LINEAR ALGEBRA. LECTURE 8

Properties of determinants

Determinants

Now halfway through the course, we leave behind rectangular matrices and focus on square ones. Our next big topics are determinants and eigenvalues.

The *determinant* is a number associated with any square matrix; we'll write it as $\det A$ or $|A|$. The determinant encodes a lot of information about the matrix; the matrix is invertible exactly when the determinant is non-zero.

Properties

Rather than start with a big formula, we'll list the properties of the determinant. We already know that $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$; these properties will give us a formula for the determinant of square matrices of all sizes.

1. $\det I = 1$
2. If you exchange two rows of a matrix, you reverse the sign of its determinant from positive to negative or from negative to positive.
3. (a) If we multiply one row of a matrix by t, the determinant is multiplied by t: $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

   (b) The determinant behaves like a linear function on the rows of the matrix:
$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}.$$

Property 1 tells us that $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$. Property 2 tells us that $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$. The determinant of a permutation matrix P is 1 or -1 depending on whether P exchanges an even or odd number of rows.

From these three properties we can deduce many others:

4. If two rows of a matrix are equal, its determinant is zero.

   This is because of property 2, the exchange rule. On the one hand, exchanging the two identical rows does not change the determinant. On the other hand, exchanging the two rows changes the sign of the determinant. Therefore the determinant must be 0.

5. If $i \neq j$, subtracting t times row i from row j doesn't change the determinant.

In two dimensions, this argument looks like:

$$\begin{aligned}
\begin{vmatrix} a & b \\ c - ta & d - tb \end{vmatrix} &= \begin{vmatrix} a & b \\ c & d \end{vmatrix} - \begin{vmatrix} a & b \\ ta & tb \end{vmatrix} && \text{property 3(b)} \\
&= \begin{vmatrix} a & b \\ c & d \end{vmatrix} - t \begin{vmatrix} a & b \\ a & b \end{vmatrix} && \text{property 3(a)} \\
&= \begin{vmatrix} a & b \\ c & d \end{vmatrix} && \text{property 4.}
\end{aligned}$$

The proof for higher dimensional matrices is similar.

6. If A has a row that is all zeros, then $\det A = 0$.

   We get this from property 3 (a) by letting $t = 0$.

7. The determinant of a triangular matrix is the product of the diagonal entries (pivots) $d_1, d_2, ..., d_n$.

   Property 5 tells us that the determinant of the triangular matrix won't change if we use elimination to convert it to a diagonal matrix with the entries d_i on its diagonal. Then property 3 (a) tells us that the determinant of this diagonal matrix is the product $d_1 d_2 \cdots d_n$ times the determinant of the identity matrix. Property 1 completes the argument.

   Note that we cannot use elimination to get a diagonal matrix if one of the d_i is zero. In that case elimination will give us a row of zeros and property 6 gives us the conclusion we want.

8. $\det A = 0$ exactly when A is singular.

   If A is singular, then we can use elimination to get a row of zeros, and property 6 tells us that the determinant is zero.

   If A is not singular, then elimination produces a full set of pivots $d_1, d_2, ..., d_n$ and the determinant is $d_1 d_2 \cdots d_n \neq 0$ (with minus signs from row exchanges).

We now have a very practical formula for the determinant of a non-singular matrix. In fact, the way computers find the determinants of large matrices is to first perform elimination (keeping track of whether the number of row exchanges is odd or even) and then multiply the pivots:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \begin{bmatrix} a & b \\ 0 & d - \frac{c}{a} b \end{bmatrix}, \text{ if } a \neq 0, \text{ so}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a(d - \frac{c}{a} b) = ad - bc.$$

9. $\det AB = (\det A)(\det B)$

   This is very useful. Although the determinant of a sum does not equal the sum of the determinants, it is true that the determinant of a product equals the product of the determinants.

For example:

$$\det A^{-1} = \frac{1}{\det A},$$

because $A^{-1}A = 1$. (Note that if A is singular then A^{-1} does not exist and $\det A^{-1}$ is undefined.) Also, $\det A^2 = (\det A)^2$ and $\det 2A = 2^n \det A$ (applying property 3 to each row of the matrix). This reminds us of volume – if we double the length, width and height of a three dimensional box, we increase its volume by a multiple of $2^3 = 8$.

10. $\det A^T = \det A$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc.$$

This lets us translate properties (2, 3, 4, 5, 6) involving rows into statements about columns. For instance, if a column of a matrix is all zeros then the determinant of that matrix is zero.

To see why $|A^T| = |A|$, use elimination to write $A = LU$. The statement becomes $|U^T L^T| = |LU|$. Rule 9 then tells us $|U^T||L^T| = |L||U|$.

Matrix L is a lower triangular matrix with 1's on the diagonal, so rule 5 tells us that $|L| = |L^T| = 1$. Because U is upper triangular, rule 5 tells us that $|U| = |U^T|$. Therefore $|U^T||L^T| = |L||U|$ and $|A^T| = |A|$.

Determinant formulas and cofactors

Now that we know the properties of the determinant, it's time to learn some (rather messy) formulas for computing it.

Formula for the determinant

We know that the determinant has the following three properties:

1. $\det I = 1$
2. Exchanging rows reverses the sign of the determinant.
3. The determinant is linear in each row separately.

Last class we listed seven consequences of these properties. We can use these ten properties to find a formula for the determinant of a 2 by 2 matrix:

$$\begin{aligned}
\begin{vmatrix} a & b \\ c & d \end{vmatrix} &= \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} \\
&= \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix} \\
&= 0 + ad + (-cb) + 0 \\
&= ad - bc.
\end{aligned}$$

Cofactor formula

The cofactor formula rewrites the big formula for the determinant of an n by n matrix in terms of the determinants of smaller matrices.

In the 3×3 case, the formula looks like:

$$\begin{aligned}
\det A &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(-a_{21}a_{33} + a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\
&= \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{vmatrix}
\end{aligned}$$

This comes from grouping all the multiples of a_{ij} in the big formula. Each element is multiplied by the *cofactors* in the parentheses following it. Note that each cofactor is (plus or minus) the determinant of a two by two matrix. That determinant is made up of products of elements in the rows and columns NOT containing a_{1j}.

In general, the cofactor C_{ij} of a_{ij} can be found by looking at all the terms in the big formula that contain a_{ij}. C_{ij} equals $(-1)^{i+j}$ times the determinant of the $n-1$ by $n-1$ square matrix obtained by removing row i and column j. (C_{ij} is positive if $i+j$ is even and negative if $i+j$ is odd.)

For $n \times n$ matrices, the cofactor formula is:

$$\boxed{\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}.}$$

Applying this to a 2×2 matrix gives us:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad + b(-c).$$

The number of parts with non-zero determinants was 2 in the 2 by 2 case, 6 in the 3 by 3 case, and will be $24 = 4!$ in the 4 by 4 case. This is because there are n ways to choose an element from the first row (i.e. a value for α), after which there are only $n - 1$ ways to choose an element from the second row that avoids a zero determinant. Then there are $n - 2$ choices from the third row, $n - 3$ from the fourth, and so on.

The big formula for computing the determinant of any square matrix is:

$$\det A = \sum_{n! \text{ terms}} \pm a_{1\alpha} a_{2\beta} a_{3\gamma} ... a_{n\omega}$$

where $(\alpha, \beta, \gamma, ...\omega)$ is some permutation of $(1, 2, 3, ..., n)$. If we test this on the identity matrix, we find that all the terms are zero except the one corresponding to the trivial permutation $\alpha = 1$, $\beta = 2$, ..., $\omega = n$. This agrees with the first property: $\det I = 1$. It's possible to check all the other properties as well, but we won't do that here.

Applying the method of elimination and multiplying the diagonal entries of the result (the pivots) is another good way to find the determinant of a matrix.

Cramer's rule, inverse matrix, and volume

We know a formula for and some properties of the determinant. Now we see how the determinant can be used.

Formula for A^{-1}

We know:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Can we get a formula for the inverse of a 3 by 3 or n by n matrix? We expect that $\frac{1}{\det A}$ will be involved, as it is in the 2 by 2 example, and by looking at the cofactor matrix $\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$ we might guess that cofactors will be involved.

In fact:

$$A^{-1} = \frac{1}{\det A}C^T$$

where C is the matrix of cofactors – please notice the transpose! Cofactors of row one of A go into column 1 of A^{-1}, and then we divide by the determinant.

The determinant of A involves products with n terms and the cofactor matrix involves products of $n-1$ terms. A and $\frac{1}{\det A}C^T$ might cancel each other. This is much easier to see from our formula for the determinant than when using Gauss-Jordan elimination.

To more formally verify the formula, we'll check that $AC^T = (\det A)I$.

$$AC^T = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \cdots & C_{nn} \end{bmatrix}.$$

The entry in the first row and first column of the product matrix is:

$$\sum_{j=1}^{n} a_{1j}C_{j1} = \det A.$$

(This is just the cofactor formula for the determinant.) This happens for every entry on the diagonal of AC^T.

To finish proving that $AC^T = (\det A)I$, we just need to check that the off-diagonal entries of AC^T are zero. In the two by two case, multiplying the entries in row 1 of A by the entries in column 2 of C^T gives $a(-b) + b(a) = 0$. This is the determinant of $A_s = \begin{bmatrix} a & b \\ a & b \end{bmatrix}$. In higher dimensions, the product of the first row of A and the last column of C^T equals the determinant of a matrix whose first and last rows are identical. This happens with all the off diagonal matrices, which confirms that $A^{-1} = \frac{1}{\det A}C^T$.

This formula helps us answer questions about how the inverse changes when the matrix changes.

Cramer's Rule for $\mathbf{x} = A^{-1}\mathbf{b}$

We know that if $A\mathbf{x} = \mathbf{b}$ and A is nonsingular, then $\mathbf{x} = A^{-1}\mathbf{b}$. Applying the formula $A^{-1} = C^T / \det A$ gives us:

$$\mathbf{x} = \frac{1}{\det A} C^T \mathbf{b}.$$

Cramer's rule gives us another way of looking at this equation. To derive this rule we break $\mathbf{x}$ down into its components. Because the i'th component of $C^T\mathbf{b}$ is a sum of cofactors times some number, it is the determinant of some matrix B_j.

$$x_j = \frac{\det B_j}{\det A},$$

where B_j is the matrix created by starting with A and then replacing column j with $\mathbf{b}$, so:

$$\begin{aligned} B_1 &= \begin{bmatrix} \mathbf{b} & \begin{array}{c}\text{last n-1} \\ \text{columns} \\ \text{of } A\end{array} \end{bmatrix} \qquad \text{and} \\ B_n &= \begin{bmatrix} \begin{array}{c}\text{first n-1} \\ \text{columns} \\ \text{of } A\end{array} & \mathbf{b} \end{bmatrix}. \end{aligned}$$

This agrees with our formula $x_1 = \frac{\det B_1}{\det A}$. When taking the determinant of B_1 we get a sum whose first term is b_1 times the cofactor C_{11} of A.

Computing inverses using Cramer's rule is usually less efficient than using elimination.

$|\det A|$ = volume of box

Claim: $|\det A|$ is the volume of the box (*parallelepiped*) whose edges are the column vectors of A. (We could equally well use the row vectors, forming a different box with the same volume.)

If $A = I$, then the box is a unit cube and its volume is 1. Because this agrees with our claim, we can conclude that the volume obeys determinant property 1.

If $A = Q$ is an orthogonal matrix then the box is a unit cube in a different orientation with volume $1 = |\det Q|$. (Because Q is an orthogonal matrix, $Q^TQ = I$ and so $\det Q = \pm 1$.)

Swapping two columns of A does not change the volume of the box or (remembering that $\det A = \det A^T$) the absolute value of the determinant (property 2). If we show that the volume of the box also obeys property 3 we'll have proven $|\det A|$ equals the volume of the box.

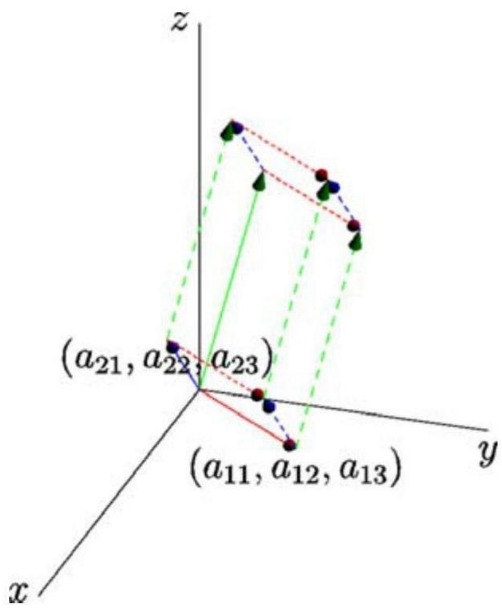

Figure 1: The box whose edges are the column vectors of A.

If we double the length of one column of A, we double the volume of the box formed by its columns. Volume satisfies property 3(a).

Property 3(b) says that the determinant is linear in the rows of the matrix:

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}.$$

Figure 2 illustrates why this should be true.

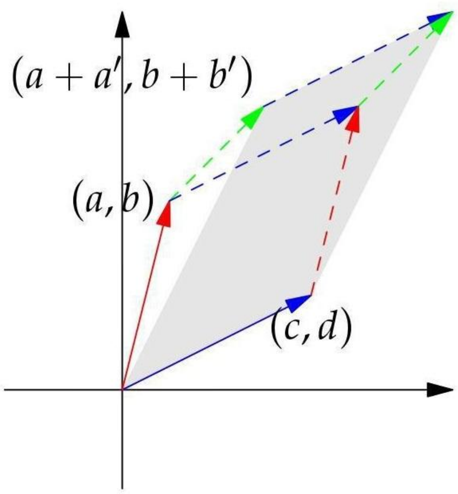

Figure 2: Volume obeys property 3(b).

Although it's not needed for our proof, we can also see that determinants obey property 4. If two edges of a box are equal, the box flattens out and has no volume.

Important note: If you know the coordinates for the corners of a box, then computing the volume of the box is as easy as calculating a determinant. In particular, the area of a parallelogram with edges $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ d \end{bmatrix}$ is $ad - bc$.

The area of a triangle with edges $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ d \end{bmatrix}$ is half the area of that parallelogram, or $\frac{1}{2}(ad - bc)$. The area of a triangle with vertices at (x_1, y_1), (x_2, y_2) and (x_3, y_3) is:

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$