Probability Theory & Statistics

Innopolis University, BS-I,II

Spring Semester 2016-17

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Part I

PROBABILITY TREE

Recall Multiplication Rule

- From lectures for week $:

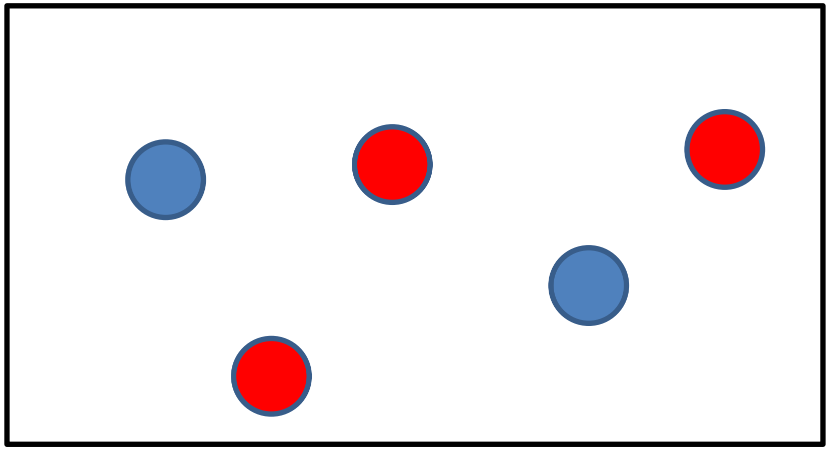
$$P(A_1 \cap A_2 \cap \ldots \cap A_n) =$$

$$= P(A_1) * P(A_2|A_1) * P(A_3|A_1 \cap A_2) *$$

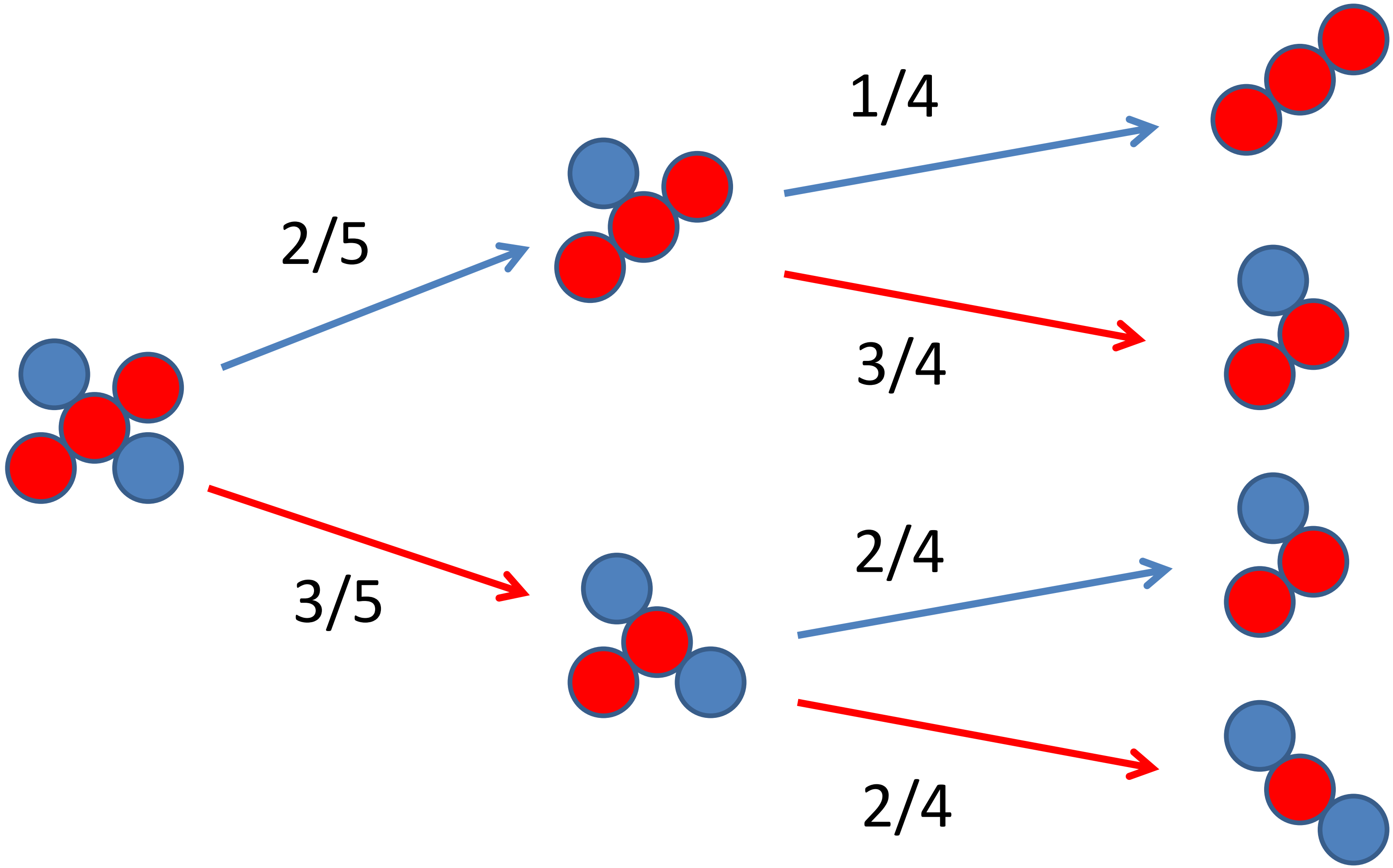
$$P(A_4|A_1 \cap A_2 \cap A_3) * \ldots * P(A_n|A_1 \cap A_2 \cap \ldots \cap A_{(n-1)})$$

Example: Urn with Balls

There is an urn (bag, etc.) with 2 blue and 3 red balls. What is probability to get first blue than red balls?



A Tree Representation

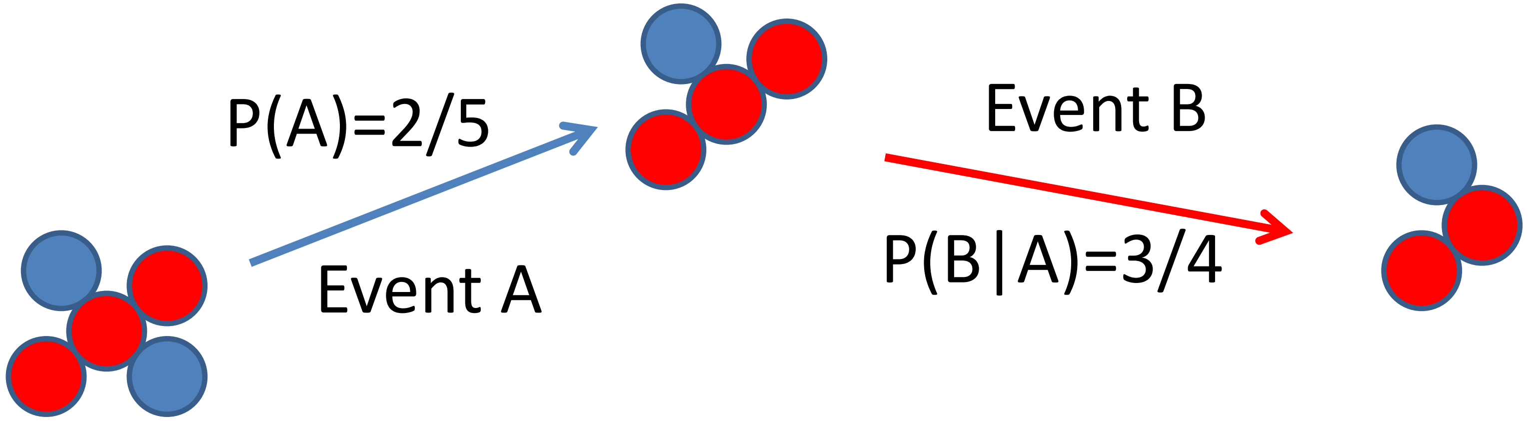


Calculations

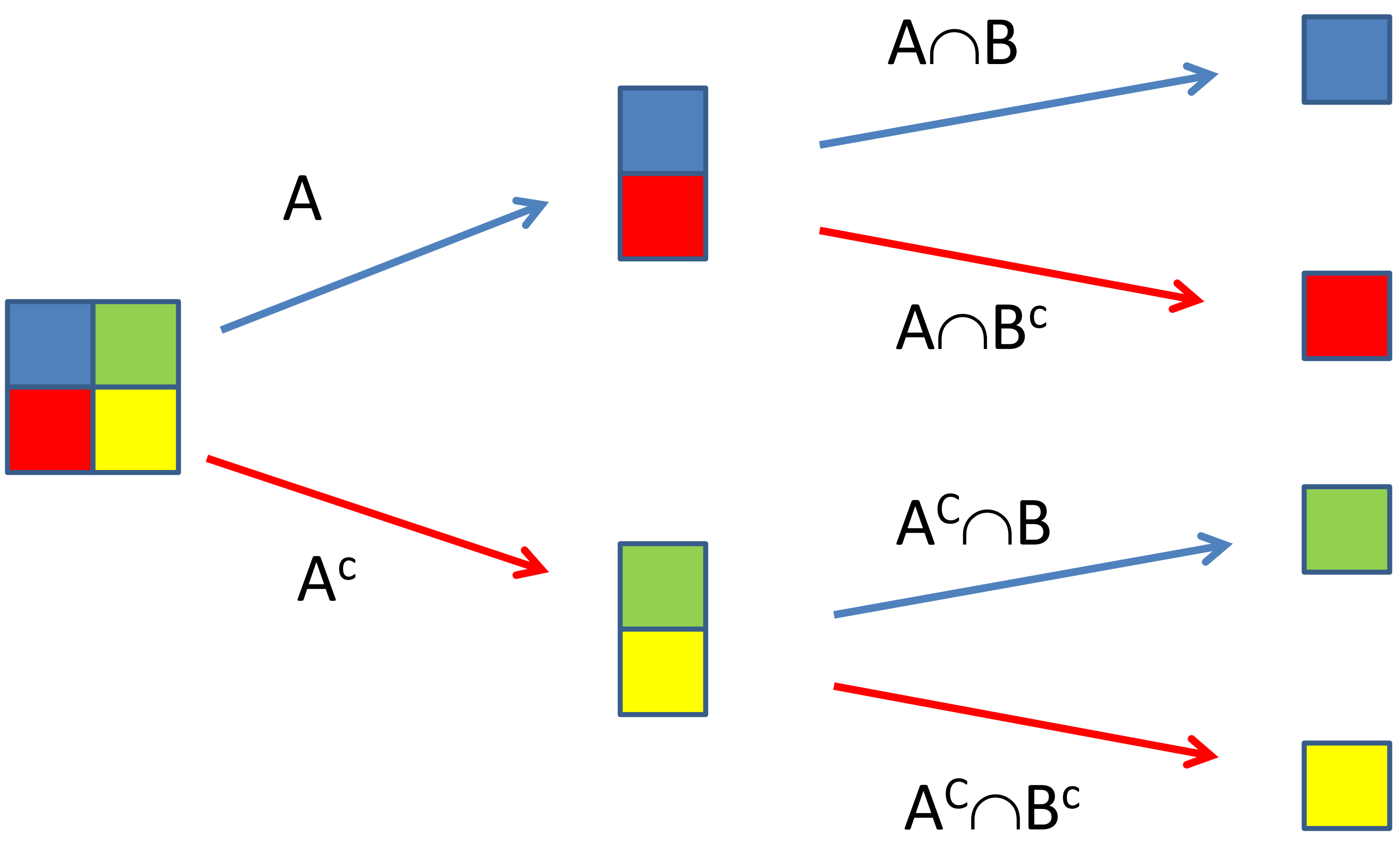
- Event A: the first ball is blue;
- Event B*: the second ball is read;

*Disclaimer: Event B is not Event-B, one of formal method study/teaching by Nestor Catano and Victor Riviera in Innopolis University!

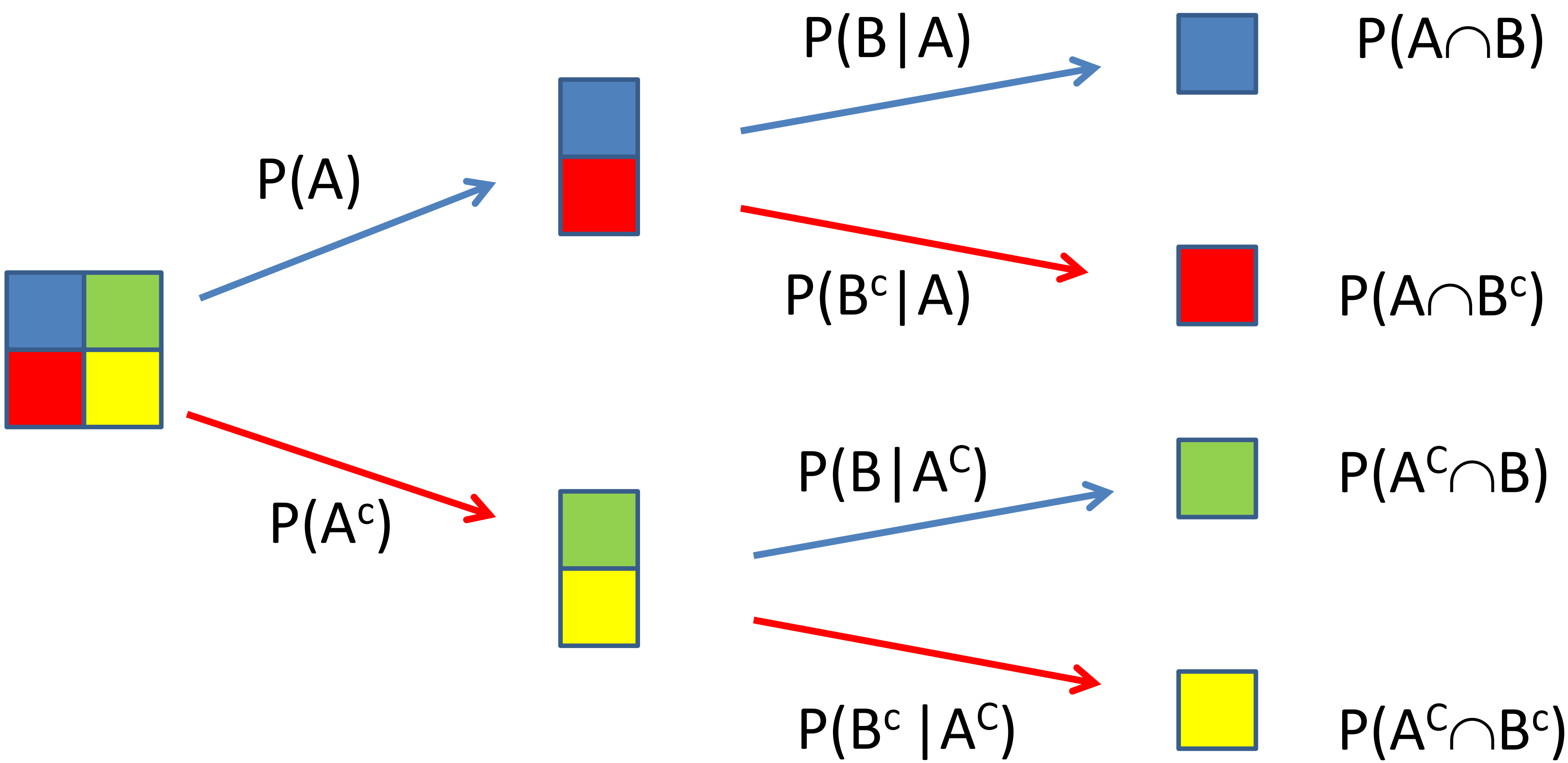
- $P(A \cap B) = (2*3*P_3)/P_5 = 6/20 = P(A)*P(B|A)$



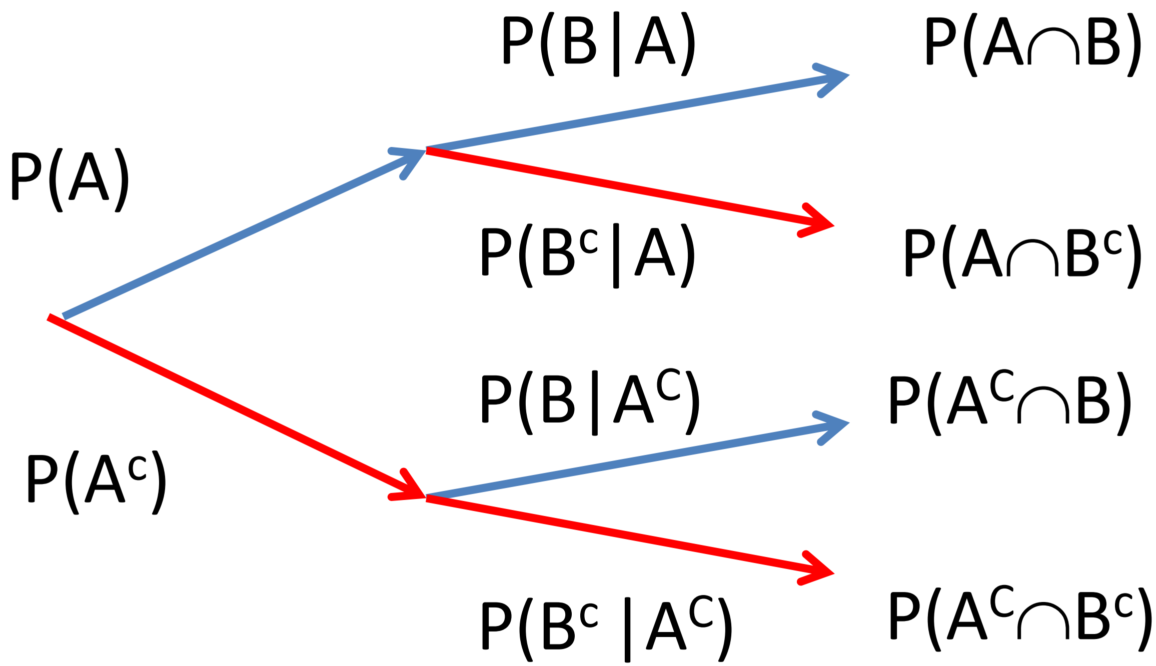
Dichotomy Event Tree



Dichotomy Probability Tree

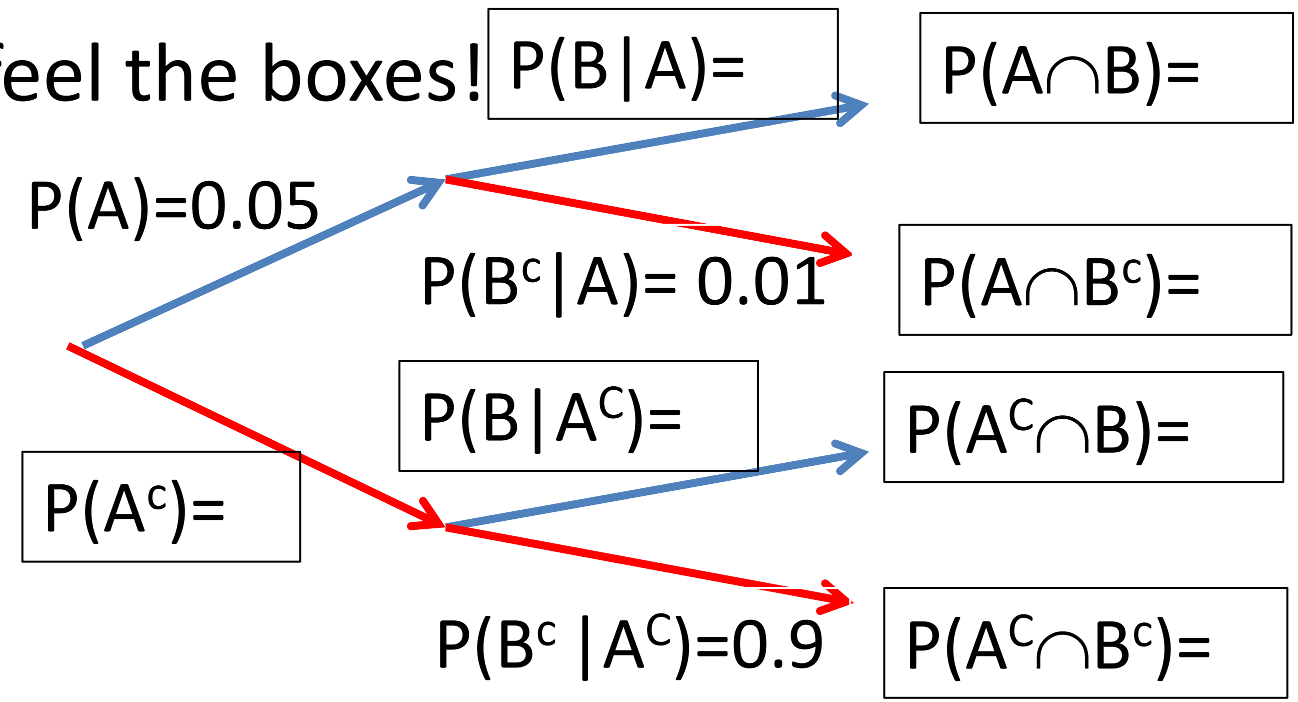


Dichotomy Probability Tree (cont.)



Example: plane & radar

- Event A: an alien plane is in radar control area;
- Event B: an alien signal on radar screen;
- Problem: feel the boxes!



Part II

INDEPENDENT EVENTS

Definition & Multiplication Property

- Events A and B are *independent* if

$$P(A) = P(A|B) \text{ and/or } P(B) = P(B|A);$$

  otherwise the events are *dependent*.

- Product of a pair of independent events A and B enjoys the multiplication rule for independent events

$$P(A \cap B) = P(A)*P(B).$$

Does intuition work?

- Let us say that events are *intuitively independent* if the probability of one does not depend on happening or not-happening of another
- and formalize this property by the following equalities:
  - $P(A) = P(A|B) = P(A|B^c)$,
  - $P(B) = P(B|A) = P(B|A^c)$.

Does intuition work? (cont.)

- What do you think how concepts of *independence* and *intuitive independence* relate to each other? Are they
  - equivalent;
  - nested one in another (but not equal);
  - not related?

More questions on Independence

- Assume that events A, B and C are pair-wise independent.
- What events
  - A and B^c,
  - A^c and B^c,
  - A and $(B \cup C)$,
  - A and $(B \cap C)$

  are independent?

Mutually Independent Events

- A set of events is *mutually independent* if each event in the set is independent with every product of other events in the set.
- Product of mutually independent events enjoys the multiplication rule

  $$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1) * P(A_2) * \ldots * P(A_n)$$

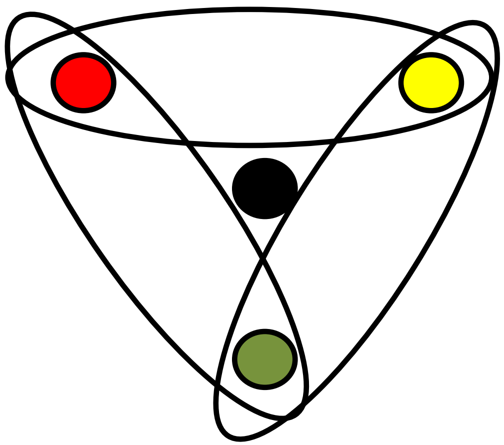
  (a corollary from the (general) multiplication rule).

Pair-wise vs. Mutual Independence

- Mutually independent events are pair-wise independent (by definition), but not vice versa.

Pair-wise vs. Mutual Independence

- Let us consider urn with 4 ball indexed as 110, 101, 011 and 000 and drawn at random and events A_1, A_2, A_3 where A_k means drawn of a ball with 1 in position k:
  - $A_1 = \{110, 101\}$, $P(A_1)=0.5$;
  - $A_2 = \{110, 011\}$, $P(A_2)=0.5$;
  - $A_3 = \{101, 011\}$, $P(A_1)=0.5$.



- A_1, A_2, A_3 are pair-wise independent but not mutually independent.

Multiplication Rule vs. Independence

- Give an example when

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)*P(A_2)*P(A_3)$$

but every pair of events is dependent?

Example: plane and missiles

- The first antiaircraft missile crew hits a plane with probability 0.8, the second – with probability 0.7.
- Each crew fires one time simultaneously but independently to alien plane.
- What is the probability that they hit the plane?

Example: plane and missiles (cont.)

- Let A be the event when the first crew hits the plane, and B – that the second.
- Since A and B are independent, then A^c and B^c are independent also.
- The probability that the plane is not hit by both crews is

$$(1-P(A^c))*(1-P(B^c)) = 0.2*0.3 = 0.06.$$

- The probability that they hit the plane is 0.94.

Part III

PROBABILITY CALCULUS: TOTAL PROBABILITY

Partitioning and Hypothesizes

Partitioning of a set is representation of the set as a union of pair-wise disjoint subsets.

Partitioning and Hypothesizes

Partition of a sample space Ω is presentation of the space as a sum of mutually exclusive events that are called hypothesizes:

$$\Omega = \cup_{n\in[0..m]} H_n = \Sigma_{n\in[0..m]} H_n$$

where $H_i \cap H_j = \varnothing$ for all $0 \leq i < j \leq m$.

Total Probability Formula

- Let a sample space be partitioned

$$\Omega = \cup_{n\in[0..m]} H_n$$

then for every event

$$P(A) = P(\cup_{n\in[0..m]}(A\cap H_n)) =$$

$$= \Sigma_{n\in[0..m]} P(A\cap H_n) = \Sigma_{n\in[0..m]} P(A|H_n) * P(H_n).$$

Example: knock the plane

- A plane hit by a single missile can survive with probability 0.2, but can not survive been hit by two or more missiles.
- In conditions of problem about plane & missiles, what is probability for the plane to survive?

Example: knock the plane (cont)

- Using same notation as for the problem plane & missiles, let
  - H_1 be $A \cap B$,
  - H_2 be $A \cap B^c$,
  - H_3 be $A^c \cap B$,
  - H_4 be $A^c \cap B^c$;
- let X be the event when the plane survive.

Example: knock the plane (cont)

- $P(X) =$

$= P(X|H_1)*P(H_1) + P(X|H_2) *P(H_2)+$

$+ P(X|H_3)*P(H_3) + P(X|H_4)*P(H_4) =$

$= 0*(0.8*0.7) + 0.2*(0.8*0.3) +$

$+ 0.2*(0.2*0.7) + 1*(0.2*0.3) =$

$= 0 + 0.048 + 0.028 + 0.06 = 0.136.$

Bayes' Formula

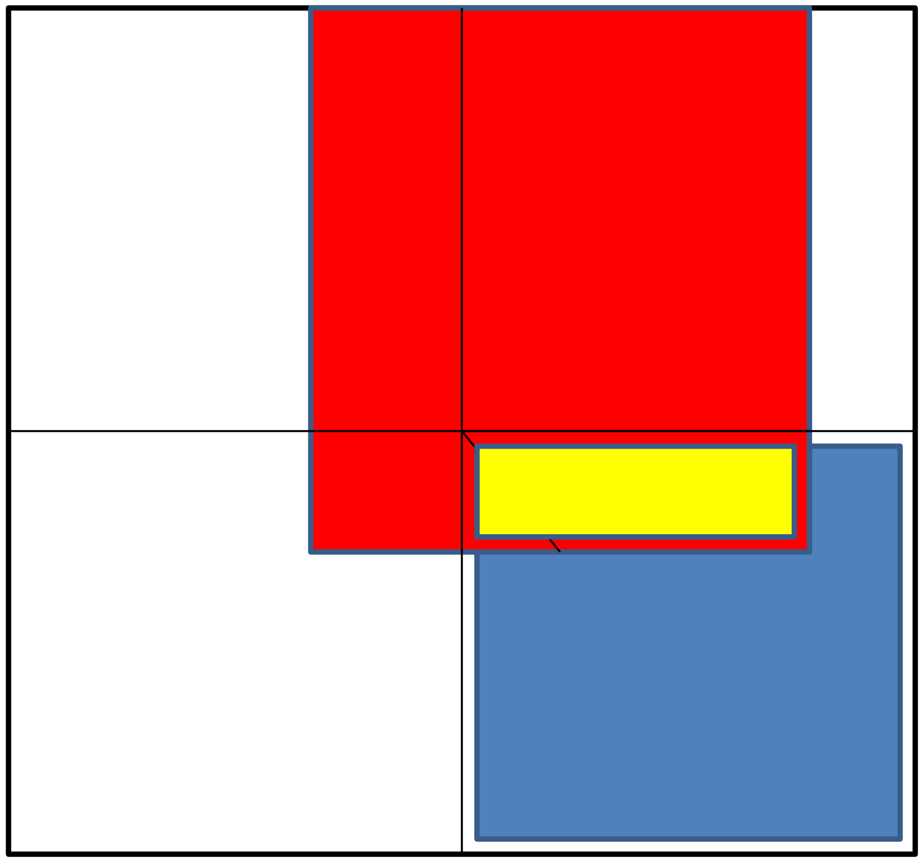
- Let a sample space be partitioned

$$\Omega = \cup_{n\in[0..m]} H_n$$

then for every event A and hypothesis H_k the *posterior probability* can be computed using *prior probabilities* as follows:

$$P(H_k|A) = P(A|H_k)^* P(H_k)/P(A) = \frac{P(A|H_k)^* P(H_k)}{\Sigma_{n\in[0..m]} P(A|H_n) * P(H_n)}.$$

Bayes' Formula



Example: knocked by one hit

- Improved missile knocks a plane after a hit.
- Two crews one time fired simultaneously but independently these missiles to an alien plane.
- The plane is knocked down by one of these two launched missiles.
- In conditions of problem about plane & missiles, what is probability the missile was fired by the first crew?