UNIVALENCE IN THE EFFECTIVE TOPOS

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Because the effective topos has complete internal categories, the category of internal groupoids in the effective topos has a model of homotopy type theory or at least of its 1-truncated part. Neither the construction of the effective topos nor the construction of the groupoid model depend on special properties of the topos of sets, we can performs them over the topos of simplical sets instead. Effective simplicial groupoids are a candidate for an effective ∞-topos although the right model structure remains a mystery.

1. Effective Toposes

This section is about the effective topos over simplicial sets and over other similar toposes. To be precise, our base topos $\mathcal{S}$ has the following properties beyond being an elementary topos:

- $\mathcal{S}$ has a natural number object $\mathbf{N}$;
- $\mathcal{S}$ has enough projectives.

The natural number object induces an internal recursion theory, which defines what *effective* means for the effective topos $\mathsf{Eff}(\mathcal{S})$ over $\mathcal{S}$.

That $\mathcal{S}$ has enough projectives means that $\mathcal{S}$ is the exact completion of its full subcategory of projectives. This allows us to construct the $\mathsf{Eff}(\mathcal{S})$ as the exact completion of the category of *partitioned assemblies.*

1.1. General recursive functions.

This section supplies the effectively computable morphisms required for building the effective topos $\mathsf{Eff}(\mathcal{S})$. Hence we are looking for the internal counterparts of the *general recursive functions.* These exist because recursion is essentially the universal property of natural number objects.

Definition 1. A natural number object is the combination of an object $\mathbf{N}$ with morphisms $z : 1 \to \mathbf{N}$ and $s : \mathbf{N} \to \mathbf{N}$ such that for each object X and each pair of morphisms $x : 1 \to X$ and $f : X \to X$ there is a unique morphism $g : \mathbf{N} \to X$ such that $g \circ z = x$ and $g \circ s = f \circ \rho(f, x)$.

$$\begin{tikzcd}
1 \arrow[r, "z"] \arrow[d, "\mathrm{id}"'] & \mathbf{N} \arrow[d, dotted, "g"'] & \mathbf{N} \arrow[l, "s"'] \arrow[d, dotted, "g"] \\
1 \arrow[r, "x"'] & X & X \arrow[l, "f"]
\end{tikzcd}$$

Remark 2. I will let $\mathbb{N}$ stand for the *set* of natural numbers which is external to $\mathcal{S}$.

Primitive recursion is basically a parametric version of this, which is available because $\mathcal{S}$ is Cartesian closed.

Lemma 3 (Primitive recursion)**.** *For each $f : X \to X$ there is a unique $\rho f : \mathbf{N} \times X \to X$ such that $\rho f \circ (z, \mathrm{id}_X) = \mathrm{id}_X$ and $\rho f \circ (s, \mathrm{id}_X) = f \circ \rho f$.*

$$\begin{tikzcd} X \arrow[r, "{(z,\mathrm{id})}"] \arrow[d, "\mathrm{id}"'] & \mathbf{N} \times X \arrow[d, "\rho f"'] & \mathbf{N} \times X \arrow[l, "{(s,\mathrm{id})}"'] \arrow[d, "\rho f"] \\ X \arrow[r, "\mathrm{id}"'] & X & X \arrow[l, "f"] \end{tikzcd}$$

Proof. There is a map $x : 1 \to X^X$ which points at id_X, and a map $f^X : X^X \to X^X$ defined by composition. Because $\mathbf{N}$ is a natural number object, there is a unique $g : \mathbf{N} \to X^X$ such that $g \circ z = x$ and $g \circ s = f^X \circ g$. The wanted morphism ρf is the transpose of g under the adjunction $- \times X \dashv -^X$. □

The natural numbers object is the initial algebra of the functor $X \mapsto 1 + X$. Because $\mathcal{S}$ is a topos, this functor also has a terminal coalgebra, which coincides with the partial map classifier of $\mathbf{N}$.

Definition 4. The *extended natural number object* $\mathbf{N}_\perp$ is the object

$$\mathbf{N}_\perp = \left\{\xi \in \Omega^{\mathbf{N}} | \xi(x) \wedge \xi(y) \to x = y\right\}$$

The extended natural numbers have a dual recursion property.

Lemma 5 (Corecursion)**.** *For each morphism $f : X \to 1 + X$ there is a least $g : X \to \mathbf{N}_\perp$ such that $g(x) = 0$ if $f(x) \in 1$ and $g(x) = s(g(f(x)))$ if $f(x) \in X$.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & X + 1 \arrow[d, "g+\mathrm{id}_1"] \\ \mathbf{N}_\perp \arrow[r, "s^{-1}"'] & \mathbf{N}_\perp + 1 \end{tikzcd}$$

Here, least means least domain.

Proof. By recursion there is a function $g^t : \mathbf{N} \to \Omega^X$ such that

$$g^t(0) = \{x \in X | f(x) \in 1\}$$
$$g^t(s(n)) = f^{-1}(g^t(n))$$

Its transpose $g : X \to \Omega^X$ is singleton valued because $g^t(m)$ and $g^t(n)$ only intersect when $m = n$ for the following reasons. The morphism f splits X into two complementary parts, $g^t(0)$ and $\neg g^t(0)$. For $n > 0$, $g^t(n) \subseteq \neg g^t(0)$ and hence $g^t(n) \cap g^t(0) = \bot$. Therefore

$$g^t(n + m) \cap \neg g^t(m) = f^{-m}(g^t(n) \cap g^t(0)) = \bot$$

□

Definition 6 (Minimization)**.** The minimization operator μ is the least function $\mathbf{N}_\perp^{\mathbf{N}} \to \mathbf{N}_\perp$ such that $\mu(f) = 0$ if $f(0) = 0$ and $\mu(f) = s(\mu(f \circ s))$ if defined. The minimization operator μ comes from corecursion with the morphism $g : \mathbf{N}_\perp^{\mathbf{N}} \to 1 + \mathbf{N}_\perp^{\mathbf{N}}$ where $g(f) : 1$ if $f(0) = 0$ and $g(f) = f \circ s$ if $f(0) \neq 0$.

The effective topos gets its name from functions $\mathbf{N} \to \mathbf{N}_\perp$ which are *effectively computable* because their definition requires only recursion and corecursion and simpler operators.

Definition 7. The class of partial recursive functions is the least class $\mathcal{P}$ of functions $\mathbf{N}^k \to \mathbf{N}_\perp$ which satisfies the following conditions.

(1) The canonical embedding $\{\cdot\} : \mathbf{N} \to \mathbf{N}_\perp$ is in $\mathcal{P}$ as are $\{z\} : 1 \to \mathbf{N}_\perp$, $\{s\} : \mathbf{N} \to \mathbf{N}_\perp$ and the projections $\vec{x} \mapsto \{x_i\} : \mathbf{N}^k \to \mathbf{N}_\perp$ for each $k \in \mathbb{N}$ and $i < k$.

(2) For each $f : \mathbf{N}^k \to \mathbf{N}_\perp$ let $f^\perp : (\mathbf{N}_\perp)^k \to \mathbf{N}_\perp$ satisfy

$$f^\perp(\vec{\xi}) = \left\{ f(\vec{x}) \in \mathbf{N} \middle| \vec{x} \in \prod_{i<k} \xi_i \right\}$$

If $f : \mathbf{N}^k \to \mathbf{N}_\perp$ and $g_i : \mathbf{N}^l \to \mathbf{N}_\perp$ for $i < k$ are in $\mathcal{P}$, then so is $f^\perp \circ (g_0, \ldots, g_{k-1}) : \mathbf{N}^l \to \mathbf{N}_\perp$.

(3) If $f_i : \mathbf{N}^k \to \mathbf{N}_\perp$ are in $\mathcal{P}$ for all $i < k$, then so are $\pi_i^\perp \circ \rho(f_0^\perp, \ldots, f_{k-1}^\perp) : \mathbf{N}^k \times \mathbf{N} \to \mathbf{N}_\perp$. Here π_i is the projection $\vec{x} \mapsto \{x_i\} : \mathbf{N}^k \to \mathbf{N}_\perp$.

(4) If $f : \mathbf{N}^k \times \mathbf{N} \to \mathbf{N}_\perp$ is in $\mathcal{P}$, then so is function $\mu \circ f^t : \mathbf{N}^k \to \mathbf{N}_\perp$, where $f^t : \mathbf{N}^k \to \mathbf{N}_\perp^{\mathbf{N}}$ is the transpose of f.

Example 8. According to the Church-Turing hypothesis, any effective algorithm that operators on natural numbers defines a partial recursive function. For those who still need conficing, I will give a few examples here.

- Applying primitive recursion to $s : \mathbf{N} \to \mathbf{N}$ gives addition: $\rho s(m, n) = m+n$

1.2. **Exact completions.** Both $\mathcal{S}$ and $\mathsf{Eff}(\mathcal{S})$ are exact completions. Carboni and Vitali 1995explains what this means. The following definition covers the most relevant notions.

Definition 9. Let $\mathcal{C}$ be a category with weak finite limits and let $\mathcal{D}$ be a regular category. For each $F : \mathcal{C} \to \mathcal{D}$ and each diagram D in $\mathcal{C}$ there is a canonical map $F(\mathrm{wlim} D) \to \lim FD$: the factorisation of the image of the weak limit cone over D in $\mathcal{C}$ through the limit cone over FD in $\mathcal{D}$. The functor $F : \mathcal{C} \to \mathcal{D}$ is left covering or *flat* if this canonical morphism is a regular epimorphism for each finite diagram D.

The *regular completion* of $\mathcal{C}$ is a universal flat functor $I : \mathcal{C} \to \mathcal{C}_{\mathrm{reg}}$. This means that for each regular $\mathcal{D}$ and each flat functor $F : \mathcal{C} \to \mathcal{D}$ there is an up to isomorphism unique regular functor $F' : \mathcal{C}_{\mathrm{ex}} \to \mathcal{D}$ such that $F'I \simeq F$.

The *exact completion* of $\mathcal{C}$ is a universal flat functor $I : \mathcal{C} \to \mathcal{C}_{\mathrm{ex}}$ to an *exact* $\mathcal{C}_{\mathrm{ex}}$. This means that for each exact $\mathcal{E}$ and each flat functor $F : \mathcal{C} \to \mathcal{E}$ there is an up to isomorphism unique regular functor $F' : \mathcal{C}_{\mathrm{ex}} \to \mathcal{E}$ such that $F'I \simeq F$.

The reason that $\mathcal{S}$ is an exact completion, is that is has enough projectives. This means that projective object cover all objects of the category. The full subcategory $\mathsf{Pro}(\mathcal{S})$ of projective objects has weak limits, and the inclusion $\mathsf{Pro}(\mathcal{S}) \to \mathcal{S}$ is an exact completion.

The effective topos $\mathsf{Eff}(\mathcal{S})$ over $\mathcal{S}$ is an exact completion too. The reason are outlined in my thesis. I will take advantage of this fact to present the effective topos as the exact completion of another category in this paper.

Definition 10. A *partitioned assembly* is a pair (X, f) where X is a projective object of $\mathcal{S}$, $f : X \to \mathbf{N}$ is an arbitrary morphism. A morphism $(X, f) \to (Y, g)$ is a morphism $h : X \to Y$ for which there exists a partial recursive $k :\to \mathbf{N}_\perp$ such that $g(h(x)) \in k(f(x))$ for all $x \in X$.

Lemma 11. *The category* $\mathsf{Pasm}(\mathcal{S})$ *has weak finite limits.*

Proof. This relies heavily on the fact that $\mathsf{Pro}(\mathcal{S})$ has weak finite limits.

The partitioned assembly $(1, z)$ is terminal.

Let (X, f) and (X, g) be two partitoned assemblies with the same underlying object X. They combine into the assembly $(X, \langle f, g\rangle)$, where $\langle f, g\rangle$ is the composition of $(f, g) : X \to \mathbf{N} \times \mathbf{N}$ with a recursive paring function $\mathbf{N} \times \mathbf{N} \to \mathbf{N}$. The new assembly $(X, \langle f, g\rangle)$ is the intersection of (X, f) and (X, g) in the sense that if (X, h) is a third assembly with the same underlying object, and $\mathrm{id} : (X, h) \to (X, f)$ and $\mathrm{id} : (X, h) \to (X, g)$ are morphisms of partitioned assemblies, then so is $\mathrm{id} : (X, h) \to (X, \langle f, g\rangle)$.

Let $k : (X, f) \to (Z, h)$ and $l : (Y, f) \to (Z, h)$ be an arbitrary pair of morphisms with the same codomain. There is a weak pullback $l' : W \to X$ and $k' : W \to Y$ of k and l in $\mathsf{Pro}(\mathcal{S})$. Now $l' : (W, \langle f\circ l', g\circ k', \rangle) \to (X, f)$ and $k' : (W, \langle f\circ l', g\circ k', \rangle) \to (Y, g)$ are morphism of assemblies, and together they form a weak pullback of k and l in $\mathsf{Pasm}(\mathcal{S})$.

Since $\mathsf{Pasm}(\mathcal{S})$ has a terminal object and weak pullbacks and hence all weak finite limits. $\square$

Definition 12. The effective topos over $\mathcal{S}$ is $\mathsf{Eff}(\mathcal{S}) = \mathsf{Pasm}(\mathcal{S})/_{\mathrm{ex}}$.

The following remarks clarify in what sense $\mathsf{Eff}(\mathcal{S})$ is similar to the effective topos Eff of Martin Hyland.

For realizability purposes, two subobjects U and V of $\mathbf{N}$ in $\mathcal{S}$ are *equivalent*, if there are partial recursive functions between U and V in both directions. The topos $\mathsf{Eff}(\mathcal{S})$ assigns an equivalence class of subobjects of $\mathbf{N}$ to each proposition of its internal language. Each subterminal object $\sigma \subseteq 1$ has a covering partitioned assembly (S, f) and the image $\exists_f(S) \subseteq \mathbf{N}$ is one of the objects of *realizers* of σ. Let numerals be global sections of $\mathbf{N}$ which have the following form for some $n \in \mathbb{N}$.

$$\underline{n} = \underbrace{s(\cdots(s}_{n\text{ times}}(0)) : 1 \to \mathbf{N}$$

With the class of morphisms selected in definition 10, $\sigma \simeq 1$ if and only if a numeral factors through $\exists_f(S)$.

For each of the following conditions on subobjects of $\mathbf{N}$ there is a topos in which more subobjects satisfy the condition than that satisfy the previous one.

(1) the existence of a numeral $\underline{n} : 1 \to U$,
(2) the existence of an arbitrary global section $x : 1 \to U$,
(3) mere inhabitation of $\exists_f(S)$ in $\mathcal{S}$, i.e. $! : U \to 1$ is epi.

Effective toposes that make propositions valid if their sets of realizers satisfy these weaker conditions are *filter quotients* of $\mathsf{Eff}(\mathcal{S})$. Take the class of subobjects $U \subseteq \mathbf{N}$ that satisfy the desired condition. For each monic $m : U' \to \mathbf{N}$ which represents U, the supports of (U', m) in $\mathsf{Eff}(\mathcal{S})$ form the filter which induces the desired effective topos.

1.3. **Generic proofs.** This sections answers the question why $\mathsf{Eff}(\mathcal{S})$ is a topos. Here we rely on the work of Matías Menni, who proved that $\mathcal{C}_{\mathrm{ex}}$ is a topos, if and only if the underlying category has a generic proof. There is a delicate connection with Kleene's recursion theorem.

2. Internal groupoid models

In this section we bring Hofmann and Streicher's groupoid interpretation of type theory to $\mathsf{Eff}(\mathcal{S})$. The major difficulty is in making everything work with the constructive internal logic of $\mathsf{Eff}(\mathcal{S})$. To achieve that, we replace families of groipoids by cloven fibrations.

The model which is worked out in the rest of this section looks roughly as follows. Cloven fibrations are a special type of functor between groupoids, which form a weak factorization system together with deformation retracts. All of this structure exists internally in $\mathsf{Eff}(\mathcal{S})$. The result is a category $\mathcal{G}$ of internal groupoids, together with a weak factorisation system.

For each internal groupoid I the category $\mathcal{F}(I)$ of cloven fibrations $F : E \to G$ is a reflective subcategory of $\mathcal{G}/I$ and therefore complete. Every fibration comes with an internal groupoid structure. Internal functors $G : I \to J$ induce change of base functors $\mathcal{F}(J) \to \mathcal{F}(I)$ which preserve these structures and this way $\mathcal{F}(-)$ becomes a model of type theory, including depedent products and identity types.

2.1. **internal groupoids.** Groupoids are categories all of whose morphisms are isomorphisms. Internalization of this notion requires nothing beyond finite limits.

Definition 13. Let $\mathcal{C}$ be any category with finite limits. An *internal groupoid* in $\mathcal{C}$ is a tuple $(G_0, G_1, d_0, d_1, r, t)$ where G_0 and G_1 are arbitrary objects, $d_0, d_1 : G_0 \to G_1$ arbirary morphisms and $r : G_0 \to G_1$ is a morphism which satisfies $d_0 \circ r = d_1 \circ r = \mathrm{id}_{G_0}$; let $d'_i : G_2 \to G_1$ be the pullbacks of d_i along d_{1-i}. The $r : G_1 \times_{G_0} G_0 \to G_1$ is a morphism such that the squares $d_i \circ t = d_i \circ d'_i$ commute and are pullbacks.

$$\begin{tikzcd} G_0 \arrow[r, "r"] \arrow[d, "r"'] \arrow[dr, "\mathrm{id}"] & G_1 \arrow[d, "d_0"] \\ G_1 \arrow[r, "d_1"'] & G_0 \end{tikzcd}$$

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