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T1 ________________	**49227**	F1 ________________
T2 ________________		F2 ________________
T3 ________________	Problem Chosen	F3 ________________
T4 ________________	**A**	F4 ________________

2016
MCM/ICM
Summary Sheet

(Your team's summary should be included as the first page of your electronic submission.)

Type a summary of your results on this page. Do not include the name of your school, advisor, or team members on this page.

Have a nice hot bath!

Summary

According to Fourier Law and Heat conduction equation, we build the Convective Heat Transfer Model, which analyzes the heat distribution of the bathtub of different shapes, thus successfully explaining the phenomenon of the unevenness of the temperature distribution throughout the bath water. In this paper, we also figure out the problem of heat dissipation affected by different shapes of bathtub and determine the best shape design to keep the temperature as close as possible to the initial temperature, based on which we further discuss the model of the most economical water consumption. Taking the person's behavior into consideration, we improve the shape parameters of the bathtub and determine the final optimized bathtub model. As for the bubble bath additive, we qualitatively analyze this situation and by changing the relevant parameters in previous models, we get a satisfying explanation that the bubbles help to maintain the temperature of bath water.

In the first model, we research the bathtub shape of rectangle，ellipse and rounded rectangle. When considering the temperature difference between bath water and air as well as the inflow of hot water, we specify the certain Dirichlet boundary condition and Neumann boundary condition. Eventually, we get the figures of temperature change in time and space through Matlab, which exactly explains the situation in our real life. We also draw a conclusion that the shape of ellipse tends to achieve a more even temperature distribution.

In the second model, we owe the heat dissipation to the water evaporation and heat conduction through bathtub walls. Through Dalton Formula and Fourier Law ,we get the heat loss rate through evaporation and the heat loss rate by heat conduction, respectively. Next, we build a nonlinear programming model of the minimum heat loss and analyze different shapes of the bathtub like cuboid, rounded edges column, semi-cylinder, inverted prismoid. We find that the shape of inverted prismoid can achieve the goal of the least heat loss.

In the third model, we set up the optimization model of water consumption. Firstly we study the variation of time-dependent temperature with hot water inflow. Based on the results, we determine the best starting time of hot water inflow and the most economical water consumption through a nonlinear programming model. Considering the conclusion in the model 2,we improve some parameters in the inverted prismoid. We conclude the person can adopt the strategy that he had better begin adding hot water after 1435s from the time when the bathtub is filled with water. On the premise that the shape of bathtub is a inverted prismoid, this strategy helps to achieve the least temperature decline and water consumption.

Finally, the sensitivity analysis and model evaluation are performed. During sensitivity analysis, we validate that our model is robust and correct. During model evaluation, strengths and weaknesses are analyzed for different models, and we compare the results in detail.

Contents

1 Introduction

1.1 Problem Preview

We all enjoy cleansing and relaxing in a bathtub filled with hot water. But if the bathtub cannot be secondarily heated and is not equipped with circulating jets, how can we keep the temperature even throughout the bathtub and as close as possible to the initial temperature without wasting too much water with just a constant trickle of hot water from the faucet to reheat the bathing water?
To solve the problem, we develop three models to determine the shape and volume of the tub, the shape/volume/temperature of the person in the bathtub, and the motions made by the person in the bathtub. We also develop a basic model to analyze the situation of bubble bath.

1.2 Notation

Table 1: Notation Table

Symbol	Physical meanings
q	thermal flux
λ	thermal conductivity
ρ	density of water
C	specific heat capcity
T	temperature
h	convective heat transfer heat
u, v, w	water flowing speed in x,y,z direction
e	air pressure at a certain height
P	pressure of water surface
$f(w)$	function of wind speed
E_m	evaporation of water
R_s	latent heat of evaporation
Φ	heat transfer rate
t	time
Q	power of heat loss
x_1, x_2, x_3	shape-determined parameters
V	volume

The physical meaning of symbol are shown in Table 1.

1.3 Assumption

Additionally, based on the condition of this problem, several general assumptions are stated here to simplify the real situation.

- The internal flowing speed of water is very slow which can be neglected.
- The bathtub can be seen as homoplasmon , so the boundary condition of it is consistent.
- The possible candidates of bathtub are regular shape like Cuboid, Semi-cylinder, Semi-cylinder with two quarter balls at each end,and Inverted prismoid.
- Heat distribution is mainly caused by heat convection from the side of bathtub and surface of water as well as heat conduction inside the bathtub.

2 Convective Heat Transfer Model

2.1 Model Introduction

To analyze the temperature distribution evenness of the water in bathtub, we build Convective Heat Transfer Model.
In the model, we consider the factors of heat loss through water surface and bathtub walls as well as heat inflow of constant trickle of hot water. The convection and diffusion of water are also discussed in the model.

2.1.1 Model Theory

When two objects at different temperature contact with each other, they will conduct heat, during which process, the heat conduction satisfies Fourier Law and Heat Conduction Equation,both of which are shown as follows.

$$q = -\lambda \cdot gradT \tag{1}$$

$$\lambda[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}]dxdydz = \frac{\partial T}{\partial t}\rho C dxdydz \tag{2}$$

Where q stands for thermal flux, stands for thermal conductivity, ρ is the density of water and C is specific heat capacity.

When it comes to water in bathtub, in addition to heat conduction, we should consider about the heat brought in and out of the infinitesimal element while water flowing.In Figure1,this is indicated. At moment t,on interface w, water of temperature T flows at the speed of u, so the quality of micro element of water dm_1 and heat flowing into an infinitesimal element per unit time q_1 can be written as

$$dm_1 = \rho u dydz \tag{3}$$

$$q_1 = \rho u T C dydz \tag{4}$$

On interface e, the speed and temperature of water are $u + \frac{\partial u}{\partial x}dx$, $T + \frac{\partial T}{\partial x}dx$,respectively,so the quality dm_2 of water micro element and heat flowing into an infinitesimal element per unit time q_2 can be written as

$$dm_2 = \rho[u + \frac{\partial u}{\partial x}dx]dydz \tag{5}$$

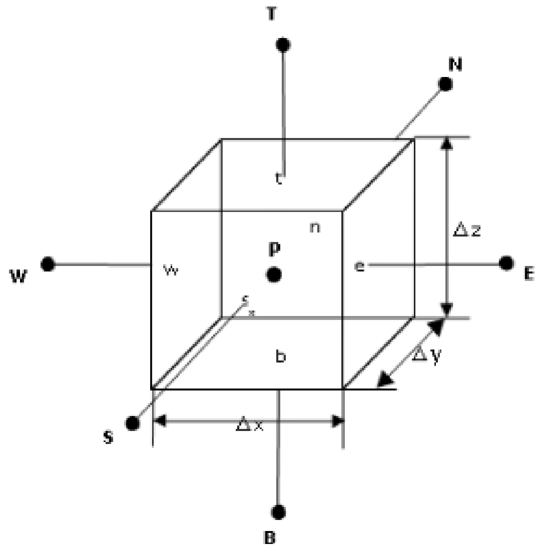

Figure 1: inflow and outflow on every surface of a differential unit

$$q_2 = \rho[u + \frac{\partial u}{\partial x}dx][T + \frac{\partial T}{\partial x}dx]Cdydz \tag{6}$$

If the speed varies in a relatively low level, we can let $\frac{\partial u}{\partial x} = 0$ .Then equation(7) can be written as

$$q_2 = \rho uTCdydz + \rho u\frac{\partial T}{\partial x}Cdxdydz \tag{7}$$

Therefore, the pure thermal stream flowing into an micro element in the x direction per unit time can be expressed as follows:

$$q_x = \rho u\frac{\partial T}{\partial x}Cdxdydz \tag{8}$$

In the y direction and z direction, the heat flux can be represented in a similar way,

$$q_y = \rho v\frac{\partial T}{\partial y}Cdxdydz \tag{9}$$

$$q_z = \rho w\frac{\partial T}{\partial z}Cdxdydz \tag{10}$$

Then, we can add equation (8),(9),(10) to the original heat conduction equation to improve the model.

$$\lambda[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}]dxdydz-[\rho Cu\frac{\partial T}{\partial x}dxdydz+\rho Cv\frac{\partial T}{\partial y}dxdydz+\rho Cw\frac{\partial T}{\partial z}dxdydz] = \frac{\partial T}{\partial t}\rho Cdxdydz \tag{11}$$

After simplification, we get Energy Equation:

$$\lambda[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}] = \frac{\partial T}{\partial t}\rho C + [\rho Cu\frac{\partial T}{\partial x} + \rho Cv\frac{\partial T}{\partial y} + \rho Cw\frac{\partial T}{\partial z}] \tag{12}$$

2.1.2 Model Modification

Equation (12) describe a situation that no internal heat source is involved. Nevertheless, in our model, we should take the heat radiation from water to atmosphere into concern, thus we make some correction and acquire a new formula:

$$-\lambda[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}]+\frac{\partial T}{\partial t}\rho C+[\rho C u\frac{\partial T}{\partial x}+\rho C v\frac{\partial T}{\partial y}+\rho C w\frac{\partial T}{\partial z}]=h(T_{ext}-T) \tag{13}$$

Where h stands for convective heat transfer coefficient, T_{ext} stands for the air temperature. When analyzing the temperature distribution of water in bathtub, we assume the internal speed of water is considerably slow, which can be neglected compared with the flowing speed caused by constant trickle of hot water, so we let $u = v = w = 0$, formula (13) changes into:

$$-\lambda[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}]+\frac{\partial T}{\partial t}\rho C=h(T_{ext}-T) \tag{14}$$

Since the unevenness of temperature distributed throughout the bathtub is mainly caused by heat dissipation into the air, as a result, we can neglect the temperature gradient in z direction, which means the three-dimensional bathtub can be seen as a two-dimensional model. In this way, we get our final equation:

$$-\lambda[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}]+\frac{\partial T}{\partial t}\rho C=h(T_{ext}-T) \tag{15}$$

And outside boundary condition (boundary of bathtub wall) should be set as below:

$$\lambda\frac{\partial T}{\partial \vec{n_1}}=hd(T_{ext}-T) \tag{16}$$

where d represents the height of bathtub.
To introduce the constant trickle of hot water into our model, we consider the edge of water inflow as round,the boundary condition of which can be set as formula (17).

$$T_{edge2}=T^{'} \tag{17}$$

Where $T^{'}$ stands for the temperature of constant trickle of hot water.

2.2 Parameters Setting

The parameters are set as in the following Table 2:

2.3 Model Solution

We use $Matlab\,2010b$ to solve the problem,the solutions are shown in Figure 2 ,Figure , and Figure 4 as above.

2.4 Solution Analysis

It is clearly seen in the figures that apart from the noticeably high temperature around the constant hot water drain, overall temperature distributes almost evenly throughout the bathtub,which exactly applies to our real life .Further information concluded from the figures is that though the water gets cooler gradually and slowly ,its temperature still remains above the room temperature.

Table 2: Parameter Set in Convective Heat Transfer Model

Physical Quantity	Value
λ	$0.599W/K$
C	$4.2 \times 10^3 J/kg \cdot K$
ρ	$1 \times 10^3 kg/m^3$
h	$600W/K \cdot m^3$
T_{ext}	$298K$
T'	$338K$

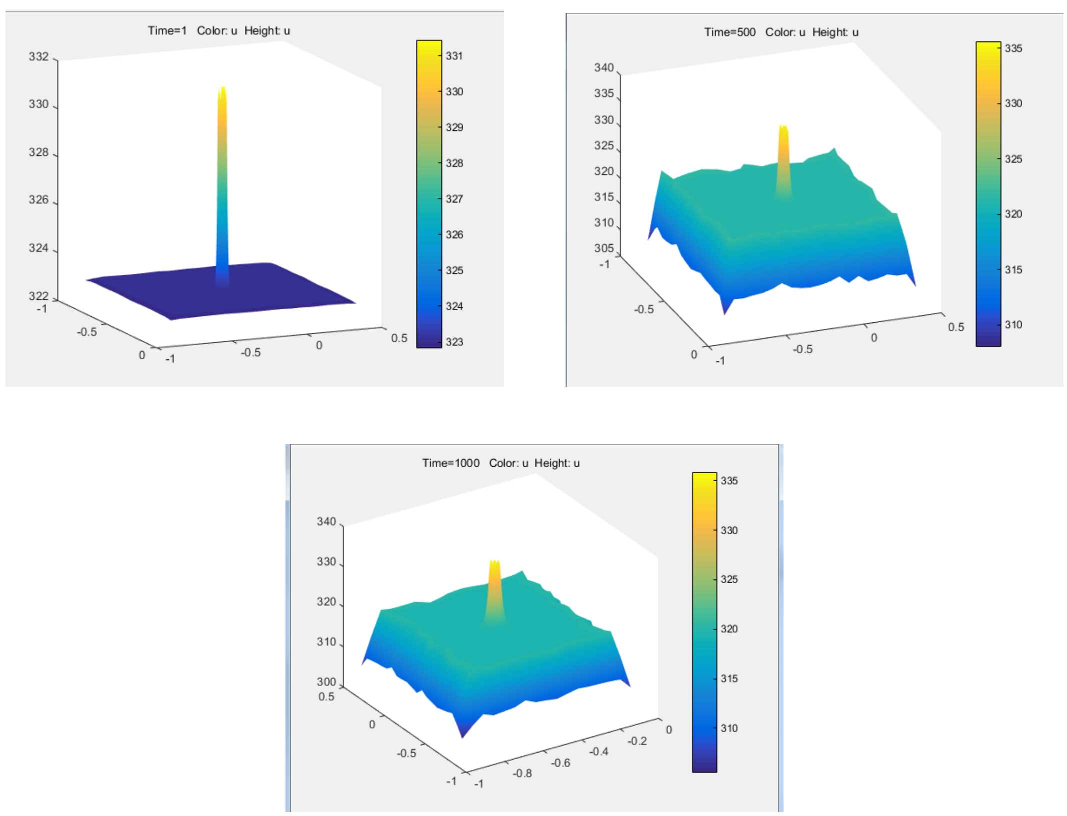

Figure 2: Temperature distribution of rectangle-surface-shaped bathtub varies with time

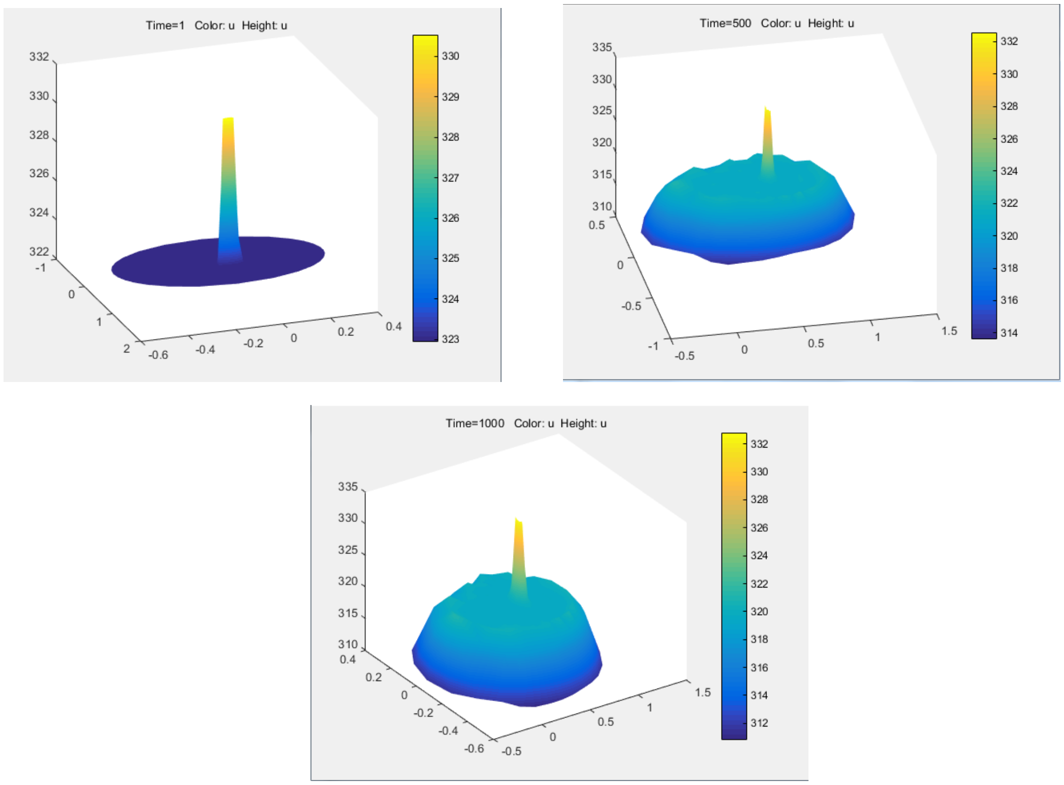

Figure 3: Temperature distribution of ellipse-surface-shaped bathtub varies with time

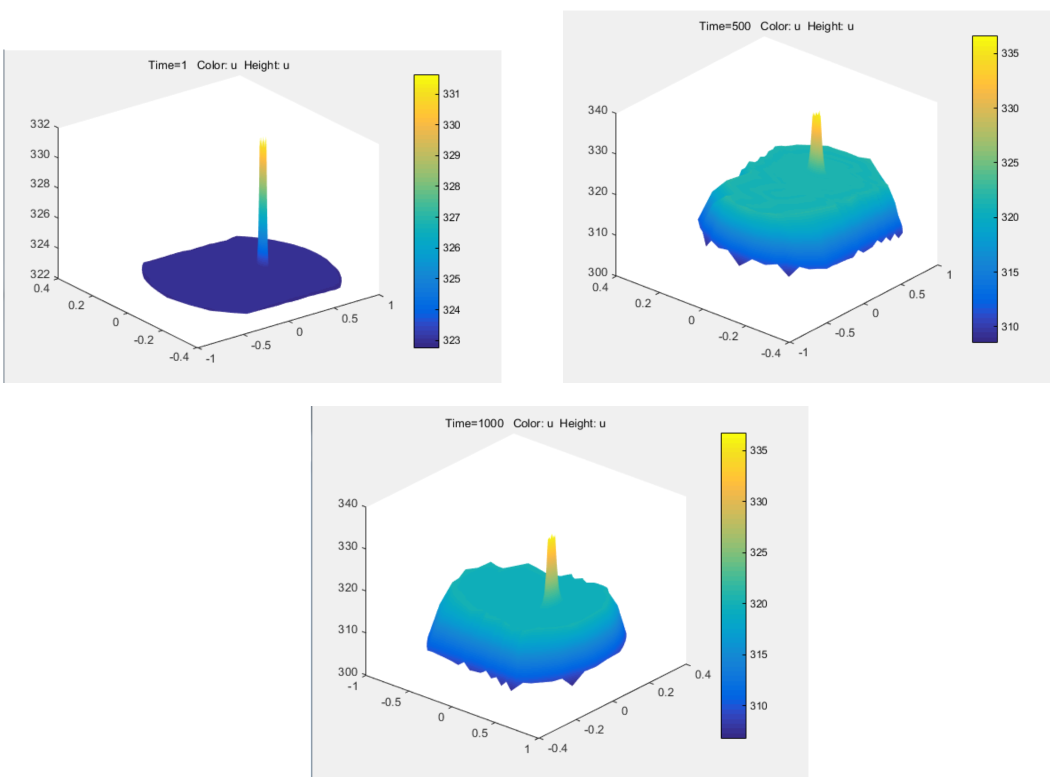

Figure 4: Temperature distribution of ellipse-surface-shaped bathtub varies with time

To show the evenness of temperature in a more obvious way,we calculate the temperature distribution variance of bathtub in different shapes.Variance can be calculated as follows:

$$\sigma^2 = \frac{\sum_{i=1}^{N}(T_i - \overline{T})}{N} \tag{18}$$

The final results are shown in Table3.

We can use the variance of temperature distribution to reflect the evenness.Apparently,the smaller variance is,the more even temperature distribution is.After analyzing the data,we draw a conclusion that ellipse-shaped bathtub is the best choice in keeping the evenness of temperature.

Table 3: temperature distribution variance of bathtub in different shapes

Bathtub surface Shape	Variance Value
$rectangle$	$31.3317K^2$
$ellipse$	$27.3059K^2$
$rounded\,rectangle$	$38.0605K^2$

3 Bathtub Optimization Design Model

In order to keep the temperature throughout the bathtub as close as possible to the initial temperature without wasting too much water, we must make efforts to lower the heat dissipation through evaporation as well as the heat conduction through bathtub walls.
To determine the water loss through evaporation, we have detailedly researched the Dalton formula which can effectively reveal the process. By searching references and analyzing the experimental data ,we eventually draw the conclusion that per hour,there is $0.08g$ water evaporates from one square centimeter of water surface. Since the evaporation latent heat of water under standard state is $2800J/K$.
We can easily conclude that the heat dissipation through evaporation from one square centimeter of water surface per hour is $224J$.
As for the heat conduction through bathtub walls, we get the heat flux which exactly quantifies the heat dissipation through this process by applying the classical Fourier law to our model.As we try to figure out the balance between economic water consumption and the least heat dissipation, we find that the shape of bathtub plays a vital important part.
In order to quantitatively analyze the relationship between different bathtub shapes and heat dissipation, assuming a certain capacity of bathtub, we establish a programming model combined some constraint conditions to determine the best shape of bathtub. In this model we explore and compare different bathtub shapes such as cuboid, rounded edges column, semi-cylinder and inverted prismoid.
Among all these shapes, we find that if the volume is certain, when the bathtub is designed in a inverted prismoid shape, the least heat dissipation can be achieved.

3.1 Dalton Formula and the heat dissipation through evaporation

Dalton Formula can be expressed as followed:

$$E = C(e_o - e_z)/P \tag{19}$$

In Equation(19),E represents the evaporation of water from the surface per unit time;C represents the coefficient associated with the wind speed;e_0 represents the saturation vapor pressure corresponding to surface water temperature; e_z represents the actual vapor pressure of air at a height of Z above the water surface;P represents the pressure of water surface.
since the poor accuracy of the evaporation observation, the effect of pressure on the water surface evaporation is negligible. So equation(19) can be written into equation(20).

$$E = C(e_o - e_z) \tag{20}$$

Equation(20) is academically acknowledged as the most famous Dalton formula. Usually, the coefficient C is regarded as the function of wind speed, so the coefficient C can be replaced by a function $f(w)$. So we get equation(21).

$$E = C(e_o - e_z) \cdot f(w) \tag{21}$$

$f(w)$ must meet the following criteria:

- Regardless of wind speed, $\frac{df(w)}{dw} > 0$
- When the wind speed is slow,$\frac{d^2f(w)}{dw^2} > 0$;when the wind speed is moderate,$\frac{d^2f(w)}{dw^2} = 0$;when the wind speed is high, $\frac{d^2f(w)}{dw^2} < 0$.

We can use a piecewise function to express $f(w)$

$$f(w) = \begin{cases} a + bw^{\alpha_1} & w \le w_1, \alpha_1 > 1 \\ c + dw & w_1 < w < w_2 \\ e + fw^{\alpha_2} & w \ge w_2, \alpha_2 < 1 \end{cases} \tag{22}$$

We define $f(w)$ of interval $w \le w_1$ as $f_1(w)$,$f(w)$ of interval $w_1 < w < w_2$ as $f_2(w)$,$f(w)$ of interval $w \ge w_2$ as $f_3(w)$. By curve fitting,we can get the final equation as follows,

$$f(w) = \begin{cases} 0.21 + 0.055w^{1.25} & w \le w_1, \alpha_1 > 1 \\ 0.18 + 0.085w & w_1 < w < w_2 \\ 0.149w^{1-0.00612(w-4)^{0.5}} - 0.076 & w \ge w_2, \alpha_2 < 1 \end{cases} \tag{23}$$

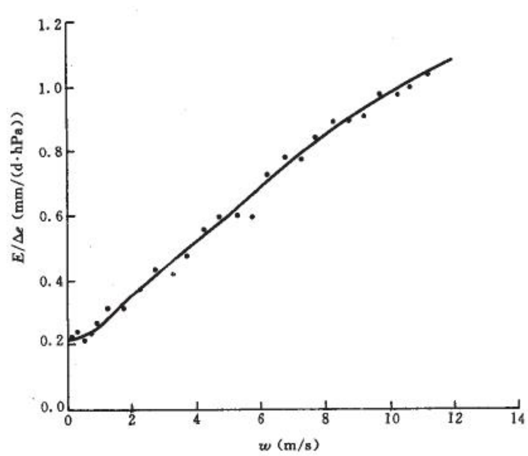

Figure 5: $E/(e_0 - e_z) \sim w$ the distribution of related point group

According to empirical study and experimental data, we eventually come to the conclusion that there is about $0.08g$ water evaporated per hour from one square centimeter of water surface.The evaporation of water E_m is $0.15g/h \cdot cm^2$,while the latent heat of evaporation of water under standard state R_s is $2800J/g$,so the heat loss through heat conduction is:

$$q_e = E_m \times R_s = 420J/h \cdot cm^2 = 1166W/m^2 \tag{24}$$

3.2 Fourier Law and the heat dissipation through heat conduction

It is apparent that there is heat dissipation caused by heat conduction through the bathtub walls. We can use Fourier law to quantitatively specify the heat loss. When we only consider the one-dimensional heat conduction,that is temperature changes in the x direction only.According to Fourier law,we can get the following formula.

$$\Phi = -\lambda A \frac{dT}{dx} \tag{25}$$

Where Φ represents the heat transfer rate,and its unit is W , λ represents the thermal conductivity,and its unit is $(W/m \cdot K)$.Negative sign indicates opposite direction of heat transfer and temperature rise direction.

$$q = \frac{\Phi}{A} = -\lambda \frac{dT}{dx} \tag{26}$$

q represents the heat flux,which means heat flow per unit time through unit area,and its unit is W/m^2 .
Assuming that δ represents the thickness of the plate,the surface temperature on both sides is maintained at t_{w1} and t_{w2} ,respectively.Under these conditions,we can calculate the value of q.
From equation(26),thus

$$q \int_0^{\delta} dx = -\lambda \int_{t_{w1}}^{t_{w2}} \frac{dT}{dx} dx \tag{27}$$

$$q = \frac{-\lambda(t_{w2} - t_{w1})}{\delta} = \frac{\lambda(t_{w1} - t_{w2})}{\delta} \tag{28}$$

By looking up the table of thermal conductivity of different materials,we find that the thermal conductivity of ceramic tile is about $0.5W/m \cdot K$. We assume that t_{w1} is the temperature of inner bathtub wall,and t_{w1} is about $323K$.Also we assume that t_{w2} is the temperature of outer bathtub wall,and t_{w2} is about $298K$.δ represents the thickness of bathtub wall and its numerical value is $0.1m$.Thus we can get the heat flux q_c . And q_c is about $125W/m^2$.

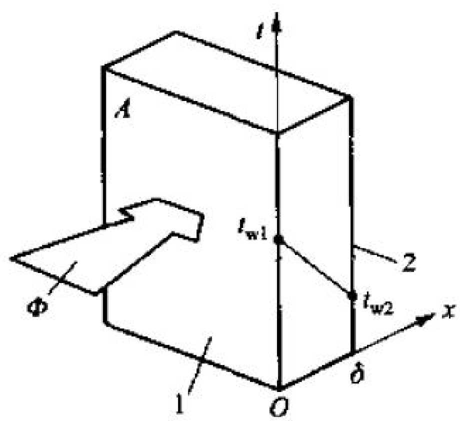

Figure 6: one-dimensional heat conduction through the tablet

3.3 The Optimization of different shapes of bathtub Model

In part 3.2,We determined the heat dissipation by evaporation q_e and heat conduction q_c through bathtub walls .

$$q_e = 1120W/m^2$$

$$q_c = 100W/m^2$$

Now we try to minimize the total heat loss per unit time. Assuming the volume of bathtub is certain, we set constraint conditions including the appropriate length, width and height, etc , which can ensure a reasonable bathtub for a person to use. After searching relevant references about the size of different bathtubs in the market, we approximately determine a scope of these constraint conditions.

Then we set up a minimum heat loss nonlinear programming model as follows:

$$\min Q(S(x_1, x_2, x_3, ...), V)$$

$$s.t. \begin{cases} x_1 \in X_1 \\ x_1 \in X_1 \\ x_1 \in X_1 \\ ... \\ V = V_0 \end{cases} \tag{29}$$

Where $S(x_1, x_2, x_3, ...)$ represents the function of bathtub shape, in which $x_1, x_2, x_3...$ represents the shape-determined parameters such as length, width and height, etc. Then we will discuss how bathtub in different shapes perform in containing heat.

3.3.1 Cuboid

Cuboid Model is shown in Figure 7

$$\min Q(S(x_1, x_2, x_3, ...), V) = 1120x_1x_2 + 2 \times 100(x_1 + x_2)x_3$$

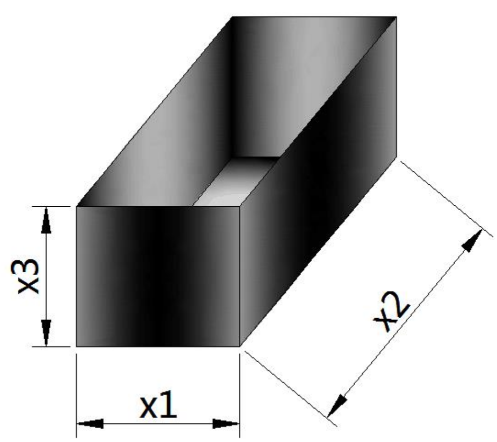

Figure 7: The cuboid model of bathtub

$$s.t.\begin{cases} 0.6m \le x_1 \le 1m \\ 1.5m \le x_2 \le 2m \\ 0.5m \le x_3 \le 0.8m \\ V = x_1x_2x_3 = 0.5m^3 \end{cases} \tag{30}$$

We use $Matlab2010b$ to set up a nonlinear programming model,and we get the solution of the model as follows.

$$\begin{cases} x_1 = 1.6m \\ x_2 = 0.5m \\ x_3 = 0.5556m \end{cases}$$

The least heat loss power of this model is:

$$Q_{1min} = 1.2413 \times 10^3 W$$

3.3.2 Cuboid combined with two semi-cylinders at each end

Cuboid Model is shown in Figure 8

$$\min Q(S(x_1, x_2, x_3, ...), V) = 1120(x_1x_2 + \frac{\pi {x_1}^2}{4}) + 100(\pi x_1 + 2x_2)x_3$$

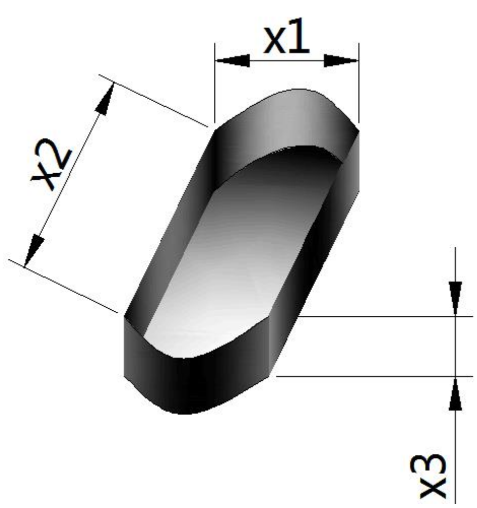

Figure 8: Cuboid combined with two semi-cylinders at each end model of bathtub

$$s.t.\begin{cases} 0.6m \leq x_1 \leq 1m \\ 1.5m \leq x_1 + x_2 \leq 2m \\ 0.5m \leq x_3 \leq 0.8m \\ V = (x_1x_2 + \frac{\pi x_1^2}{4}) = 0.5m^3 \end{cases} \tag{31}$$

We use $Matlab2010b$ to set up a nonlinear programming model,and we get the solution of the model as follows.

$$\begin{cases} x_1 = 0.6m \\ x_2 = 0.9m \\ x_3 = 0.6077m \end{cases}$$

The least heat loss power of this model is:

$$Q_{2min} = 1.2510 \times 10^3 W$$

3.3.3 Inverted prismoid

Cuboid Model is shown in Figure 9

$$\min Q(S(x_1, x_2, x_3, ...), V) = 1120x_3x_4 + 100(x_1+x_3)\sqrt{\frac{(x_2-x_4)^2}{4} + x_5^2} + 100(x_2+x_4)\sqrt{\frac{(x_3-x_1)^2}{4} + x_5^2}$$

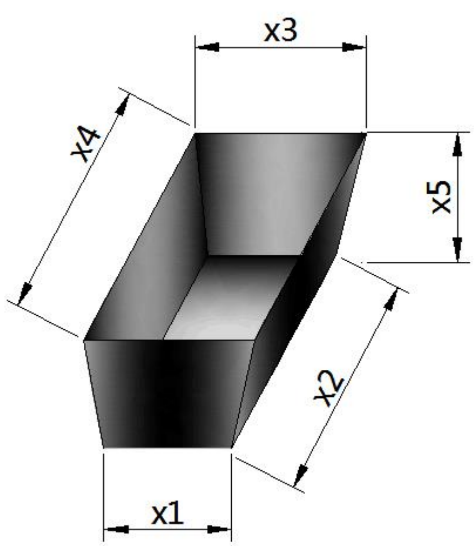

Figure 9: Inverted prismoid model of bathtub

$$s.t.\begin{cases} 0.3m \le x_1 \le 1m \\ 0.75m \le x_2 \le 2m \\ 0.6m \le x_3 \le 1m \\ 1.5m \le x_4 \le 2m \\ 0.5m \le x_5 \le 0.8m \\ V = \frac{(x_1x_2+x_3x_4+\sqrt{x_1x_2x_3x_4})x_5}{3} = 0.5m^3 \end{cases} \tag{32}$$

We use $Matlab2010b$ to set up a nonlinear programming model,and we get the solution of the model as follows.

$$\begin{cases} x_1 = 0.7901m \\ x_2 = 1.3966m \\ x_3 = 0.6m \\ x_4 = 1.5m \\ x_5 = 0.5m \end{cases}$$

The least heat loss power of this model is:

$$Q_{3min} = 1.2253 \times 10^3 W$$

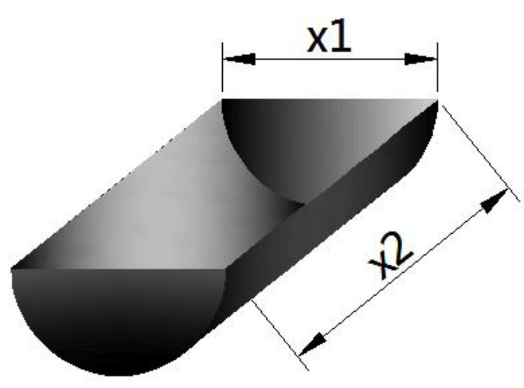

Figure 10: The Semi-cylinder model of bathtub

3.3.4 Semi-cylinder

Cuboid Model is shown in Figure 10

$$\min Q(S(x_1, x_2, x_3, ...), V) = 1120x_1x_2 + 100(\frac{\pi x_1 x_2}{2} + \frac{\pi x_1^2}{4})$$

$$s.t. \begin{cases} 0.8m \leq x_1 \leq 1.6m \\ 0.5m \leq x_2 \leq 2m \\ V = \frac{\pi x_1^2 x_2}{8} = 0.5m^3 \end{cases} \tag{33}$$

We use $Matlab2010b$ to set up a nonlinear programming model,and we get the solution of the model as follows.

$$\begin{cases} x_1 = 0.9213m \\ x_2 = 1.5m \end{cases}$$

The least heat loss power of this model is:

$$Q_{4min} = 1.8316 \times 10^3 W$$

3.3.5 Semi-cylinder with two quarter balls at each end

Cuboid Model is shown in Figure 11

$$\min Q(S(x_1, x_2, x_3, ...), V) = 1120(x_1x_2 + \frac{\pi x_1^2}{4}) + 100(\frac{\pi x_1 x_2}{2} + \frac{\pi x_1^2}{2})$$

$$s.t. \begin{cases} 0.8m \leq x_1 \leq 1.6m \\ 1.5m \leq x_1 + x_2 \leq 2m \\ V = \frac{2\pi}{3}(\frac{x_2}{2})^3 + \frac{\pi}{2}(\frac{x_1}{2})^2 x_2 = 0.5m^3 \end{cases} \tag{34}$$

We use $Matlab2010b$ to set up a nonlinear programming model,and we get the solution of the model as follows.

$$\begin{cases} x_1 = 1.0345m \\ x_2 = 0.5m \end{cases}$$

The least heat loss power of this model is:

$$Q_{5min} = 1.7701 \times 10^3 W$$

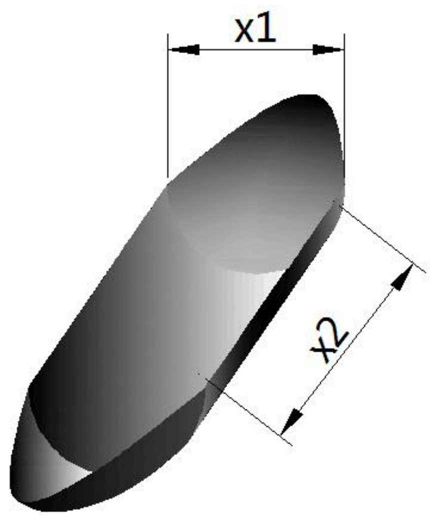

Figure 11: the Semi-cylinder with two quarter balls at each end model of bathtub

3.4 Model Solution Analysis

To summarize,the heat loss of the five models is listed as follow Table:

Table 4: Heat loss of bathtub in different shapes

Bathtub Shape	Q_{min}
Cuboid	$1.2413 \times 10^3 W$
Cuboid combined with two semi-cylinders at each end	$1.2510 \times 10^3 W$
Inverted prismoid	$1.2253 \times 10^3 W$
Semi-cylinder	$1.8316 \times 10^3 W$
Semi-cylinder with two quarter balls at each end	$1.7701 \times 10^3 W$

4 The optimization model of water consumption

As we have mentioned above,we suppose that the volume of bathtub V_0 is certain and $V_0 = 0.5m^3$.
At the initial time t_0,we can consider that the bathtub has been filled with water of temperature $323K$ with person in it.So we can calculate the initial total mass of water in the bathtub by

$$M_0 = \rho(V_0 - V^{'}) \tag{35}$$

As time passes by,if the person doesn't adopt any action such as unscrewing the hot water faucet,we can imagine that the temperature of water will decline quickly.Otherwise,at the time t_1 , if the person begins unscrewing the hot water faucet,the water temperature will drop much more slowlier.In this model,on the premise that the water temperature wonąŕt decline too much (we assume that the final water temperature only declines by $5K$,which means temperature T_2 at final time T_2 is $318K$), we should ensure the most

economical water consumption M .
To solve the problem,we take two steps to achieve the goal.

1. We consider that given the optimized shape of bathtub-the inverted prismoid whose relevant parameters have been determined in the Optimization of different shapes of bathtub Model,we set up a water consumption function M determined by the hot water inflow rate m and the starting time of hot water inflow t_1.

2. We fix the hot water inflow rate m whose optimized solution can be determined by step1,and make the parameters of inverted prismoid as well as t_1 vary to optimize the water consumption function M .

4.1 The differential equation model of time-dependent water temperature

By designing the differential equation model,we can get the variation of time-dependent temperature. According to the energy conservation,we know the heat loss must equal to the heat increase. That is,

$$cM_0dT + Qd\tau = cm(T^{'} - T)d\tau \tag{36}$$

At time τ,when the time changes slightly by $d\tau$,the water temperature of the bathtub reaches T,and its variation is dT . $T^{'}$ represents the constant temperature of the hot water inflow from the faucet.Q represents the heat loss rate of evaporation and heat conduction,which has been calculated in in the Optimization of different shapes of bathtub Model. After the simplification, we get

$$\frac{dT}{d\tau} + \frac{m}{M_0}T = \frac{cmT^{'} - Q}{cM_0} \tag{37}$$

To solve this linear ordinary differential equation of the first order, we can find the special solution and general solution of it.
General solution,

$$T = (T_1 - T^{'} + \frac{Q}{cm})e^{-\frac{m}{M_0}(t-t_1)} \tag{38}$$

Special solution,

$$T^{*} = \frac{cmT^{'} - Q}{cm} \tag{39}$$

So,the complete solution is

$$T(t) = (T_1 - T^{'} + \frac{Q}{cm})e^{-\frac{m}{M_0}(t-t_1)} + \frac{cmT^{'} - Q}{cm} \tag{40}$$

T_1 represents the water temperature of time t_1 . Therefore,we can get

$$t_2 - t_1 = -\frac{M_0}{m}ln(\frac{T_2 - T^{'} + \frac{Q}{cm}}{T_1 - T^{'} + \frac{Q}{cm}}) \tag{41}$$

Where T_2 represents the final temperature. Meanwhile,when we suppose the initial time $t_0 = 0$, and temperature of t_0 is T_0,we can get

$$T_1 = T_0 - \frac{Q}{cM_0}t_1 \tag{42}$$

So,

$$t_2 - t_1 = -\frac{M_0}{m}ln(\frac{T_2 - T^{'} + \frac{Q}{cm}}{T_0 - \frac{Q}{cM_0}t_1 - T^{'} + \frac{Q}{cm}}) \tag{43}$$

The parameters are set as follows:

$$\begin{cases} Q = 1225.00W \\ T^{'} = 338K \\ T_2 = 320K \\ m_0 = 100kg/s \end{cases}$$

4.1.1 The nonlinear programming model 1 of water consumption

The variation is hot water inflow rate m and the starting time of hot water inflow t_1 .

$$\min M(m, t_1) = M_0 - M_0 ln(\frac{T_2 - T^{'} + \frac{Q}{cm}}{T_0 - \frac{Q}{cM_0}t_1 - T^{'} + \frac{Q}{cm}}) \tag{44}$$

$$s.t. \qquad t_2 = t_1 - \frac{M_0}{m} ln(\frac{T_2 - T^{'} + \frac{Q}{cm}}{T_0 - \frac{Q}{cM_0}t_1 - T^{'} + \frac{Q}{cm}}) \geq 1500s \tag{45}$$

The result is:

$$\begin{cases} m = 0.430kg/s \\ t_1 = 1454.5s \\ M = 101.9641kg \end{cases}$$

4.1.2 The nonlinear programming model 2 of water consumption

We have calculated the optimized hot water inflow rate m in model $2.4.2$,,so in this model,the variation is the parameters of inverted prismoid x_1, x_2, x_3, x_4, x_5 and the starting time of hot water inflow t_1 .In this model,we try to improve the inverted prismoid model thus reaching the goal of the optimized water as well as the most economical water consumption.

$$\min M(Q(x_1, x_2, x_3, x_4, x_5), t_1) = M_0 - M_0 ln(\frac{T_2 - T^{'} + \frac{Q}{cm}}{T_0 - \frac{Q}{cM_0}t_1 - T^{'} + \frac{Q}{cm}})$$

$$s.t. \begin{cases} 0.3m \leq x_1 \leq 1m \\ 0.75m \leq x_2 \leq 2m \\ 0.6m \leq x_3 \leq 1m \\ 1.5m \leq x_4 \leq 2m \\ 0.5m \leq x_5 \leq 0.8m \\ 0s \leq t_1 \leq 1500s \\ t_2 = t_1 - \frac{M_0}{m} ln(\frac{T_2 - T^{'} + \frac{Q}{cm}}{T_0 - \frac{Q}{cM_0}t_1 - T^{'} + \frac{Q}{cm}}) \geq 1500s \end{cases} \tag{46}$$

The final solution is:

$$\begin{cases} x_1 = 0.433m \\ x_2 = 0.84m \\ x_3 = 0.6m \\ x_4 = 0.84m \\ x_5 = 0.48m \\ t_1 = 1435s \end{cases}$$

5 Bubble bath

When you put bubble bath additive into water , the thermal quality of water as well as the substance construction at the interface of water and air would be changed , which can affect the result of our model . The bubble floating on the surface of water formed a mixture of gas and liquid , which can be an isolation . The specific heat capacity of mixture is bigger than the pure water , but the influence of this change is slighter compared with the change of substance construction . Thus convective heat transfer coefficient will become smaller , and assume the specific heat capacity stays constant.We set h in equation(14)as 0 . And we get equation (47) as follows.

$$-\lambda[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}] + \frac{\partial T}{\partial t}\rho C = 0 \tag{47}$$

After modifying the model , the solution is shown in Figure 12.

6 Sensitivity Analysis

When parameter changes, the result of our model will change. For example, in the first model, if h getter smaller, the temperature will distribute more uneven in the edges but keep more close to initial temperature. In the second model, if the parameters associated with shape varies in a wider range, the ideal shape would still be inverted prismoid. In Optimization Model of Water Consumption, if parameter changes, the strategy will almost be the same.

7 Advantage And Limitation

7.1 Advantages

We develop Convective Heat Transfer Model,Bathtub Optimization Design Model, Nonlinear Programming Model and the Optimization Model of Water Consumption considering factors like hot water inflow, heat dissipation through bathtub wall and water surface, successfully determining a best strategy involving how to choose the shape and volume of bathtub, when to add hot water and how should a person react in the water. The strategy is easily comprehensive according to our real life. By changing some parameters, our model can also apply to the bath situation with bubble.

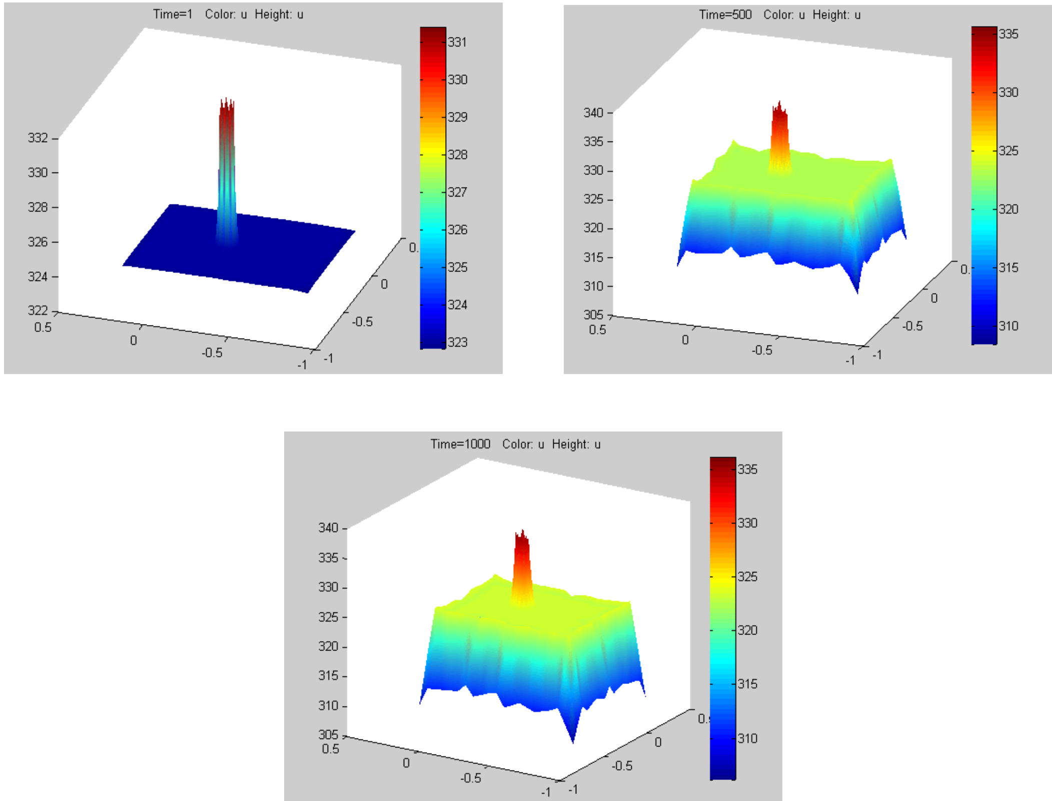

Figure 12: Temperature distribution of bubble bath

7.2 Limitation

- To simplify the model we didn't take the effect of water flowing inside for temperature distribution.
- We only discuss some reasonable shapes of bathtub, didn't traverse all possible situations.
- We make some simplification of people's motion.

8 A non-technical explanation

You must be eager for a pleasant and relaxing hot bath in your life. Are you always wishing to have the water temperature distribute evenly in the bathtub and stay warm as long as possible without wasting too much water? Don't be upset anymore, we have already found the best strategy to solve your problem. Follow the strategy we offer and then you can enjoy the most comfortable hot bath.

Firstly, choose the right bathtub in shape and volume. When you take a bath, the heat loses from water surface and the bathtub wall, while the only heat inflow is the hot water stream which is always located at the top of one side of the bathtub. Because of the intrinsic uneven heat distribution, it is difficult to realize completely evenly maintained temperature throughout the bath water. But we can try to realize more even temperature distribution. A regularly shaped surface and inverted prismoid shape will be greatly helpful to keep the temperature high and evenly distributed. So, next time when you shop for a bathtub, pay attention to the ones that have a elliptical surface and are shaped similar to inverted prismoid. In addition, the volume should be reasonable to save water.Less than $300L$ will be an ideal choice.

Secondly, your motion in the bathtub matters a lot as well. Try to keep yourself staying static in the bathtub,and add hot water more than $1300s$ from the time you start bathing without stopping.

Finally, if you want to use bubble bath additive,you can add hot water a little later,since the bubble helps

to maintain the temperature.
Thank you for finish reading and wish you a comfortable hot bath!

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