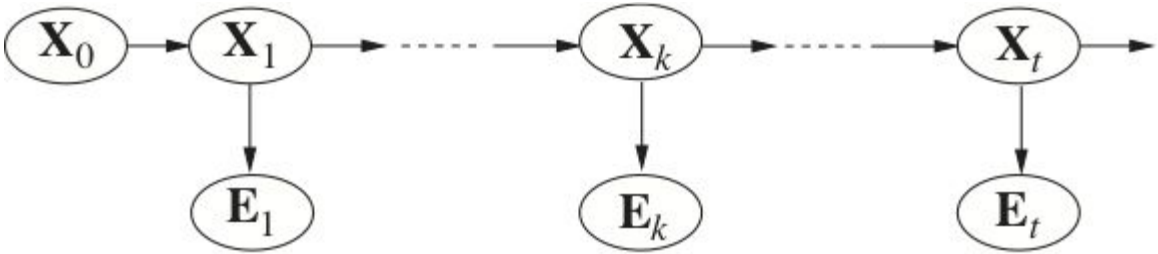
# Smoothing

CS 3600
Intro to Artificial Intelligence

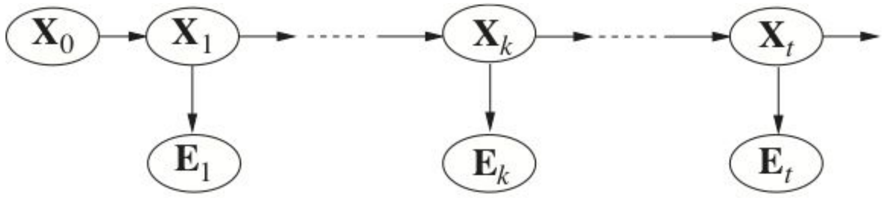
# Smoothing

What if we want to know the probability of the state variable at a given point in time in the past?




If we want to know about $X_k$ and we have evidence from $E_{1:t}$, we should incorporate that (rather than just using $E_{1:k}$)

# Forward-Backward (1)




The new probability we care about is $p(X_k|e_{1:t})$ which we can split into two pieces

$$
\begin{aligned}
p(X_k \mid e_{1:t}) &= p(X_k \mid e_{1:k}, e_{k+1:t}) \\
&= \alpha \cdot p(e_{k+1:t} \mid X_k, e_{1:k}) \cdot p(X_k \mid e_{1:k}) \\
&= \alpha \cdot p(e_{k+1:t} \mid X_k) \cdot \underline{p(X_k \mid e_{1:k})}
\end{aligned}
$$

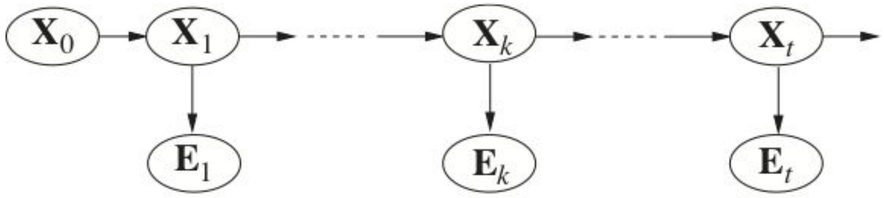
Bayes rule

Conditional Independence

Let's take a closer look at this term

We already know how to compute this!

# Forward-Backward (2)




$$p(e_{k+1:t} \mid X_k) = \sum_h p(e_{k+1:t}, X_{k+1} = h \mid X_k)$$

Marginalize out $X_{k+1}$

Definition of conditional prob

$$= \sum_h p(e_{k+1:t} \mid X_k, X_{k+1} = h) \cdot p(X_{k+1} = h \mid X_k)$$

Cond. Indep.

$$= \sum_h p(e_{k+1:t} \mid X_{k+1} = h) \cdot p(X_{k+1} = h \mid X_k)$$

Split $e_{k+1:t}$ into $e_{k+1}$ and $e_{k+2:t}$

$$= \sum_h p(e_{k+1}, e_{k+2:t} \mid X_{k+1} = h) \cdot p(X_{k+1} = h \mid X_k)$$

Cond. Indep.

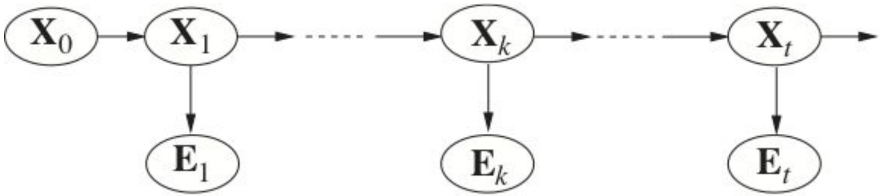
$$= \sum_h \underbrace{p(e_{k+1} \mid X_{k+1} = h)}_{\text{Sensor Model}} \cdot \underbrace{p(e_{k+2:t} \mid X_{k+1} = h)}_{\text{Recurrence!}} \cdot \underbrace{p(X_{k+1} = h \mid X_k)}_{\text{Transition Model}}$$

**Sensor Model**

Recurrence!

**Transition Model**

# Forward-Backward (3)




So our equation for smoothing is

$$p(X_k \mid e_{1:t}) = \alpha \cdot p(X_k \mid e_{1:k}) \cdot p(e_{k+1:t} \mid X_k)$$
$$= \alpha \cdot \mathbf{f}_{1:k} \odot \mathbf{b}_{k+1:t}$$

Where **f** is the “forward” variable

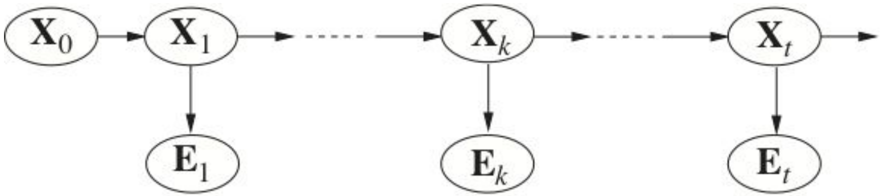
$$\mathbf{f}_{1:t} = p(X_t \mid e_{1:t})$$
$$= \alpha \cdot p(e_t \mid X_t) \sum_h p(X_t \mid X_{t-1} = h) \cdot p(X_{t-1} = h \mid e_{1:t-1})$$

And **b** is the “backward” variable

$$\mathbf{b}_{k+1:t} = p(e_{k+1:t} \mid X_k)$$
$$= \sum_h p(e_{k+1} \mid X_{k+1} = h) \cdot p(e_{k+2:t} \mid X_{k+1} = h) \cdot p(X_{k+1} = h \mid X_k)$$

Base case: $\mathbf{b}_{t+1:t}$ = p( $e_{t+1:t}|X_t$) = **1**

# Forward-Backward (4)




So we have an equation for how to smooth a sequence of evidence for a **single** state, how do we do this for **all** the states?

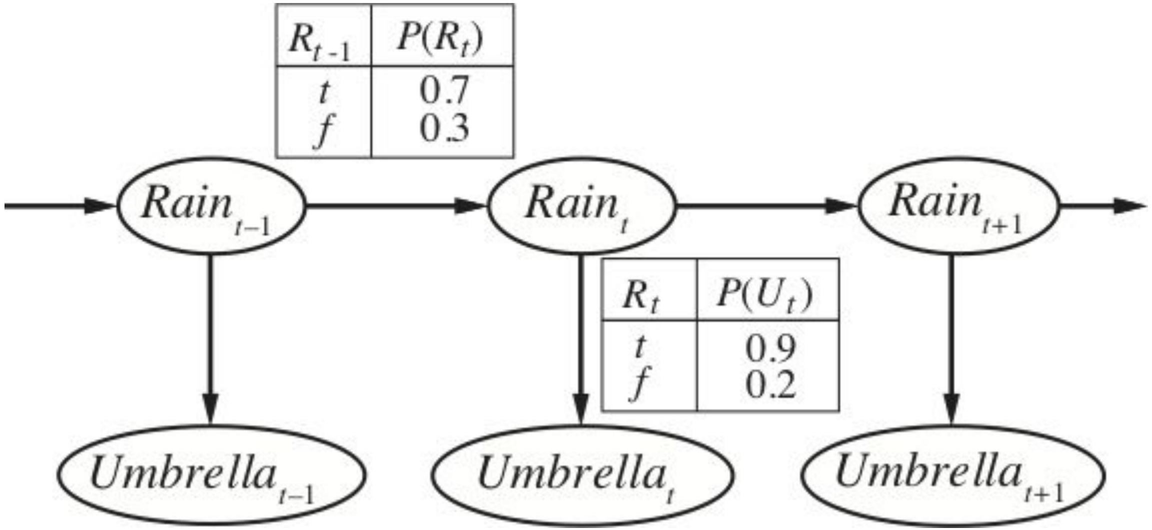
$$p(X_k \mid e_{1:t}) = \alpha \cdot p(X_k \mid e_{1:k}) \cdot p(e_{k+1:t} \mid X_k)$$
$$= \alpha \cdot \mathbf{f}_{1:k} \odot \mathbf{b}_{k+1:t}$$

```
def forward_backward(sensor_m, transition_m, prior, evidence):
      fv[0] = prior
      b = numpy.ones(len(prior))
      for i in range(1,t+1):
            fv[i] = forward(fv[i-1],evidence[i],sensor_m,transition_m)
      for i in range(t,0,-1):
            smoothed[i] = normalize(fv[i]*b)
            b = backward(b,evidence[i],sensor_m,transition_m)
      return smoothed
```

Key Idea: save the forward pass computations for use during the backward pass

# Weather example (1)

Example problem: a security guard would like to know about the weather. They can see people entering/leaving with umbrellas, but can't see directly whether it's raining or not.



# Weather example (2)

**Observations**: ($U_1$=True, $U_2$=True)

**Forward pass**

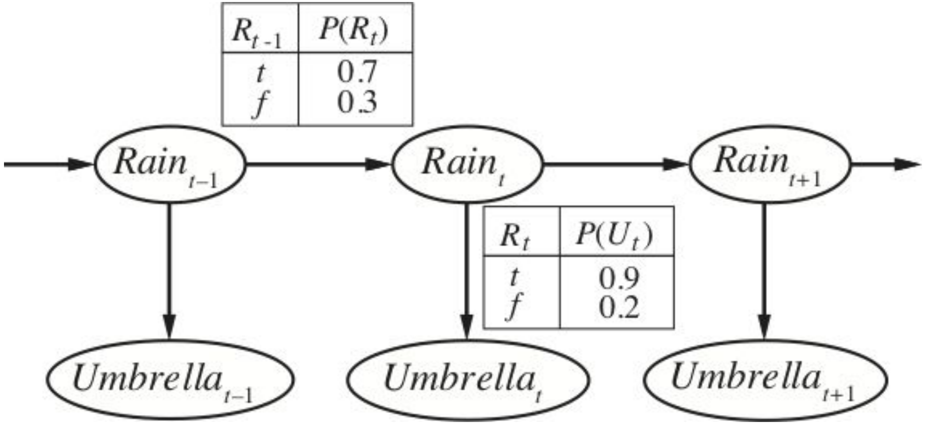
$\mathbf{f}_{1:0} = p(R_0) = \langle 0.5, 0.5\rangle$

$\mathbf{f}_{1:1} = \alpha \langle 0.9,0.2\rangle*(\langle 0.7,0.3\rangle*0.5 + \langle 0.3,0.7\rangle*0.5)$

$= \alpha \langle 0.45,0.1\rangle = \langle 0.818, 0.182\rangle$

$\mathbf{f}_{1:2} = \alpha \langle 0.9,0.2\rangle*(\langle 0.7,0.3\rangle*.818+\langle 0.3,0.7\rangle*.182)$

$= \alpha \langle 0.565, 0.075\rangle = \langle 0.883, 0.117\rangle$




**Backward pass**

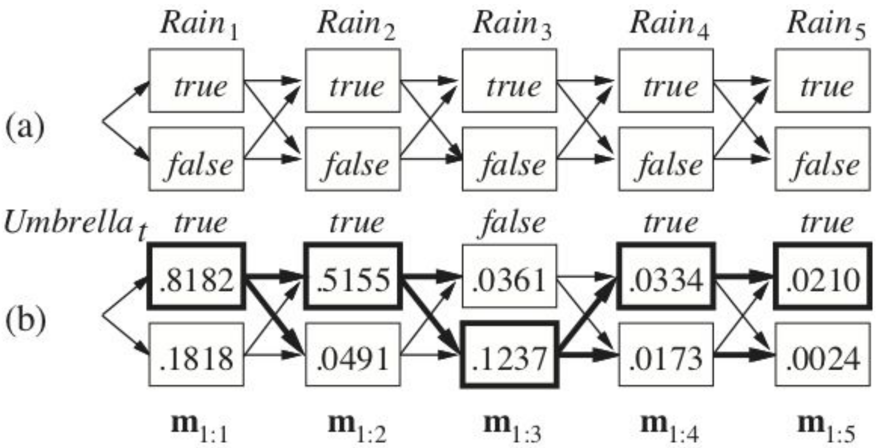
$\mathbf{b}_{3:2}=1$

$\mathbf{b}_{2:2}=(.9*1*\langle .7,.3\rangle + .2*1*\langle .3,.7\rangle) = \langle 0.69,0.41\rangle$, **smoothed** = $\langle 0.927,0.073\rangle$

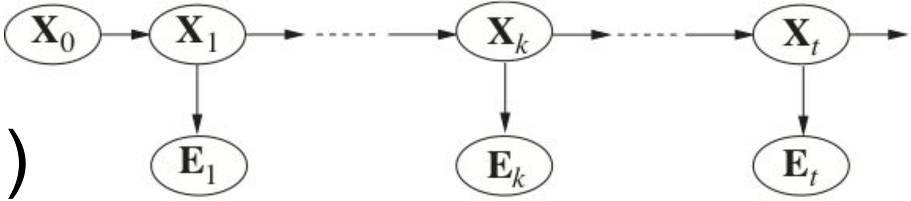
$\mathbf{b}_{1:2}=(.9*.69*\langle .7,.3\rangle + .2*.41*\langle .3,.7\rangle) = \langle .459,.243\rangle$, **smoothed** = $\langle 0.894,0.106\rangle$

# The most likely sequence (1)

What is the most likely sequence of states?



# The most likely sequence (2)




Rephrase: what is the **probability** of the last state $X_t$ in the most likely sequence?

Bayes' Rule on $e_t$ and $x_{1:t-1}$, $X_t$

$$\max_{x_{1:t-1}} p(x_{1:t-1}, X_t \mid e_{1:t}) = \max_{x_{1:t-1}} \alpha \cdot p(e_t \mid x_{1:t-1}, X_t, e_{1:t-1}) \cdot p(x_{1:t-1}, X_t \mid e_{1:t-1})$$

Sensor Markov

$$= \max_{x_{1:t-1}} \alpha \cdot p(e_t \mid X_t) \cdot p(x_{1:t-1}, X_t \mid e_{1:t-1})$$

Product rule on $x_{1:t-1}$, $X_t$

$$= \max_{x_{1:t-1}} \alpha \cdot p(e_t \mid X_t) \cdot p(X_t \mid x_{1:t-1}, e_{1:t-1}) \cdot p(x_{1:t-1} \mid e_{1:t-1})$$

Transition Markov

$$= \max_{x_{1:t-1}} \alpha \cdot p(e_t \mid X_t) \cdot p(X_t \mid x_{t-1}) \cdot p(x_{1:t-1} \mid e_{1:t-1})$$

$$= \alpha \cdot \underbrace{p(e_t \mid X_t)}_{\text{Sensor Model}} \cdot \max_{x_{t-1}} \underbrace{p(X_t \mid x_{t-1})}_{\text{Transition Model}} \cdot \underbrace{\max_{x_{1:t-2}} p(x_{1:t-2}, X_{t-1} = x_{t-1} \mid e_{1:t-1})}_{\text{Recurrence!}}$$

# The most likely sequence (3)

Define the **max** variable

$$\begin{aligned}\mathbf{m}_{1:t} &= \max_{x_{1:t-1}} p(x_{1:t-1}, X_t \mid e_{1:t}) \\ &= \alpha \cdot p(e_t \mid X_t) \cdot \max_{x_{t-1}} p(X_t \mid x_{t-1}) \odot \mathbf{m}_{1:t-1}\end{aligned}$$

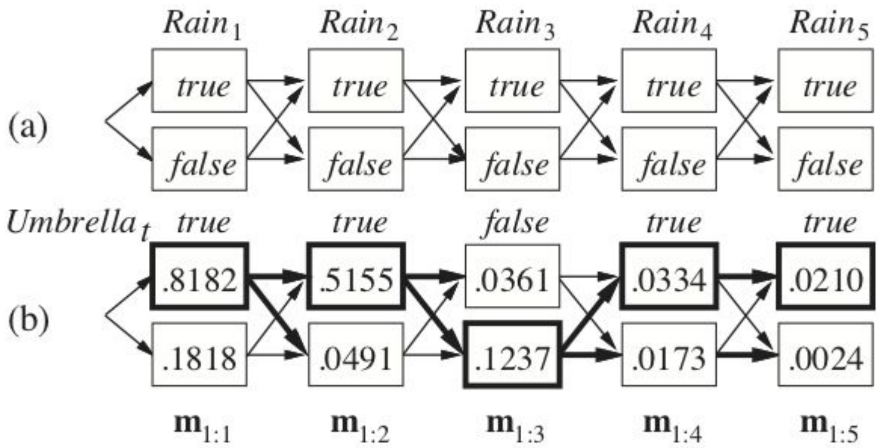
Compare with the **forward** variable

$$\begin{aligned}\mathbf{f}_{1:t} &= p(X_t \mid e_{1:t}) \\ &= \alpha \cdot p(e_t \mid X_t) \sum_h p(X_t \mid X_{t-1} = h) \odot \mathbf{f}_{1:t-1}\end{aligned}$$

Swapped **sum** for **max**

# The Viterbi algorithm

1. Init with $\mathbf{m}_{1:0} = p(X_0)$ (prior)
2. For each i in 1:t
   a. Compute $\mathbf{m}_{1:i}$
   b. Store the best state that leads to $X_i$ (bold arrows)
3. max($\mathbf{m}_{1:t}$) is the probability of the most likely sequence
4. The actual sequence can be recovered by following backpointers from the most likely final state



# Filtering, smoothing, Viterbi

**Exact Filtering**

$\mathbf{f}_{1:T}$: space O(|S|), time O(|S|*T), Online

**Smoothing** (forward-backward)

$\mathbf{f}_{1:T}$: space O(|S|*T), time O(|S|*T)
$\mathbf{b}_{1:T}$: space O(|S|), time O(|S|*T)
Offline (fixed-lag smoothing online version)

**Most Likely Sequence** (Viterbi)

$\mathbf{m}_{1:T}$: space O(|S|*T), time O(|S|*T), Offline

# Summary and preview

Wrapping up

- Two more inference algorithms: Smoothing, and Viterbi
- All of these inference algorithms can be modified to work with Bayes nets with different structures
- Additionally, for some Bayes nets, we can actually **learn** the parameters given sequences of observations (Expectation-Maximization)

Next time:

- Help for project 3