Discrete Mathematics

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Do not worry about your difficulties in Mathematics. I can assure you mine are still greater!

-Albert Einstein-

About me

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Research interests:

- Machine Learning, Computer Vision, Smartphone-based Computing.

Outline

- What is DM and when can it be used?
- Why study DM?
- Topics
- Grading
- Today’s Lecture

Discrete Mathematics

- Study of $\underbrace{\text{Discrete}}_{\text{Consisting of Distinct Objects}}$ Objects

- Problems Solved Using Discrete Math

  - How many ways are there to choose a valid password?
  - What's the probability of winning a lottery?
  - How to encrypt a message?
  - What is the shortest path b/w two cities?
  - How to sort a list of integers?
  - How to prove that an algorithm works correctly?
  - .
  - .
  - .

Why Study Discrete Mathematics???

- Ability to understand and create mathematical arguments
- Gateway to more advanced courses
  - Algorithms
  - Database theory
  - Automata theory
  - Compiler theory
  - Computer security
  - Operating system

Topics we'll study

- ❑ Logic and Proofs
- ❑ Mathematical Induction
- ❑ Sequences and Recursion
- ❑ Set Theory
- ❑ Functions
- ❑ Relations
- ❑ Counting and Probability
- ❑ Graphs and Trees

Course organization

- **Class Schedule**
  - **Lecture:** Thursdays: 09:00 to 10:30
  - **Tutorials:** Thursdays: 10:40 to 15:20

- Class will be conducted using Slides

- **Text Book**
  - Susana Epp, ***Discrete Mathematics with its Applications***, (4th Edition)

Grading

- **Four One Hour Tests (OHTs) (60%)**
- **Final Exam (40%)**

Other Info

- Cooperation Policy
- Feedback
- Bonus Points

Today's Lecture

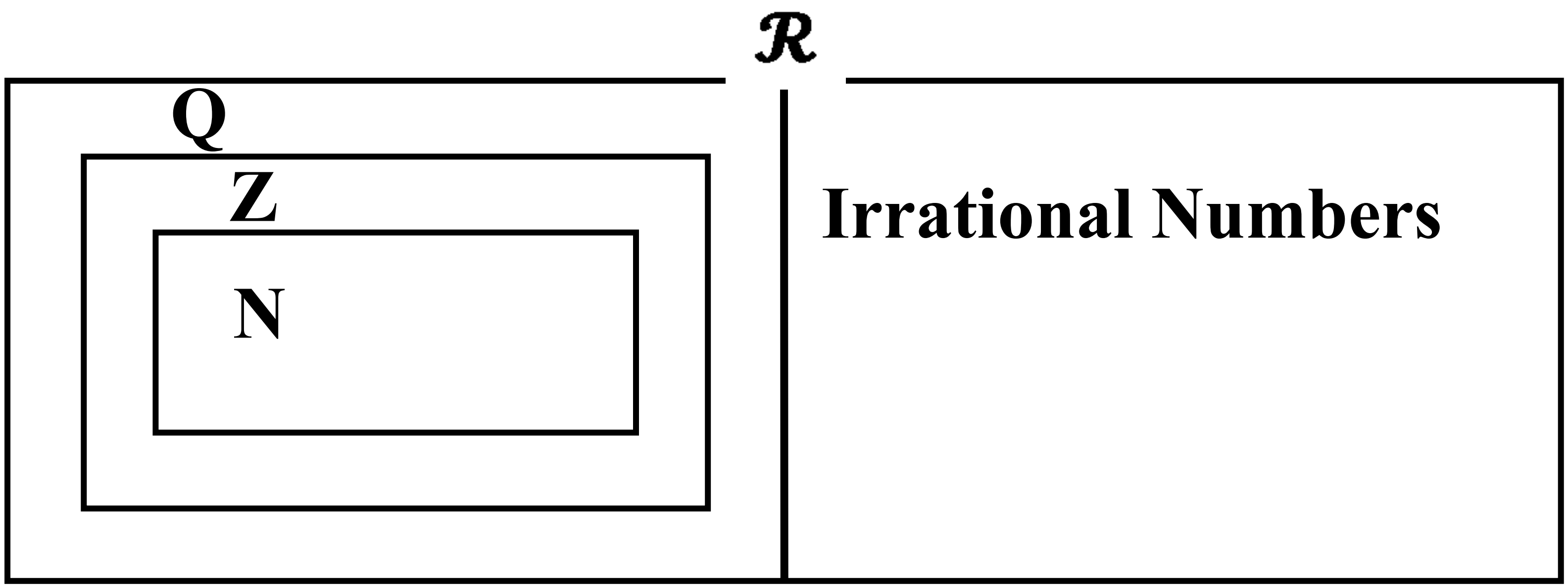
- **Integers:**
  - Arithmetic Properties
  - Powers
  - Divisibility
  - Primes of Composite Numbers
- **Rational Numbers**
  - Equivalent fractions
  - Operating with fractions
  - Decimals
- **Irrational Numbers**
- **Real Numbers**
  - Square roots
  - N-th roots
  - Logarithms
  - Inequalities
- **Oder of Operations**

Numbers

- $N = \{0,1,2,3, \ldots\}$ The set of Natural Numbers

- $Z = \{\ldots, -2,-1,0,1,2, \ldots\}$ The set of Integers

- $Q = \{p/q \mid p \in Z, \text{ and } q \neq 0\}$ The set of rational numbers
- $\mathcal{R}$, the set of real number. e.g. Real Space



Integers

Simple rule of Addition

- For an integer a,
- $0+a = a+0=a$
- $a+(-a)=0$, and $(-a)+a=0$
- $-a$ is the **additive inverse** of a.

We use “**Minus a**” rather than “**Negative a**”

Integers

- **Rules of Addition**
  - **Commutativity**
    - If a and b are integers, then
      - $a + b = b + a$
  - **Associativity**
    - If a , b and c are integers, then
    - $(a + b) + c = a + (b + c)$

Integers

❑ Rules of Addition

- If $a + b = 0$, then $b = -a$ and $a = -b$

- Proof

$$a + b = 0$$

Add $-a$ to both sides

$$-a + a + b = 0 - a$$

$$0 + b = 0 - a$$

$$\mathbf{b = -a}$$

As desired.

Similarly we can find ---- $a = -b$

Integers

- **Rules of Addition**
  - If a , b are positive integers, then a + b is also positive integer.
  - If a , b are negative integers, then a + b is also negative integer.

  - If we have the relationship b/w three integers.
    - $a + b = c$

  Then we can drive other relationships b/w them.

  $$\mathbf{a = c - b \qquad b = c - a}$$

- Example: Solve for x.

$$\mathbf{x + 3 = 5}$$
$$\mathbf{x = 5 - 3}$$
$$\mathbf{x = 2}$$

Integers

- **Rules of Addition**

  - Cancellation rule for addition

    - If $a + b = a + c$, then $b = c$

Exercise:

Prove that if $a + b = a$, then $b = 0$?

Integers

Rules of Multiplication

Commutativity

- If a and b are integers, then
  - A * b = b * a

Associativity

- If a , b and c are integers, then
  - (a * b) * c = a * (b * c)

- For any integer a
  - 1 * a = a and 0 * a = 0

Integers

- **Rules of Multiplication**

  - **Distributivity**

    - $a * (b + c) = a * b + a * c$
    - $(b + c) * a = b * a + c * a$

    Using all these properties

    - $-1 * a = -a$
    - $-(a * b) = (-a) * (b)$ **or** $-(a * b) = a * (-b)$
    - $(-a) * (-b) = a * b$

Integers

- **Powers**
  - An exponent is used to indicate repeated multiplication.
  - Tells how many times the base is used as a factor.
    - $\mathbf{a * a = a^2}$
    - $\mathbf{a * a * a = a^3}$

In general if n is a positive integer,

- $\mathbf{a^n = a * a * a \ldots a}$ (product is taken n times)

We say a^n is the n-th power of a.

If m , n are positive integers, then

- $\mathbf{a^{m+n} = a^m * a^n}$

Integers

- **Powers**

  - $(a^m)^n = a^{m * n}$

  Some important formulas

  - $(a + b)^2 = a^2 + b^2 + 2ab$
  - $(a - b)^2 = a^2 + b^2 - 2ab$
  - $(a + b)(a - b) = a^2 - b^2$

Integers

Even and Odd integers

- An even integer is an integer which can be written in the form **2n** for some integer **n**
  - $2 = 2 * 1$
  - $4 = 2 * 2$
  - $6 = 2 * 3$
- An odd integer is an integer that differs from an even integer by 1.
- It can be written in the form **2m ± 1** for some integer **m.**
  - $1 = (2 * 1) - 1$
  - $3 = (2 * 2) - 1$
  - $7 = (2 * 3) + 1$

Integers

- **Theorem**

  - Let a , b be integers,

    - If a is even and b is also even, then a + b is also even

    - If a is even and b is odd, then a + b is odd

    - If a is odd and b is even, then a + b is odd

    - If a is odd and b is also odd, then a + b is also even

- **Exercise.**

  - Let’s prove the Second statement ??

Integers

- **Divisibility**

  - Given two integers a and b, with $a \neq 0$, we say that **a divides b**, or that **b is divisible by a** if there is an integer c, such that **b = a * c**.

  - Remember that every integer is divisible by 1 because we can always write

    - $n = 1 * n$

- Also, every positive integer is divisible by itself.

Rational Numbers

- By a rational numbers, we mean a fraction as $\frac{m}{n}$, where m and n are integers, $n \neq 0$.
  - m is called numerator
  - n is called denominator

- Improper fraction
  - m larger than or equal to n

- Proper fraction
  - m smaller than n

Rational Numbers

❑ Equivalent Fractions

- ❑ Two fractions that represent the same value.

  - $\frac{1}{2} = \frac{2}{4}$

  How can we know whether two fractions are equivalent?

- ❑ Rule for cross-Multiplication

  - Let m ,n ,r ,s be integers and assume that n $\neq 0$ and s $\neq 0$. Then
  - $\left(\frac{m}{n}\right) = \left(\frac{r}{s}\right)$, iff $m * s = r * n$

Rational Numbers

- **Simplifying Fractions**

  - We can simplify four special fractions forms

    - Fractions that have the same numerator and denominator.
      - $1 = \frac{1}{1} = \frac{2}{2} = \frac{3}{3} = \frac{4}{4} = \ ...$
    - Fractions that have a denominator of 1.
      - $\frac{5}{1} = 5,\ \frac{24}{2} = 24,\ \frac{-6}{1} = -6$
    - Fractions that have a numerator of 0.
      - $\frac{0}{8} = 0,\ \frac{0}{71} = 0,\ \frac{0}{-10} = 0$
    - Fractions that have a denominator of 0
      - $\frac{7}{0} = \infty,\ \frac{-17}{0} = \infty,$ (∞=Infinity=Undefined Value)

Rational Numbers

- **Simplifying Fractions**
  - Cancellation Rule for Fractions
    - Let a be a non-zero integer. Let m , n be integers, and n ≠ 0, then
      - $\frac{am}{an} = \frac{m}{n}$
    - Proof: By applying the rule for cross-multiplication and using the associativity and commutativity laws.

Rational Numbers

- **Simplifying Fractions**

  - A fraction is in **simplest form** when the numerator and denominator have no common factors (or divisors) other than 1.

  - Theorem:

    - "Any positive rational number has an expression as a fraction in the lowest form."

Rational Numbers

❑ Operating with Fractions

- ❑ Addition (or Subtraction) with same denominator.

  - $\frac{a}{d}+\frac{b}{d}=\frac{a+b}{d}$ **<u>or</u>** $\frac{a}{d}-\frac{b}{d}=\frac{a-b}{d}$

- ❑ With different denominator:

  - $\frac{m}{n}+\frac{r}{s}=\frac{ms+rn}{ns}$ **<u>or</u>** $\frac{m}{n}+\frac{r}{s}=\frac{ms+rn}{ns}$

- ❑ Follows the same basic rules as addition of integers (commutativity and association)

Rational Numbers

- **Multiplication:**

  - Let $a = \frac{m}{n}$

    - Then for any positive integer k, such that

      - $a^k = \left(\frac{m}{n}\right)^k = \frac{m^k}{n^k}$

  - Follows the same basic rules as multiplication of integers.

Rational Numbers

- **Division:**

  - If a is a rational number and $a \neq 0$, then there exists ($\exists$) a rational number, denoted by
    - a^{-1} such that
    - $a^{-1} * a = a * a^{-1} = 1$
  - Note that if $a = \frac{m}{n}$ then $a^{-1} = \frac{n}{m}$
  - a^{-1} is called the multiplicative inverse of a.

Rational Numbers

❑ Decimals:

❑ Finite decimals (and periodic) give us examples of rational numbers.

- $1.4 = \frac{14}{10}$
- $1.41 = \frac{141}{100}$
- $0.2 = \frac{1}{5}$
- $0.75 = \frac{3}{4}$
- $0.3333 = 0.\overline{3} = \frac{4}{3}$
- $\ldots$

Irrational Numbers

- A number that cannot be expressed as fraction of $\frac{p}{q}$ for any integers p and q.

- Have decimal expressions that neither terminate nor become periodic

  - $\sqrt[2]{2} = 1.41421356237 \ldots$
  - $\sqrt[2]{3} = 1.73205080757\ldots$
  - $\pi = 3.14159265359 \ldots$
  - $\ldots$

Irrational Numbers

- Is $\sqrt[2]{25}$ an irrational number?
  - No!
  - Because $\sqrt[2]{25} = \pm 5$
- Is $\sqrt[2]{-1}$ an irrational number?
  - No or Yes??? In both cases HOW?

Real Numbers

- Integers, Rational and Irrational Numbers are part of a larger system.

- Real Numbers can be described as all the numbers that consist of a decimal expansion, possibly infinite.

Real Numbers

Properties of Real Numbers:

Addition:

- $a + b = b + a$
- $a + (b + c) = (a + b) + c$
- For all (∀) real numbers a, b, and c.

Multiplication

- $a * b = b * a$
- $a * (b * c) = (a * b) * c$
- ∀ real numbers a, b , c.

Also

- $a * (b + c) = a * b + a * c$
- $(b + c) * a = b * a + c * a$

Real Numbers

- **Absolute Value**

  - The non-negative values of a real number without regard to it sign.

    - $|a| = a$ for a positive a.
    - $|a| = -a$ for a negative a
      (in which case –a is positive).
    - $|0| = 0$

Real Numbers

- **Square Roots**
  - If $a > 0$, then there exists ($\exists$) a number b such that (s.t).
    - $b^2 = a$
- **N-th Roots**
  - There exists a unique real number r such that
    - $r^n = a$

      It is called the n-th root of a, and is denoted by
    - $a^{1/n}$ or $\sqrt[n]{a}$

Logarithms

- Can be seen as the reverse operation of the exponentiation.
- The logarithm of a number is the exponent to which another fixed value, the **Base** must be raised to produce that number.
  - $\log_{10}(10000) = 4$, because $10^4=10000$
  - $\log_2(16) = 4$, because $2^4=16$
  - $\log_3\left(\frac{1}{3}\right) = -1$, because $3^{-1}=\frac{1}{3}$

Logarithms

- **Properties of Logarithms:**
  - Product:
    - $\log_b(x * y) = \log_b(x) + \log_b(y)$
  - Quotient:
    - $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$
  - Power:
    - $\log_b(x^p) = p * \log_b(x)$
  - Change of Base:
    - $\log_b(x) = \frac{\log_k(x)}{\log_k(b)}$

Inequalities

Symbol	Meaning	Example
$>$	Greater Than	$(X + 3) > 2$, for any X
$<$	Less Than	$(7X) < 28$, $X= \{\ldots, -2, -1, 0 , 1 , 2 , 3\}$
$\geq$	Greater Than or Equal	$5 \geq (X - 1)$, $X= \{\ldots, -2, -1, 0, 1, \ldots, 5 ,6\}$
$\leq$	Less Than or Equal	$(2Y + 1) \leq 7$, $Y= \{\ldots, -2, -1, 0, 1, 2, 3\}$

- Let a, b, c be real numbers,

  - If $a > b$ and $b > c$ then $a > c$. (Transitivity)
  - If $a > b$ and $c > 0$ then $a*c > b*c$.
  - If $a > b$ and $c < 0$ then $a*c < b*c$.