A* with a $\boxed{\text{single}}$ element in *B*.

- **Example:**
- f: from *X to Y*



Functions – Defining a Function

- The function definition should have enough details to unambiguously define

  - The domain and codomain
  - The output for every input

- **For Example**

- $f{:}N \rightarrow N$, where $f(n)=n^2$.

- On the other hand $f(x) = x^2$ is a bit less precise.

Functions – Defining a Function

- Why the function definition $f(x) = x^2$ is less precise?

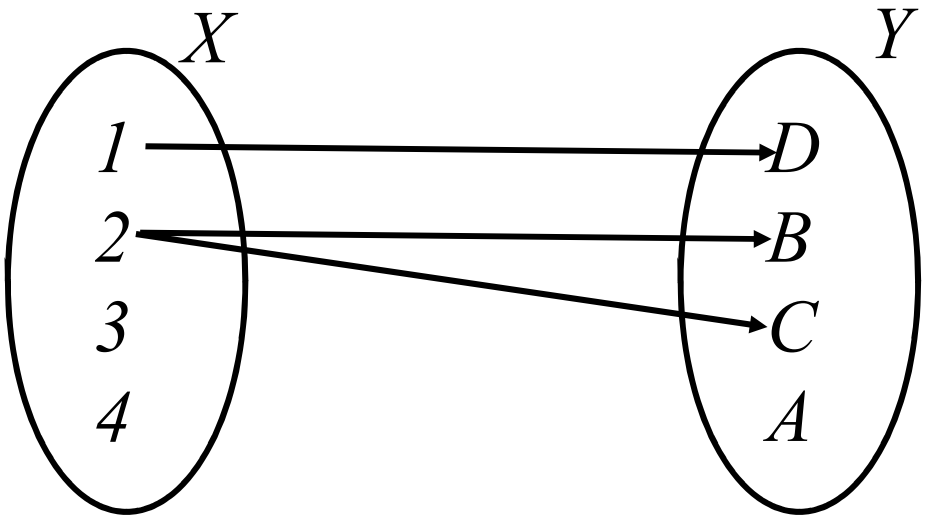
$f{:}N \to N \qquad f{:}Z \to N$

$f{:}Z \to Z \qquad f{:}R \to R$

- All four are valid, however, the properties of the function will be widely different
- For example, if $f{:}N \to N$, then f has the property that if $f(x) < f(y)$, then $x < y$
- Isn't true for $f{:}Z \to Z$

Functions – Example

- Another Example:



- What do you think?
- Can this mapping be produced by a function?

- What about the **function $\sqrt{x}$** from $R \rightarrow R$?

Functions – Basic Definitions—Cont.

- Sometimes, you will hear the term “range” of a function

  - **Range: set of all possible outputs of a function**

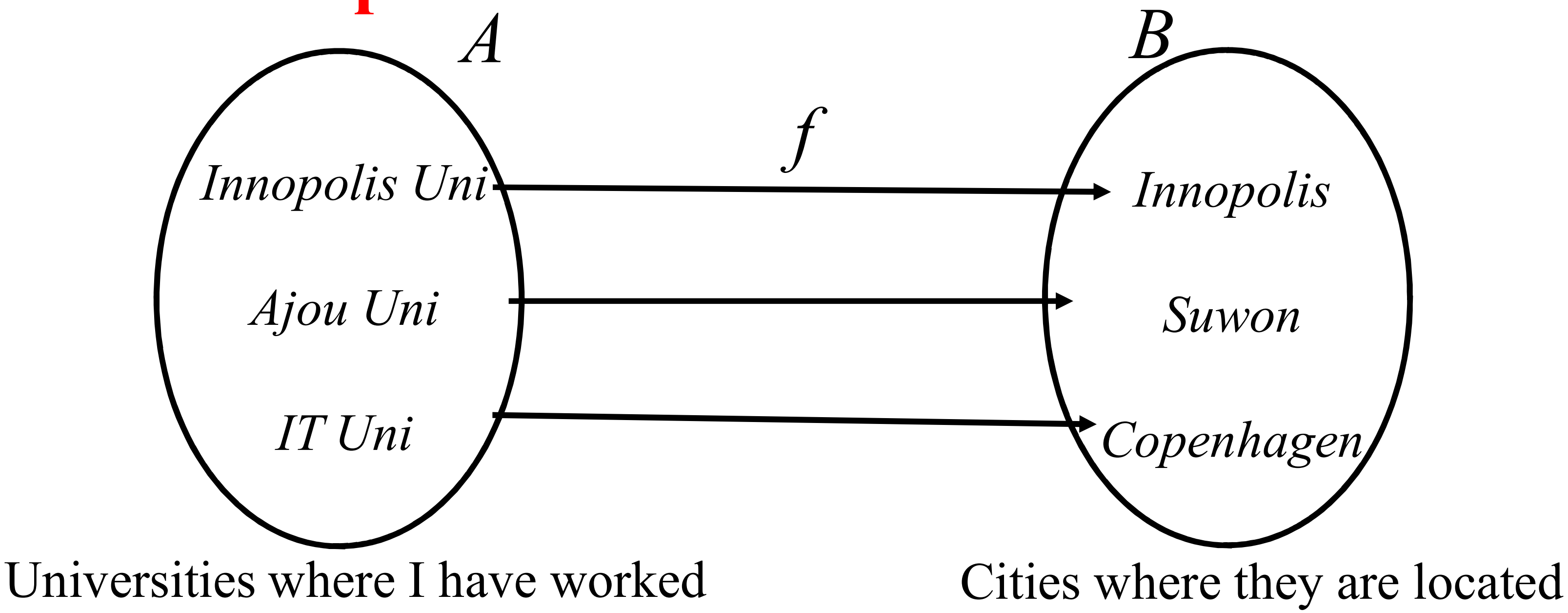
- We will touch this topic a little later in this lecture

Defining Functions by a Picture

- When the domain and codomains are finite sets

- We can often define the function by drawing a picture

- **For Example:**



Defining Function – Cont.

- Another way often used to define a function is by specifying a variety of different rules to the input giving conditions under which each rule should be applied.

- These are often called piecewise functions.

- **For Example:**

$$|x| = \begin{cases} x & if\ x > 0 \\ -x & otherwise \end{cases}$$

Piecewise Functions

- When defining such functions, it is important to ensure that

- Every possible input falls into at least one of the cases

- If an input falls into multiple cases, each case produces the same output.

- **For Example:**

$$|x| = \begin{cases} x & if\ x \geq 0 \\ -x & if \leq 0 \end{cases} \quad \textbf{VS} \quad |x| = \begin{cases} x & if\ x > 0 \\ -x & if < 0 \end{cases}$$

Functions with multiple inputs

- When programming we often use functions like these

- ```
  int raiseToPower (int x, int y) {
          int result = 1;
          for (int i = 0; i < y, i++) {
              result *= x;
          }
          return result;
    }
  ```
- How can we define such functions mathematically? – because in our definition, a function takes only one argument, i.e., an element of the domain
  ```

Functions with multiple inputs - Cont

- Lets assume only natural numbers as input
- We can think of the above function, which appears to take in $\boxed{\text{two arguments}}$, as a functions that takes in just $\boxed{\text{one argument}}$.

"An ordered pair of natural numbers"

- **Mathematically;**

Raise To Power: $\mathbb{N} * \mathbb{N} \rightarrow \mathbb{N}$ where

Raise To Power $\big((x, y)\big) = x^y$

Functions with multiple inputs-Cont

- More generally,
- We can always model an $\boxed{\text{n} - \text{argument}}$ function as a function containing a $\boxed{\text{single argument}}$.

n-way Cartesian product of an appropriate group of sets

Functions with multiple inputs-Cont

- More generally,
- We can always model an n – argument function as a function containing a single argument.

n-way Cartesian product of an appropriate group of sets

- How will you represent a function that adds together three real numbers and an integer?

Functions with multiple inputs-Cont

- More generally,
- We can always model an $\boxed{n - \text{argument}}$ function as a function containing a $\boxed{\text{single argument}}$.

n-way Cartesian product of an appropriate group of sets

- How will you represent a function that $\boxed{\text{adds}}$ together $\boxed{\text{three real numbers}}$ and $\boxed{\text{an integer}}$?

- **Final Comment:**

  if $f: A_1 * \ldots * A_n \rightarrow B$, then we denote $f((x_1, \ldots, x_n))$ by $f(x_1, \ldots, x_n)$

- How can we classify functions into different types?

Injection, Surjection, and Bijection

- Functions come in different shapes and size

- But these are certain types of functions that appear more frequently than others

1. Surjections (**Onto**)
2. Injections (**One to One**)
3. Bijections (**Both**)

Surjections

- Lets consider a problem

- You are in charge of distributing a bunch of $\boxed{\text{fruit baskets}}$ among $\boxed{\text{student groups}}$ at IU.

  Student groups {BS1, BS2, BS3, BS4, MS1, MS2}.

- You want to do it such that every group gets at least one fruit basket, and all the baskets get distributed.

Surjections – Cont.

- Mathematically, you can think of this as a function
- $f: B \rightarrow G$, where ***B*** be the set of fruit baskets and ***G*** be the set of student groups.
- "for every $\boxed{g \in G}$, there is some fruit basket $\boxed{b \in B}$ such that $\boxed{f(b) = g}$"
- Such a function is called surjection.

Surjections – Cont.

- More generally;

- $\boldsymbol{f}: \boldsymbol{A} \rightarrow \boldsymbol{B}$, is a surjection if for any $b \in B$, there is some $a \in A$ such that

$$f(a) = b$$

- Also called an **Onto function**.

If we represent such a function with a picture, what will it look like?

Surjections – Cont.

- Which of these are Surjections??

- $\boldsymbol{f(x) = x}$, over real numbers

- $\boldsymbol{f(x) = x^2}$, over real numbers

Injections (One to One)

- Now suppose you are the head of a student group
- You get a fruit basket
- Now you want to distribute among students
- In other words, you want to find a function $f : F \to S$

where F and S represents the set of fruits and set of students.

Injections (One to One)

- Unfortunately, there are not enough fruits, so you want to be fair,

- Thus you define

- $\boldsymbol{f}: \boldsymbol{F} \rightarrow \boldsymbol{S}$
    - With the condition that every one should get at most one fruit.

- Such a function is called injection.

Injections (One to One)

- More generally,

- $f: A \to B$ is an injection if for any $x_1, x_2 \in A$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$

- Equivalently,

  for any $x_1, x_2 \in A$ if $x_1 \neq x_2$ then
  $$f(x_1) \neq f(x_2)$$

- Also called a **One to One** function.

If we represent such a function with a picture, what will it look like?

Injections – Cont.

- Which of these are Injections??

- $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}$, over real numbers

- $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}^{\mathbf{2}}$, over real numbers

Some more concepts related to Surjections and Injections, before we move on to Bijections!

Functions and Sets

❖ Images:

- If $f: A \rightarrow B$ and $X \subseteq A$, the image of X under f is the set

$$f[x] = \{f(x)/x \in X\}$$

- Set of elements that we would get if we applied f to every element of X.

Functions and Sets – Cont.

- **Images:**

- What is the image of $X = [-1, 3]$ under

$$f: R \to R \text{ where } f(x) = x^2 \text{ ??}$$

Functions and Sets – Cont.

❖ Image of the Entire Domain:

- $f : A \to B$

$$f[A] = \{f(a)/a \in A\}$$

where $\{f(a)/a \in A\}$ consists of all the possible outputs of a function.

Functions and Sets – Cont.

- $f[A]$ is the same as codomain of f??

- Not necessarily!

- For Example: $f: R \rightarrow R$ where $f(a) = \sin(a)$

  then $f[R] = ??$

- Also referred to as the $\boxed{\text{range}}$ of the function.

<u>Range</u> $\rightarrow$ Values in the codomain that can actually be produced by the function.

Functions and Sets – Cont.

❖ **Another Important Question:**

- When are the range and codomain the same and when are they different??

  - Range = Codomain

- When every possible value of the codomain can be produced by the function **as its output on some input**.

Which functions have this property?

Functions and Sets – Cont.

- **Theorem:** Let $f: A \to B$, $f[A] = B$ iff f is surjective.

Functions and Sets – Cont.

- **Theorem:** Let $f: A \to B$, $f[A] = B$ iff f is surjective.

  We will prove this by proving both sides of implication.

  (a): If $f[A] = B$ then f is surjective.

  (b): If f is surjective, then $f[A] = B$

Functions and Sets – Cont.

- **Theorem:** Let $f: A \to B$, $f[A] = B$ iff f is surjective.

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Functions and Sets – Cont.

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  consider any $b \in B$

  Since $f[A] = B$ then $b \in f[A]$

Functions and Sets – Cont.

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Functions and Sets – Cont.

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Functions and Sets – Cont.

- **Proof: (b):** If f is surjective, then $f[A] = B$

Functions and Sets – Cont.

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  - We have $f : A \to B$,

Functions and Sets – Cont.

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  - We have $f: A \rightarrow B$,
  - We know that every element of $f[A]$ is an element of B. Thus $f[A] \subseteq B$. What we need to show now is that $B \subseteq f[A]$.

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Functions and Sets – Cont.

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Functions and Sets – Cont.

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From **(a)** and **(b)**: For $f: A \to B$, $f[A] = B$ iff f is surjective.

Functions and Sets – Cont.

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From **(a)** and **(b)**: For $f: A \to B$, $f[A] = B$ iff f is surjective.

Functions and Sets – Cont.

❖ Preimage:

- If $f: A \rightarrow B$ and $Y \subseteq B$, then the preimage of Y under f is the set

  - $f^{-1}[Y] = \{x \in A / f(x) \in Y\}$

    where $\{x \in A / f(x) \in Y\}$ is a set of all the element of A (domain) that map into set Y, where $Y \subseteq B$.

Functions and Sets – Cont.

❖ Preimage – Cont.

- What is $f^{-1}[Y]$ in the following case?

  - If $f: R \to R$, where $f(x) = 2x$,

    $$Y = [1, 3]$$

  - If $f: R \to R$, where $f(x) = x^2$,

    $$Y = [4, 9]$$

  - If $f: R \to R$, where $f(x) = x^2 + 2$,

    $$Y = [0, 1], Y = [0, 2]$$

Functions and Sets – Cont.

❖ Preimage and Injections

- Just as images and surjections are related, so are preimages and injections.

Functions and Sets – Cont.

❖ Preimage and Injections

- Just as images and surjections are related, so are preimages and injections.
- Let $f: A \to B$ be an injection
  - This means that every $b \in B$ has either $\boxed{0}$ or $\boxed{1}$ elements mapping to it.
  - Therefore $f^{-1}[\{b\}]$ should either contain $\boxed{0}$ or $\boxed{1}$ elements.

Functions and Sets – Cont.

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  - Therefore $f^{-1}[\{b\}]$ should either contain $\boxed{0}$ or $\boxed{1}$ elements.
  - **In other words if f is injective then,**

$$|f^{-1}[\{b\}]| \leq 1$$

Functions and Sets – Cont.

❖ **Bijections**

- A function is called a bijection if it is **injection** and **surjection**.

- For every element of the codomain, there is a unique element of the domain mapping to it.

Functions and Sets – Cont.

❖ Bijections

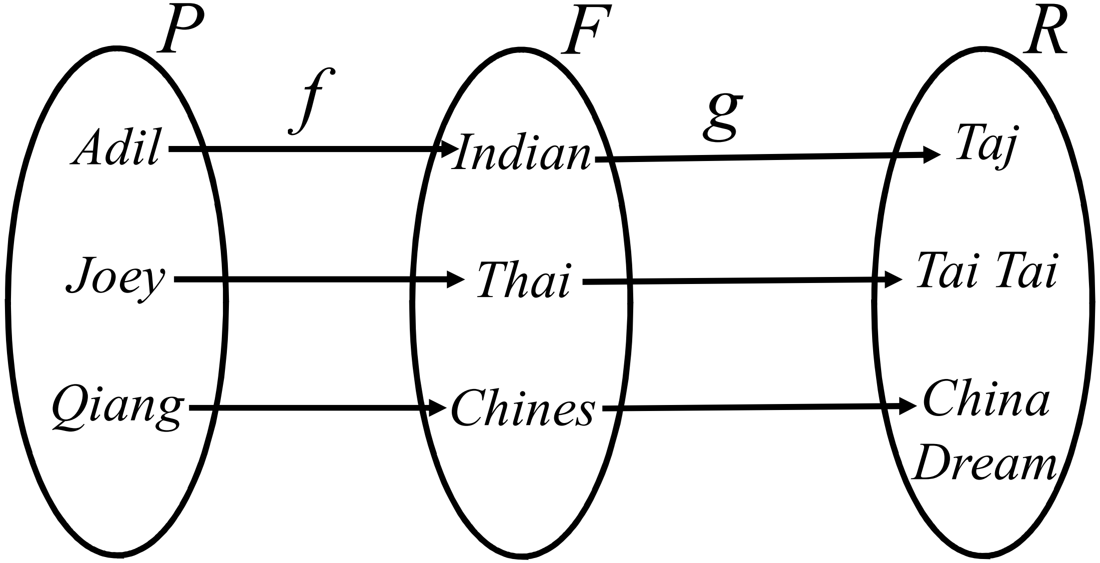
- For Example:
  - $f: R \rightarrow R$, where $f(x) = x^3$
  - $f: S \rightarrow S$, where $f(x) = x$
- What about
  - $f: R \rightarrow R$, where $f(x) = x^2$

- How can we define the new functions from the existing ones?

Functions and Sets – Cont.

❖ Transformations on Functions

- P: set of people
- F: Set of different types of food
- R: Set of restaurants



- $f: P \rightarrow F$
- $g: F \rightarrow R$

Functions and Sets – Cont.

❖ Transformations on Functions – Cont.

- We want to tell people in which restaurant they can find their favorite food.

- This is, we want to find a new function

    - $\boxed{m: P \to R}$

  - How to define this function??

Functions and Sets – Cont.

❖ Transformations on Functions – Cont.

- $\boxed{m}$ must glue together $\boxed{f}$ and $\boxed{g}$.

- That is
  - $M(p) = b\big(f(p)\big), p \in P$

- It is very common to join function like this. It is called composition of the functions.

Functions and Sets – Cont.

❖ Transformations on Functions – Cont.

- More formally,
  - Let $f: A \to B$ and $g: B \to C$.
  - Define a new function $g \circ f: A \to C$ as follows
  - $(g \circ f)(a) = g(f(a))$ for all $a \in A$

Functions and Sets – Cont.

❖ Transformations on Functions – Cont.

- Given two functions *f* and *g*, is $\boxed{g \circ f}$ or $\boxed{f \circ g}$ always guaranteed??
- NO!
- For Example:
  - Let $g: F \to R$ (form previous example)
  - Let $h: R \to R$ where $h(x) = x^3$
  - Can we do $\boxed{g \circ f}$ or $\boxed{f \circ g}$??
  - Their domains and codomains are incomparable!

Functions and Sets – Cont.

- **Composition of Injections, Surjections, and Bijections.**

Theorem: Let $f: A \to B$ and $g: B \to C$ be injections. Then $g \circ f: A \to C$ is an injection.

Functions and Sets – Cont.

❖ **Composition of Injections, Surjections, and Bijections.**

Theorem: Let $f: A \to B$ and $g: B \to C$ be injections. Then $g \circ f: A \to C$ is an injection.

Proof: Consider any $x, y \in A$ where $x \neq y$

Functions and Sets – Cont.

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- Since $x \neq y$, f(x) $\neq f(y)$ as *f* is an injection.

Functions and Sets – Cont.

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Proof: Consider any $x, y \in A$ where $x \neq y$

- Since $x \neq y$, f$(x) \neq f(y)$ as *f* is an injection.
- Since f$(x) \neq f(y)$ and *g* is an injection, therefore,

$$(g \circ f)(x) \neq (g \circ f)(y)$$

Functions and Sets – Cont.

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- Since f(x) $\neq f(y)$ and *g* is an injection, therefore,
$$(g \circ f)(x) \neq (g \circ f)(y)$$
- Since our choice of x and y was arbitrary, This means that for any $x, y \in A$ where, $x \neq y, (g \circ f)(x) \neq (g \circ f)(y)$, so $g \circ f$ is injective.

Functions and Sets – Cont.

- **Composition of Injections, Surjections, and Bijections.**

Theorem: Let $f: A \to B$ and $g: B \to C$ are surjections. Then $g \circ f: A \to C$ is an surjection.

Functions and Sets – Cont.

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Functions and Sets – Cont.

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Theorem: Let $f: A \to B$ and $g: B \to C$ are surjections. Then $g \circ f: A \to C$ is an surjection.

Proof: Consider any $c \in C$

- Since g is surjective, there exists, some $b \in B$ such that $g(b) = c$.

Functions and Sets – Cont.

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Proof: Consider any $c \in C$

- Since g is surjective, there exists, some $b \in B$ such that $g(b) = c$.
- Similarly, since f is surjective, there exists, some $a \in A$ such that $f(a) = b$.

Functions and Sets – Cont.

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- Similarly, since f is surjective, there exists, some $a \in A$ such that $f(a) = b$.
- Then $(g \circ f)(a) = g\big(f(a)\big) = g(b) = c$

Functions and Sets – Cont.

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Theorem: Let $f: A \to B$ and $g: B \to C$ are surjections. Then $g \circ f: A \to C$ is an surjection.

Proof: Consider any $c \in C$

- Since g is surjective, there exists, some $b \in B$ such that $g(b) = c$.
- Similarly, since f is surjective, there exists, some $a \in A$ such that $f(a) = b$.
- Then $(g \circ f)(a) = g\big(f(a)\big) = g(b) = c$
- Thus, for any $c \in C$, there is an $a \in A$. Therefore $\boxed{g \circ f}$ is surjective.

Functions and Sets – Cont.

❖ **Composition of Injections, Surjections, and Bijections.**

- **Theorem:** Let $f : A \to B$ and $g : B \to C$ are bijections, then $g \circ f : A \to C$ is a bijection.

- **Proof:**

"What do you guys think??"

"What is the proof?"