

The space of greedy list-colourings

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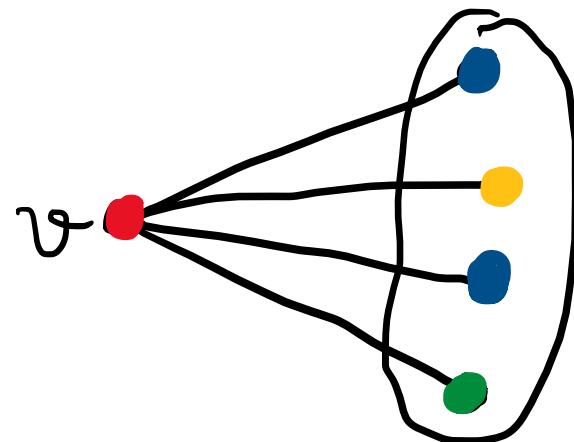
Based on joint works with Stijn Cambie, Daniel Cranston,
Ewan Davies, Jan van den Heuvel and Ross Kang.

Ottawa, CANADAM 2025, May 2025.

Basic greedy bound

Graph G has maximum degree Δ , chromatic number χ

$$\chi \leq \Delta + 1$$



Proof

Induction: $\exists (\Delta+1)$ -colouring c of $G-v$. $|N(v)| \leq \Delta$
Then colour v greedily from $[\Delta+1] \setminus c(N(v)) \neq \emptyset$

□

This also works for list-colouring.

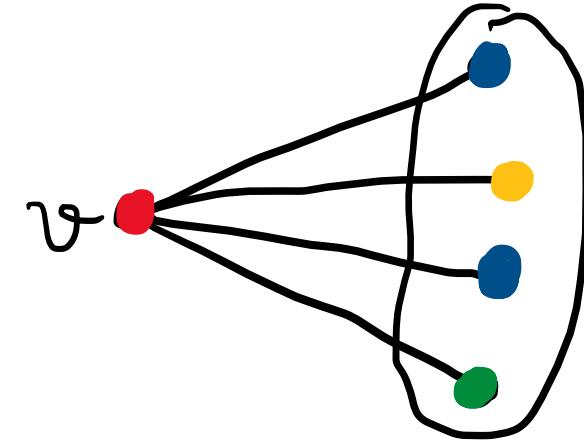
$L: V(G) \rightarrow 2^{\mathbb{N}}$ is a list-assignment.

An L -colouring is a proper colouring

$c: V(G) \rightarrow \mathbb{N}$ s.t. $c(v) \in L(v)$, $\forall v \in V(G)$.

By same argument...

\exists L-colouring of \mathcal{G} if



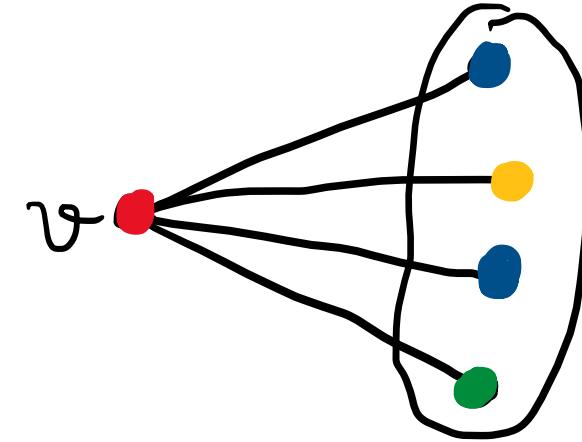
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$$|L(v)| \geq \Delta + 1$$

$$\forall v \in V(\mathcal{G}).$$

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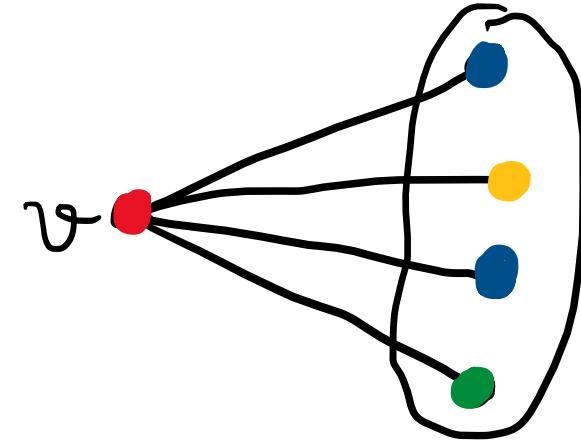
$$|N(v)| \leq \Delta$$

$$|L(v)| \geq \Delta + 1 \quad \forall v \in V(\mathcal{G}).$$

& also if

$$|L(v)| \geq \deg(v) + 1 \quad \forall v \in V(\mathcal{G}).$$

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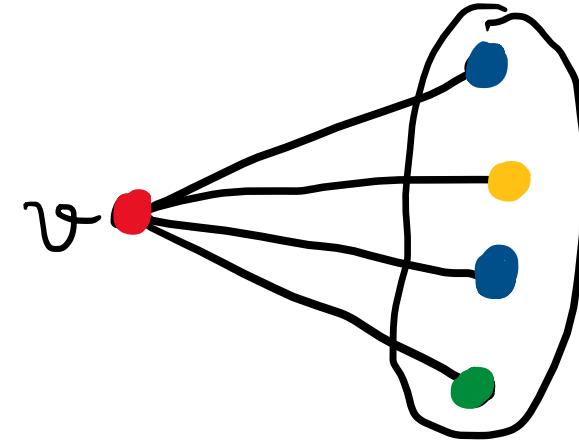
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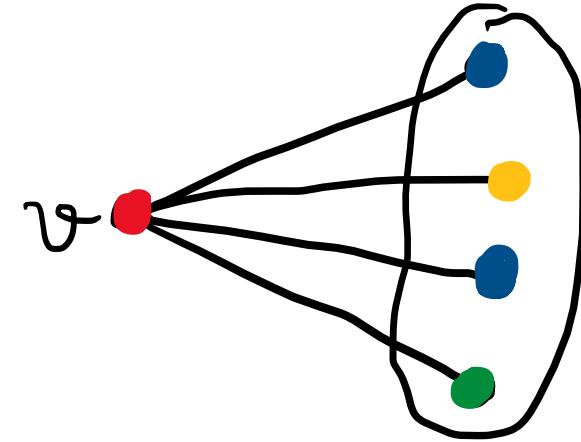
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$$|N(v)| \leq \Delta$$

Choose $c(v)$
from
non-empty
 $L(v) \setminus c(N(v))$.

By same argument...



\exists L-colouring of \mathcal{G} if

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$$|L(v)| \geq \deg(v) + 1 \quad \forall v \in V(\mathcal{G}).$$

← deg + 1
assignment

So far obtained existence of \geq one L-colouring.

But ...

Want to understand entire space of L-colourings.

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But ...

Want to understand entire space of L-colourings.

- ① count # L-colourings?
- ② flexible/balanced everywhere?
- ③ How similar/close are the L-colourings to each other?

Count # L-colourings

Recall : $|L(v)| \geq \deg(v) + 1 \quad \forall v \in V(g).$

Observation : g connected $\Rightarrow \exists$ exponentially many L-colourings

Count # L-colourings

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 n vertices

Proof

Count # L-colourings

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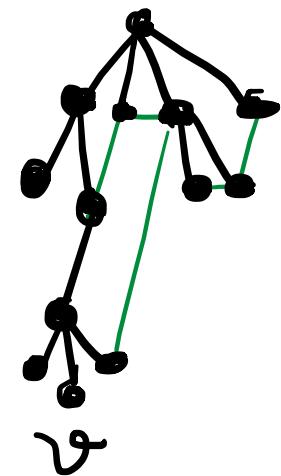
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 n vertices

Proof Choose spanning tree T .

Choose leaf v of T .

$\exists \geq \deg(v) + 1 \geq 2$ choices for colour $c(v)$.

Apply induction to $g-v$, removing $c(v)$ from the lists of v 's neighbours. $\Rightarrow 2^{n-2}$ L-colourings of $g-v$. \square



So \exists many L -colourings.

Are they also "flexible / balanced / tweakable" everywhere?

So \exists many L -colourings.

Are they also „flexible / balanced / tweakable” everywhere?

yes

Lemma (Cambie, CvB, Davies, Kang, 2023)

\exists probability distribution on L -colourings s.t.

$$\mathbb{P}(c(v) = x) = \frac{1}{|L(v)|}$$

$$\forall v \in V(G) \quad \forall x \in L(v).$$

„ (G, L) admits a fractional list packing“

At each vertex, every colour is equally likely.

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- **True** if $|L(v)| \geq \deg(v) + 1$ for all v .
- **False** $|L(w)| = \deg(w)$ for just one w .
- **True** $|L(v)| \geq \text{pathwidth} + 1$
- Unknown $|L(v)| \geq \text{treewidth} + 1$.
- **False** $|L(v)| \geq \text{degeneracy} + 1$.

\exists probability distribution on L -colourings s.t.

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„ (G, L) admits a fractional list packing“

Problem characterize (G, L) with

$$|L(w)| = \deg(w) \quad \text{for some } w$$

$$|L(v)| = \deg(v) + 1 \quad \text{for all other } v$$

that do not admit a fractional list packing.

e.g.
cliques

\exists probability distribution on L -colourings s.t.

$$\mathbb{P}(c(v) = x) = \frac{1}{|L(v)|}$$

$$\forall v \in V(G) \quad \forall x \in L(v).$$

③ How similar / close are the L-colourings to each other?

③ How similar / close are the L -colourings to each other?

Def.

Reconfiguration graph $R_L(G)$ of G has

vertices: $V(R_L) = \{L\text{-colourings of } G\}.$

edges: $c_1, c_2 \in E(R_L) \iff c_1 \text{ & } c_2 \text{ differ on}$
precisely one vertex

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Q₁ $R_L(G)$ connected?

Q₂ if so, how small is its diameter?

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Def.

Reconfiguration graph $R_K(G)$ of G has vertices: $V(R_K) = \{ K\text{-colourings of } G \}$.

$K \in \mathbb{N}$

edges: $c_1, c_2 \in E(R_K) \iff c_1 \text{ & } c_2 \text{ differ on precisely one vertex}$

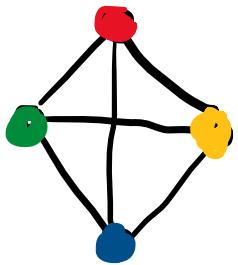
Q₁ $R_K(G)$ connected?

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Q₁ →

NOT always connected
if $k \leq \Delta + 1$ available colours.

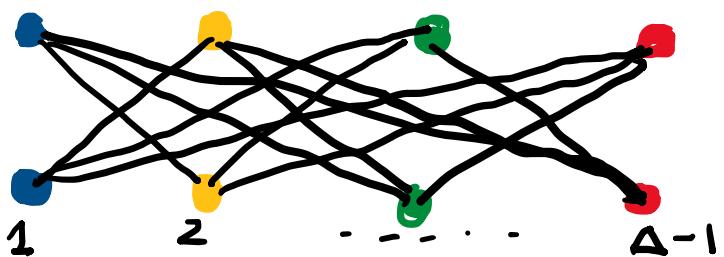
Exp.



= frozen 4-colouring of K_4
= isolated vertex of $R_4(K_4)$.

Exp.

$G := K_{\Delta, \Delta}$ - perfect matching.



= frozen $(\Delta + 1)$ -colouring
= isolated vertex of $R_{\Delta+1}(G)$.

So need $k \geq \Delta+2$ colours,

Thm (Jerrum, 1995)

$R_{\Delta+2}(G)$ is connected (with diameter $O(2^n)$).

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Thm (Cerf-Cedra, 2007)

$\text{diam } (R_{\Delta+2}(G)) = \Theta(n^2)$

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Thm (Cambie, CvB, Cranston, 2023)

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If $|L(v)| \geq \deg(v) + 2$ then

$$\text{diam } (R_L(G)) \leq 2n.$$

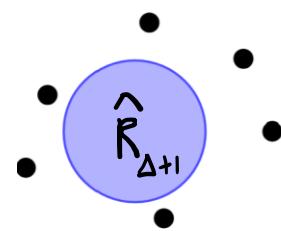
Returning to $\Delta+1$... despite frozen colourings...

Thm (Feghali, Johnson, Paulusma, 2016)

If G connected, not path or cycle, then

$R_{\Delta+1}(G)$ is union of isolated vertices

and at most one nontrivial component $\hat{R}_{\Delta+1}(G)$
with diameter $O(n^2)$.



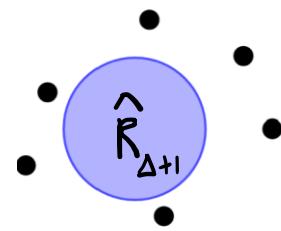
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Thm (Bousquet, Feuilloley, Heinrich, Rabie, 2024)

If also min degree ≥ 3 then diameter $O(\Delta^\Delta \cdot n)$.

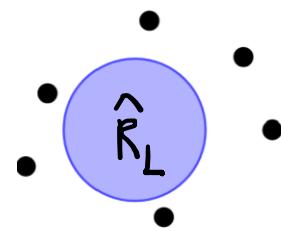
Our work : generalising to local lists

Thm (Cambie, CvB, Cranston, vd Heuvel, Kang, 2025+)

If G connected, not path or cycle,
and $|L(v)| \geq \deg(v) + 1 \quad \forall v$, then

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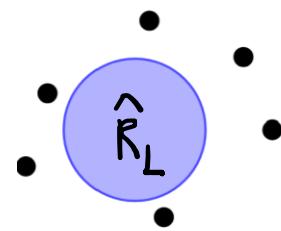
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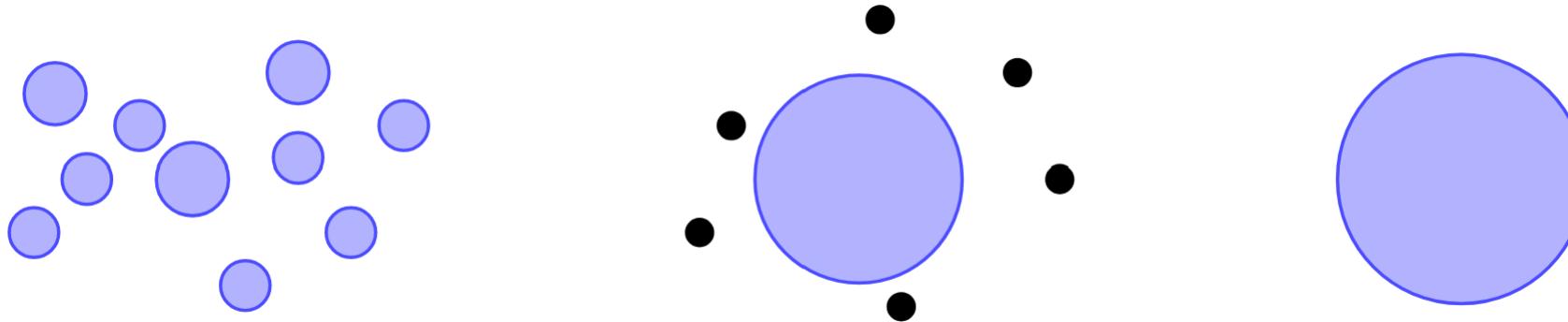
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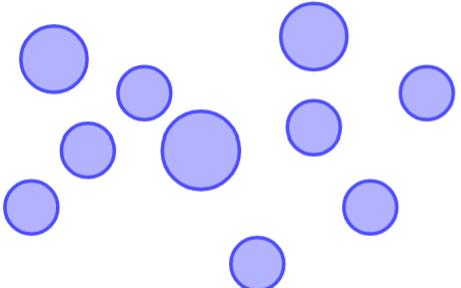


and # isolated vertices is negligible if $\Delta \ll n$.

Sensitive to the local constraints

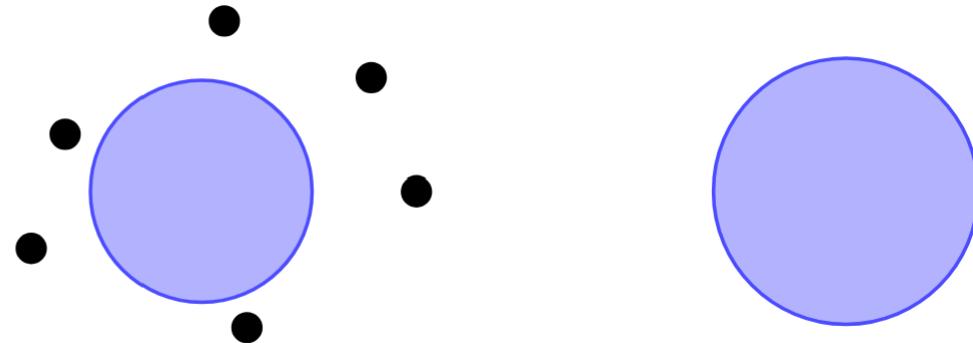


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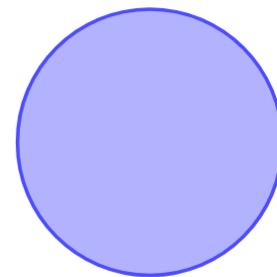
$|L(w)| \leq \deg(w)$
for some w

$R_L(g)$ may shatter
into many large
components.



$|L(v)| = \deg(v)+1$
for all v

$R_L(g)$ connected
up to isolated vertices



$|L(w)| \geq \deg(w)+2$
for some w

$R_L(g)$ connected

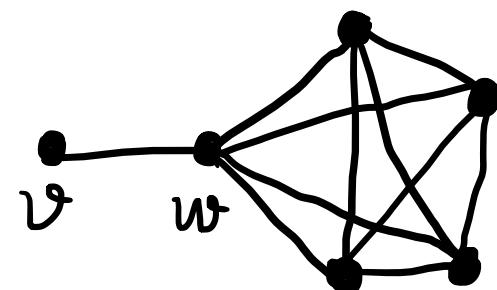
Shattering observation

If $|L(w)| = \deg(w)$ for some w
and $|L(v)| \geq \deg(v) + 1$ for all other v ,

Then $R_L(G)$ could have many large components.

Exp. Take K_n + edge vw ,

- Lists $\{1, 2, \dots, n\}$ on K_n
- List $\{n+1, \dots, n+z\}$ on v .



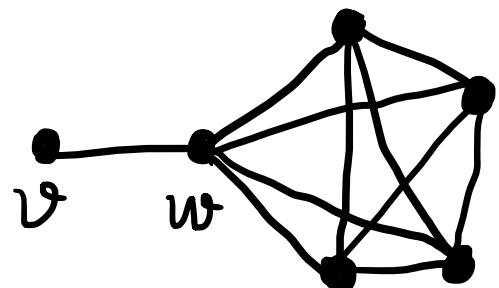
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Exp. Take K_n + edge vw ,

- Lists $\{1, 2, \dots, n\}$ on K_n
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\Rightarrow All L -colourings are frozen on K_5 but $\exists z$ choices for colour of w .
 $\Rightarrow R_L(G)$ has $n!$ components of size ≈ 1 \square .

Key Lemma

If $|L(w)| \geq \deg(w) + 2$ for some w

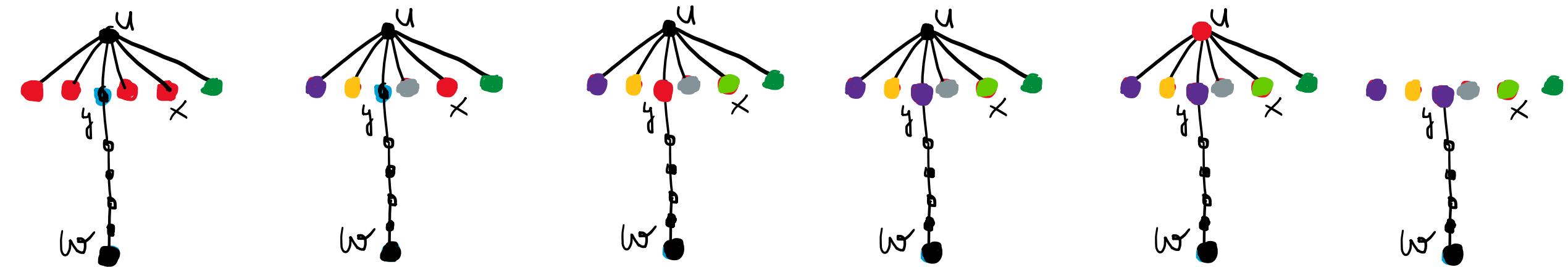
and $|L(v)| \geq \deg(v) + 1$ for all other v ,

then $\text{diam}(R_L(G)) \leq \left(\frac{3}{2} + o(1)\right) n^2$.

Key Lemma

If $|L(w)| \geq \deg(w) + 2$ for some w
and $|L(v)| \geq \deg(v) + 1$ for all other v ,
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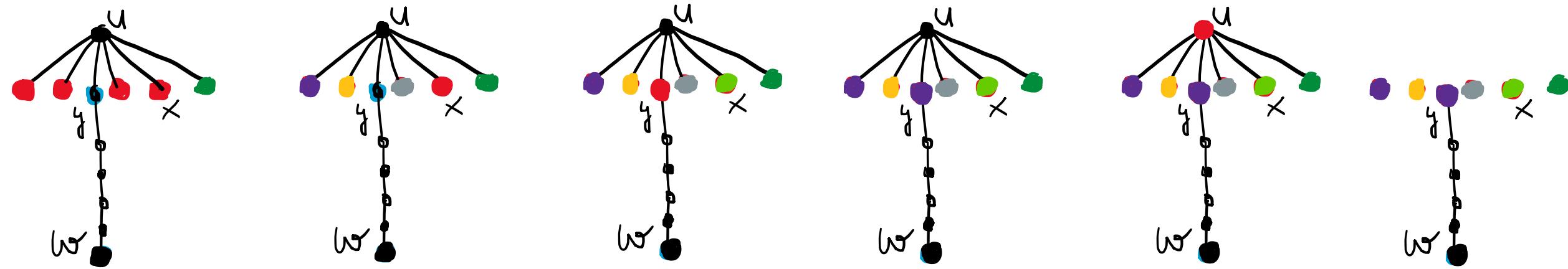
Proof sketch: Choose $u \neq w$. Goal recolour u to 



Proof sketch of Key Lemma

Goal recolour u to \bullet

Can always recolour along shortest w, u path P



many
neighbours
with
colour \bullet

$\leq \deg(u)$ steps

x unique
neighbour of
 u with
colour \bullet

$\lesssim \#P$ steps

y unique
neighbour of
 u with
colour \bullet

$\lesssim \#P_1$ steps

NO
neighbour
with
colour \bullet

1 step

recolour
 u to
 \bullet

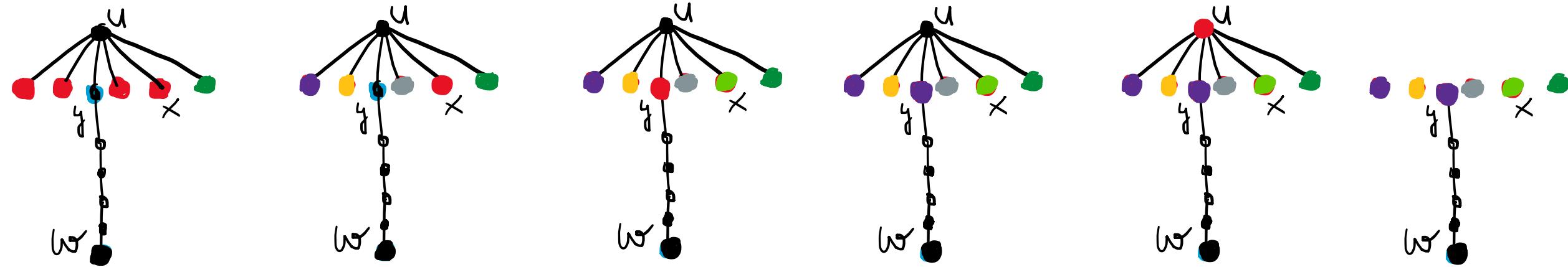
Induction
on
 $G - u$.

(& Remove \bullet
from lists of
neighbours of u)

Proof sketch of Key Lemma

Goal recolour u to \bullet

Can always recolour along shortest w, u path P



\Rightarrow need $\leq 3n$ steps to recolour u

\Rightarrow need $\lesssim \sum_{i=1}^n 3i \approx \frac{3}{2}n^2$ steps to recolour all vertices

□

Remark

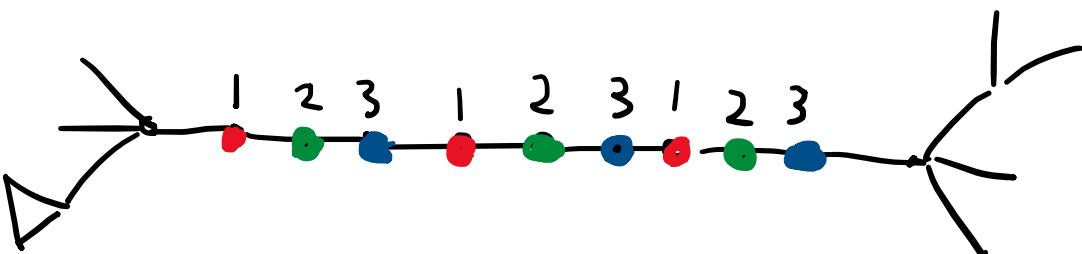
diameter $O(n^2)$ is optimal:

(Both in Key Lemma
and Main Theorem)

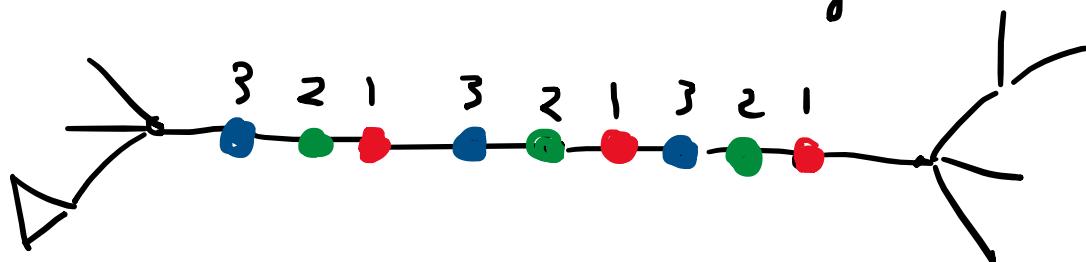
Remark

diameter $O(n^2)$ is optimal:

Proof G could have long induced path P_t of degree 2 vertices with $t = \sqrt{n}$, and list $\{1, 2, 3\}$ on each vertex, so that



↑ need many
recolorings



$$\text{diam } (\hat{R}_L(G)) \geq$$

$$\text{diam } (R_3(P_t)) \geq \frac{1}{4}t^2$$

$$= \Omega(n^2).$$

□

Open Problems

"easy"

$$\left. \begin{array}{l} |L(w)| \geq \deg(w) + 2 \\ |L(v)| \geq \deg(v) + 1 \quad \forall v \neq w \\ \text{& minimum degree} \geq 3 \end{array} \right\} \Rightarrow \text{diam } (R_L(G)) = G(n) ?$$



hard

$$\left. \begin{array}{l} |L(v)| \geq \deg(v) + 1 \quad \forall v \\ \text{& minimum degree} \geq 3 \end{array} \right\} \Rightarrow \text{diam } (\hat{R}_L(G)) = G(n) ?$$

$$\left. \begin{array}{l} |L(v)| \geq \deg(v) + 1 \quad \forall v \\ \text{& no path of } t \text{ consecutive} \\ \text{degree - 2 vertices} \end{array} \right\} \Rightarrow \text{diam } (\hat{R}_L(G)) = O((t+1) \cdot n) ?$$

Summary

If $|L(v)| \geq \deg(v) + 1$, then

- ① \exists exponentially many L -colourings
- ② (G, L) admits a fractional list-packing
- ③ the reconfiguration graph of L -colourings is essentially connected, with diameter $O(n^2)$.

Bonus

$R_L(g)$ connected



"Glauber dynamics" (a natural Markov chain from statistical physics)

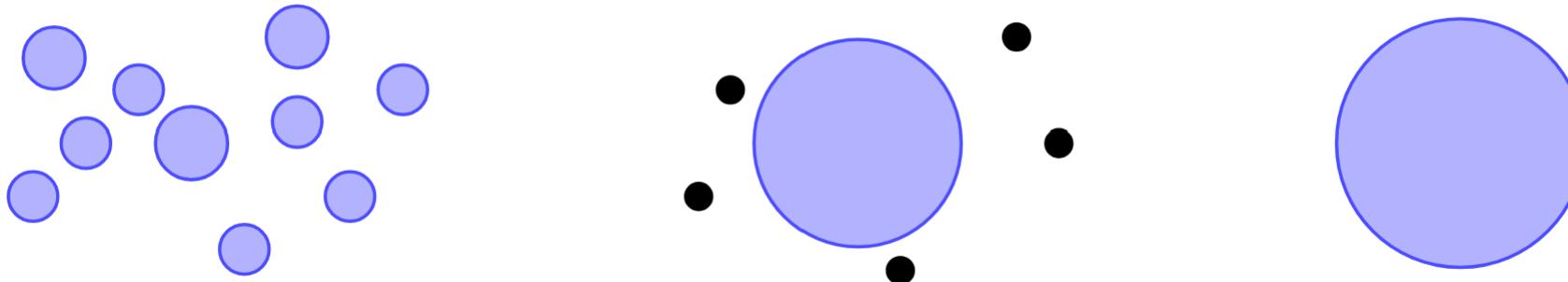
Converges to uniform distribution over all L -colourings.



Can sample uniformly random L -colouring &

can approximately count # L -colourings.

Thank you for
your attention !



Reconfiguration of list colourings, [arxiv:2505.08020](https://arxiv.org/abs/2505.08020)

Fractional list packing for layered graphs, [arxiv:2410.02695](https://arxiv.org/abs/2410.02695)

List packing number of bounded degree graphs, [arxiv:2303.01246](https://arxiv.org/abs/2303.01246)

Optimally reconfiguring list and correspondence colourings, [arxiv:2204.07928](https://arxiv.org/abs/2204.07928)

Slides available at woutercvb.github.io

Thm (Cambie, C.vB, Cranston, 2023)

$$\text{diam } (R_{\Delta+2}(G)) \leq 2n$$

Conj.

$$\text{diam } (R_{\Delta+2}(G)) = n + N \quad \text{where } N = \\ \text{maximum size of a} \\ \text{matching of } G.$$

Thm (De Meyer, 2025+)

Conjecture is True for subcubic graphs &
complete multipartite graphs.

Glauber dynamics

Initialize with any L -colouring, then repeat:

- (i) Choose uniformly random vertex v .
- (ii) Choose " " random colour $x \in L(v)$,
- (iii) Recolour v to x if it yields proper L -colouring.
o/w keep current colouring.

When $R_L(\gamma)$ is connected, this (irreducible symmetric) Markov chain converges to the uniform distribution over all L -colourings. Hence this process can be used to sample a \approx uniformly random L -colouring.