Simulation of Multivariate Hawks Process

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June 8, 2019

1 Introduction

Multivariate Hawkes process is a multi-dimensional extension of the univariate case. Apart from the self-excitation feature, it also has the mutual-exciting between events in different dimensions. In this experiment, we utilize the thinning algorithm proposed by to simulate an inhomogeneous multivariate Hawks processes. We also explore the effects of different rates of exponential decay. Next we will first introduce the principle of our experiments and then illustrate our experiment results.

2 Multivariant Hawks Process

Let $N(t) = (N^1(t), N^2(t), \ldots, N^M(t))$ be a multivariante point process, it satisfies:

- for $m = 0$, $N(0) = 0$
- for $m = 1, 2, \ldots, m$:
  1. $\lambda^m(t)$ is a left-continuous stochastic process given by Stieltjes integral:

$$\begin{aligned} \lambda^m(t) &= \mu_m + \sum_{n=1}^{M} \int \alpha_{mn} e^{-\beta_{mn}(t-s)} dN^n(s) \\ &= \mu_m + \sum_{n=1}^{M} \sum_{\{k: t_k^n < t\}} \alpha_{mn} e^{-\beta_{mn}(t-t_k^n)} \end{aligned} \tag{1}$$

     where $\mu_m > 0, \alpha_{mn} \geq 0$ and $\beta_{mn} \geq 0 \ for \ m, n = 1, 2, \ldots, M$
  2. $\lambda^m(t)$ is the stochastic intensity of the marginal point process $N^m(t)$ indipendently for each m.

$$P\{N^m(t+h) - N^m(t) = 1 | F_{t^-}^N\} = \lambda^m(t)h + o(h)$$

  3. The point process is orderly:

$$P\{N(t+h) - N(t) \geq 2 | F_{t^-}^N\} = o(h)$$

     where $F_{t^-}^N$ is the natural filtration of the process.

In this experiment, the exponential decay β_{mn} is given by a constant value across all dimensions. i.e. $\beta_{mn} = w \ for \ m, n \ in \ 1, 2, \ldots M$

3 Thinning Algorithm

Thinning Algorithm was originally proposed by Lewis and Shedler for the simulation of non-homogeneous Poisson processes. The method is applicable for any rate function and is based on controlled deletion of points in a Poisson process whose rate function dominates the given rate function. Ogata applied the Thinning Algorithm onto the intensity functions of Hawks Processes.

The core of thinning is that the interarrival times are independent identical random variables that obeys exponential distribution. The we can sample the arrival time according to an exponential distribution. From the paper it demonstrates the following preposition:

Let the point process N be an inhomogenous Poisson process with intensity function $\lambda(t)$, then $N(t)$ follows a Poisson distribution with parameter $\int_0^t \lambda(s)ds$. i.e.

$$P\{N(t) = n\} = \frac{e^{-\int_0^t \lambda(s)ds}(\int_0^t \lambda(s)ds)^n}{n!} \tag{2}$$

Therefore, Let $n = 0$ and the number of arrivals in the interval $[a, b]$ can be expressed more precisely:

$$P\{N(a, b] = 0\} = P\{N(b) - N(a) = 0\} = e^{-\int_a^b \lambda(s)ds} \tag{3}$$

We set the rate of homogenous Poisson process $\hat{\lambda} = sup\{\lambda(t)\}$ which dominates $\lambda(t)$. A larger $\hat{\lambda}$ leads to more generated points and correspond to a smaller probability of point acceptance. We can sample the interval of arrival time from a uniform distribution by computing the inverse of the CDF of exponential distribution with rate parameter $\hat{\lambda}$.

$$\begin{aligned} y &= F(x) = 1 - e^{-\lambda x} \\ x &= F^{-1}(y) = -\frac{ln(1-y)}{\hat{\lambda}} \end{aligned} \tag{4}$$

Since y is uniform distribution on $[0, 1]$ and so $1 - y$ is also uniform distribution on $[0, 1]$. Then (4) can be simplified to $x = -ln(y)/\hat{\lambda}$.

4 Simulation on Hawks Process

The simulation of a multivariate Hawks Process with exponential decay on a fixed interval can be summarized as follow. It first decides next event arrival time t, then performs *reject sampling* to decide whether it is accepted by which dimension or is rejected.

4.1 Pseudo Code

The procedure of generating points from given parameters of a Hawks Process can be summarized as follow.

1. Set the set of accepted arrival time for each dimension $T^1 = T^2 = \cdots = T^M = \emptyset$, $t = 0$
2. Repeat until $t > T$:
   (a) Set $\hat{\lambda} = \sum_{m=1}^{M}(\mu_m + \sum_{\tau \in T^n} \alpha_{mn} e^{-w(t-\tau)})$
   (b) Sample y from uniform distribution on $[0, 1]$
   (c) Set the new arrival time as $t = t - ln(y)/\hat{\lambda}$
   (d) Sample u from uniform distribution on $[0, 1]$.
      i. If $u > \sum_{m=1}^{M} \lambda^m(t)/\hat{\lambda}$: Then Reject.
      ii. Else
         A. find the smallest m such that
$$\sum_{m=1}^{k-1} \lambda^m(t)/\hat{\lambda} < u \leq \sum_{m=1}^{k} \lambda^m(t)/\hat{\lambda}$$
         B. Set $T^m = T^m \cup \{t\}$

$T^m\ for\ m \in 1, 2, \ldots, M$ contain the simulated points.

4.2 Code

The experiment code(simulation.py) has been uploaded to FTP. Some figures are also attached.

5 Experiment

By setting the Infective Matrix A and exponential decay $w = 0.6$ as given by homework sheet. We simulate a 10-dim Multivariate Hawks Process by thinning algorithm mentioned above. Here we show some figures during simulation.

Setting1 : Z=10, w=0.6, #sampled points=20

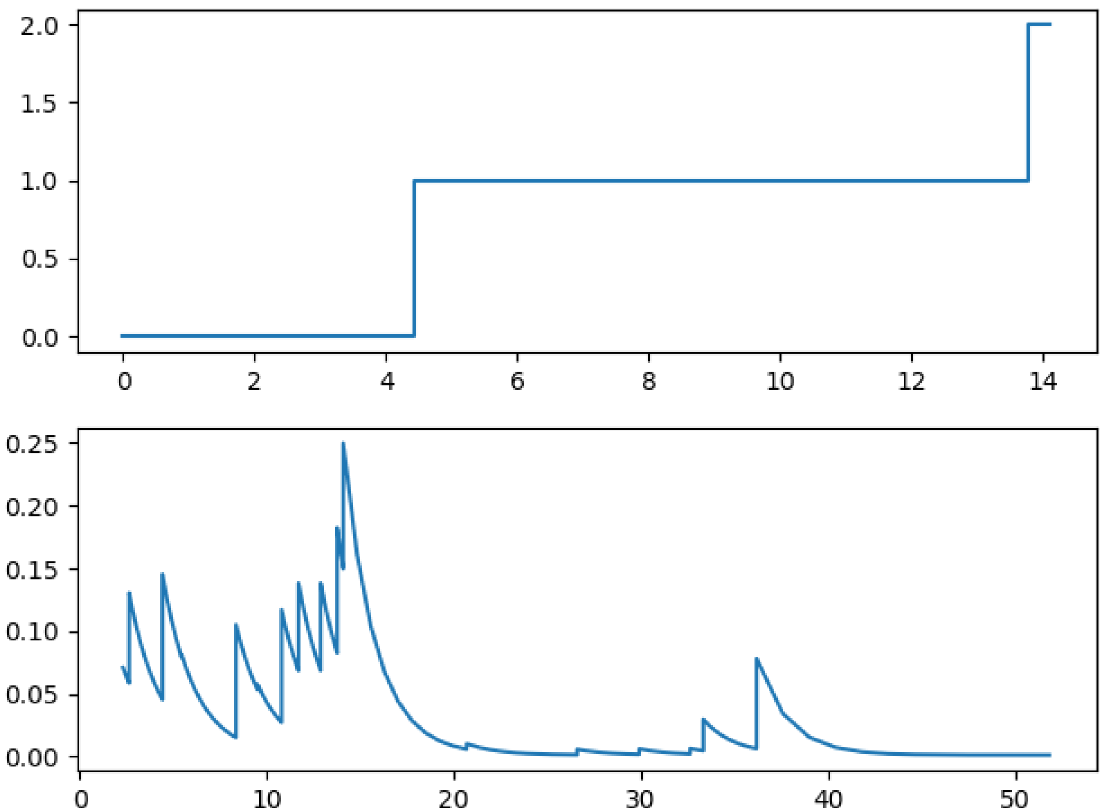

Figure 1: Counting Process $N(t)$(Above) and Intensity Function $\lambda^0(t)$ over time(Bottom) in dimension 0. After Each occurrence, the intensity jumps and then starts to decay. There are also some intensity jumps that does not corresponding to any occurrence in dim-0 because of mutual-exciting by other dimensions.

5.1 Influence of exponential decay

We change the value of $w = 0.6$ *and* 0.01 and find interesting results.

Setting 2 : Z=10, w=0.6, **#sampled points=100** (**More sampled points**)

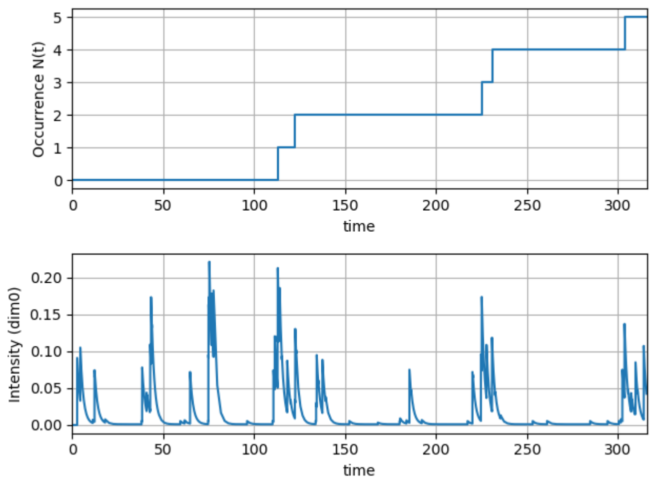

Figure 2: Zoom out. Sampling more points.

Setting3 : Z=10, **w=0.01**, #sampled points=100 (**Smaller Decay Rate**)

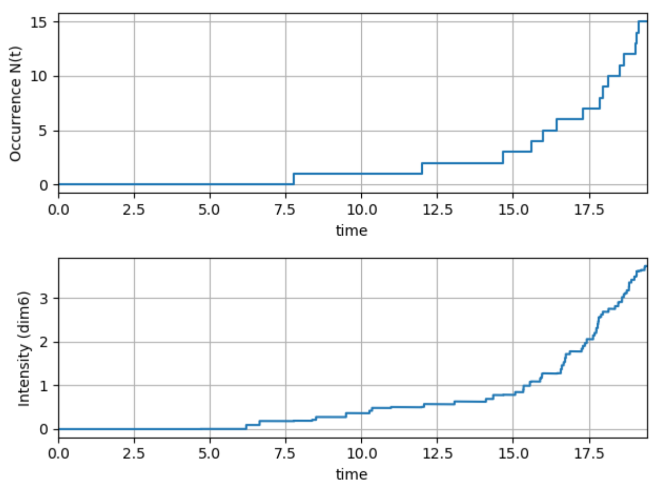

Figure 3: Notice that when the exponential decay factor w becomes small, the time interval of two adjacent events becomes short. The frequency of mutual-exciting has exceeded the decay rate, which leads to an intensity explosion.

6 Appendix

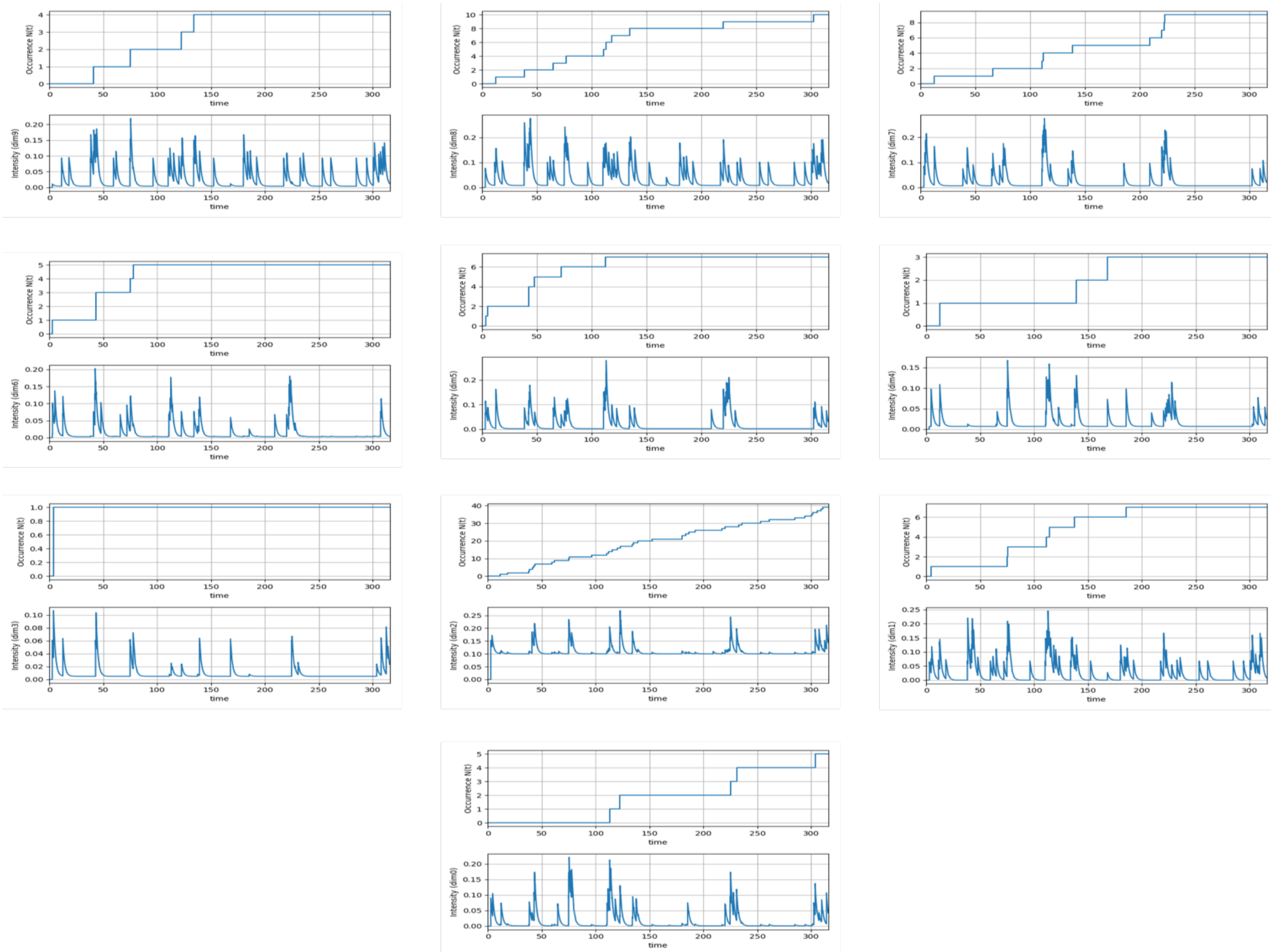

Figure 4: Z=10, w=0.01, #sampled points=100

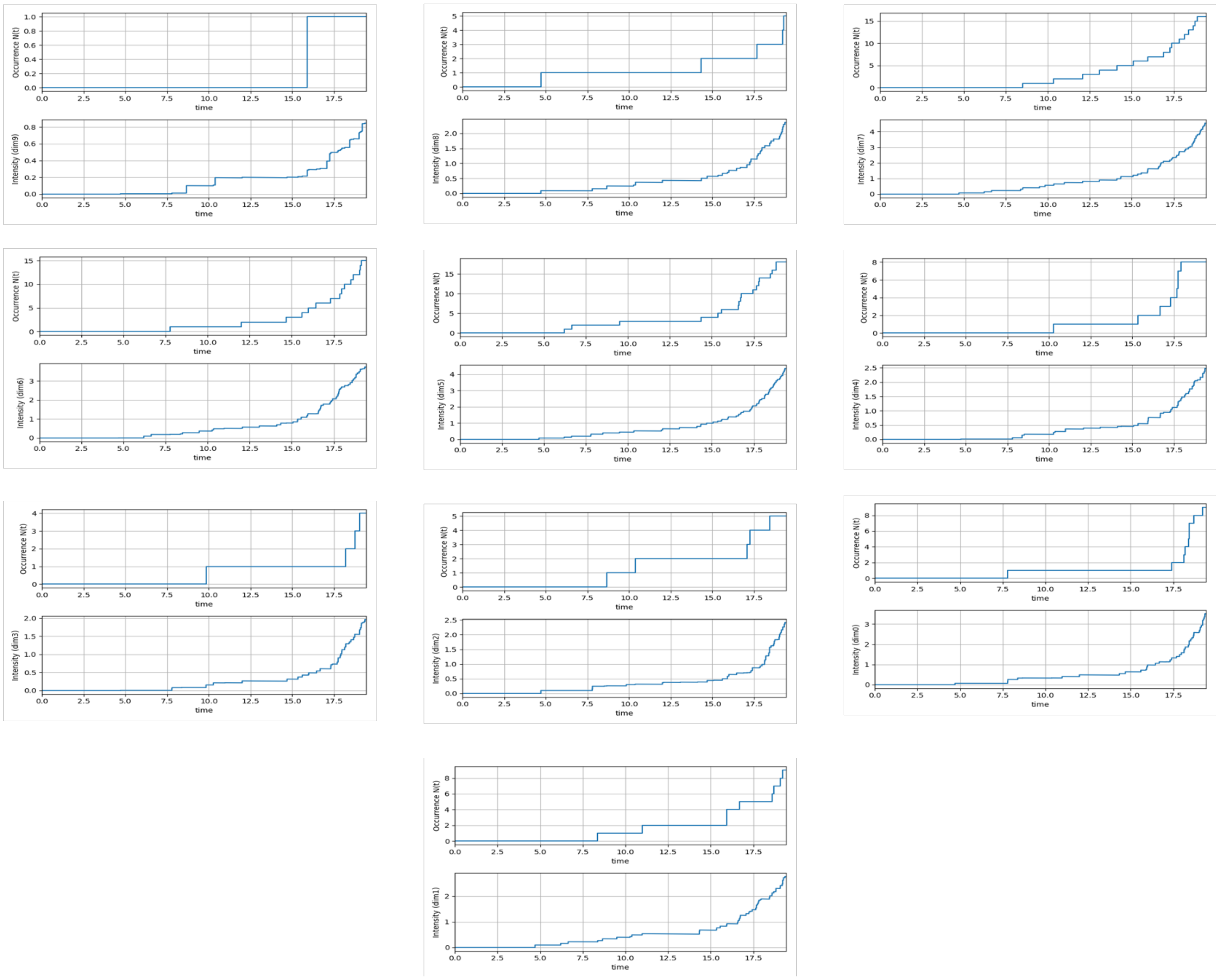

Figure 5: Z=10, w=0.6, #sampled points=100

6.1 How to run?

Please refer to **Readme.md** for more information about running the code.