Short transformations between list colourings

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Dutch Day of Combinatorics
Eindhoven, 12 May 2022

Graph colouring

Given: a graph G and a positive integer k.

Proper k-colouring: colouring of the vertices using k colours, s.t. adjacent vertices receive distinct colours.

Q1. Does G have a proper k-colouring?

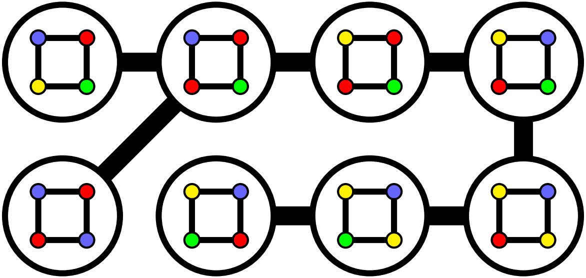
Q2. Can any two proper k-colourings of G be transformed into each other through a sequence of simple modifications?

Q3. How 'close' are the k-colourings of G to each other?

Definition

The *reconfiguration graph* $\mathcal{C}_k(G)$ has

- **vertices**: the proper k-colourings of G;
- **edges**: two proper k-colourings are adjacent iff their corrresponding colourings differ on exactly one vertex of G.



Example: Part of the reconfiguration graph $\mathcal{C}_4(C_4)$

Three Questions formalized

Given: a graph G and a positive integer k.

Q1. Does G have a proper k-colouring?
Is $\mathcal{C}_k(G)$ non-empty?

Q2. Can any two proper k-colourings of G be transformed into each other through a sequence of simple modifications?
Is $\mathcal{C}_k(G)$ connected?

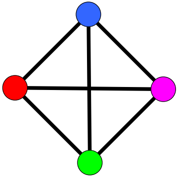
Q3. How 'close' are the k-colourings of G to each other?
What is the diameter of $\mathcal{C}_k(G)$?

Frozen colourings

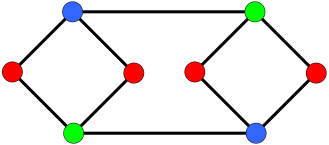
Observation

If for each vertex v of G, all k colours appear in the closed neighbourhood $N[v]$, then the colouring is *frozen*; an isolated vertex of $\mathcal{C}_k(G)$.

Examples:



$k = 4$

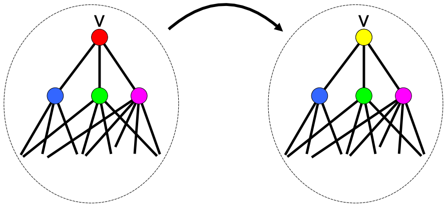


$k = 3$

Degeneracy

Definition

The *degeneracy* $degen(G)$ of G is the smallest integer d such that each subgraph of G contains a vertex v of degree at most d.



If $k \geq degen(G) + 2$, then G has no frozen k-colouring.
(Indeed: at least one colour does not appear on $N[v]$.)

Non-empty? Connected?

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Q2. Is $\mathcal{C}_k(G)$ connected?

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Yes if $k \geq degen(G) + 1$.

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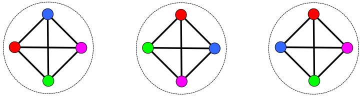
Yes if $k \geq degen(G) + 2$.



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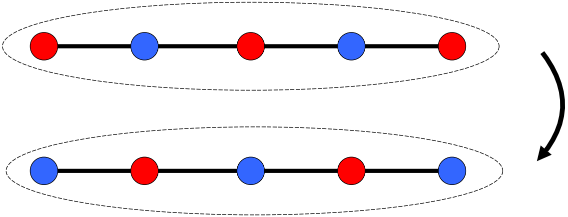

Example: if G is a path, then $\text{diam}(\mathcal{C}_3(G)) = \Omega(n^2)$.

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Theorem (Bousquet, Heinrich, 2022)

If $k \geq degen(G) + 2$, then $\text{diam}(\mathcal{C}_k(G)) = O(n^{degen(G)+1})$

What about maximum degree?

$\Delta(G) :=$maximum degree of G.

Theorem (Bousquet et al, 2022+)

If $k \geq \Delta(G) + 2$, then $\mathrm{diam}(\mathcal{C}_k(G)) = O(\Delta(G) \cdot n)$.

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Theorem (Cambie,C.,Cranston, 2022+)

If $k \geq \Delta(G) + 2$, then $\mathrm{diam}(\mathcal{C}_k(G)) \leq 2n$.

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But can we do even better?

Recall our theorem: $\mathrm{diam}(\mathcal{C}_k(G)) \leq 2n$, for all $k \geq \Delta(G) + 2$.

Remark: $n \leq \mathrm{diam}(\mathcal{C}_k(G))$, for all k.

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Remark: $n \leq \mathrm{diam}(\mathcal{C}_k(G))$, for all k.

Indeed: consider two k-colourings α and β of G, such that $\alpha(v) \neq \beta(v)$, for every vertex v. Then every vertex needs to be recoloured at least once. $\square$

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Conjecture (Cambie, C., Cranston, 2022)

If G is a graph on n vertices, then for every $k \geq \Delta(G) + 2$,

$$\mathrm{diam}(\mathcal{C}_k(G)) \leq \left\lfloor \frac{3n}{2} \right\rfloor.$$

True with equality for the complete graph (Bonamy and Bousquet, 2018).

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Conjecture (Cambie, C., Cranston, 2022+)

If G is a graph on n vertices with matching number $\mu(G)$, then for every $k \geq \Delta(G) + 2$,

$$\mathrm{diam}(\mathcal{C}_k(G)) = n + \mu(G) \leq \left\lfloor \frac{3n}{2} \right\rfloor.$$

True for the complete graph (Bonamy and Bousquet, 2018).

Intuition lower bound $n + \mu(G)$

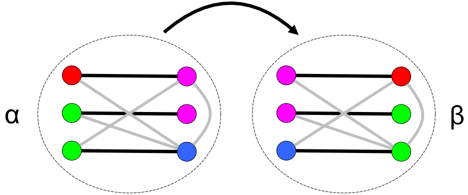
Observation

Consider a maximum matching M of G, with edges $v_1 v_2$, $v_3 v_4, \ldots$
Suppose there exist two proper k-colourings α, β of G such that their colours are swapped on each edge of the matching. I.e. for all i:

$$\alpha(v_{2i-1}) = \beta(v_{2i}) \text{ and } \beta(v_{2i-1}) = \alpha(v_{2i}).$$

Then

$$\mathrm{diam}(\mathcal{C}_k(G)) \geq \mathrm{dist}(\alpha, \beta) \geq n + \mu(G).$$



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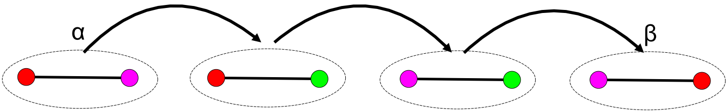
Proof: To transform α into β, we need at least three recolourings on $\{v_{2i-1}, v_{2i}\}$, for all i. So in total we need $\geq n + \mu(G)$ recolourings. □



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E.g. directly implies that $\mathrm{diam}(C_k(G)) \geq n + \mu(G)$ *if* G *complete bipartite. Also...*

Theorem (CCC, 2022+)

For every $k \geq \Delta(G) + 2$ we have

$$\mathrm{diam}_k(G) \geq n + \mu(G)$$

in each of the following cases:

- $\Delta(G) \leq 3$;
- G triangle-free with $\Delta(G)$ suff. large;
- $G = G_{n,p}$ the random graph with $p \in (0, 1)$ fixed, (a.a.s. as $n \to \infty$).

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Furthermore, for large enough k we achieve equality:

Theorem (CCC,2022+)

For every graph G and every $k \geq 2\Delta(G) + 1$,

$$\mathrm{diam}_k(G) = n + \mu(G)$$

Generalization to list-colouring

A *list-assignment* L provides each vertex v of G with a list $L(v)$ of possible colours. An L-colouring, is a proper colouring such that each vertex v receives a colour from $L(v)$. For fixed L, we can again define a *reconfiguration graph* $\mathcal{C}_L(G)$ for all L-colourings of G.

List Conjecture (CCC, 2022)

If $|L(v)| \geq \deg(v) + 2$ for every vertex v, then

$$\operatorname{diam}(\mathcal{C}_L(G)) \leq n + \mu(G).$$

- Best possible for each graph G, if true.

List Conjecture (CCC, 2022+)

If $|L(v)| \geq \deg(v) + 2$ for every vertex v, then $\mathrm{diam}(\mathcal{C}_L(G)) \leq n + \mu(G)$.

We proved the List Conjecture for all *trees, cycles, bipartite cubic graphs, complete bipartite graphs and complete graphs.* Furthermore,

Theorem (CCC, 2022+)

- If $|L(v)| \geq \deg(v) + 2$ for every v, then $\mathrm{diam}(\mathcal{C}_L(G)) \leq n + 2\mu(G)$.
- If $|L(v)| \geq 2\deg(v) + 1$ for every v, then $\mathrm{diam}(\mathcal{C}_L(G)) \leq n + \mu(G)$.

The first step towards the $n + 2\mu(G)$ upper bound

Lemma

Let G be an n-vertex graph and L a list-assignment s.t.$|L(v)| \geq \deg(v) + 2$ for all $v \in V(G)$. Then $\text{dist}(\alpha, \beta) \leq 2n$ for any two L-colourings α, β.

Proof:

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Proof: Induction on $|\beta(V(G))|$, the number of distinct colours under β.

By pigeon hole principle, there exists a colour c such that

$$|\alpha^{-1}(c)| \leq |\beta^{-1}(c)|.$$

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- Recolour each $v \in \alpha^{-1}(c)$ to some colour different from c.
- Then recolour $\beta^{-1}(c)$ to c.

This takes $|a^{-1}(c)| + |b^{-1}(c)| \leq 2|\beta^{-1}(c)|$ recolouring steps.

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This takes $|a^{-1}(c)| + |b^{-1}(c)| \leq 2|\beta^{-1}(c)|$ recolouring steps.
Now apply induction to $G - \beta^{-1}(c)$, with colour c removed from all lists.
In total we use $\leq \sum_{c \in \beta(V(G))} |2\beta^{-1}(c)| = 2n$ steps. □

Open problems

- Prove the List Conjecture for more graph classes. *Bipartite, complete multipartite, subcubic, outerplanar, planar, ...*
- Is it true that $\operatorname{diam}(\mathcal{C}_k(G)) \geq n + \mu(G)$ for every $k \geq \Delta(G) + 2$?
- Bonus: *Correspondence Conjecture*

Thank you for your attention!

Theorem (Cambie, C., Cranston, 2022)

$\text{diam}_k(G) \geq n + \mu(G)$ in each of the following cases:

- $k \geq 2\Delta(G)$;
- $k \geq \Delta(G) + 2 = 5$;
- $k \geq \Delta(G) + 2$ and G triangle-free with $\Delta(G)$ suff. large;
- $k \geq \Delta(G) + 2$ and $G = G_{n,p}$ the random graph (a.a.s. as $n \to \infty$).

On the other hand:

Theorem (Cambie, C., Cranston, 2022)

$\text{diam}_k(G) \leq n + 2\mu(G)$ if

- $k \geq \Delta(G) + 2$,

and $\text{diam}_k(G) = n + \mu(G)$ in each of the following cases:

- $k \geq 2\Delta(G) + 1$;
- $k \geq \Delta(G) + 2$ and G complete bipartite, complete, cycle or a tree.