Disjoint list-colorings for planar graphs

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Joint work with Stijn Cambie and Xuding Zhu

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List-packing

Introduced in 2021, the **list-packing number** of a graph G has several equivalent definitions and interpretations, e.g. in terms of

- Chromatic number of certain blow-ups of G, or
- Perfect matchings of certain hypergraphs, or
- Disjoint independent transversals, or
- Disjoint list-colorings.

Recent papers studied well-definedness of the list-packing number, bounds in terms of maximum degree and chromatic number, inverse problems, counting, computability, ...

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This talk focuses on **planar graphs** (Cambie, CvB and Zhu, 2023+)

Plan for this talk:
Motivate list-packing (of planar graphs) from the bottom-up.

Starting from...

1. coloring
2. list-coloring
3. counting list-colorings
4. list-colorings with special requests
5. balanced probability distributions on list-colorings

...we will end up with a definition of the list-packing number and see that it can be used to strengthen some of the literature on the above concepts.

Chromatic number

$\chi(G)$ the chromatic number of a graph G.

Theorem *(Appel and Haken, 1977; Grötzsch, 1959)*

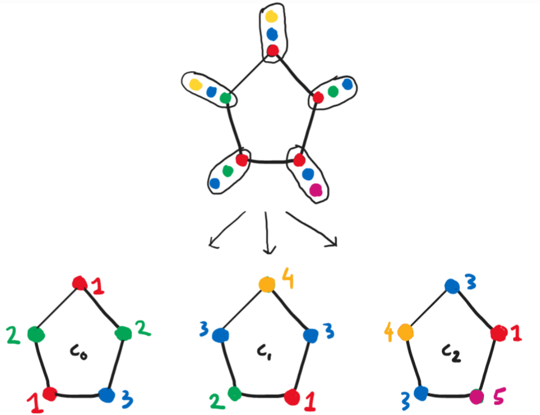
For G planar:

$$\chi(G) \leq \begin{cases} 4 \\ 3 & \text{if } G \text{ triangle-free.} \end{cases}$$

List-coloring

Definition *(Vizing, 1976; Erdős, Rubin and Taylor, 1979)*

The **list-chromatic number** $\chi_\ell(G)$ of a graph G is the smallest integer k such that *for every* k-fold list-assignment $L : V(G) \to \binom{\mathbb{N}}{k}$, there exists an L-colouring, i.e. a proper vertex-coloring c s.t. $c(v) \in L(v)$ for all v.



1990s: List-coloring planar graphs

Recall:

Theorem *(Appel and Haken, 1977; Grötzsch, 1959)*

For G planar:

$$\chi(G) \leq \begin{cases} 4 & \\ 3 & \text{if } G \text{ triangle-free.} \end{cases}$$

However, this does not generalize to list-coloring.

Theorem *(Thomassen, 1994, 1995; Voigt, 1993, 1995; Mirzakhani, 1996)*

For G planar, the optimal bounds are:

$$\chi_\ell(G) \leq \begin{cases} 5 & \\ 4 & \text{if } G \text{ triangle-free} \\ 3 & \text{if } G \text{ girth} \geq 5. \end{cases}$$

2000s: Exponentially many L-colorings

Results from the 1990s on previous slide guarantee existence of at least *one* L-coloring. In fact there exist exponentially many, i.e. $\geq c^{\#V(G)}$ for some uniform $c > 1$.

Theorem *(Thomassen, 2007; Kelly and Postle, 2008)*

For G planar, a k-fold list-assignment L admits exponentially many L-colorings in each of the following cases:

$$\begin{cases} k = 5 & \\ k = 4 & \text{and } G \text{ triangle-free} \\ k = 3 & \text{and } G \text{ girth} \geq 5. \end{cases}$$

2010s and 2020s: Flexible list-colorings

Since there exist many L-colorings, can we guarantee a very nice one?

Suppose each vertex v requests a preferred color $R(v)$ from its list. Does there exist an L-coloring that respects a large fraction of the requests?

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Suppose each vertex v requests a preferred color $R(v)$ from its list. Does there exist an L-coloring that respects a large fraction of the requests?

Definition (Dvořák, Norin and Postle, 2019)

Graph G is *ϵ-flexible* wrt list-assignment L if for every collection of requests $(R(v) \in L(v))_{v \in V(G)}$, there exists an L-coloring c s.t.

$$c(v) = R(v)$$

for at least $\epsilon \cdot \# V(G)$ of the vertices v.

Example

In special case that all vertices have the **same list** $L(v) = [k]$, it easily follows that G is $\frac{1}{k}$-flexible wrt L. (Provided $k \geq \chi_\ell(G)$)

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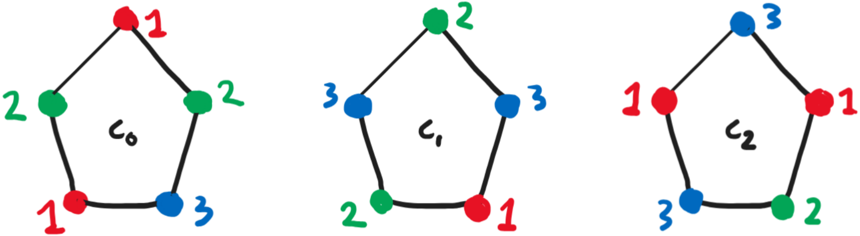
Proof sketch

Fix a k-coloring c_0, and cyclically permute it to obtain k colorings. By pigeon hole, at least one of them satisfies $\geq \frac{1}{k} \cdot \#V(G)$ requests. □



Example with $k = 3$. Three permuted 3-colorings of C_5.

Stronger property: weighted ϵ-flexible

Definition (Dvořák, Norin and Postle, 2019)

Graph G is **weighted ϵ-flexible** wrt list-assignment L if there exists a probability distribution on L-colorings c s.t. $\forall v \in V(G), \forall x \in L(v)$:

$$\mathbb{P}(c(v) = x) \geq \epsilon.$$

Remark: **weighted ϵ-flexible** implies **ϵ-flexible**:

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Remark: **weighted ϵ-flexible** implies **ϵ-flexible**:

Proof

The expected value of the number of satisfied requests is

$$\sum_{v \in V(G)} \mathbb{P}(c(v) = R(v)) \geq \sum_{v \in V(G)} \epsilon.$$

So there exists a coloring satisfying $\geq \epsilon \cdot \# V(G)$ requests. $\square$

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Remark: wrt k-fold L, the **highest value we can hope for is $\epsilon = \frac{1}{k}$**.

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Remark: wrt k-fold L, the **highest value we can hope for is $\epsilon = \frac{1}{k}$**.

Proof

Otherwise pick any vertex v. We get contradiction:

$$1 = \sum_{x \in L(v)} \mathbb{P}(c(v) = x) \geq \epsilon \cdot \#L(v) > \frac{1}{k} \cdot k = 1.$$

□

Theorem *(Dvořák, Norin and Postle, 2019; Dvořák, Masařik, Musílek, Prangrác, 2020 and 2021*; Bi and Bradshaw, 2023)

A planar G is (weighted) ϵ-flexible wrt all k-fold list-assignments L for:

$$\begin{cases} k = 7 & \\ k = 4 & \text{if } G \text{ triangle-free} \\ k = 3 & \text{if } G \text{ girth} \geq 6. \end{cases}$$

For respectively $\epsilon = 7^{-36}, \epsilon = 2^{-186}$ and $\epsilon = 2^{-30}$.

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Theorem *(Cambie, CvB., Zhu, 2023+)*

A planar G is (weighted) $\frac{1}{k}$-flexible wrt all k-fold list-assignments L for:

$$\begin{cases} k = 8 \\ k = 5 & \text{if } G \text{ triangle-free} \\ k = 4 & \text{if } G \text{ girth} \geq 5 \text{ (=optimal)} \\ k = 3 & \text{if } G \text{ girth} \geq 6. \text{ (=optimal)} \end{cases}$$

So... what about the title of this talk? Why disjoint list-colorings?

Well...

k **disjoint** L**-colorings**

implies

weighted $\frac{1}{k}$**-flexible.**

Disjoint list-colorings

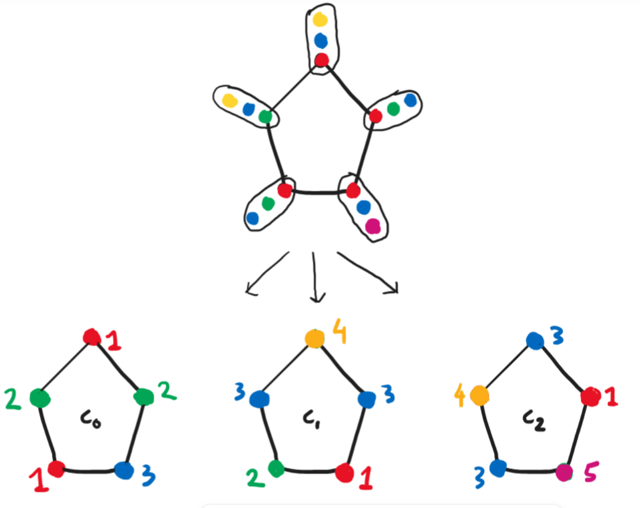
Definition (Cambie, CvB., Davies, Kang, 2021)

The **list-packing number** $\chi_\ell^\star(G)$ of a graph G is the smallest integer k such that for every k-fold list-assignment $L : V(G) \to \binom{\mathbb{N}}{k}$, there are k disjoint L-colourings $c_1, \ldots, c_k$.

I.e.: for every vertex v and every color $x \in L(v)$, there is precisely one $i \in [k]$ such that $c_i(v) = x$.

Note: $\chi_\ell(G) \leq \chi_\ell^\star(G)$, since at least one L-colouring required.

Example: $\chi_\ell^\star \leq 3$ for cycles



Top: a 3-fold L for C_5. Bottom: three disjoint L-colorings.

From list-packing to flexibility

Observation

If $\chi_\ell^\star(G) \le k$ then G is weighted $\frac{1}{k}$-flexible w.r.t every k-fold L-assignment.

Proof

Let $c_1, \ldots, c_k$ be k disjoint L-colorings. Among them, choose a uniformly random coloring. Then at every vertex v, every color $x \in L(v)$ has equal probability $\frac{1}{k}$ of being assigned to v. $\square$

Thus... our results on flexibility follow from the theorem on the next slide.

Theorem (Cambie, C., Zhu 23+, and Cranston, Smith-Roberge 24+)

For every planar graph G:

$$\chi_\ell^\star(G) \le \begin{cases} 8 & \\ 5 & \text{if triangle-free.} \\ 4 & \text{if girth at least five. (4=optimal, even for larger girth!)} \end{cases}$$

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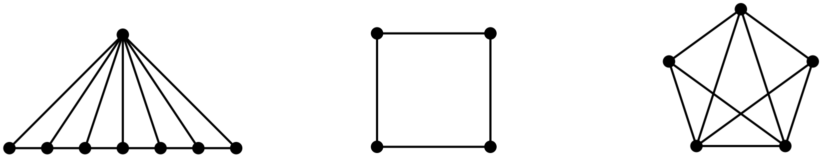
Questions

- *Does there exist a planar graph G with $\chi_\ell^\star(G) > 5$?*
- *Does there exist a triangle-free planar graph G with $\chi_\ell^\star(G) = 5$?*



$\chi_\ell = 3 < 4 = \chi_\ell^\star$ $\chi_\ell = 2 < 3 = \chi_\ell^\star$ $\chi_\ell = 4 = \chi_\ell^\star$

Going mad

Definition maximum average degree

$\text{mad}(G) := \max\{\text{average degree of } H \mid H \text{ subgraph of } G\}$

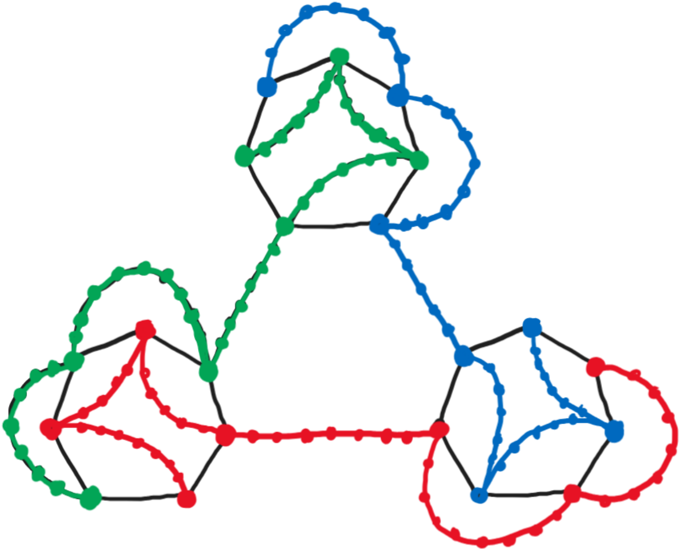
Planar graphs (and also K_5-minor-free graphs) satisfy $\text{mad}(G) < 6$.
In this way, our results follow from

Theorem (Cambie, CvB, Zhu 23+)

For every graph G,

$$\chi_\ell^\star(G) \le \begin{cases} 8 & \text{if } \text{mad}(G) < 6 \\ 5 & \text{if } \text{mad}(G) < 4 \\ 4 & \text{if } \text{mad}(G) < 10/3 \end{cases}$$

Planar graph with arbitrarily large girth, yet $\chi_\ell^* > 3$



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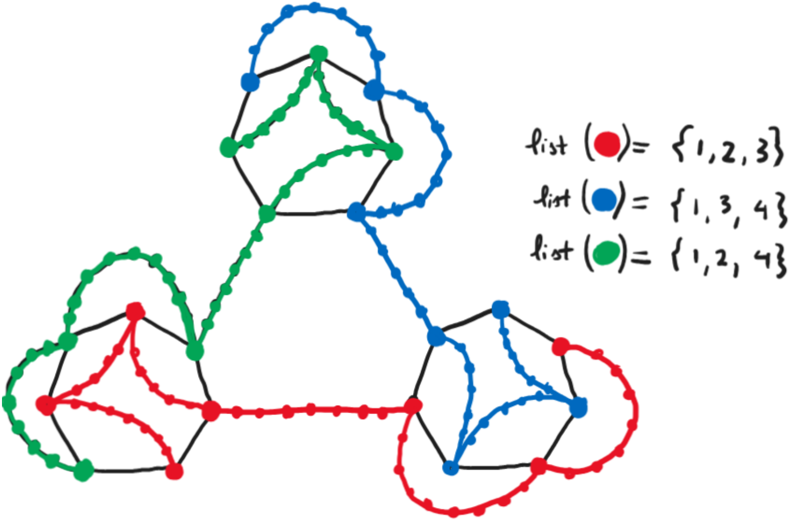

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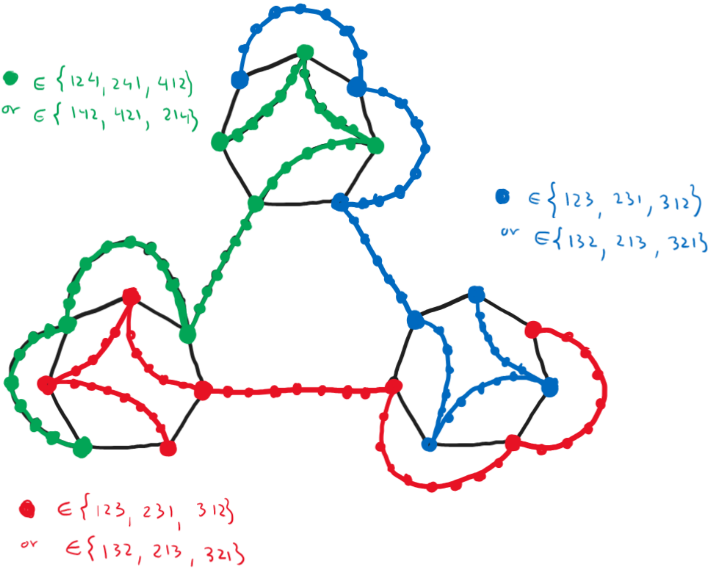
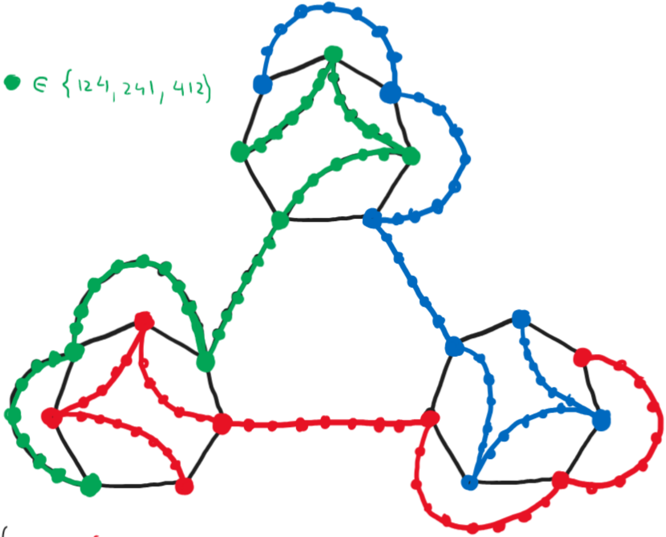


Planar graph with arbitrarily large girth, yet $\chi^*_\ell > 3$

If ● $\in \{124, 241, 412\}$



If ● $\in \{123, 231, 312\}$

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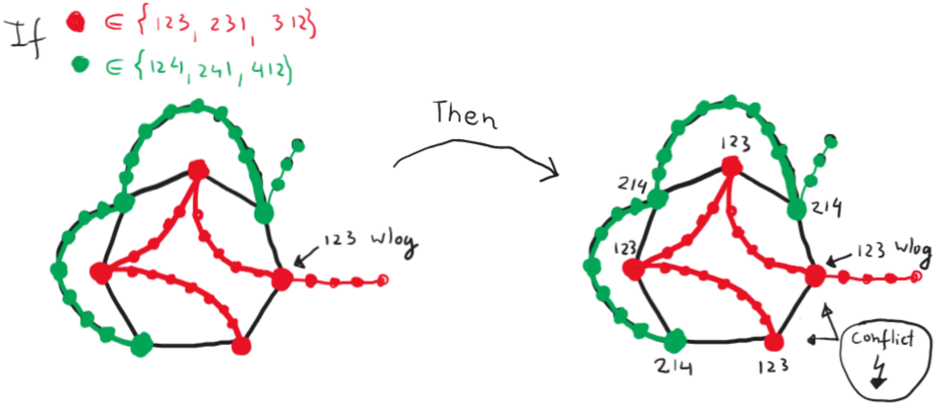

Planar girth ≥ 6 graphs

Even though $\chi_\ell^\star \leq 3$ does NOT hold for every planar large girth graph, we still have...

Theorem (Cambie, CvB, Zhu, 2023+)

Every planar girth ≥ 6 graph is weighted $\frac{1}{3}$-flexible wrt every 3-fold L.

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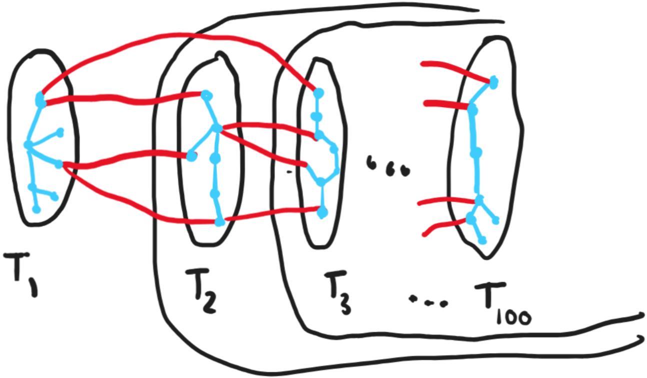
Theorem (Cambie, CvB, Zhu, 2023+)

Every planar girth ≥ 6 graph is weighted $\frac{1}{3}$-flexible wrt every 3-fold L.

Proof sketch

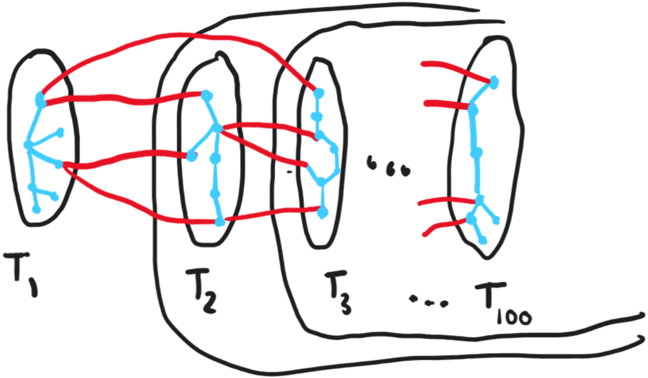
- By a Technical Lemma, it suffices to prove that G contains an induced subtree T of which every vertex has at most one neighbour in $G - V(T)$.
- Euler's formula and a subtle global discharging argument yield T. $\square$

Technical lemma; a tree-layering



Suppose G has a layering into **induced subtrees** $T_1, T_2, \ldots,$ such that each vertex in T_i has *at most one neighbour* in the layers $T_1, \ldots, T_{i-1}$ to its left. Then the whole graph is **weighted $\frac{1}{3}$-flexible**.

Technical lemma; a tree-layering



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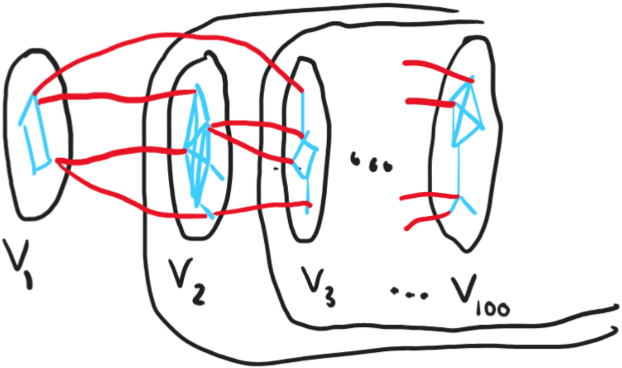
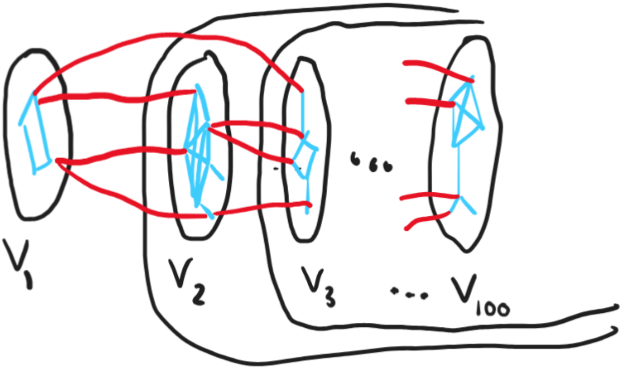
This works because trees are weighted $\frac{1}{2}$-flexible wrt every 2-fold L

Technical lemma; more general



A layering of the vertices of G. If each layer V_i is **weighted $\frac{1}{k-1}$-flexible**, and each vertex in V_i has *at most one neighbour* in the layers $V_1, \ldots, V_{i-1}$ to its left, then the whole graph is **weighted $\frac{1}{k}$-flexible**.

Technical lemma; more general



(Can be applied to any graph class with a nice layered structure;
Cartesian products, graphs with bounded treedepth, ...)

Summary

List-packing lies at the base of a sequence of implications.

$$\chi_\ell^\star(G) \le k \Leftrightarrow G \text{ admits } k \text{ disjoint } L\text{-colorings}_{\text{ wrt every } k\text{-fold } L}$$

$$\Rightarrow G \text{ is weighted } \frac{1}{k}\text{-flexible}_{\text{ wrt every } k\text{-fold } L}$$

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$$\Rightarrow G \text{ has an } L\text{-coloring}_{\text{ for every } k\text{-fold } L} \quad \Leftrightarrow \chi_\ell(G) \le k$$

$$\Rightarrow \chi(G) \le k.$$

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$$\begin{aligned}
\chi_\ell^\star(G) \le k &\Leftrightarrow G \text{ admits } k \text{ disjoint } L\text{-colorings} \text{ {\scriptsize wrt every k-fold L}} \\
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&\Rightarrow G \text{ is } \frac{1}{k}\text{-flexible} \text{ {\scriptsize wrt every k-fold L}} \\
&\Rightarrow G \text{ has an } L\text{-coloring} \text{ {\scriptsize for every k-fold L}} \qquad \Leftrightarrow \chi_\ell(G) \le k \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Rightarrow \chi(G) \le k.
\end{aligned}$$

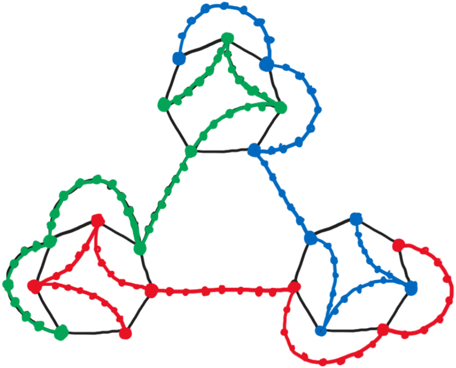
- Through bounding $\chi_\ell^\star$, we improved results on weighted ϵ-flexibility, with optimal value $\epsilon = \frac{1}{k}$.
- In some cases we directly proved weighted $\frac{1}{k}$-flexibility, via a graph layering argument.

Many open problems

What is the optimal upper bound on $\chi_\ell^\star(G)$ if G is ...

- Planar. $5, 6, 7$ or 8?
- Planar triangle-free. 4 or 5?
- Planar bipartite. $3, 4$ or 5?
- Bounded treewidth?

Thank you!



Cambie, S., Cames van Batenburg, W. and Zhu, X., Disjoint list-colorings for planar graphs, arxiv:2312.17233

Technical lemma- exact statement of special case

The following holds for *every* graph G:

Key technical lemma

Let $k \geq 2$. If there is an induced subgraph T of G s.t.

1. Every vertex of T has at most one neighbour in $G - V(T)$;
2. T is weighted $\frac{1}{k-1}$-flexible;
3. $G - V(T)$ is weighted $\frac{1}{k}$-flexible;

Then G is weighted $\frac{1}{k}$-flexible.

Mad question

Question

Let $k \in \mathbb{N}$. Does $\mathrm{mad}(G) < k$ imply $\chi_\ell^\star(G) \leq k + 1$?