LINEAR ALGEBRA. LECTURE 13

Linear transformations and their matrices

In older linear algebra courses, linear transformations were introduced before matrices. This geometric approach to linear algebra initially avoids the need for coordinates. But eventually there must be coordinates and matrices when the need for computation arises.

Without coordinates (no matrix)

Example 1: Projection

We can describe a projection as a *linear transformation* T which takes every vector in $\mathbb{R}^2$ into another vector in $\mathbb{R}^2$. In other words,

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2.$$

The rule for this *mapping* is that every vector $\mathbf{v}$ is projected onto a vector $T(\mathbf{v})$ on the line of the projection. Projection is a linear transformation.

Definition of *linear*

A transformation T is *linear* if:

$$T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$$

and

$$T(c\mathbf{v}) = cT(\mathbf{v})$$

for all vectors $\mathbf{v}$ and $\mathbf{w}$ and for all scalars c. Equivalently,

$$T(c\mathbf{v} + d\mathbf{w}) = cT(\mathbf{v}) + dT(\mathbf{w})$$

for all vectors $\mathbf{v}$ and $\mathbf{w}$ and scalars c and d. It's worth noticing that $T(\mathbf{0}) = \mathbf{0}$, because if not it couldn't be true that $T(c\mathbf{0}) = cT(\mathbf{0})$.

Non-example 1: Shift the whole plane

Consider the transformation $T(\mathbf{v}) = \mathbf{v} + \mathbf{v}_0$ that shifts every vector in the plane by adding some fixed vector $\mathbf{v}_0$ to it. This is *not* a linear transformation because $T(2\mathbf{v}) = 2\mathbf{v} + \mathbf{v}_0 \neq 2T(\mathbf{v})$.

Non-example 2: $T(\mathbf{v}) = ||\mathbf{v}||$

The transformation $T(\mathbf{v}) = ||\mathbf{v}||$ that takes any vector to its length is not a linear transformation because $T(c\mathbf{v}) \neq cT(\mathbf{v})$ if $c < 0$.

We're not going to study transformations that aren't linear. From here on, we'll only use T to stand for linear transformations.

Example 2: Rotation by 45°

This transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ takes an input vector $\mathbf{v}$ and outputs the vector $T(\mathbf{v})$ that comes from rotating $\mathbf{v}$ counterclockwise by 45° about the origin. Note that we can describe this and see that it's linear without using any coordinates.

The big picture

One advantage of describing transformations geometrically is that it helps us to see the big picture, as opposed to focusing on the transformation's effect on a single point. We can quickly see how rotation by 45° will transform a picture of a house in the plane. If the transformation was described in terms of a matrix rather than as a rotation, it would be harder to guess what the house would be mapped to.

Frequently, the best way to understand a linear transformation is to find the matrix that lies behind the transformation. To do this, we have to choose a basis and bring in coordinates.

With coordinates (matrix!)

All of the linear transformations we've discussed above can be described in terms of matrices. In a sense, linear transformations are an abstract description of multiplication by a matrix, as in the following example.

Example 3: $T(\mathbf{v}) = A\mathbf{v}$

Given a matrix A, define $T(\mathbf{v}) = A\mathbf{v}$. This is a linear transformation:

$$A(\mathbf{v} + \mathbf{w}) = A(\mathbf{v}) + A(\mathbf{w})$$

and

$$A(c\mathbf{v}) = cA(\mathbf{v}).$$

Example 4

Suppose $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. How would we describe the transformation $T(\mathbf{v}) = A\mathbf{v}$ geometrically?

When we multiply A by a vector $\mathbf{v}$ in $\mathbb{R}^2$, the x component of the vector is unchanged and the sign of the y component of the vector is reversed. The transformation $\mathbf{v} \mapsto A\mathbf{v}$ reflects the xy-plane across the x axis.

Example 5

How could we find a linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ that takes three dimensional space to two dimensional space? Choose any 2 by 3 matrix A and define $T(\mathbf{v}) = A\mathbf{v}$.

Describing $T(\mathbf{v})$

How much information do we need about T to to determine $T(\mathbf{v})$ for all $\mathbf{v}$? If we know how T transforms a single vector $\mathbf{v}_1$, we can use the fact that T is a linear transformation to calculate $T(c\mathbf{v}_1)$ for any scalar c. If we know $T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$ for two independent vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, we can predict how T will transform any vector $c\mathbf{v}_1 + d\mathbf{v}_2$ in the plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$. If we wish to know $T(\mathbf{v})$ for all vectors $\mathbf{v}$ in $\mathbb{R}^n$, we just need to know $T(\mathbf{v}_1), T(\mathbf{v}_2), ..., T(\mathbf{v}_n)$ for any basis $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ of the input space. This is because any $\mathbf{v}$ in the input space can be written as a linear combination of basis vectors, and we know that T is linear:

$$\begin{aligned} \mathbf{v} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \\ T(\mathbf{v}) &= c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_nT(\mathbf{v}_n). \end{aligned}$$

This is how we get from a (coordinate-free) linear transformation to a (coordinate based) matrix; the c_i are our coordinates. Once we've chosen a basis, every vector $\mathbf{v}$ in the space can be written as a combination of basis vectors in exactly one way. The coefficients of those vectors are the *coordinates* of $\mathbf{v}$ in that basis.

Coordinates come from a basis; changing the basis changes the coordinates of vectors in the space. We may not use the standard basis all the time – we sometimes want to use a basis of eigenvectors or some other basis.

The matrix of a linear transformation

Given a linear transformation T, how do we construct a matrix A that represents it?

First, we have to choose two bases, say $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ of $\mathbb{R}^n$ to give coordinates to the input vectors and $\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_m$ of $\mathbb{R}^m$ to give coordinates to the output vectors. We want to find a matrix A so that $T(\mathbf{v}) = A\mathbf{v}$, where $\mathbf{v}$ and $A\mathbf{v}$ get their coordinates from these bases.

The first column of A consists of the coefficients $a_{11}, a_{21}, ..., a_{m1}$ of $T(\mathbf{v}_1) = a_{11}\mathbf{w}_1 + a_{21}\mathbf{w}_2 + \cdots + a_{m1}\mathbf{w}_m$. The entries of column i of the matrix A are determined by $T(\mathbf{v}_i) = a_{1i}\mathbf{w}_1 + a_{2i}\mathbf{w}_2 + \cdots + a_{mi}\mathbf{w}_m$ Because we've guaranteed that $T(\mathbf{v}_i) = A\mathbf{v}_i$ for each basis vector $\mathbf{v}_i$ and because T is linear, we know that $T(\mathbf{v}) = A\mathbf{v}$ for all vectors $\mathbf{v}$ in the input space.

In the example of the projection matrix, $n = m = 2$. The transformation T projects every vector in the plane onto a line. In this example, it makes sense to use the same basis for the input and the output. To make our calculations as simple as possible, we'll choose $\mathbf{v}_1$ to be a unit vector on the line of projection and $\mathbf{v}_2$ to be a unit vector perpendicular to $\mathbf{v}_1$. Then

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1\mathbf{v}_1 + \mathbf{0}$$

and the matrix of the projection transformation is just $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

$$A\mathbf{v} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}.$$

This is a nice matrix! If our chosen basis consists of eigenvectors then the matrix of the transformation will be the diagonal matrix Λ with eigenvalues on the diagonal.

To see how important the choice of basis is, let's use the standard basis for the linear transformation that projects the plane onto a line at a 45° angle. If we choose $\mathbf{v}_1 = \mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \mathbf{w}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we get the projection matrix $P = \frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$. We can check by graphing that this is the correct matrix, but calculating P directly is more difficult for this basis than it was with a basis of eigenvectors.

Example 6: $T = \dfrac{d}{dx}$

Let T be a transformation that takes the derivative:

$$T(c_1 + c_2x + c_3x^2) = c_2 + 2c_3x. \tag{1}$$

The input space is the three dimensional space of quadratic polynomials $c_1 + c_2x + c_3x^2$ with basis $\mathbf{v}_1 = 1$, $\mathbf{v}_2 = x$ and $\mathbf{v}_3 = x^2$. The output space is a two dimensional subspace of the input space with basis $\mathbf{w}_1 = \mathbf{v}_1 = 1$ and $\mathbf{w}_2 = \mathbf{v}_2 = x$.

This is a linear transformation! So we can find $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and write the transformation (1) as a matrix multiplication (2):

$$T\left(\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}\right) = A \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_2 \\ 2c_3 \end{bmatrix}. \tag{2}$$

Conclusion

For any linear transformation T we can find a matrix A so that $T(\mathbf{v}) = A\mathbf{v}$. If the transformation is invertible, the inverse transformation has the matrix A^{-1}. The product of two transformations $T_1 : \mathbf{v} \mapsto A_1\mathbf{v}$ and $T_2 : \mathbf{w} \mapsto A_2\mathbf{w}$ corresponds to the product A_2A_1 of their matrices. This is where matrix multiplication came from!

Left and right inverses; pseudoinverse

Although pseudoinverses will not appear on the exam, this lecture will help us to prepare.

Two sided inverse

A *2-sided inverse* of a matrix A is a matrix A^{-1} for which $AA^{-1} = I = A^{-1}A$. This is what we've called the *inverse* of A. Here $r = n = m$; the matrix A has full rank.

Left inverse

Recall that A has full column rank if its columns are independent; i.e. if $r = n$. In this case the nullspace of A contains just the zero vector. The equation $A\mathbf{x} = \mathbf{b}$ either has exactly one solution $\mathbf{x}$ or is not solvable.

The matrix A^TA is an invertible n by n symmetric matrix, so $(A^TA)^{-1}A^TA = I$. We say $A_{\text{left}}^{-1} = (A^TA)^{-1}A^T$ is a *left inverse* of A. (There may be other left inverses as well, but this is our favorite.) The fact that A^TA is invertible when A has full column rank was central to our discussion of least squares.

Note that AA_{left}^{-1} is an m by m matrix which only equals the identity if $m = n$. A rectangular matrix can't have a two sided inverse because either that matrix or its transpose has a nonzero nullspace.

Right inverse

If A has full row rank, then $r = m$. The nullspace of A^T contains only the zero vector; the rows of A are independent. The equation $A\mathbf{x} = \mathbf{b}$ always has at least one solution; the nullspace of A has dimension $n - m$, so there will be $n - m$ free variables and (if $n > m$) infinitely many solutions!

Matrices with full row rank have right inverses A_{right}^{-1} with $AA_{\text{right}}^{-1} = I$. The nicest one of these is $A^T(AA^T)^{-1}$. Check: A times $A^T(AAT)^{-1}$ is I.

Pseudoinverse

An invertible matrix ($r = m = n$) has only the zero vector in its nullspace and left nullspace. A matrix with full column rank $r = n$ has only the zero vector in its nullspace. A matrix with full row rank $r = m$ has only the zero vector in its left nullspace. The remaining case to consider is a matrix A for which $r < n$ and $r < m$.

If A has full column rank and $A_{\text{left}}^{-1} = (A^TA)^{-1}A^T$, then

$$AA_{\text{left}}^{-1} = A(A^TA)^{-1}A^T = P$$

is the matrix which projects $\mathbb{R}^m$ onto the column space of A. This is as close as we can get to the product $AM = I$.

Similarly, if A has full row rank then $A^{-1}_{\text{right}}A = A^T(AA^T)^{-1}A$ is the matrix which projects $\mathbb{R}^n$ onto the row space of A.

It's nontrivial nullspaces that cause trouble when we try to invert matrices. If $A\mathbf{x} = \mathbf{0}$ for some nonzero $\mathbf{x}$, then there's no hope of finding a matrix A^{-1} that will reverse this process to give $A^{-1}\mathbf{0} = \mathbf{x}$.

The vector $A\mathbf{x}$ is always in the column space of A. In fact, the correspondence between vectors $\mathbf{x}$ in the (r dimensional) row space and vectors $A\mathbf{x}$ in the (r dimensional) column space is one-to-one. In other words, if $\mathbf{x} \neq \mathbf{y}$ are vectors in the row space of A then $A\mathbf{x} \neq A\mathbf{y}$ in the column space of A. (The proof of this would make a good exam question.)

Proof that if $\mathbf{x} \neq \mathbf{y}$ then $A\mathbf{x} \neq A\mathbf{y}$

Suppose the statement is false. Then we can find $\mathbf{x} \neq \mathbf{y}$ in the row space of A for which $A\mathbf{x} = A\mathbf{y}$. But then $A(\mathbf{x} - \mathbf{y}) = \mathbf{0}$, so $\mathbf{x} - \mathbf{y}$ is in the nullspace of A. But the row space of A is closed under linear combinations (like subtraction), so $\mathbf{x} - \mathbf{y}$ is also in the row space. The only vector in both the nullspace and the row space is the zero vector, so $\mathbf{x} - \mathbf{y} = \mathbf{0}$. This contradicts our assumption that $\mathbf{x}$ and $\mathbf{y}$ are not equal to each other.

We conclude that the mapping $\mathbf{x} \mapsto A\mathbf{x}$ from row space to column space is invertible. The inverse of this operation is called the *pseudoinverse* and is very useful to statisticians in their work with linear regression – they might not be able to guarantee that their matrices have full column rank $r = n$.

Finding the pseudoinverse A^+

The *pseudoinverse* A^+ of A is the matrix for which $\mathbf{x} = A^+A\mathbf{x}$ for all $\mathbf{x}$ in the row space of A. The nullspace of A^+ is the nullspace of A^T.

We start from the singular value decomposition $A = U\Sigma V^T$. Recall that Σ is a m by n matrix whose entries are zero except for the singular values $\sigma_1, \sigma_2, ..., \sigma_r$ which appear on the diagonal of its first r rows. The matrices U and V are orthonormal and therefore easy to invert. We only need to find a pseudoinverse for Σ.

The closest we can get to an inverse for Σ is an n by m matrix Σ^+ whose first r rows have $1/\sigma_1, 1/\sigma_2, ..., 1/\sigma_r$ on the diagonal. If $r = n = m$ then $\Sigma^+ = \Sigma^{-1}$. Always, the product of Σ and Σ^+ is a square matrix whose first r diagonal entries are 1 and whose other entries are 0.

If $A = U\Sigma V^T$ then its pseudoinverse is $A^+ = V\Sigma^+\mathbf{U}^T$. (Recall that $Q^T = Q^{-1}$ for orthogonal matrices U, V or Q.)

We would get a similar result if we included non-zero entries in the lower right corner of Σ^+, but we prefer not to have extra non-zero entries.