Modeling Intelligent Reconfigurable Surfaces:

We consider that we have a fixed coordinates system (x, y, z) defined by the location of the metasurface. This surface will be lying in the x-y plane of our coordinates system. The first element of this metasurface (upper left element) will be located at point (0, 0, 0) the origin of our coordinate system. The width of the surface will be spanning along the x direction, and the height will go along the y direction.
The z-axis will be the axis perpendicular to the surface, so the normal vector to the surface will be the unit vector $u_z = [0, 0, 1]$ along the z-axis.

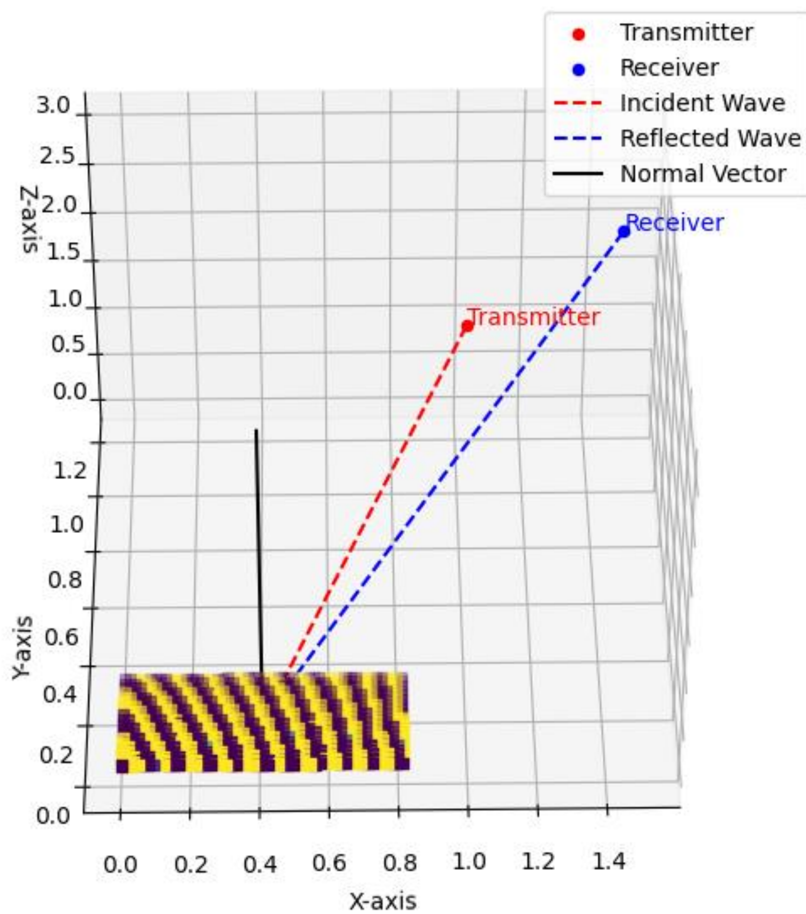

Figure 1: Metasurface model

We start by giving the following inputs to the model:

- Transmitter coordinates: $t_x = [x_i, y_i, z_i]$
- Receiver coordinates: $r_x = [x_r, y_r, z_r]$

- Index of refraction of free space n_i

- For the transmitted signal:
  - Transmitted Signal Frequency $f\ (in\ Hz)$
  - Transmitted Signal Amplitude $A_i\ (in\ V)$
  - Transmitted Signal Phase $\varphi_i\ (in\ rad)$

- For the varactor:
  - Element Resistance value $R\ (in\ \Omega)$
  - Element bottom layer inductance $L_1\ (in\ H)$
  - Element top layer inductance $L_2\ (in\ H)$
  - Element effective capacitance $C\ (in\ F)$
  - Capacitance range that the varactor can produce.

- For the metasurface:
  - Surface dimensions (w_x, h_y)
    Where w_x is the number of elements along the width of the surface (in the x direction)
    h_y is the number of elements along the height of the surface (in the y direction)
  - Elements size $e\ (in\ m)$
    Where the elements are considered as square with edge length e.
  - Element spacing $\Delta_e\ (in\ m)$
    Where element spacing Δ_e is the spacing between the edge of the first element and the edge of the second element.
    Element spacing is the same in both x and y directions.

Then using the inputs, we previously mentioned, we can calculate some parameters as follows:

- Wavelength $\lambda = \frac{c}{f}\ (in\ m)$
  Where $c = 3 \times 10^8$ is the speed of light
- $Angular\ frequency\ \omega = 2\pi f\ (in\ \ rad/s)$
- $Wave\ nunmber\ k_0 = \frac{2\pi}{\lambda}$

- The distance between the middle of 2 consecutive elements of the surface in both x and y directions: $\Delta = e + \Delta_e\ (in\ m)$

The aim of this work is to create a Reconfigurable intelligent surface that can be reprogramed instantaneously in order to reflect an input signal from a given transmitter toward a known receiver. The main working principle of this surface will follow the generalized Snell's law for anomalous reflection in 3D space. This law is modeled by the following equations:

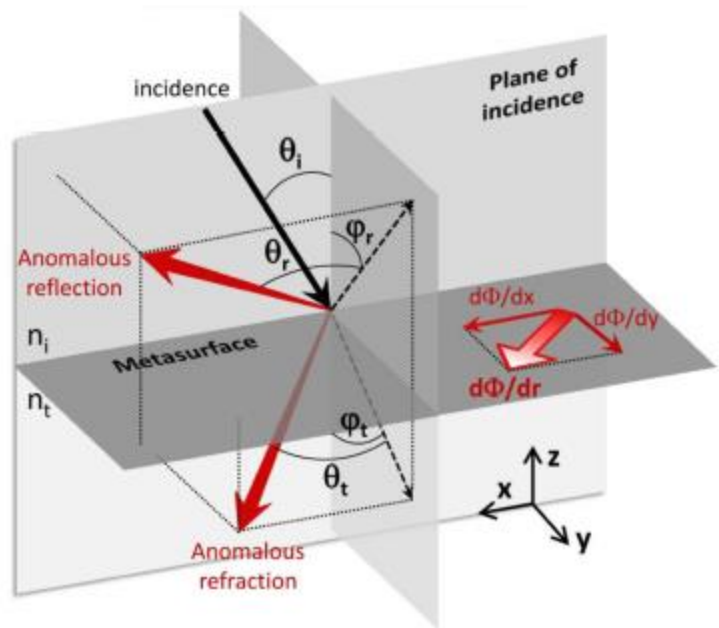

Figure 2: figure showing the angles used in the generalized Snell's law equations.

$$\sin(\theta_r) - \sin(\theta_i) = \frac{1}{n_i k_0}\frac{\partial \phi(x, y)}{\partial x}$$
$$\cos(\theta_r)\sin(\varphi_r) = \frac{1}{n_i k_0}\frac{\partial \phi(x, y)}{\partial y}$$

We denote by $v_{r,proj}$ the projection of the reflected vector v_r onto the plane perpendicular to the incident plane of incidence.
φ_r is the angle between the vector $v_{r,proj}$ projection of the reflected vector and the z-axis.
θ_r is the angle between the reflected vector v_r and its projection vector $v_{r,proj}$

We know the location of the transmitter and the receiver, and we want to properly reflect the signal from toward the receiver.
Based on this information, we can geometrically calculate the values of the reflection angles θ_r and φ_r to be able to calculate later the phase shifts needed for every element.
We consider that the transmitter in an omnidirectional antenna radiating in all directions. In our model we will discretize the propagation sphere by modeling it using equidistant rays generated from the transmitter and hitting each element of our surface. So, our model will only consider the part of the transmission sphere that will reach our surface, and it will be modeled as a ray hitting each element of our surface. So, the number of considered rays is equal to the number of elements on the surface. These rays will be represented by a (w_x, h_y) matrix each entry of this matrix contain $v_i = [x_i, y_i, z_i]$ the coordinates of the incident ray hitting the corresponding element. These vectors will be calculated geometrically for every element using the following formula:
$$v_i = element[x, y, z] - t_x$$

Since our goal is to reach the receiver with all the reflected rays, we will calculate the theoretical reflected rays considering all of them will reach the receiver. These rays will be represented by a (w_x, h_y)

matrix each entry of this matrix contain $v_r = [x_r, y_r, z_r]$ the coordinates of the reflected ray hitting the receiver. These vectors will be calculated geometrically for every element using the following formula:

$$v_r = r_x - element[x, y, z]$$

After calculating the incident and the reflected vectors, the next step is to calculate the incident and the reflection angles. These angles are θ_i, θ_r and φ_r shown in the Figure 1 above.

For the incident angles, they will also be represented by a (w_x, h_y) matrix, where each entry will represent the incident angle of the corresponding vector v_i.

We know that θ_i is the angle between the incident vector v_i and the normal to the plane which is in our case unit vector $u_z = [0, 0, 1]$ along the z-axis.

Then to calculate θ_i we will use the dot product equation:

$$-v_i \cdot u_z = |v_i||u_z| \cos \theta_i$$

$$\theta_i = \cos^{-1} \frac{-v_i \cdot u_z}{|v_i||u_z|}$$

We added a minus sign (-) to the incident vector $(-v_i)$ to reverse the direction of the vector to have both v_i and u_z in the same direction so we can calculate the small angle between them which is θ_i.

Similarly, for θ_r and φ_r, we use the dot product notation to calculate these angles between the reflected vector v_r and $v_{r,proj}$, and $v_{r,proj}$ and u_z respectively. We remind you that $v_{r,proj}$ is the projection of the reflected vector v_r onto the plane perpendicular to the incident plane of incidence.

$$v_{r,proj} \cdot u_z = |v_{r,proj}||u_z| \cos \varphi_r$$

$$\varphi_r = \cos^{-1} \frac{v_{r,proj} \cdot u_z}{|v_{r,proj}||u_z|}$$

and

$$v_r \cdot v_{r,proj} = |v_r||v_{r,proj}| \cos \theta_r$$

$$\theta_r = \cos^{-1} \frac{v_r \cdot v_{r,proj}}{|v_r||v_{r,proj}|}$$

To calculate $v_{r,proj}$ the projection of the reflected vector v_r onto the plane perpendicular to the incident plane of incidence, we will perform the following steps:

1. Find n_i the normal vector of the plane of incidence.
   We know that the incident vector v_i is inside the plane of incidence, we also know that the unit vector $u_z = [0, 0, 1]$ along the z-axis is parallel to the plane of incidence. Then to find the normal vector to the plane of incidence, we will perform the cross product of the mentioned vectors.
   $$n_i = v_i \times u_z$$
2. Find $n_{r,proj}$ the normal vector of the plane perpendicular to the plane of incidence (we will call this plane $P_{r,proj}$).
   We know that the unit vector $u_z = [0, 0, 1]$ along the z-axis is parallel to the plane $P_{r,proj}$. Additionally, since $P_{r,proj}$ is perpendicular to the plane of incidence, then the vector n_i normal to the plane of incidence will be also parallel to the plane $P_{r,proj}$. Moreover, from the previous step we know that vectors u_z and n_i are perpendicular to each other. Then the normal to the plane $P_{r,proj}$ is the vector perpendicular to both u_z and n_i. To calculate this vector, we will perform the cross product between vectors u_z and n_i.
   $$n_{r,proj} = v_i \times u_z$$
3. Calculate the projection of the reflected vector v_r onto the plane $P_{r,proj}$.

To do so, we first must calculate the projection of the reflected vector v_r onto the vector $n_{r,proj}$ which is the normal vector to the plane:

$$v_{proj-v_r-n_{r,proj}} = \frac{v_r \cdot n_{r,proj}}{\left|n_{r,proj}\right|^2} \, n_{r,proj}$$

Then we calculate $v_{r,proj}$, the projection of the reflected vector v_r onto the plane $P_{r,proj}$:

$$v_{r,proj} = v_r - v_{proj-v_r-n_{r,proj}}$$

This process will be performed $w_x \times h_y$ times computing the coordinates of vector $v_{r,proj}$ the projection of the reflected vector v_r onto the plane perpendicular to the incident plane for every incident vector. So, in the end we will have a $\left(w_x, h_y\right)$ matrix containing $v_{r,proj}$ to the corresponding incident vector v_i.

After successfully calculating the angles θ_i, θ_r and φ_r geometrically, now we can calculate the gradient phase shifts needed to reflect the transmitted signal toward the receiver location. To compute the phase shift gradient in both x and y direction we will use the generalized Snell's law that we presented previously.

$$\frac{\partial \phi}{\partial x}(x, y) = n_i k_0 (\sin(\theta_r) - \sin(\theta_i))$$

$$\frac{\partial \phi}{\partial y}(x, y) = n_i k_0 \cos(\theta_r) \sin(\varphi_r)$$

After this step we will have 2 $\left(w_x, h_y\right)$ matrices first is the phase gradient along x direction, the second is the phase gradient along the y direction.

The next step is to use the $\frac{\partial \phi}{\partial x}(x, y)$ and the $\frac{\partial \phi}{\partial y}(x, y)$ matrices to find the phase shift $\phi(x, y)$ for every element of the surface. To do so, we have to start thinking in the forward direction on how the gradients are calculated before thinking how to recover the phase shift function $\phi(x, y)$ from its gradients. Then we will analyze the results of the forward thinking and apply it to the process of recovering $\phi(x, y)$.

The following equations are used to calculate the gradients of a function in both x and y direction:

$\text{In the x direction } \frac{\partial \phi}{\partial x}(x, y):$

$$First\ column\ (x = 0): \frac{\partial \phi}{\partial x}(x, y) = \frac{\phi(x + 1, y) - \phi(x, y)}{\Delta x}$$

$$Last\ column\ (x = -1): \frac{\partial \phi}{\partial x}(x, y) = \frac{\phi(x, y) - \phi(x - 1, y)}{\Delta x}$$

$$Middle\ columns\ (x = [1, \dots, -2]): \frac{\partial \phi}{\partial x}(x, y) = \frac{\phi(x + 1, y) - \phi(x - 1, y)}{2 \times \Delta x}$$

In the y direction $\frac{\partial\phi}{\partial y}(x,y)$:

$First\ column\ (y=0)$: $\frac{\partial\phi}{\partial y}(x,y) = \frac{\phi(x,y+1)-\phi(x,y)}{\Delta y}$

$Last\ column\ (y=-1)$: $\frac{\partial\phi}{\partial y}(x,y) = \frac{\phi(x,y)-\phi(x,y-1)}{\Delta y}$

$Middle\ columns\ (y=[1,\dots,-2])$: $\frac{\partial\phi}{\partial y}(x,y) = \frac{\phi(x,y+1)-\phi(x,y+1)}{2\times\Delta y}$

$\phi(x,y)$ will be a 2D array of size (w_x,h_y) as follows:

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	…
b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	…
c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	…
d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	…
e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	…
f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	…
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	…

To be able to calculate $\phi(x,y)$ from its gradients $\frac{\partial\phi}{\partial x}(x,y)$ and $\frac{\partial\phi}{\partial y}(x,y)$, we have to fix a starting point in the $\phi(x,y)$ and used it along with the $\frac{\partial\phi}{\partial x}(x,y)$ and $\frac{\partial\phi}{\partial y}(x,y)$ values to calculate the rest of the $\phi(x,y)$ values.

So, we fix the first value of $\phi(x,y)$ and we assume it to be 0. $\phi(0,0)=0$

Now to calculate the rest of the values of the phase shifts matrix $\phi(x,y)$ we can use an altered version of the derivatives formulas as follows:

In the x direction $\frac{\partial\phi}{\partial x}(x,y)$:

$First\ column\ (x=0)$: $\phi(x+1,y) = \left(\frac{\partial\phi}{\partial x}(x,y)\times\Delta x\right)+\phi(x,y)$

$Last\ column\ (x=-1)$: $\phi(x,y) = \left(\frac{\partial\phi}{\partial x}(x,y)\times\Delta x\right)+\phi(x-1,y)$

$Middle\ columns\ (x=[1,\dots,-2])$: $\phi(x+1,y) = \left(\frac{\partial\phi}{\partial x}(x,y)\times 2\times\Delta x\right)+\phi(x-1,y)$

In the y direction $\frac{\partial\phi}{\partial y}(x,y)$:

$First\ column\ (y=0)$: $\phi(x,y+1) = \left(\frac{\partial\phi}{\partial y}(x,y)\times\Delta y\right)+\phi(x,y)$

$Last\ column\ (y=-1)$: $\phi(x,y) = \left(\frac{\partial\phi}{\partial y}(x,y)\times\Delta y\right)+\phi(x,y-1)$

$Middle\ columns\ (y=[1,\dots,-2])$: $\phi(x,y+1) = \left(\frac{\partial\phi}{\partial y}(x,y)\times 2\times\Delta y\right)+\phi(x,y-1)$

Now we will write some of the equation to calculate the elements of the phase shifts $\phi(x,y)$ matrix:

Row 1	*Column 1*
$a_1 = 0$	$a_1 = 0$
$a_2 = \left(\frac{\partial\phi}{\partial x}(0,0) \times \Delta x\right) + a_1$	$b_1 = \left(\frac{\partial\phi}{\partial y}(0,0) \times \Delta y\right) + a_1$
$a_3 = \left(\frac{\partial\phi}{\partial x}(1,0) \times 2 \times \Delta x\right) + a_1$	$c_1 = \left(\frac{\partial\phi}{\partial y}(0,1) \times 2 \times \Delta y\right) + a_1$
$a_4 = \left(\frac{\partial\phi}{\partial x}(2,0) \times 2 \times \Delta x\right) + a_2$	$d_1 = \left(\frac{\partial\phi}{\partial x}(0,2) \times 2 \times \Delta y\right) + b_1$
$a_5 = \left(\frac{\partial\phi}{\partial x}(3,0) \times 2 \times \Delta x\right) + a_3$	$e_1 = \left(\frac{\partial\phi}{\partial x}(0,3) \times 2 \times \Delta y\right) + c_1$
$a_6 = \left(\frac{\partial\phi}{\partial x}(4,0) \times 2 \times \Delta x\right) + a_4$	$f_1 = \left(\frac{\partial\phi}{\partial y}(0,4) \times 2 \times \Delta y\right) + d_1$

If we notice from the equations above that row 1 can be calculating uniquely starting with a_1 and using only the gradient along x; $\left(\frac{\partial\phi}{\partial x}\right)$.
Similarly, column 1 can be calculating uniquely starting with a_1 and using only the gradient along y; $\left(\frac{\partial\phi}{\partial y}\right)$.
Then by calculating the unique values of row 1 from $\frac{\partial\phi}{\partial x}$ and column 1 from $\frac{\partial\phi}{\partial y}$, we can use these values to calculates the rest of the phase shift values successively.
To calculate the other values, we will take as examples b_2 and c_3

	b_2	c_3
In x direction $\left(Using\ \frac{\partial\phi}{\partial x}\right)$	$b_2 = \left(\frac{\partial\phi}{\partial x}(0,1) \times \Delta x\right) + b_1$	$c_3 = \left(\frac{\partial\phi}{\partial x}(1,2) \times 2 \times \Delta x\right) + c_1$
In y direction $\left(Using\ \frac{\partial\phi}{\partial y}\right)$	$b_2 = \left(\frac{\partial\phi}{\partial y}(1,0) \times \Delta y\right) + a_2$	$c_3 = \left(\frac{\partial\phi}{\partial y}(2,1) \times 2 \times \Delta y\right) + a_3$

As we can see we can calculate the same value of the phase shift function in two different ways using $\frac{\partial\phi}{\partial x}$ and $\frac{\partial\phi}{\partial y}$ but since we only have a single phase shift function so both ways should give the same result. In other words, in order for the phase shift gradient $\frac{\partial\phi}{\partial x}$ and $\frac{\partial\phi}{\partial y}$ matrices that we have to be the gradient of the phase shift function $\phi(x,y)$, then it does not matter which gradient it is used to calculate a given value since with both we will get the same results.
After this analysis, our strategy to find the phase shifts function matrix of $\phi(x,y)$ from its gradients $\frac{\partial\phi}{\partial x}$ and $\frac{\partial\phi}{\partial y}$ is to calculate first two phase shifts matrices $\phi_x(x,y)$ and $\phi_y(x,y)$ using $\frac{\partial\phi}{\partial x}$ and $\frac{\partial\phi}{\partial y}$ respectively. Noting that in both matrices will have the same first column and first rows, where the first column is calculated starting with a_1 and using only the gradient along y; $\left(\frac{\partial\phi}{\partial y}\right)$ and the first row is calculated starting with a_1 and using only the gradient along x; $\left(\frac{\partial\phi}{\partial x}\right)$.

Theoretically after these calculations, $\phi_x(x,y)$ and $\phi_y(x,y)$ should be exacly similar to each other's, but given that we also have some imperfections when calculating initially the phase gradients $\frac{\partial\phi}{\partial x}$ and $\frac{\partial\phi}{\partial y}$, we expect some error margin which will be shown by some differences in $\phi_x(x,y)$ and $\phi_y(x,y)$. To solve this issue and to have finally one unique phase shift matrix, we will take the average of both matrices by summing them on element basis and dividing by 2. In the end, we will have the final phase shift function.

$$\phi(x,y) = \frac{\phi_x(x,y) + \phi_y(x,y)}{2}$$

This function represents the phase shift that each element should apply on the incoming signal in order to have the desired reflection. $\phi(x,y)$ will be a 2D matrix of size (w_x, h_y), where ech entry represents the phase shift required to be produced by the corresponding element of the metasurface.

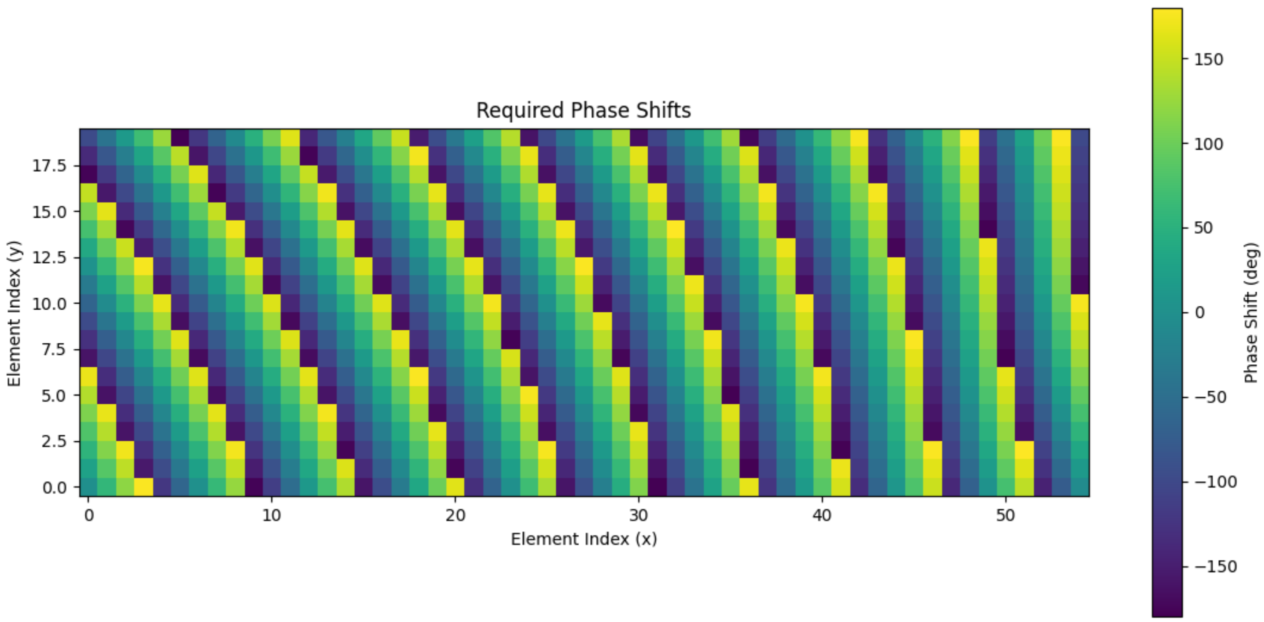

Figure 3: Required phase shifts.

Now that we have calculated the phase shift required by every element of the surface, we should procced in calculating the required capacitance for the element to produce this phase shift, and ultimately calculate the bias voltage that should be supplied to the varactor in order the produce the required capacitance.

We can find the phase shift of an element by checking its reflection coefficient Γ.

$$\Gamma = \frac{Z_n - Z_0}{Z_n + Z_0} = A_n e^{j\phi_n}$$

Where:

- A_n is the amplitude which practically will be in the range $[0, \ldots, 1]$ and considered as the resistive loss that the signal will endure when hitting a given element.
- ϕ_n is the actual phase shift calculated previously.
- Z_0 is the impedance of free space. $Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \approx 377\Omega$
- Z_n is the electrical impedance of the element. This impedance depends on the electronic model of a single element.
  Below we will see the reflecting element electronic model and we will calculate Z_n.

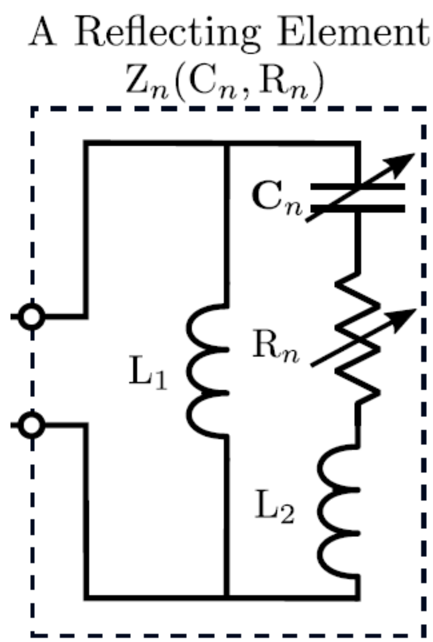

Figure 4: reflecting Element electronic model

We can see in Figure 4 the electronic model for a reflecting element. The impedance of this element can be calculated as follows:

$$Z_n = \frac{j\omega L_1 \left(j\omega L_2 + \frac{1}{j\omega C_n} + R_n\right)}{j\omega L_1 + \left(j\omega L_2 + \frac{1}{j\omega C_n} + R_n\right)}$$

Where:

- R_n: effective resistance of element
- L_1: bottom layer inductance of the element
- L_2: top layer inductance of the element
- C_n: effective capacitance of an element
- ω: angular frequency ($\omega = 2\pi f$)

Then to calculate the required capacitance value that will create the desired phase shift, we have to guess a C_n value that will give the element a certain impedance Z_n and plugging this Z_n in the reflection coefficient equation Γ we should have the angle equal to the desired phase shift. The range of C_n values is decided by the model of the varactor used in the element and the capacitance range that it is able to produce when given different voltages. This varactor should be sized to correspond to the predicted frequencies that will be used on this surface. In other words, the capacitance range that this varactor should produce should be in the exact range to be able to cover all the angles between $[-180° \ 180°]$.

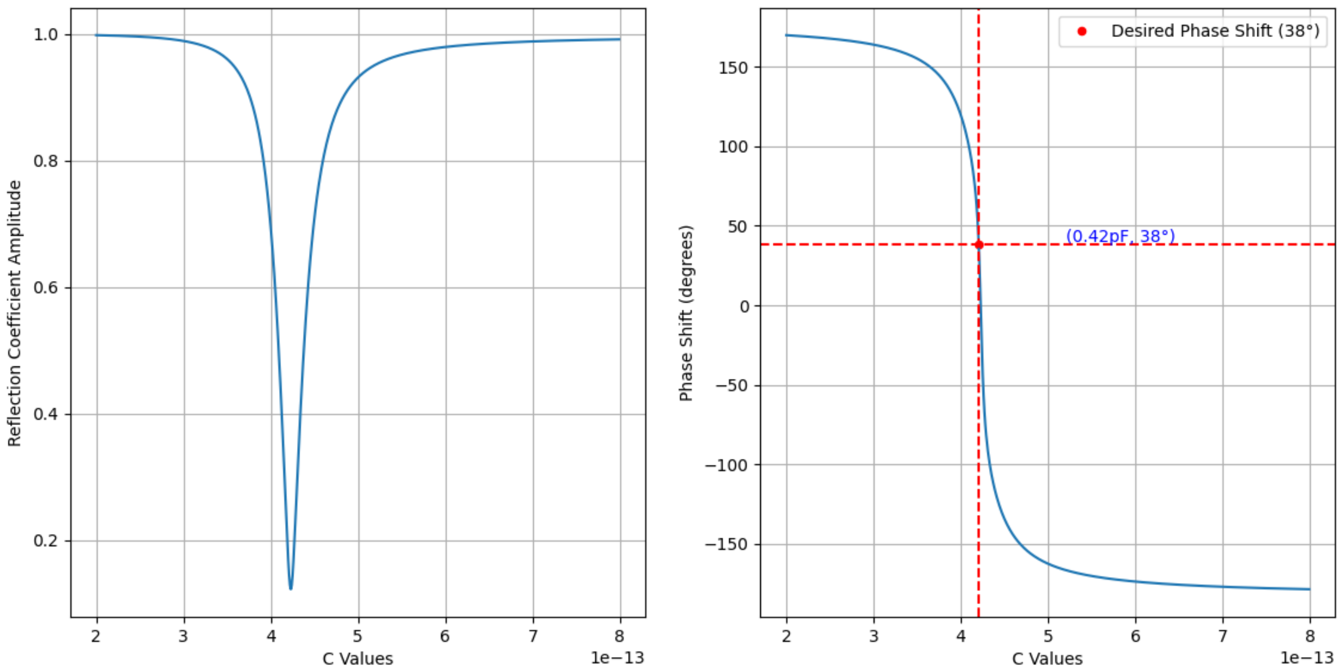

Figure 5: Amplitudes and Angles of the reflection coefficient with different C values (for a frequency f=10GHz)

In the Figure 5 above we can see the amplitudes and the angles of the reflection coefficient with different C values for a frequency $f = 10GHz$. We can see that the capacitances are chosen in the right range for the reflection coefficient angle to cover a range very close to the $[-180° \ 180°]$ range.

In what follows I will describe the strategy used to estimate the capacitance needed to create the desired phase shift:

1. First, as we spoke earlier, we will identify the range of that capacitance C_n available. We should create a 1D matrix containing these capacitance values.
2. Using the available capacitance matrix, we will calculate the element impedance Z_n that could be created for each value of C_n. (Knowing that the values of R_n, L_1, L_2 are constants).
   Then the result will be an array of impedances having the same length as the array of available capacitances, and the value of the impedance on a given location of the array will corresponds to the value of the capacitance form the available capacitances array in the same location.
3. Using the achievable element impedances calculated earlier, we will now calculate the reflection coefficient Γ of the element that could be achieved by the given element impedances.

The result will also be an array of reflection coefficients having the same length as the array of available capacitances, and the value of the reflection coefficient on a given location of the array will corresponds to the value of the capacitance form the available capacitances array in the same location.

4. In this step we will calculate the angles of every reflection coefficient $angle_{\Gamma} = angle(\Gamma) = \phi_n$. We have in the elements achievable reflection coefficients array calculated in the previous step. The result will also be an array of reflection coefficients angles having the same length as the array of available capacitances, and the value of the reflection coefficient angle on a given location of the array will corresponds to the value of the capacitance form the available capacitances array in the same location.
5. Now we have a connection between the capacitance and the phase shifts angles (which is basically the reflection coefficients angles array). The last thing left to do is to estimate the value of the capacitance C for a given phase shift. This estimation will be done by interpolation. (Note: the more capacitance values we have in the initial available capacitance matrix, the more accurate the estimated capacitance will be)

So finally, we take the phase shifts matrix $\phi(x, y)$ that we calculated previously and apply the capacitance estimation method that we discussed earlier to find in the end the required capacitance for every element to achieve the desired phase shift. The result after this method will be a 2D matrix of size (w_x, h_y), where each entry represents the required capacitance to be tuned in the corresponding element of the metasurface to achieve its desired phase shift $\phi(x, y)$.

In the next step, we will calculate the real phase shift that will be actually produced by the surface based on the estimated capacitance value we calculated in the previous step. To do so, we must follow the following process:

1. Using the estimated elements capacitance matrix calculated earlier, we will calculate the real element impedance Z_n using its equation that we provided earlier. (Knowing that the values of R_n, L_1, L_2 are constants).
   Then the result will be a 2D matrix of size (w_x, h_y), where each entry represents the actual impedance of the corresponding element on the metasurface.
2. Using the elements impedances calculated earlier, we will now calculate the reflection coefficient Γ of the elements given their impedances.
   Then the result will be a 2D matrix of size (w_x, h_y), where each entry represents the actual reflection coefficient of the corresponding element on the metasurface.
3. Now to find the real phase shifts, all what is left to do is to calculate the angle of each reflection coefficient in the real reflection coefficient matrix.
   Then the result will be a 2D matrix of size (w_x, h_y), where each entry represents the real phase shifts $\phi_{real}(x, y)$ that will be introduced by the corresponding element on the metasurface.

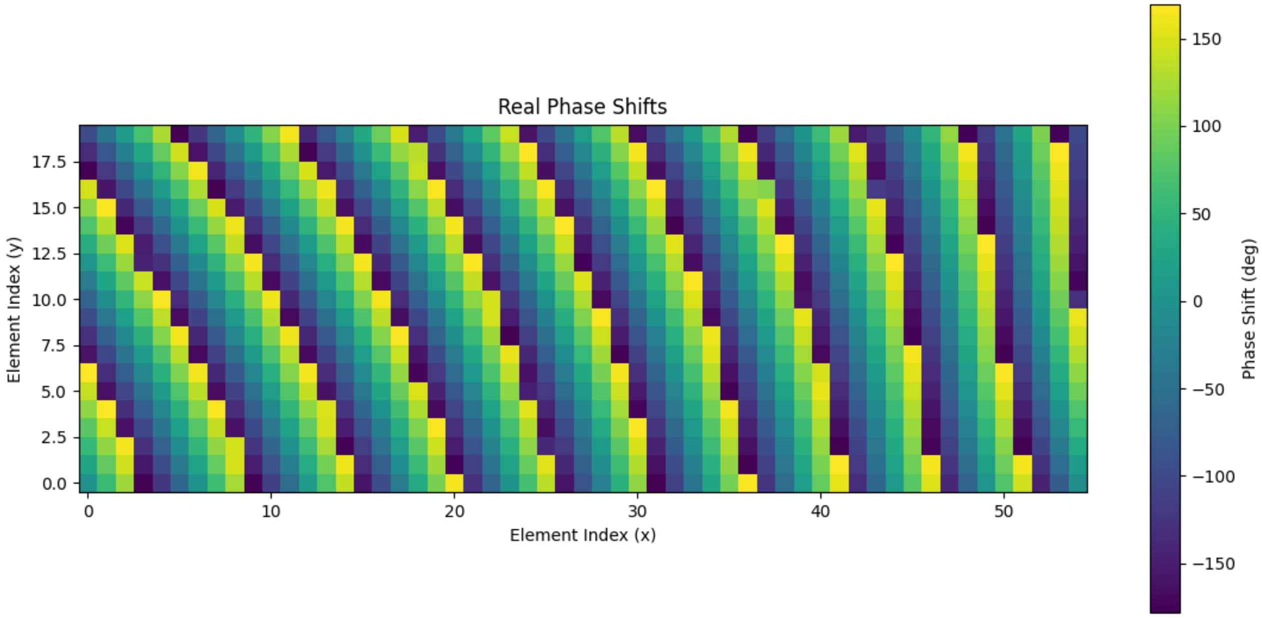

Figure 6: Real Phase shifts

Then from the calculated real phase shifts of each element we can calculate the real reflection angles θ_r and φ_r of the reflected signal. To do so, we will derive $\phi_{real}(x,y)$ in both x and y directions to get the gradient of the real phase shift used in the generalized Snell's law.

To derive real phase shifts $\phi_{real}(x,y)$ we will be using the following derivation formulas, considering the fact that the phase shifts should always be in the $[-\pi\ \pi]$ range, so even when subtracting 2 values of the $\phi_{real}(x,y)$, we should make sure that it belongs to the range $[-\pi\ \pi]$.

As an example:

$$[\phi_{real}(x+1,y)-\phi_{real}(x,y)]mod[-\pi\ \pi]$$

This equation means that the resultant of $\phi_{real}(x+1,y)-\phi_{real}(x,y)$ will be moved to the $[-\pi\ \pi]$ range.

Taking all of the above into account we will have the following equations to perform the derivatives:

In the x direction $\frac{\partial\phi_{real}}{\partial x}(x,y)$:

$$First\ column\ (x=0): \frac{\partial\phi_{real}}{\partial x}(x,y)=\frac{[\phi_{real}(x+1,y)-\phi_{real}(x,y)]mod[-\pi\ \pi]}{\Delta x}$$

$$Last\ column\ (x=-1): \frac{\partial\phi_{real}}{\partial x}(x,y)=\frac{[\phi_{real}(x,y)-\phi_{real}(x-1,y)]mod[-\pi\ \pi]}{\Delta x}$$

$$Middle\ columns\ (x=[1,\dots,-2]): \frac{\partial\phi_{real}}{\partial x}(x,y)=\frac{[\phi_{real}(x+1,y)-\phi_{real}(x-1,y)]mod[-\pi\ \pi]}{2\times\Delta x}$$

In the y direction $\frac{\partial \phi_{real}}{\partial y}(x,y)$:

$First\ column\ (y=0)$: $\frac{\partial \phi_{real}}{\partial y}(x,y) = \frac{[\phi_{real}(x,y+1) - \phi_{real}(x,y)] mod[-\pi\ \pi]}{\Delta y}$

$Last\ column\ (y=-1)$: $\frac{\partial \phi_{real}}{\partial y}(x,y) = \frac{[\phi_{real}(x,y) - \phi_{real}(x,y-1)] mod[-\pi\ \pi]}{\Delta y}$

$Middle\ columns\ (y=[1,\dots,-2])$: $\frac{\partial \phi_{real}}{\partial y}(x,y) = \frac{[\phi_{real}(x,y+1) - \phi_{real}(x,y+1)] mod[-\pi\ \pi]}{2 \times \Delta y}$

Then after successfully finding the gradients of the real phase shifts $\phi_{real}(x,y)$, $\frac{\partial \phi_{real}}{\partial x}$ and $\frac{\partial \phi_{real}}{\partial y}$. We can now use them to calculate the real reflected angles θ_r and φ_r based on the generalized Snell's law of reflection. Then:

$$\theta_r = \sin^{-1}\left(\frac{1}{n_i k_0}\frac{\partial \phi(x,y)}{\partial x} + \sin(\theta_i)\right)$$

$$\varphi_r = \sin^{-1}\frac{1}{n_i k_0 \cos(\theta_r)}\frac{\partial \phi(x,y)}{\partial y}$$

The results after this method will be two 2D matrix of size (w_x, h_y) each, the first is the angle θ_r of the reflected ray from each element. The second is the angle φ_r of the reflected ray from each element.

By finding the real reflection angles θ_r and φ_r, and knowing the location of the transmitter, the location of the receiver and the location of every element of the metasurface, we can now calculate the real reflected vector. Then after calculating the reflected vector, we will check if it hits the antenna of the receiver successfully by checking if the endpoint of the reflected vector is inside the receiver antenna defined by its shape, its dimensions and its center which is the coordinates of the receiver that we defined in the beginning.

The first step is to find the real $v_{r,proj}$ the projection of the reflected vector v_r onto the plane perpendicular to the incident plane of incidence. We denote this plane by $P_{r,proj}$ with its normal vector $n_{r,proj}$. The coordinates of this vector will be denoted by $n_{r,proj} = [a,b,c]$

We will also denote the unit vector of the vector $v_{r,proj}$ projection of the reflected vector v_r onto the plane $P_{r,proj}$ by $u_{v_{r,proj}} = [x_{p_r}, y_{p_r}, z_{p_r}]$.

The origin of the vector $u_{v_{r,proj}}$ is the point with the coordinates $[x_1, y_1, z_1]$, and its extremity is the point with coordinates $[x_2, y_2, z_2]$.

Then we can also denote the vector $u_{v_{r,proj}} = [(x_2 - x_1), (y_2 - y_1), (z_2 - z_1)]$

We know the coordinates of the origin $[x_1, y_1, z_1]$ which are the coordinates of the element of the metasurface. Now we will be looking to find the coordinates of the extremity point of the unit vector $[x_2, y_2, z_2]$.

We also know that φ_r is the angle between $u_{v_{r,proj}}$ and u_z therefore we can find z_2:

$$u_{v_{r,proj}} \cdot u_z = z_{p_r} = \cos\ \varphi_r$$
$$z_2 - z_1 = \cos\ \varphi_r$$
$$\mathrm{z}_2 = \cos\ \varphi_r + z_1$$

Now we can write an equation for the plane $P_{r,proj}$:

$$ax + by + cz + d = 0$$

We know that the point $[x_2, y_2, z_2]$ is in the plane $P_{r,proj}$, then we can write:

$$ax + by + cz_2 + d = 0 \qquad (1)$$

This is equation 1 which will be used the find the variables $x\ and\ y$ which are basically $x_2\ and\ y_2$.
We can write this equation as:

$$y = -\tfrac{1}{b}(ax + cz_2 + d)$$
$$y = -\tfrac{1}{b}(ax + A) \qquad\qquad where\ A = cz_2 + d$$

To find the second equation we will use the formula of the norm of the vector $u_{v_{r,proj}}$ which is a unit vector, then we have:

$$\sqrt{{x_{p_r}}^2 + {y_{p_r}}^2 + {z_{p_r}}^2} = 1$$

$${x_{p_r}}^2 + {y_{p_r}}^2 = 1 - {z_{p_r}}^2$$
$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = 1 - \cos^2\ \varphi_r$$
$$x^2 - 2x_1x + {x_1}^2 + y^2 - 2y_1y + {y_1}^2 + \cos^2\ \varphi_r - 1 = 0 \qquad (2)$$

Combining these two equations, we get:

$$x^2 - 2x_1x + {x_1}^2 + \frac{1}{b^2}(ax + A)^2 - \frac{2y_1}{b}(ax + A) + {y_1}^2 + \cos^2\ \varphi_r - 1 = 0$$
$$\left(\frac{a^2}{b^2} + 1\right)x^2 + \left(\frac{2aA}{b^2} + \frac{2ay_1}{b} - 2x_1\right)x + \left(\frac{A^2}{b^2} + \frac{2Ay_1}{b} + {x_1}^2 + {y_1}^2 \cos^2\ \varphi_r - 1\right) = 0$$

Assigning the following:

$$\alpha = \frac{a^2}{b^2} + 1$$
$$\beta = \frac{2aA}{b^2} + \frac{2ay_1}{b} - 2x_1$$
$$\gamma = \frac{A^2}{b^2} + \frac{2Ay_1}{b} + {x_1}^2 + {y_1}^2 \cos^2\ \varphi_r - 1$$

Now we have:

$$\alpha x^2 + \beta x + \gamma = 0$$

Solving this quadratic equation, we will have:

$$x = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$$

We have here 2 solutions for x, to choose between these solutions we will perform the following:

- Project the receiver point $r_x = [x_r, y_r, z_r]$ onto the plane $P_{r,proj}$.
- Compare the x component of the origin point x_1 of the vector $u_{v_{r,proj}}$, to the x component of the projection of the receiver.

  $if\ x_1 > x_{r,proj}$:

  $\qquad \min(sol_1, sol_2)$

  $if\ x_1 < x_{r,proj}$:

  $\qquad \max(sol_1, sol_2)$

now we found x. Replace it in the y equation $y = -\frac{1}{b}(ax + cz_2 + d)$

we found x_2, y_2

then we found the vector $u_{v_{r,proj}} = [(x_2 - x_1), (y_2 - y_1), (z_2 - z_1)]$ which is the unit vector of the vector $v_{r,proj}$ that we are looking for. To find this vector from its unit vector, first we have to write $u_{v_{r,proj}}$ in parametric form (knowing that the point $[x_1, y_1, z_1]$ is the origin of the vector $u_{v_{r,proj}}$):

$x = x_1 + (x_2 - x_1)t$

$y = y_1 + (y_2 - y_1)t$

$z = z_1 + (z_2 - z_1)t$

We are looking for the point of the vector that is on the same plane as the receiver, in other words, we are looking for the point on the line having unit vector $u_{v_{r,proj}}$ where $z = z_{receiver}$

$$t = \frac{z_{receiver} - z_1}{(z_2 - z_1)}$$

Now we replace this t in the equations of x and y to find their values.

Finally, we can find the reflected vector $v_{r,proj} = [(x - x_1), (x - y_1), (z_{receiver} - z_1)]$

After finding the projection of the reflected vector $v_{r,proj}$, the next step is to find the reflected vector v_r using its reflection $v_{r,proj}$.

In this section we consider $v_{r,proj} = [a, b, c]$

The reflected vector $v_r = [x, y, z]$

We want to find the coordinates $[x, y, z]$ of the reflected vector v_r

We also know that the z coordinate of the vectors $v_{r,proj}$ and v_r will be the same: $z = c = z_r$

We also know that $|v_{r,proj}| = |v_r| \cos\theta_r$

$$\Rightarrow |v_r| = \frac{|v_{r,proj}|}{\cos\theta_r} = L$$

$$|v_r| = \sqrt{x^2 + y^2 + z_r^2} = L$$

$$x^2 + y^2 + z_r^2 - L^2 = 0 \qquad (1)$$

This is equation 1 used to find the coordinates of the reflected vector $v_r = [x, y, z]$

To find the second equation we will use the dot product between $v_{r,proj}$ and v_r

$$v_{r,proj} \cdot v_r = ax + by + cz = |v_{r,proj}||v_r| \cos\theta_r$$

$$ax + by + z^2 = |v_{r,proj}||v_r| \cos\theta_r$$

$$ax + by = |v_{r,proj}||v_r| \cos\theta_r - z^2$$

$$ax + by = X \qquad where\ X = |v_{r,proj}||v_r| \cos\theta_r - z^2$$

$$\Rightarrow y = \frac{X - ax}{b} \qquad (2)$$

Combining equations (1) and (2):

$$x^2 + \frac{1}{b^2}(X - ax)^2 + {z_r}^2 - L^2 = 0$$

$$x^2 + \frac{X^2}{b^2} + \frac{a^2}{b^2}x^2 - \frac{2Xa}{b}x + {z_r}^2 - L^2 = 0$$

$$\left(\frac{a^2}{b^2} + 1\right)x^2 - \frac{2Xa}{b}x + \left(\frac{X^2}{b^2} + {z_r}^2 - L^2\right) = 0$$

Assigning the following:

$$\alpha = \frac{a^2}{b^2} + 1$$

$$\beta = -\frac{2Xa}{b}$$

$$\gamma = \frac{X^2}{b^2} + {z_r}^2 - L^2$$

Now we have:

$$\alpha x^2 + \beta x + \gamma = 0$$

Solving this quadratic equation, we will have:

$$x = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$$

We have here 2 solutions for x, to choose between these solutions we will perform the following:

- $est_{sol_1} = sol_1 + x_1$
  $est_{sol_2} = sol_2 + x_1$
  Where x_1 is the x component of the origin of the reflected vector which is known (the x coordinate of the element of the metasurface).
- Then we choose the solution which is closer to the x coordinate of the receiver, $x_{receiver}$.
  $if\ |est_{sol_1} - x_{receiver}| > |est_{sol_2} - x_{receiver}|$:
  $\quad \mathrm{sol}_1$
  $else$:
  $\quad \mathrm{sol}_2$

now we found x. Replace it in the y equation $y = \frac{|v_{r,proj}||v_r| \cos\theta_r - z^2 - ax}{b}$

we found x, y of the reflected vector $v_r = [x, y, z_r]$

then to find the extremity point of this vector, or the point that should hit the plane of the receiver, we will do the following:

$p_r = [(x + x_1), (y + y_1), (z_r + z_1)]$

where the point$[x_1, y_1, z_1]$ is the origin of the reflected vector, which is the coordinates of point center of the element on the metasurface.

The next and the final step is to check if the reflected ray hits the receiver successfully. To do so we can simply define the receiver antenna shape, dimensions, and center point, then we can check it the point p_r is inside or outside the area occupied by the antenna.

In the end, the result here will be a 2D Boolean "successful reflections (SR)" matrix of size $\left(w_x, h_y\right)$, where each entry will have a Boolean "True" value indicating that the reflected ray by the corresponding element is received corrected by the receiver antenna, or a Boolean "False" value indicating that the reflected ray by corresponding element misses the receiver antenna.

After determining the reflection model, by calculating the reflection angles that will dictate the path of the ray. We can now know if a given reflected ray will reach the receiver or not. The next step is to create the power model by calculating the effective power received by the receiver antenna.

The power model to be used in our case is similar to the 2-ray model, where we calculate the power received by a receiver coming from 2 paths, one line of sight path and another non-line of sight path where the signal is reflected by a surface and the reflected ray reaches the receiver. The 2-ray power model is the following:

$$P_r = P_t\left(\frac{\lambda}{4\pi}\right)^2 \times \left|\frac{\sqrt{G_{los}} \times e^{-j2\pi l/\lambda}}{l} + \Gamma(\theta)\sqrt{G_{surface}}\frac{e^{-j2\pi(x+x')n_i/\lambda}}{x + x'}\right|^2$$

$where$:

P_r: $The\ power\ received$

P_t: $The\ power\ transmitted$

$P_t = \frac{{A_i}^2}{2}, A_i\ is\ the\ amplitude\ of\ the\ transmitted\ signal$

G_{los}: $The\ gain\ of\ the\ antenna$

$G_{surface}$: $the\ gain\ of\ ths\ surface\ of\ reflection$

Γ: The reflection coefficient of the surface

l: $The\ line\ of\ sight\ distance\ between\ the\ transmitter\ and\ the\ receiver$

$(x + x')$: $The\ non\ line\ of\ sight\ distance\ between\ the\ transmitter\ and\ the\ receiver$

n_i: $The\ index\ of\ reflection\ of\ freespace$

In the power model for the IRS, we only care about the reflected part of the signal and not the line-of-sight part, since in case there is a line of sight between the transmitter and the receiver, the power reflected signal part will not be a huge addition to the total power received. So, we consider only the non-line-of-sight case where the received signal will only come from the reflected part of the transmitted signal by the metasurface. In this case, we will have the line-of-sight distance ($l = +\infty$). Adapting the power equation, we will have:

$$P_r = P_t G_t \left(\frac{\lambda}{4\pi}\right)^2 \times \sum_{n=1}^{N} \left(\Gamma_n \sqrt{G_{surface}} \frac{e^{-j2\pi(x_n + x'_n)n_i/\lambda}}{x_n + x'_n} \right)^2$$

$Where$:

$N\ is\ the\ number\ of\ elements\ on\ the\ metasurface$
$x_n\ is\ the\ distance\ from\ the\ transmitter\ to\ the\ n^{th}\ element\ of\ the\ metasurface$
$x'_n\ is\ the\ distance\ from\ the\ distance\ form\ the\ n^{th}\ element\ to\ the\ receiver$
$(x_n + x'_n)\ is\ the\ non\ line\ of\ sight\ distance\ between\ the\ transmitter\ and\ the\ receiver$
$through\ the\ n^{th}\ element\ of\ the\ metasurface$
$\Gamma_n\ is\ the\ reflection\ coefficient\ of\ the\ n^{th}\ element\ of\ the\ metasurface$

In the equation above, we are summing over the powers of all reflected rays reaching the receiver.

The only missing thing from this equation is the fact that we have to disregard the power coming from the rays that will be reflected in an incorrect way and will not reach the receiver antenna. To take this issue into account, we will add the successful reflections (SR) matrix that we calculated previously. The power model will become:

$$P_r = P_t G_t \left(\frac{\lambda}{4\pi}\right)^2 \times \sum_{n=1}^{N} \left(SR_n \Gamma_n \sqrt{G_{surface}} \frac{e^{-j2\pi(x_n + x'_n)n_i/\lambda}}{x_n + x'_n} \right)^2$$

$Where$:

$SR_n\ is\ an\ indice\ representing\ if\ the\ ray\ reflected\ by\ the\ n^{th}\ element\ of\ the\ metasurface$
$will\ reach\ the\ receiver\ antenna\ successfully$

When the reflected ray by the n[th] element hits the receiver successfully, $SR_n = "True"\ or\ "1"$ so we include the power of this ray when calculating the received power. But when the reflected ray by the n[th] element misses the receiver successfully, $SR_n = "False"\ or\ "0"$ so we multiply the power of this ray by "0" to ignore it and not take it into account when calculating the total power at the receiver.

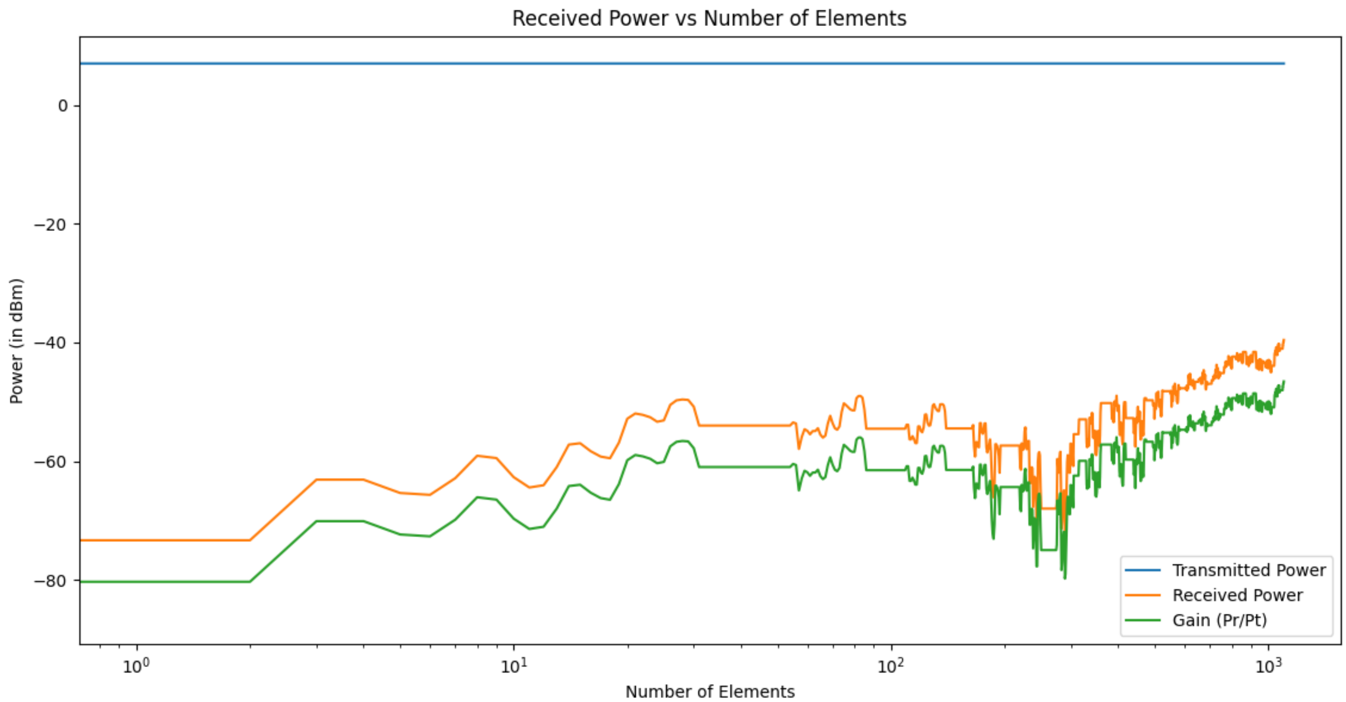

Figure 7: Power received vs number of elements

Finally, by calculating the reflection model and the power model, we have completed the Intelligent Reconfigurable Surface model. Now we have the power of the signal received.

In the next part, we will calculate the power received by the receiver antenna when we do not have an intelligent metasurface and compare it with the IRS case to check the additional gain that the metasurface introduces. In this case the reflection will be governed by the original Snell's law $\theta_r = \theta_i$ and the transmitted and the reflected rays will be in the same plane, so we have $\varphi_r = 0$.

Given the location of the transmitter $t_x = [x_i, y_i, z_i]$, the location of the receiver $r_x = [x_r, y_r, z_r]$, the normal to the plane of incidence, (considering our initial assumption for a fixed coordinates system where the transmitter and the receiver are points in space, and the surface of incidence in the xy plane, (z=0) so the normal is the unit vector parallel to the z-axis $u_z = [0, 0, 1]$) we can calculate the angles θ_r and θ_i when a transmitted ray reaches the receiver.

We define the location of incidence at point $p_0\ (x, y)$. We have to find p_0.

$Incidence\ vector$: $v_i = [(x_i - x), (y_i - y), z_i]$

$Reflected\ vector$: $v_r = [(x - x_r), (y - y_r), z_r]$

Then we have:

$$v_i \cdot u_z = z_i$$
$$= |v_i||u_z| \cos\theta_i$$
$$\Rightarrow \theta_i = \cos^{-1}\left(\frac{v_i \cdot u_z}{|v_i||u_z|}\right) = \cos^{-1}\left(\frac{z_i}{|v_i||u_z|}\right)$$

and

$$v_r \cdot u_z = z_r$$
$$= |v_r||u_z| \cos\theta_r$$
$$\Rightarrow \theta_r = \cos^{-1}\left(\frac{v_r \cdot u_z}{|v_r||u_z|}\right) = \cos^{-1}\left(\frac{z_r}{|v_r||u_z|}\right)$$

$$\theta_i = \theta_r$$

$$\cos^{-1}\left(\frac{z_i}{|v_i||u_z|}\right) = \cos^{-1}\left(\frac{z_r}{|v_r||u_z|}\right)$$

$$\frac{z_i}{|v_i|} = \frac{z_r}{|v_r|}$$

$$\frac{z_i}{\sqrt{(x_i - x)^2 + (y_i - y)^2 + {z_i}^2}} = \frac{z_r}{\sqrt{(x - x_r)^2 + (y - y_r)^2 + {z_r}^2}}$$

$$\frac{(x_i - x)^2 + (y_i - y)^2 + {z_i}^2}{(x - x_r)^2 + (y - y_r)^2 + {z_r}^2} = \left(\frac{z_i}{z_r}\right)^2$$

$$\frac{(x_i - x)^2 + (y_i - y)^2 + {z_i}^2}{(x - x_r)^2 + (y - y_r)^2 + {z_r}^2} - \left(\frac{z_i}{z_r}\right)^2 = 0$$

Then we should find the point $p_0\ (x, y)$ so the function above is closest to 0.

After finding the point $p_0\ (x, y)$, to find the angles $\theta_i = \theta_r$:

$$v_i = t_x - p_0$$

$$\theta_i = \cos^{-1}\left(\frac{v_i \cdot u_z}{|v_i||u_z|}\right)$$

The final part is to calculate the power received by the receiver antenna in a system without a metasurface, and we do not have a line-of-sight between the transmitter and the receiver. In this case the power will be:

$$P_r = P_t G_t \left(\frac{\lambda}{4\pi}\right)^2 \times \left(\Gamma \sqrt{G_{surface}} \frac{e^{-j2\pi(x+x')n_i/\lambda}}{x + x'}\right)^2$$

Where Γ is the reflection coefficient of the surface, this reflection coefficient will depend on the material of the surface, and it will be different for signals with perpendicular or parallel polarizations.

$$\Gamma_{\perp} = \frac{\cos\theta_i - \sqrt{\varepsilon_r - \sin^2\theta_i}}{\cos\theta_i + \sqrt{\varepsilon_r - \sin^2\theta_i}}$$

$$\Gamma_{\parallel} = \frac{(\varepsilon_r \times \cos\theta_i) - \sqrt{\varepsilon_r - \sin^2\theta_i}}{(\varepsilon_r \times \cos\theta_i) + \sqrt{\varepsilon_r - \sin^2\theta_i}}$$

$where$

ε_r $is\ the\ permittivity\ of\ the\ material\ composing\ the\ surface\ on\ which\ the\ signal\ will\ reflect\ back$
θ_i $is\ angle\ of\ incidence\ of\ the\ signal\ into\ the\ surface\ calculated\ in\ the\ previous\ section$
$section\ using\ the\ original\ Snell's\ law\ (\theta_i = \theta_r)$

The result of this function will be the power of the signal received by the receiver antenna in the system without Intelligent Reconfigurable Surface.