Hyperkähler varieties and their relation to Shimura stacks

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Contents

1

Introduction

1.1 The spinor norm and determinant of monodromy operators on K3 surfaces and elliptic curves

Let $n \geq 3$ be an integer. There are no elliptic curves over $\mathbf{Q}$ whose $\mathbf{Q}$-rational n-torsion points are isomorphic to $(\mathbf{Z}/n\mathbf{Z})^{\oplus 2}$. More precisely, the following result holds.

Proposition 1.1.1. *Let E be an elliptic curve over a field k of characteristic 0, and $n \geq 3$ an integer. If E has maximal n-torsion, that is, if $E[n](k) \cong (\mathbf{Z}/n\mathbf{Z})^{\oplus 2}$, then k contains a primitive nth root of unity.*

In this thesis we prove, among other things, the following analogue of Proposition 1.1.1 for K3 surfaces.

Proposition 1.1.2. *Let S be a K3 surface over a field k of characteristic 0, and let ℓ be an odd prime number. If the lattice $\Lambda := \mathrm{Pic}(S)$ has rank 11, and if the ℓ-part of its discriminant $\Lambda^\vee/\Lambda$ has length 11, then k contains a square root of $(-1)^{\frac{\ell-1}{2}}\ell$.*

We prove a stronger version of this proposition in Chapter 5, namely Theorem 5.6.1. Moreover, it is possible to prove a similar theorem for $\ell = 2$, involving the biquadratic field $\mathbf{Q}(i, \sqrt{2})$, see Remark 5.6.2.

Proposition 1.1.1 follows from the following more general theorem.

Theorem 1.1.3. *Let E be an elliptic curve over a field k of characteristic 0, and ℓ a prime number. Then the diagram*

$$\begin{tikzcd} \mathrm{Gal}_k \arrow[r, "\rho"] \arrow[dr, "\chi_\ell^{-1}"'] & \mathrm{GL}(\mathrm{H}^1_{\text{ét}}(E_{\overline{k}}, \mathbf{Z}_\ell)) \arrow[d, "\det"] \\ & \mathbf{Z}_\ell^\times \end{tikzcd}$$

commutes, where $\chi_\ell \colon \mathrm{Gal}_k \to \mathbf{Z}_\ell^\times$ denotes the cyclotomic character, and ρ is the natural action of Gal_k on $\mathrm{H}^1_{\text{ét}}(E_{\overline{k}}, \mathbf{Z}_\ell)$.

Proposition 1.1.1 is derived from this as follows. For convenience, we take $n = \ell$. When $E[\ell](k) \cong (\mathbf{Z}/\ell\mathbf{Z})^{\oplus 2}$, then Gal_k acts trivially on $\mathrm{H}^1_{\text{ét}}(E_{\overline{k}}, \mathbf{Z}/\ell\mathbf{Z})$. It is easy to see that the image of

$$\left\{ g \in \mathrm{GL}(\mathrm{H}^1_{\text{ét}}(E_{\overline{k}}, \mathbf{Z}_\ell)) \mid g \otimes \mathbf{Z}/\ell\,\mathbf{Z} = \mathrm{id} \right\}$$

under $\det\colon \mathrm{GL}(\mathrm{H}^1_{\text{ét}}(E_{\overline{k}}, \mathbf{Z}_\ell)) \to \mathbf{Z}_\ell^\times$ is trivial. It follows from Theorem 1.1.3 that the cyclotomic character χ_ℓ has trivial image, which implies that k contains a primitive root of unity.

One of the main results of this thesis is the following analogue of Theorem 1.1.3 for K3 surfaces, see Theorem 5.2.1. It can be used to prove Proposition 1.1.2 via an argument similar to the derivation of Proposition 1.1.1 from Theorem 1.1.3. It makes use of a description of the image of the spinor norm for $\mathbf{Z}_\ell$-lattices, which can be found in Section 5.1.3

Theorem 1.1.4. *Let S be a K3 surface over a field k of characteristic* 0. *Then the diagram*

$$\begin{tikzcd}
\mathrm{Gal}_k \arrow[r, "\rho"] \arrow[dr, "\chi_\ell"'] & \mathrm{O}(\mathrm{H}^2_{\text{ét}}(S_{\overline{k}}, \mathbf{Z}_\ell(1))) \arrow[d, "\nu \cdot \det"] \\
& \mathbf{Z}_\ell^\times/2
\end{tikzcd}$$

commutes, where $\chi_\ell\colon \mathrm{Gal}_k \to \mathbf{Z}_\ell^\times$ denotes the cyclotomic character, ρ is the natural action of Gal_k *on $\mathrm{H}^2_{\text{ét}}(S_{\overline{k}}, \mathbf{Z}_\ell(1))$, and ν denotes the spinor norm.*

It may not be immediately clear how the spinor norm in Theorem 1.1.4 is related to the determinant in Theorem 1.1.3. The theory of Shimura stacks provides us with a link.

1.2 Shimura stacks and moduli spaces

The moduli space of complex elliptic curves is isomorphic to the quotient of the upper half plane $\mathcal{H}^+$ by an action of $\mathrm{SL}_2(\mathbf{Z})$. The global Torelli theorem for complex K3 surfaces gives a similar description of the moduli space of polarized K3 surfaces.

Let (S, λ) be a complex polarized K3 surface of degree $2d$, and let Λ be its primitive second cohomology group. That is, Λ is the orthogonal complement of $c_1(\lambda)$ in $\mathrm{H}^2(S, \mathbf{Z}(1))$. Then Λ is a $\mathbf{Z}$-lattice of signature $(2, 19)$. Up to isomorphism, Λ does not depend on (S, λ). Let Ω be the Hermitian symmetric domain parametrizing the Hodge structures of K3 type on Λ (see Section 2.3 for a definition). The complex structure of S induces a Hodge structure on Λ, which yields a point of Ω. Mapping a polarized K3 surface of degree $2d$ to its primitive degree 2 cohomology group defines an open immersion from the moduli space of complex polarized K3 surfaces to the quotient of Ω by the action of some arithmetic group Γ.

The theory of Shimura varieties permits us to extend these descriptions to the moduli spaces of elliptic curves and polarized K3 surfaces over $\mathbf{Q}$.

Let $\mathbf{A}_{\mathrm{f}}$ be the ring of finite adèles, let G be the algebraic group GL_2 over $\mathbf{Q}$, let $\mathcal{K}$ be the compact open subgroup $\mathrm{GL}_2(\widehat{\mathbf{Z}})$ of $G(\mathbf{A}_{\mathrm{f}})$, and let $\mathcal{H}$ be the double half-plane $\mathbf{C} \setminus \mathbf{R}$. Then there is an isomorphism of complex Deligne-Mumford stacks

$$[\mathrm{SL}_2(\mathbf{Z}) \backslash \mathcal{H}^+] \xrightarrow{\sim} \mathrm{Sh}_{\mathcal{K}}[G, \mathcal{H}]_{\mathbf{C}} := [G(\mathbf{Q}) \backslash \mathcal{H} \times G(\mathbf{A}_{\mathrm{f}}) / \mathcal{K}],$$

where the square brackets indicate that the quotients are taken stackily (or orbifoldily). The stack $\mathrm{Sh}_{\mathcal{K}}[G, \mathcal{H}]_{\mathbf{C}}$ is known as a Shimura stack. Let $\mathbf{Ell}$ be the moduli stack of elliptic curves over $\mathbf{Q}$. We have an isomorphism

$$\mathbf{Ell}_{\mathbf{C}} \xrightarrow{\sim} \mathrm{Sh}_{\mathcal{K}}[G, \mathcal{H}]_{\mathbf{C}}. \tag{1.1}$$

The theory of canonical models of Shimura varieties shows that $\mathrm{Sh}_{\mathcal{K}}[G, \mathcal{H}]_{\mathbf{C}}$ descends to a Deligne-Mumford stack over $\mathbf{Q}$, and that the morphism in (1.1) descends to an isomorphism over $\mathbf{Q}$.

Work of Rizov and Madapusi-Pera (see [R3] and [MP1]), refined in [T1], shows that a similar statement holds for the moduli stack $\mathbf{K3}_{2d}$ of degree $2d$ polarized K3 surfaces over $\mathbf{Q}$. That is, the open immersion $\mathbf{K3}_{2d,\mathbf{C}} \hookrightarrow [\Gamma \backslash \Omega]$ descends to an open immersion of Deligne-Mumford stacks

$$\mathbf{K3}_{2d} \hookrightarrow \mathrm{Sh}_{\mathcal{K}}[G, \Omega] \tag{1.2}$$

over $\mathbf{Q}$, where G is the special orthogonal group $\mathrm{SO}(\Lambda \otimes \mathbf{Q})$.

In this thesis, we give a detailed exposition of the descent of (1.2), and generalize it to moduli spaces of polarized hyperkähler varieties. These are higher-dimensional analogues of K3 surfaces, also referred to as irreducible holomorphic symplectic manifolds. One of our results is the following theorem (see Theorem 4.5.2).

Theorem 1.2.1. *Let $\mathbf{M}_{\mathrm{or}}$ be a connected component of the moduli stack of polarized oriented hyperkähler varieties over $\mathbf{Q}$. There exists an orthogonal Shimura stack $\mathrm{Sh}_{\mathcal{K}}[G, \Omega]$ and an étale morphism*

$$\mathbf{M}_{\mathrm{or}} \longrightarrow \mathrm{Sh}_{\mathcal{K}}[G, \Omega],$$

defined over $\mathbf{Q}$.

A key ingredient of the proof of Theorem 1.2.1 is a theorem of Deligne and Milne which states that many Shimura varieties are moduli spaces of abelian motives (see [M3]). In order to effectively use this result, we give a more Tannakian approach to the statement and proof of the result of Deligne and Milne.

Note that Theorem 1.2.1 is weaker than the analogous statement for K3 surfaces in two ways.

The first difference is that it is a result about a moduli stack of polarized *oriented* hyperkähler varieties (see Section 4.3). This is a degree 2 étale covering of the moduli stack of polarized hyperkähler varieties. For hyperkähler varieties with even second Betti number (for example, K3 surfaces), we can follow the arguments in [T1] to refine Theorem 1.2.1 to get rid of the orientations. See Theorem 4.6.2.

The second difference is that the morphism $\mathbf{M}_{\mathrm{or}} \to \mathrm{Sh}_{\mathcal{K}}[G, \Omega]$ is an étale morphism, rather than an open immersion. For hyperkähler varieties of $\mathrm{K3}^{[n]}$-type, we are able to refine Theorem 1.2.1 to obtain an open immersion, for suitably chosen $\mathcal{K}$ (see Theorem 4.7.18). The existence of such $\mathcal{K}$ is a priori not obvious. We prove its existence by extending a result of Markman on the monodromy of complex $\mathrm{K3}^{[n]}$-type hyperkähler varieties to $\mathrm{K3}^{[n]}$-type hyperkähler varieties over $\mathbf{Q}$. This result may be of independent interest, see Theorem 4.7.12.

1.3 Deligne's reciprocity law

To link the results in Section 1.2 to the ones in Section 1.1, we use a result of Deligne on the connected components of Shimura varieties.

Let (G, X) be a Shimura datum with reflex field E. When (G, X) is either $(\mathrm{GL}_2, \mathcal{H})$ or $(\mathrm{SO}(\Lambda \otimes \mathbf{Q}), \Omega)$, then $E = \mathbf{Q}$. Consider the projective system of Shimura varieties $\mathrm{Sh}(G, X) = (\mathrm{Sh}_{\mathcal{K}}(G, X))_{\mathcal{K}}$, where $\mathcal{K}$ ranges over all compact open subgroups of $G(\mathbf{A}_{\mathrm{f}})$, and let $\pi_0(\mathrm{Sh}(G, X)_{\overline{E}})$ be the limit $\lim_{\mathcal{K}} \pi_0(\mathrm{Sh}_{\mathcal{K}}(G, X)_{\overline{E}})$. Then Deligne gives an expression of the profinite set $\pi_0(\mathrm{Sh}(G, X)_{\overline{E}})$ as a quotient of $G(\mathbf{A}_{\mathrm{f}})$. Moreover, he gives an explicit description of the Gal_E-action on $\pi_0(\mathrm{Sh}(G, X)_{\overline{E}})$ in terms of the group $G(\mathbf{A}_{\mathrm{f}})$ and the class field theory of E. The full statement of Deligne's result can be found in Section 5.3.1.

When $(G, X) = (\mathrm{GL}_2, \mathcal{H})$, it can be shown that the determinant $\det\colon G(\mathbf{A}_{\mathrm{f}}) \to {\mathbf{A}_{\mathrm{f}}}^{\times}$ induces an isomorphism from $\pi_0(\mathrm{Sh}(G, X)_{\overline{\mathbf{Q}}})$ to $\widehat{\mathbf{Z}}^{\times}$. Moreover, if we endow $\widehat{\mathbf{Z}}^{\times}$ with the natural $\mathrm{Gal}_{\mathbf{Q}}$-action coming from the Kronecker-Weber theorem, this isomorphism is $\mathrm{Gal}_{\mathbf{Q}}$-equivariant.

In the orthogonal case, the *spinor norm* $G(\mathbf{A}_{\mathrm{f}}) \to {\mathbf{A}_{\mathrm{f}}}^{\times}/2$ induces an isomorphism from the $\mathrm{Gal}_{\mathbf{Q}}$-set $\pi_0(\mathrm{Sh}(G, X)_{\overline{\mathbf{Q}}})$ to $\widehat{\mathbf{Z}}^{\times}/2$, endowed with the $\mathrm{Gal}_{\mathbf{Q}}$-action coming from the Kronecker-Weber theorem. A proof of this fact can be found in Section 5.3.2.

A more careful analysis, which combines these results with the ones in Section 1.2, yields Theorem 1.1.3 and Theorem 1.1.4.

We have not been able to generalize Theorem 1.1.4 to higher-dimensional hyperkähler varieties. It seems plausible that this can be approached with similar methods. However, our proof of Theorem 1.1.4 uses that we can get rid of the orientations in Theorem 1.2.1 for K3 surfaces, and that the second cohomology group of K3 surfaces is self-dual.

1.4 Overview

In this section, we give a brief overview of the chapters in this thesis. More detailed descriptions of each chapters' contents can be found in their respective introductions.

In Chapter 2, we recall the basic theory of Shimura varieties and motives, and we give a Tannakian exposition of Milne's results relating the canonical models of Shimura varieties to moduli spaces of abelian motives. In Chapter 3, we prove that the moduli stack of polarized hyperkähler varieties is, among other things, a Deligne-Mumford stack. The next chapter contains various generalizations of (1.2) to hyperkähler varieties of higher dimension. Finally, in Chapter 5, we apply the existence of (1.2) and Deligne's results on the connected components of a Shimura stack to prove Theorem 1.1.4.

2

Shimura varieties and motives

This chapter is an exposition of the fact that certain Shimura varieties are moduli spaces of abelian motives. This was originally proved in [D2] and [M3]. We focus in particular on Shimura varieties of orthogonal type, as these play an important role in the moduli theory of hyperkähler varieties, which we will see in Chapter 4.

In the first section we briefly go over the basics of the theory of Shimura varieties, primarily to set up notation. Then, in Section 2.2, we collect the features of André's category of motives [A2] which we need. The two main facts we will need in the sequel are Deligne's result that Hodge cycles on abelian varieties are motivated (Theorem 2.2.2), and Milne's theorem that the Hodge structures parametrized by Shimura varieties of Hodge and orthogonal type come from abelian motives (Proposition 2.2.4).

In Section 2.3, we give the main result of this chapter, namely a description of the complex points of an orthogonal Shimura variety in terms of abelian motives endowed with a symmetric bilinear form and a trivialization of the determinant, Theorem 2.3.3. In particular we will show that this description is compatible with the action of $\mathrm{Aut}(\mathbf{C})$. We prove this theorem by describing the complex points of a Hodge type Shimura variety $\mathrm{Sh}(G,X)$ in terms of tensor functors from the category of G-representations to the category of abelian motives, in Theorem 2.3.16. The reader familiar with [D2] and [M4] will note that all results in this section go back to Deligne and Milne, with the possible exception of our more Tannakian treatment of the proof and statement of the theorem.

2.1 Shimura varieties

In this section we establish basic notation regarding Shimura varieties, and in particular about orthogonal Shimura varieties. See also [M4] and [D2] for a more detailed account. For the part about orthogonal Shimura varieties, [MP2], [D3], and [A1] are excellent references.

2.1.1 Generalities

Let (G,X) be a Shimura datum, see [M4, Definition 5.5]. In particular, G is a connected reductive group over $\mathbf{Q}$, and $X \subseteq \mathrm{Hom}(\mathbf{S}, G_{\mathbf{R}})$ is a $G(\mathbf{R})$-conjugacy class, where $\mathbf{S} = \operatorname{Res}_{\mathbf{C}/\mathbf{R}} \mathbf{G}_m$ denotes the Deligne torus. Let Z be the center of G. We assume throughout this thesis that (G, X) satisfies condition SV5 in [M4]. That is, we assume that $Z(\mathbf{Q})$ is discrete in $G(\mathbf{A}_{\mathrm{f}})$, where $\mathbf{A}_{\mathrm{f}}$ is the ring of finite adèles.

For a commutative $\mathbf{C}$-algebra A, the map $A \otimes_{\mathbf{R}} \mathbf{C} \to A \times A$, $a \otimes z \mapsto (za, \bar{z}a)$ is an isomorphism of rings. This yields an isomorphism $\mathbf{G}_{m,\mathbf{C}} \times \mathbf{G}_{m,\mathbf{C}} \to \mathbf{S}_{\mathbf{C}}$, which we will use to identify these two group schemes. For $h \in X$, we define $\mu_h\colon \mathbf{G}_{m,\mathbf{C}} \to G_{\mathbf{C}}$ as $z \mapsto h_{\mathbf{C}}(z, 1)$. The reflex field E of (G, X) is by definition the unique smallest subfield of $\mathbf{C}$ such that the $G(\mathbf{C})$-conjugacy class of μ_h is defined over E.

There is an inverse system $(\mathrm{Sh}_{\mathcal{K}}(G, X))_{\mathcal{K}}$ of varieties over E associated with (G, X), where $\mathcal{K}$ ranges over all compact open subgroups of $G(\mathbf{A}_{\mathrm{f}})$. The varieties $\mathrm{Sh}_{\mathcal{K}}(G, X)$ are called **Shimura varieties**. The set of $\mathbf{C}$-points of $\mathrm{Sh}_{\mathcal{K}}(G, X)$ is the double coset

$$G(\mathbf{Q})\backslash X \times G(\mathbf{A}_{\mathrm{f}})/\mathcal{K}.$$

Here, $G(\mathbf{Q})$ acts on X by conjugation, and on $G(\mathbf{A}_{\mathrm{f}})$ by left multiplication. The group $\mathcal{K}$ acts trivially on X, and on $G(\mathbf{A}_{\mathrm{f}})$ by right multiplication.

The limit of the inverse system $(\mathrm{Sh}_{\mathcal{K}}(G, X))_{\mathcal{K}}$ is denoted $\mathrm{Sh}(G, X)$. Proposition 2.1.10 in [D4], combined with the fact that $Z(\mathbf{Q})$ is discrete in $G(\mathbf{A}_{\mathrm{f}})$, implies that the set of $\mathbf{C}$-points of $\mathrm{Sh}(G, X)$ is

$$\mathrm{Sh}(G, X)(\mathbf{C}) = G(\mathbf{Q})\backslash X \times G(\mathbf{A}_{\mathrm{f}}).$$

The action of $G(\mathbf{A}_{\mathrm{f}})$ on $\mathrm{Sh}(G, X)(\mathbf{C})$ via right multiplication descends to an action of $G(\mathbf{A}_{\mathrm{f}})$ on $\mathrm{Sh}(G, X)$ defined over E.

Example 2.1.1. Let (V, ψ) be a symplectic $\mathbf{Q}$-vector space of dimension $2d$. The group of symplectic similitudes associated with V is defined to be the algebraic group

$$\mathrm{GSp}(V) = \big\{(g, c) \in \mathrm{GL}(V) \times \mathbf{G}_m \mid \forall v, w \in V \;\; \psi(gv, gw) = c\psi(v, w)\big\}.$$

We define $\mathcal{H}_V$ to be the complex manifold consisting of $h\colon \mathbf{S} \to \mathrm{GSp}(V)_{\mathbf{R}}$ such that h endows V with a Hodge structure of type $d(0,1) + d(1,0)$. Now $(\mathrm{GSp}(V), \mathcal{H}_V)$ is a Shimura datum with reflex field $\mathbf{Q}$, known as a **Siegel Shimura datum**. When there is no possibility for confusion to arise, we use GSp and $\mathcal{H}$ to denote $\mathrm{GSp}(V)$ and $\mathcal{H}_V$, respectively. The Shimura varieties associated with $(\mathrm{GSp}, \mathcal{H})$ are moduli spaces for polarized abelian varieties of dimension d with level structure, as is shown in [D2, § 4].

Definition 2.1.2. A Shimura datum (G, X) is said to be of **Hodge type** if there exists a Siegel Shimura datum $(\mathrm{GSp}, \mathcal{H})$ as in Example 2.1.1 and a morphism of Shimura data $(G, X) \to (\mathrm{GSp}, \mathcal{H})$ such that $G \to \mathrm{GSp}$ is a closed immersion.

2.1.2 Orthogonal Shimura varieties

In this section we go over the basics of Shimura varieties associated with certain quadratic spaces.

Let V be a quadratic space over $\mathbf{Q}$ of signature $(2, n)$ with $n \geq 1$. To V we can associate a Shimura datum $(\mathrm{SO}(V), \Omega_V)$, known as a Shimura datum **of orthogonal type**, as follows. We let $\mathrm{SO}(V)$ be the group of orthogonal transformations of V with determinant 1. The Hermitian symmetric domain $\Omega_V \subseteq \mathrm{Hom}_{\mathbf{R}\,\mathbf{Grp}}(\mathbf{S}, \mathrm{SO}(V)_{\mathbf{R}})$ is defined to be the set of Hodge structures of K3 type on V. That is, it consists of those Hodge structures on V for which

- V has type $(1, -1)$, $(0, 0)$, $(-1, 1)$,
- $V^{1,-1}$ and $V^{-1,1}$ are one-dimensional and orthogonal to $V^{0,0}$,
- the space $(V \otimes \mathbf{R}) \cap (V^{1,-1} \oplus V^{-1,1})$ is positive-definite.

The space Ω_V has two connected components, interchanged by mapping a Hodge structure on $V \otimes \mathbf{R}$ to the one whose $(1, -1)$, $(0, 0)$, and $(-1, 1)$ parts are $V^{-1,1}$, $V^{0,0}$, and $V^{1,-1}$, respectively. When there is no possibility for confusion to arise, we will use SO and Ω to denote $\mathrm{SO}(V)$ and Ω_V, respectively.

There is a natural central extension $\mathrm{GSpin}(V)$ of $\mathrm{SO}(V)$, called the **Clifford group** of V, which is constructed using the even Clifford algebra of V. It fits in a short exact sequence

$$1 \to \mathbf{G}_m \longrightarrow \mathrm{GSpin}(V) \longrightarrow \mathrm{SO}(V) \to 1,$$

and comes endowed with a homomorphism $N\colon \mathrm{GSpin}(V) \to \mathbf{G}_m$ whose kernel is the **spin group** $\mathrm{Spin}(V)$. Again, when there is no possibility of confusion, we will use GSpin and Spin to denote $\mathrm{GSpin}(V)$ and $\mathrm{Spin}(V)$, respectively.

For each $h\colon \mathbf{S} \to \mathrm{SO}_{\mathbf{R}}$ in Ω, there exists a unique $h'\colon \mathbf{S} \to \mathrm{GSpin}_{\mathbf{R}}$ making the diagram

$$\begin{tikzcd} \mathbf{S} \arrow[r, "h"] \arrow[dr, dashed, "h'"'] & \mathrm{SO}_{\mathbf{R}} \\ \mathbf{G}_{m,\mathbf{R}} \arrow[u, "w"] \arrow[r] & \mathrm{GSpin}_{\mathbf{R}} \arrow[u] \end{tikzcd}$$

commute, where $w\colon \mathbf{G}_{m,\mathbf{R}} \to \mathbf{S}$ is the weight homomorphism. The set of such h' is a $\mathrm{GSpin}(\mathbf{R})$-conjugacy class Ω' in $\mathrm{Hom}(\mathbf{S}, \mathrm{GSpin}_{\mathbf{R}})$. The pair $(\mathrm{GSpin}, \Omega')$ is a Shimura datum, and the homomorphisms $\mathrm{GSpin} \to \mathrm{SO}$ and $N\colon \mathrm{GSpin} \to \mathbf{G}_m$ induce morphisms of Shimura data

$$(\mathrm{SO}, \Omega) \longleftarrow (\mathrm{GSpin}, \Omega') \longrightarrow (\mathbf{G}_m, \{\mathbf{Q}(-1)\}). \tag{2.1}$$

Note that the map $\Omega' \to \Omega$ is a bijection. As can be seen in [MP2, Lemma 3.6], the Shimura datum $(\mathrm{GSpin}, \Omega')$ is of Hodge type.

Lemma 2.1.3. *The Shimura data* $(\mathrm{GSpin}, \Omega')$ *and* (SO, Ω) *have reflex field* $\mathbf{Q}$.

Proof. The fact that $(\mathrm{GSpin}, \Omega')$ has reflex field $\mathbf{Q}$ can be found in [S4]. Applying [D4, 2.2.1] to the morphism $(\mathrm{GSpin}, \Omega') \to (\mathrm{SO}, \Omega)$ now shows that the reflex field of (SO, Ω) is $\mathbf{Q}$. □

2.2 Motives

We will work with the category of motives as constructed by André in [A2]. In this section we summarize its salient features.

2.2.1 Generalities

Let k be a field of characteristic 0. Let $\mathbf{SmPr}_k$ be the category of smooth projective varieties over k, and let $\mathbf{Mot}_k$ be the category of motives over k defined using motivated cycles. Then $\mathbf{Mot}_k$ is a $\mathbf{Q}$-linear semisimple neutral Tannakian category endowed with a functor $\mathfrak{h}\colon \mathbf{SmPr}_k^{\mathrm{opp}} \to \mathbf{Mot}_k$, sending a variety to its associated motive. We use $\mathbf{1}$ to denote the unit motive, and for a motive M and $n \in \mathbf{Z}$, we denote by $M(n)$ its nth Tate twist.

Let σ be an embedding of fields $k_0 \to k_1$. Then pullback of schemes yields a covariant functor from $\mathbf{SmPr}_{k_0}$ to $\mathbf{SmPr}_{k_1}$, mapping a smooth projective variety X over k_0 to $\sigma^* X$, defined by the cartesian diagram

$$\begin{tikzcd} \sigma^* X \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec}(k_1) \arrow[r, "\operatorname{Spec}(\sigma)"'] & \operatorname{Spec}(k_0) \end{tikzcd} \tag{2.2}$$

This extends uniquely to the category of motives. That is, for each motive $M \in \mathbf{Mot}_{k_0}$ we have a motive $\sigma^* M$ over k_1 such that $\mathfrak{h}(\sigma^* X) = \sigma^* \mathfrak{h}(X)$ for each $X \in \mathbf{SmPr}_{k_0}$.

Singular cohomology with coefficients in $\mathbf{Q}$ induces a fiber functor $\mathrm{H}_{\mathrm{B}}\colon \mathbf{Mot}_{\mathbf{C}} \to \mathbf{Q}\text{-}\mathbf{Vect}$, known as the Betti realization functor. For an embedding $\sigma\colon k \hookrightarrow \mathbf{C}$, this gives rise to a fiber functor $\mathrm{H}_\sigma\colon \mathbf{Mot}_k \to \mathbf{Q}\text{-}\mathbf{Vect}$, defined as the composition

$$\mathbf{Mot}_k \xrightarrow{\sigma^*} \mathbf{Mot}_{\mathbf{C}} \xrightarrow{\mathrm{H}_{\mathrm{B}}} \mathbf{Q}\text{-}\mathbf{Vect}\,.$$

Similarly, when k is algebraically closed, étale cohomology with coefficients in the ring of finite adèles $\mathbf{A}_{\mathrm{f}}$ induces a fiber functor $\mathrm{H}_{\text{ét}}\colon \mathbf{Mot}_k \to \mathbf{A}_{\mathrm{f}}\text{-}\mathbf{Mod}$, known as the étale realization functor. We use M_σ and $M_{\text{ét}}$ as shorthands for the images of a motive M under H_σ and $\mathrm{H}_{\text{ét}}$, respectively. Artin's comparison isomorphism [AGV, Théorème XI.4.4] between étale and singular cohomology allows us to identify $\mathrm{H}_{\text{ét}}$ with $\mathrm{H}_\sigma \otimes \mathbf{A}_{\mathrm{f}}$ as functors from $\mathbf{Mot}_k$ to $\mathbf{A}_{\mathrm{f}}\text{-}\mathbf{Mod}$.

For a smooth projective variety X over $\mathbf{C}$, the cohomology groups $\mathrm{H}^i(X, \mathbf{Q})$ are canonically endowed with a polarizable Hodge structure. This extends to the Betti realization of motives, leading to a tensor functor which we abusively denote $\mathrm{H}_\sigma\colon \mathbf{Mot}_k \to \mathbf{Q}\text{-}\mathbf{HS}$, where σ is an embedding $k \hookrightarrow \mathbf{C}$, and $\mathbf{Q}\text{-}\mathbf{HS}$ is the category of polarizable $\mathbf{Q}$-Hodge structures.

Let k_0 and k_1 be algebraically closed fields of characteristic 0, and let $\sigma\colon k_0 \to k_1$ be an embedding of fields. For a smooth projective variety X over k_0, this induces a functorial isomorphism on étale cohomology $\overline{\sigma}^*\colon \mathrm{H}^i_{\text{ét}}(X, \mathbf{A}_{\mathrm{f}}) \to \mathrm{H}^i_{\text{ét}}(\sigma^* X, \mathbf{A}_{\mathrm{f}})$.

This extends to the category of motives to give a functorial isomorphism

$$\overline{\sigma}^* \colon \mathrm{H}_{\text{ét}}(M) \longrightarrow \mathrm{H}_{\text{ét}}(\sigma^* M) \tag{2.3}$$

for motives M over k_0.

For later use, it is convenient to phrase the case $k_0 = k_1 = \mathbf{C}$ in terms of the following 2-commutative diagram:

$$\begin{tikzcd} \mathbf{Mot}_{\mathbf{C}} \arrow[rr, "\sigma^*"] \arrow[dr, "\mathrm{H}_{\text{ét}}"'] & \arrow[d, phantom, "\overset{\sigma^*}{\Longrightarrow}"] & \mathbf{Mot}_{\mathbf{C}} \arrow[dl, "\mathrm{H}_{\text{ét}}"] \\ & \mathbf{A}_{\mathrm{f}}\text{-}\mathbf{Mod} & \end{tikzcd} \tag{2.4}$$

That is, σ^* is an isomorphism of tensor functors $\mathrm{H}_{\text{ét}} \to \mathrm{H}_{\text{ét}}\, \sigma^*$ from $\mathbf{Mot}_{\mathbf{C}}$ to $\mathbf{A}_{\mathrm{f}}$-$\mathbf{Mod}$.

2.2.2 Abelian motives

Let $\mathbf{Mot}_{\mathrm{ab},k}$ be the Tannakian subcategory of $\mathbf{Mot}_k$ generated by the motives of abelian varieties. The objects of $\mathbf{Mot}_{\mathrm{ab},k}$ are called abelian motives.

Example 2.2.1. Using the Kuga-Satake construction, André has shown that if X is a hyperkähler variety with $b_2(X) > 3$, then $\mathfrak{h}^2(X)$ is an abelian motive. In particular, the motive of a K3 surface is abelian. See [A1, Theorem 1.5.1].

For the remainder of the chapter, we only consider motives over $\mathbf{C}$.

The Betti realization functor restricts to a fiber functor $\mathrm{H}_{\mathrm{B}} \colon \mathbf{Mot}_{\mathrm{ab},\mathbf{C}} \to \mathbf{Q}$-$\mathbf{Vect}$. Let $\mathcal{G}_{\mathrm{ab}} = \mathrm{Aut}^{\otimes}(\mathrm{H}_{\mathrm{B}})$, so that H_{B} identifies $\mathbf{Mot}_{\mathrm{ab},\mathbf{C}}$ with $\mathcal{G}_{\mathrm{ab}}$-$\mathbf{Rep}$. Similarly, we use $\mathcal{G}_{\mathrm{Hdg}}$ to denote the Tannakian fundamental group associated with the forgetful functor $\mathbf{Q}$-$\mathbf{HS} \to \mathbf{Q}$-$\mathbf{Vect}$. The functor $\mathbf{Mot}_{\mathrm{ab},\mathbf{C}} \to \mathbf{Q}$-$\mathbf{HS}$ yields a homomorphism $\mathcal{G}_{\mathrm{Hdg}} \to \mathcal{G}_{\mathrm{ab}}$. We denote its restriction to $\mathbf{S} \subseteq \mathcal{G}_{\mathrm{Hdg},\mathbf{R}}$ by $h_{\mathrm{ab}} \colon \mathbf{S} \to \mathcal{G}_{\mathrm{ab},\mathbf{R}}$.

The following is a restatement of a fundamental result of Deligne, which says that Hodge cycles on complex abelian varieties are motivated.

Theorem 2.2.2 ([A2, Théorème 0.6.2])**.** *The Betti realization functor restricts to a fully faithful functor* $\mathrm{H}_{\mathrm{B}} \colon \mathbf{Mot}_{\mathrm{ab},\mathbf{C}} \to \mathbf{Q}$-$\mathbf{HS}$.

Corollary 2.2.3. *The homomorphism* $\mathcal{G}_{\mathrm{Hdg}} \to \mathcal{G}_{\mathrm{ab}}$ *is surjective.*

Proof. According to Theorem 2.2.2, the functor $\mathrm{H}_{\mathrm{B}} \colon \mathbf{Mot}_{\mathrm{ab},\mathbf{C}} \to \mathbf{Q}$-$\mathbf{HS}$ is fully faithful. Moreover, when M is an abelian motive, then every subobject of $\mathrm{H}_{\mathrm{B}}(M)$ is isomorphic to the image of a subobject of M, by the semisimplicity of $\mathbf{Q}$-$\mathbf{HS}$. This implies that the corresponding homomorphism $\mathcal{G}_{\mathrm{Hdg}} \to \mathcal{G}_{\mathrm{ab}}$ is surjective. ☐

Milne has shown for a large class of Shimura varieties of abelian type that the Hodge structures they parameterize are the Betti realizations of abelian motives, see [M3, Theorem 1.34]. We will only need the following specific instance.

Proposition 2.2.4. *Let (G, X) be a Shimura datum of Hodge or of orthogonal type, and let $h \in X$. Then there exists a unique homomorphism $\widetilde{h}\colon \mathcal{G}_{\mathrm{ab}} \to G$ such that the diagram*

$$\begin{tikzcd} \mathbf{S} \arrow[r, "h"] \arrow[d, "h_{\mathrm{ab}}"'] & G_{\mathbf{R}} \\ \mathcal{G}_{\mathrm{ab},\mathbf{R}} \arrow[ur, "\widetilde{h}_{\mathbf{R}}"'] & \end{tikzcd}$$

commutes.

Proof. The uniqueness is an immediate consequence of Corollary 2.2.3.

We first prove the existence part of the proposition for (G, X) of the form $(\mathrm{GSp}(V), \mathcal{H})$. Let $h\colon \mathbf{S} \to \mathrm{GSp}(V)_{\mathbf{R}}$ be an element of $\mathcal{H}$. Then Riemann's theorem [D2, Théorème 4.7] shows that there exists an abelian variety A with $\mathrm{H}^1(A, \mathbf{Q})$ isomorphic to (V, h) as a $\mathbf{Q}$-Hodge structure. Moreover, the symplectic form ψ is a morphism of $\mathbf{Q}$-Hodge structures $\bigwedge^2 V \to \mathbf{Q}(-1)$. By Theorem 2.2.2, ψ yields a morphism of motives $\bigwedge^2 \mathfrak{h}^1(A) \to \mathbf{1}(-1)$. It follows that $h\colon \mathbf{S} \to \mathrm{GSp}(V)_{\mathbf{R}}$ lifts to a morphism $\mathcal{G}_{\mathrm{ab}} \to \mathrm{GSp}(V)$.

For general (G, X) of Hodge type, pick an embedding $(G, X) \hookrightarrow (\mathrm{GSp}, \mathcal{H})$. Then by the preceding argument, the composition $\mathcal{G}_{\mathrm{Hdg}} \to G \to \mathrm{GSp}$ factors through $\mathcal{G}_{\mathrm{ab}}$. Since $\mathcal{G}_{\mathrm{Hdg}} \to \mathcal{G}_{\mathrm{ab}}$ is surjective by Corollary 2.2.3, the image of $\mathcal{G}_{\mathrm{ab}} \to \mathrm{GSp}$ is contained in G, yielding the desired lift $\widetilde{h}\colon \mathcal{G}_{\mathrm{ab}} \to G$ of h.

Now let (SO, Ω) be a Shimura datum of orthogonal type. Consider the morphism of Shimura data $(\mathrm{GSpin}, \Omega') \to (\mathrm{SO}, \Omega)$ as in (2.1), and let $h'\colon \mathbf{S} \to \mathrm{GSpin}_{\mathbf{R}}$ be the unique element of Ω' lifting h. Since $(\mathrm{GSpin}, \Omega')$ is of Hodge type, there exists a homomorphism $\widetilde{h}'\colon \mathcal{G}_{\mathrm{ab}} \to \mathrm{GSpin}$ with $h' = \widetilde{h}' \circ h_{\mathrm{ab}}$. The composition $\mathcal{G}_{\mathrm{ab}} \xrightarrow{\widetilde{h}'} \mathrm{GSpin} \to \mathrm{SO}$ is the desired lift of h. □

2.3 Orthogonal Shimura varieties as moduli of motives

This section contains the main theorem of this chapter, which gives a description of the complex points of a Shimura variety of orthogonal type in terms of motives endowed with a motivic bilinear form and a motivic trivialization of its determinant.

Let (V, b_V) be a quadratic space over $\mathbf{Q}$ of signature $(2, n)$, with $n \geq 1$, and let $\omega_V\colon \mathbf{Q} \to \det V$ be an isomorphism of vector spaces.

As in Section 2.3 we associate with (V, b_V) a Shimura datum (SO, Ω) with reflex field $\mathbf{Q}$. In particular, we have a set $\mathrm{Sh}(\mathrm{SO}, \Omega)(\mathbf{C})$ endowed with a left $\mathrm{Aut}(\mathbf{C})$-action and a right $\mathrm{SO}(\mathbf{A}_{\mathrm{f}})$-action which commute, an $(\mathrm{Aut}(\mathbf{C}), \mathrm{SO}(\mathbf{A}_{\mathrm{f}}))$-set, for short.

Definition 2.3.1. Let $\mathbf{Mot}(V)$ be the set of isomorphism classes of tuples (M, b, ω, α), where

- M is an abelian motive over $\mathbf{C}$,
- b is a morphism $\operatorname{Sym}^2 M \to \mathbf{1}$,
- ω is an isomorphism $\mathbf{1} \to \det M$,
- α is an isomorphism of $\mathbf{A}_{\mathrm{f}}$-modules $V \otimes \mathbf{A}_{\mathrm{f}} \to M_{\text{ét}}$ mapping b_V to $b_{\text{ét}}$ and ω_V to $\omega_{\text{ét}}$.

Two tuples $(M_1, b_1, \omega_1, \alpha_1)$ and $(M_2, b_2, \omega_2, \alpha_2)$ are said to be isomorphic if there is an isomorphism of motives $\varphi\colon M_1 \to M_2$ mapping b_1 and ω_1 to b_2 and ω_2 and such that the diagram

$$\begin{tikzcd} M_{1,\text{ét}} \arrow[rr, "\varphi_{\text{ét}}"] & & M_{2,\text{ét}} \\ & V \otimes \mathbf{A}_{\mathrm{f}} \arrow[ul, "\alpha_1"] \arrow[ur, "\alpha_2"'] & \end{tikzcd}$$

of $\mathbf{A}_{\mathrm{f}}$-modules commutes.

Pullback of motives as in (2.4) defines a left $\operatorname{Aut}(\mathbf{C})$-action on $\mathbf{Mot}(V)$. Moreover, by precomposing α with $g \in \mathrm{SO}(\mathbf{A}_{\mathrm{f}})$, we obtain a right $\mathrm{SO}(\mathbf{A}_{\mathrm{f}})$-action. It is easy to verify that these two actions commute, making $\mathbf{Mot}(V, b_V, \omega_V)$ an $(\operatorname{Aut}(\mathbf{C}), \mathrm{SO}(\mathbf{A}_{\mathrm{f}}))$-set.

Definition 2.3.2. Let $\mathbf{Mot}(V, \Omega)$ be the subset of $\mathbf{Mot}(V)$ consisting of those tuples (M, b, ω, α) such that there exists an isomorphism

$$\alpha^{\mathrm{B}}\colon V \longrightarrow M_{\mathrm{B}} \tag{2.5}$$

mapping b_V and ω_V to b_{B} and ω_{B}, and such that the Hodge structure on V induced by α^{B} is an element of Ω.

It is clear that the $\mathrm{SO}(\mathbf{A}_{\mathrm{f}})$-action on $\mathbf{Mot}(V)$ restricts to one on $\mathbf{Mot}(V, \Omega)$. The following theorem shows that the $\operatorname{Aut}(\mathbf{C})$-action restricts to one on $\mathbf{Mot}(V, \Omega)$ as well.

For $h \in \Omega$, we denote by $\widetilde{h}$ the unique lift of h to a homomorphism $\mathcal{G}_{\mathrm{ab}} \to \mathrm{SO}$, as in Proposition 2.2.4.

Theorem 2.3.3. *The map* $\mathrm{Sh}(\mathrm{SO}, \Omega)(\mathbf{C}) \to \mathbf{Mot}(V)$ *given by*

$$[h, g] \longmapsto ((V, \widetilde{h}),\, b_V,\, \omega_V, g)$$

is $(\operatorname{Aut}(\mathbf{C}), \mathrm{SO}(\mathbf{A}_{\mathrm{f}}))$*-equivariant, and defines a bijection from* $\mathrm{Sh}(\mathrm{SO}, \Omega)(\mathbf{C})$ *to* $\mathbf{Mot}(V, \Omega)$.

The proof of this theorem will be given in subsection 2.3.4. The $\operatorname{Aut}(\mathbf{C})$-equivariance will be deduced from the Shimura-Taniyama formula [D2, Théorème 4.19] for abelian varieties of CM type.

We will now give a corollary of Theorem 2.3.3 which will be more convenient in our treatment of moduli stacks of polarized hyperkähler varieties.

Let (W, b_W) be a quadratic space over $\mathbf{Q}$ of signature $(3, n)$, with $n \geq 1$, let $\lambda_W \in W$ be an element of positive length, and let $\omega_W \colon \mathbf{Q} \to \det W$ be an isomorphism of vector spaces.

Definition 2.3.4. Let $\mathbf{Mot}(W, \lambda_W)$ be the set of isomorphism classes of tuples $(M, b, \lambda, \omega, \alpha)$, where

- M is an abelian motive over $\mathbf{C}$,
- b is a morphism $\operatorname{Sym}^2 M \to \mathbf{1}$,
- λ is a morphism $\mathbf{1} \to M$,
- ω is an isomorphism $\mathbf{1} \to \det M$,
- α is an isomorphism of $\mathbf{A}_\mathrm{f}$-modules $W \otimes \mathbf{A}_\mathrm{f} \to M_{\text{ét}}$ mapping b_W, λ_W, and ω_W to $b_{\text{ét}}$, $\lambda_{\text{ét}}$, and $\omega_{\text{ét}}$, respectively.

Two tuples $(M_1, b_1, \omega_1, \lambda_1, \alpha_1)$ and $(M_2, b_2, \lambda_2, \omega_2, \alpha_2)$ are said to be isomorphic if there is an isomorphism of motives $\varphi \colon M_1 \to M_2$ mapping b_1, λ_1, and ω_1 to b_2, λ_2, and ω_2, respectively, and such that the diagram

$$\begin{tikzcd} M_{1,\text{ét}} \arrow[rr, "\varphi_{\text{ét}}"] & & M_{2,\text{ét}} \\ & W \otimes \mathbf{A}_\mathrm{f} \arrow[ul, "\alpha_1"] \arrow[ur, "\alpha_2"'] & \end{tikzcd}$$

of $\mathbf{A}_\mathrm{f}$-modules commutes.

Define V to be the orthogonal complement of λ_W in W, and b_V the pairing on V induced by b_W. Then V is a quadratic space of signature $(2, n)$, and hence gives rise to an orthogonal Shimura datum (SO, Ω). Let $\rho \colon \mathrm{SO} \to \mathrm{SO}(W)$ be the homomorphism sending g to $g \oplus \mathrm{id}_{\mathbf{Q}\lambda_W}$. Note that similarly to $\mathbf{Mot}(V)$, the set $\mathbf{Mot}(W, \lambda_W)$ comes with an $\mathrm{Aut}(\mathbf{C})$-action, and ρ induces a right $\mathrm{SO}(\mathbf{A}_\mathrm{f})$-action on $\mathbf{Mot}(W, \lambda_W)$.

The following corollary follows immediately from Theorem 2.3.3.

Corollary 2.3.5. *The map* $\mathrm{Sh}(\mathrm{SO}, \Omega)(\mathbf{C}) \to \mathbf{Mot}(W, \lambda_W)$ *given by*

$$[h, g] \longmapsto ((W, \widetilde{\rho h}),\, b_W,\, \lambda_W,\, \omega_W, g)$$

is $(\mathrm{Aut}(\mathbf{C}), \mathrm{SO}(\mathbf{A}_\mathrm{f}))$*-equivariant.*

2.3.1 $\mathbf{Mot}(G)$

Before we start with the proof of Theorem 2.3.3, it will be useful to put the construction of $\mathbf{Mot}(V)$ and $\mathbf{Mot}(V, \Omega)$ in a more Tannakian framework.

Let G be an affine group scheme over $\mathbf{Q}$. We denote by G-**Rep** the category of finite-dimensional representations of G, and by $\omega_G \colon G\text{-}\mathbf{Rep} \to \mathbf{Q}\text{-}\mathbf{Vect}$ the forgetful functor.

Definition 2.3.6. Let G be an affine group scheme over $\mathbf{Q}$. Then we denote by $\mathbf{Mot}(G)$ the set of isomorphism classes of pairs (F, η), where F is a tensor functor from G-**Rep** to $\mathbf{Mot}_{\mathrm{ab},\mathbf{C}}$, and $\eta \colon \omega_G \otimes \mathbf{A}_{\mathrm{f}} \to \mathrm{H}_{\text{ét}}\, F$ is an isomorphism of tensor functors from G-**Rep** to $\mathbf{A}_{\mathrm{f}}$-**Mod**. It will be convenient to represent such a pair (F, η) as the 2-commutative diagram

$$\begin{tikzcd} G\text{-}\mathbf{Rep} \arrow[rr, "F"] \arrow[dr, "\omega_G \otimes \mathbf{A}_{\mathrm{f}}"'] \arrow[rr, phantom, shift right=5ex, "\overset{\eta}{\Longrightarrow}"] & & \mathbf{Mot}_{\mathrm{ab},\mathbf{C}} \arrow[dl, "\mathrm{H}_{\text{ét}}"] \\ & \mathbf{A}_{\mathrm{f}}\text{-}\mathbf{Mod} & \end{tikzcd}$$

Here, two pairs (F_1, η_1) and (F_2, η_2) are said to be isomorphic if there exists an isomorphism of tensor functors $\varphi \colon F_1 \to F_2$ from G-**Rep** to $\mathbf{Mot}_{\mathrm{ab},\mathbf{C}}$ for which the diagram

$$\begin{tikzcd} \mathrm{H}_{\text{ét}}\, F_1 \arrow[rr, "\mathrm{H}_{\text{ét}}(\varphi)"] & & \mathrm{H}_{\text{ét}}\, F_2 \\ & \omega_G \otimes \mathbf{A}_{\mathrm{f}} \arrow[ul, "\eta_1"] \arrow[ur, "\eta_2"'] & \end{tikzcd}$$

of functors $G\text{-}\mathbf{Rep} \to \mathbf{A}_{\mathrm{f}}\text{-}\mathbf{Mod}$ commutes.

Remark 2.3.7. For $(F, \eta) \in \mathbf{Mot}(G)$, the exactness of fiber functors implies that F is exact.

There is an alternative description of $\mathbf{Mot}(G)$ in terms of G-torsors on $\mathbf{Q}_{\text{ét}}$, which we now give.

Definition 2.3.8. We define $\mathbf{Mot}'(G)$ to be the set of isomorphism classes of tuples (T, h, α), where

- T is a G-torsor on $\mathbf{Q}_{\text{ét}}$,
- $h \colon \mathcal{G}_{\mathrm{ab}} \to \underline{\mathrm{Aut}}_G(T)$ is a homomorphism of group schemes,
- $\alpha \in T(\mathbf{A}_{\mathrm{f}})$.

Two tuples (T_1, h_1, α_1) and (T_2, h_2, α_2) are said to be isomorphic if there exists an isomorphism of G-torsors $T_1 \to T_2$ mapping h_1 and α_1 to h_2 and α_2, respectively.

Remark 2.3.9. Note that the automorphism scheme $\underline{\mathrm{Aut}}_G(T)$ is a pure inner form of G, which is isomorphic to G if T has a $\mathbf{Q}$-valued point.

We define a map $f\colon \mathbf{Mot}(G) \to \mathbf{Mot}'(G)$. Let $(F, \eta) \in \mathbf{Mot}(G)$. The isomorphism sheaf $T := \underline{\mathrm{Isom}}^{\otimes}(\omega_G, \mathrm{H_B} \circ F)$ is a G-torsor on $\mathbf{Q}_{\text{ét}}$, satisfying $\underline{\mathrm{Aut}}_G(T) = \underline{\mathrm{Aut}}^{\otimes}(\mathrm{H_B} \circ F)$ by the equivalence between G-torsors and fiber functors on $G\text{-}\mathbf{Rep}$ (see [S1, Proposition III.3.2.5.3]). Consider the canonical homomorphism $\underline{\mathrm{Aut}}^{\otimes}(\mathrm{H_B}) \to \underline{\mathrm{Aut}}^{\otimes}(\mathrm{H_B} \circ F)$. Since $\underline{\mathrm{Aut}}^{\otimes}(\mathrm{H_B}) = \mathcal{G}_{\mathrm{ab}}$ and $\underline{\mathrm{Aut}}^{\otimes}(\mathrm{H_B} \circ F) = \underline{\mathrm{Aut}}_G(T)$, this gives a homomorphism $h\colon \mathcal{G}_{\mathrm{ab}} \to \underline{\mathrm{Aut}}_G(T)$. By definition of T, the isomorphism of tensor functors η is an $\mathbf{A}_{\mathrm{f}}$-valued point of T. We now set $f(F, \eta) = (T, h, \eta)$.

Lemma 2.3.10. *The map $f\colon \mathbf{Mot}(G) \to \mathbf{Mot}'(G)$ is a bijection.*

Proof. Let $\mathbf{Mot}''(G)$ be the set of isomorphism classes of tuples (ω, h, α), where

- $\omega\colon G\text{-}\mathbf{Rep} \to \mathbf{Q}\text{-}\mathbf{Vect}$ is a fiber functor,
- $h\colon \mathcal{G}_{\mathrm{ab}} \to \underline{\mathrm{Aut}}^{\otimes}(\omega)$ is a homomorphism of group schemes,
- $\alpha\colon \omega_G \otimes \mathbf{A}_{\mathrm{f}} \to \omega \otimes \mathbf{A}_{\mathrm{f}}$ is an isomorphism of tensor functors from $G\text{-}\mathbf{Rep}$ to $\mathbf{A}_{\mathrm{f}}\text{-}\mathbf{Mod}$.

From a fiber functor $\omega\colon G\text{-}\mathbf{Rep} \to \mathbf{Q}\text{-}\mathbf{Vect}$ we obtain a G-torsor $\underline{\mathrm{Isom}}^{\otimes}(\omega_G, \omega)$, which yields an equivalence between the stack of fiber functors on $G\text{-}\mathbf{Rep}$ and the stack of G-torsors, see [S1, Proposition III.3.2.5.3] for more details. This equivalence yields a bijection $\mathbf{Mot}''(G) \to \mathbf{Mot}'(G)$.

It follows that we need to show that the map $\mathbf{Mot}(G) \to \mathbf{Mot}''(G)$ given by

$$(F, \eta) \longmapsto \big(\mathrm{H_B} \circ F,\ h\colon \mathcal{G}_{\mathrm{ab}} \to \underline{\mathrm{Aut}}^{\otimes}(\mathrm{H_B} \circ F),\ \eta\big)$$

is a bijection.

For the surjectivity, consider a tuple $(\omega, h, \alpha) \in \mathbf{Mot}''(G)$. Then ω lifts to an equivalence $\omega\colon G\text{-}\mathbf{Rep} \to \underline{\mathrm{Aut}}^{\otimes}(\omega)$, and α fits in the 2-commutative diagram

$$\begin{tikzcd}
G\text{-}\mathbf{Rep} \arrow[rr, "\omega"] \arrow[dr, "\omega_G \otimes \mathbf{A}_{\mathrm{f}}"'] & & \underline{\mathrm{Aut}}^{\otimes}(\omega)\text{-}\mathbf{Rep} \arrow[dl, "\omega_{\underline{\mathrm{Aut}}^{\otimes}(\omega)} \otimes \mathbf{A}_{\mathrm{f}}"] \\
& \mathbf{A}_{\mathrm{f}}\text{-}\mathbf{Mod} \arrow[u, phantom, "\overset{\alpha}{\Longrightarrow}"] &
\end{tikzcd}$$

Moreover, $h\colon \mathcal{G}_{\mathrm{ab}} \to \underline{\mathrm{Aut}}^{\otimes}(\omega)$ gives rise to a functor $h^*\colon \underline{\mathrm{Aut}}^{\otimes}(\omega) \to \mathbf{Mot}_{\mathrm{ab},\mathbf{C}}$ compatible with the natural fiber functors. It is easily checked that the outer triangle in the diagram

$$\begin{tikzcd}
G\text{-}\mathbf{Rep} \arrow[r, "\omega"] \arrow[dr, "\omega_G \otimes \mathbf{A}_{\mathrm{f}}"'] & \underline{\mathrm{Aut}}^{\otimes}(\omega)\text{-}\mathbf{Rep} \arrow[r, "h^*"] \arrow[d] & \mathbf{Mot}_{\mathrm{ab},\mathbf{C}} \arrow[dl, "\mathrm{H}_{\text{ét}}"] \\
\arrow[r, phantom, "\overset{\alpha}{\Longrightarrow}"] & \mathbf{A}_{\mathrm{f}}\text{-}\mathbf{Mod} \arrow[r, phantom, "\overset{\mathrm{id}}{\Longrightarrow}"] &
\end{tikzcd}$$

defines an element of $\mathbf{Mot}(G)$ mapping to the tuple (ω, h, α).

For the injectivity, let $(F_1, \eta_1), (F_2, \eta_2) \in \mathbf{Mot}(G)$, and assume that the associated tuples $(\mathrm{H_B} \circ F_1, h_1, \eta_1)$ and $(\mathrm{H_B} \circ F_2, h_2, \eta_2)$ are isomorphic. That is, assume there is an isomorphism of tensor functors $\varphi\colon \mathrm{H_B} \circ F_1 \to \mathrm{H_B} \circ F_2$ such that the diagram of group schemes

$$\begin{tikzcd} \underline{\mathrm{Aut}}^{\otimes}(\mathrm{H_B} \circ F_1) \arrow[rr, "\varphi"] & & \underline{\mathrm{Aut}}^{\otimes}(\mathrm{H_B} \circ F_2) \\ & \mathcal{G}_{\mathrm{ab}} \arrow[ul, "h_1"] \arrow[ur, "h_2"'] & \end{tikzcd} \tag{2.6}$$

and the diagram of tensor functors from G-$\mathbf{Rep}$ to $\mathbf{A}_{\mathrm{f}}$-$\mathbf{Mod}$

$$\begin{tikzcd} (\mathrm{H_B} \circ F_1) \otimes \mathbf{A}_{\mathrm{f}} \arrow[rr, "\varphi \otimes \mathbf{A}_{\mathrm{f}}"] & & (\mathrm{H_B} \circ F_2) \otimes \mathbf{A}_{\mathrm{f}} \\ & \omega_G \otimes \mathbf{A}_{\mathrm{f}} \arrow[ul, "\eta_1"] \arrow[ur, "\eta_2"'] & \end{tikzcd} \tag{2.7}$$

commute. From (2.6) we obtain that for any $V \in G$-$\mathbf{Rep}$, the homomorphism of vector spaces $\varphi_V\colon \mathrm{H_B} \circ F_1(V) \to \mathrm{H_B} \circ F_2(V)$ is $\mathcal{G}_{\mathrm{ab}}$-equivariant. This implies that φ lifts to an isomorphism of tensor functors $\varphi\colon F_1 \to F_2$ from G-$\mathbf{Rep}$ to $\mathbf{Mot}_{\mathrm{ab},\mathbf{C}}$. Since $\mathrm{H_{ét}} = \mathbf{A}_{\mathrm{f}} \otimes \mathrm{H_B}$, Equation (2.7) says that φ is compatible with η_1 and η_2, and hence that φ defines an isomorphism from (F_1, η_1) to (F_2, η_2). □

Let (V, b_V) be a quadratic space over $\mathbf{Q}$ of signature $(2, n)$ with $n \geq 1$, and $\omega_V\colon \mathbf{Q} \to \det V$ an isomorphism of vector spaces. We now relate $\mathbf{Mot}(\mathrm{SO}(V))$ to the set $\mathbf{Mot}(V)$ defined in Definition 2.3.1. Note that if we endow $\mathbf{Q}$ with the trivial $\mathrm{SO}(V)$-action, then $b_V\colon \mathrm{Sym}^2 V \to \mathbf{Q}$ and $\omega_V\colon \mathbf{Q} \to \det V$ are both morphisms in $\mathrm{SO}(V)$-$\mathbf{Rep}$. As such, we obtain a map $\Psi\colon \mathbf{Mot}(\mathrm{SO}(V)) \to \mathbf{Mot}(V)$ given by

$$(F, \eta) \longmapsto (F(V), F(b_V), F(\omega_V), \eta_V).$$

Lemma 2.3.11. *The map $\Psi\colon \mathbf{Mot}(\mathrm{SO}(V)) \to \mathbf{Mot}(V)$ is a bijection.*

Proof. Let (V', b') be a quadratic space over $\mathbf{Q}$ of dimension $2 + n$, let $\omega'\colon \mathbf{Q} \to \det V'$ be an isomorphism of vector spaces, and let $\alpha\colon V \otimes \mathbf{A}_{\mathrm{f}} \to V' \otimes \mathbf{A}_{\mathrm{f}}$ be an isometry which maps ω_V to ω'. We will show that the isomorphism sheaf $\mathrm{Isom}((V, b_V, \omega_V), (V', b', \omega'))$ is an $\mathrm{SO}(V)$-torsor on $\mathbf{Q}_{\mathrm{ét}}$. To do this, it suffices to show that there is an isomorphism $V \otimes \overline{\mathbf{Q}} \to V' \otimes \overline{\mathbf{Q}}$ mapping b_V to b' and ω_V to ω'.

First we assign an invariant to the tuple (V, b_V, ω_V). Let $\det(b_V)\colon (\det V)^{\otimes 2} \to \mathbf{Q}$ be the isomorphism of vector spaces given by mapping $(v_1 \wedge \ldots \wedge v_{2+n}) \otimes (w_1 \wedge \ldots \wedge w_{2+n})$ to the determinant of the matrix $(b_V(v_i, w_j))_{i,j}$. Now composing the isomorphism $\omega_V^{\otimes 2}\colon \mathbf{Q} \to (\det V)^{\otimes 2}$ with $\det(b_V)$ yields an element of $\mathbf{Q}^{\times}$ which we denote with λ. Similarly one defines $\det(b')\colon (\det V')^{\otimes 2} \to \mathbf{Q}$ and $\lambda' \in \mathbf{Q}^\times$ using b' and ω'. Note that the existence of the isomorphism α and the injectivity of $\mathbf{Q}^\times \to \mathbf{A}_\mathrm{f}{}^\times$ imply that $\lambda = \lambda'$.

Any two quadratic spaces of the same dimension over an algebraically closed field are isometric, so there exists an isometry $\varphi\colon V \otimes \overline{\mathbf{Q}} \to V' \otimes \overline{\mathbf{Q}}$. Let $\rho \in \mathbf{Q}^\times$ be such that $\varphi(\omega) = \rho\omega'$. Taking the tensor square and composing with $\det(b')$ yields the equality $\det(b')\varphi(\omega^{\otimes 2}) = \rho^2 \det(b')(\omega')^{\otimes 2}$. The right-hand side is equal to $\rho^2\lambda'$, and since $\det(b') = \det(\varphi(b'))$, the left-hand side is equal to λ. From the fact that $\lambda = \lambda'$, it follows that $\rho^2 = 1$ and hence that $\rho = \pm 1$. Composing φ with an element of $\mathrm{O}(V')$ with determinant -1 if necessary, we obtain an isomorphism $V \otimes \overline{\mathbf{Q}} \to V' \otimes \overline{\mathbf{Q}}$ mapping b_V to b' and ω_V to ω'. This proves that $\mathrm{Isom}((V, b_V, \omega_V), (V', b', \omega'))$ is an $\mathrm{SO}(V)$-torsor on $\mathbf{Q}_{\text{ét}}$.

This yields an equivalence between tuples (V', b', ω') endowed with an isomorphism $\alpha\colon V \otimes \mathbf{A}_\mathrm{f} \to V' \otimes \mathbf{A}_\mathrm{f}$ and $\mathrm{SO}(V)$-torsors T endowed with an $\mathbf{A}_\mathrm{f}$ point $\alpha \in T(\mathbf{A}_\mathrm{f})$. Since an element of $\mathbf{Mot}(V)$ consists of such a tuple (V', b', ω') endowed with a homomorphism $\mathcal{G}_\mathrm{ab} \to \mathrm{SO}(V')$, this equivalence gives a bijection $f'\colon \mathbf{Mot}(V) \to \mathbf{Mot}'(\mathrm{SO}(V))$.

It is now easily verified that the composition $f'\Psi$ is the bijection $f\colon \mathbf{Mot}(\mathrm{SO}(V)) \to \mathbf{Mot}'(\mathrm{SO}(V))$ from Lemma 2.3.10, proving that Ψ is itself a bijection. For example, for $(F, \eta) \in \mathbf{Mot}(\mathrm{SO}(V))$, we have an $\mathrm{SO}(V)$-equivariant map

$$\underline{\mathrm{Isom}}^\otimes(\omega_{\mathrm{SO}(V)}, \mathrm{H_B} \circ F) \longrightarrow \mathrm{Isom}((V, b_V, \omega_V), (F(V), F(b_V), F(\omega_V))$$

given by $\varepsilon \mapsto \varepsilon_V$. This is an isomorphism since the source and target are $\mathrm{SO}(V)$-torsors. $\square$

The set $\mathbf{Mot}(G)$ comes with a right $G(\mathbf{A}_\mathrm{f})$-action, which we now define. An element $g \in G(\mathbf{A}_\mathrm{f})$ yields an automorphism of $\omega_G \otimes \mathbf{A}_\mathrm{f}$, which we also denote by g. Such g then acts on a pair $(F, \eta) \in \mathbf{Mot}(G)$ by $(F, \eta)g := (F, \eta g)$, that is, g maps (F, η) to the outer triangle in the diagram

$$\begin{tikzcd}
G\text{-}\mathbf{Rep} \arrow[r, equal] \arrow[dr] & G\text{-}\mathbf{Rep} \arrow[r, "F"] \arrow[d] \arrow[d, phantom, shift right=5ex, "\overset{g}{\Longrightarrow}"] \arrow[d, phantom, shift left=5ex, "\overset{\eta}{\Longrightarrow}"] & \mathbf{Mot}_{\mathrm{ab},\mathbf{C}} \arrow[dl] \\
 & \mathbf{A}_\mathrm{f}\text{-}\mathbf{Mod} &
\end{tikzcd}$$

Let (G, X) be a Shimura datum of Hodge or of orthogonal type. According to Proposition 2.2.4, every $h \in X$ lifts to a unique homomorphism $\widetilde{h}\colon \mathcal{G}_\mathrm{ab} \to G$. We denote the associated tensor functor $G\text{-}\mathbf{Rep} \to \mathbf{Mot}_{\mathrm{ab},\mathbf{C}}$ with $\widetilde{h}^*$.

Lemma 2.3.12. *The assignment*

$$[h,g] \longmapsto \begin{tikzcd} G\text{-}\mathbf{Rep} \arrow[rr, "\widetilde{h}^*"] \arrow[dr] \arrow[dr, bend right=60] \arrow[dr, phantom, "\overset{g}{\Rightarrow}"] & \arrow[d, phantom, "\overset{\mathrm{id}}{\Longrightarrow}"] & \mathbf{Mot}_{\mathrm{ab},\mathbf{C}} \arrow[dl] \\ & \mathbf{A}_{\mathrm{f}}\text{-}\mathbf{Mod} & \end{tikzcd}$$

defines a $G(\mathbf{A}_{\mathrm{f}})$*-equivariant injective map*

$$\Phi\colon \operatorname{Sh}(G,X)(\mathbf{C}) \longrightarrow \mathbf{Mot}(G),$$

functorial in (G,X).

Proof. Let $(h_1,g_1) \in X \times G(\mathbf{A}_{\mathrm{f}})$, let $\gamma \in G(\mathbf{Q})$, and define $(h_2,g_2) \coloneqq \gamma(h_1,g_1)$. Since γ induces an isomorphism of tensor functors $\widetilde{h}_1^* \to \widetilde{h}_2^*$, and since $g_2 = \gamma g_1$, the map Φ is well-defined.

For the injectivity, suppose we are given $(h_1,g_1),(h_2,g_2) \in X \times G(\mathbf{A}_{\mathrm{f}})$ and an isomorphism $\varphi\colon (\widetilde{h}_1^*, g_1) \to (\widetilde{h}_2^*, g_2)$. Then from the compatibility of φ with g_1 and g_2 we find that for $(V, \rho\colon G \to \mathrm{GL}(V))$ in G-**Rep**, the map $\varphi_{(V,\rho)}\colon V \to V$ is $\rho(g_2 g_1^{-1})$. By taking (V,ρ) to be a faithful representation we obtain that $\gamma \coloneqq g_2 g_1^{-1}$ is an element of $G(\mathbf{Q})$, and that $\gamma h_1 = h_2$, proving the injectivity. □

Remark 2.3.13. We denote by $\mathbf{Mot}(G,X)$ the subset of $\mathbf{Mot}(G)$ consisting of those pairs (F,η) for which there exists an isomorphism of tensor functors $\eta^{\mathrm{B}}\colon \omega_G \to \mathrm{H}_{\mathrm{B}} \circ F$, that is,

$$\begin{tikzcd} G\text{-}\mathbf{Rep} \arrow[rr, "F"] \arrow[dr, "\omega_G"'] & \arrow[d, phantom, "\overset{\eta^{\mathrm{B}}}{\Longrightarrow}"] & \mathbf{Mot}_{\mathrm{ab},\mathbf{C}} \arrow[dl, "\mathrm{H}_{\mathrm{B}}"] \\ & \mathbf{Q}\text{-}\mathbf{Vect} & \end{tikzcd} \tag{2.8}$$

such that if $\psi\colon \mathcal{G}_{\mathrm{ab}} \to G$ is the homomorphism corresponding to (F,η^{B}), then $\psi h_{\mathrm{ab}}\colon \mathbf{S} \to G_{\mathbf{R}}$ is an element of X.

Proposition 2.3.14. *Let* (G,X) *be a Shimura datum of Hodge or of orthogonal type. The map* $\Phi\colon \operatorname{Sh}(G,X)(\mathbf{C}) \to \mathbf{Mot}(G)$ *from Lemma 2.3.12 has* $\mathbf{Mot}(G,X)$ *as its image.*

Proof. If $[h,g] \in \operatorname{Sh}(G,X)(\mathbf{C})$, then $\mathrm{H}_{\mathrm{B}}\,\widetilde{h}^* = \omega_G$. Therefore we can take $\eta^{\mathrm{B}}\colon \omega_G \to \mathrm{H}_{\mathrm{B}}\,\widetilde{h}^*$ to be the identity. This shows that $\Phi([h,g]) \in \mathbf{Mot}(G,X)$.

To prove that Φ has $\mathbf{Mot}(G,X)$ as its image, let $(F,\eta) \in \mathbf{Mot}(G,X)$, and let $\eta^{\mathrm{B}}\colon \omega_G \to \mathrm{H}_{\mathrm{B}}$ be an isomorphism of tensor functors as in (2.8). Then the pair (F,η^{B}) gives rise to a unique homomorphism $\psi\colon \mathcal{G}_{\mathrm{ab}} \to G$ such that η^{B}

lifts to an isomorphism of tensor functors $\varepsilon\colon \widetilde{\psi}^* \to F$. Now the composition $h := \psi_{\mathbf{R}} h_{\mathrm{ab}}\colon \mathbf{S} \to G_{\mathbf{R}}$ is an element of X satisfying $\widetilde{h} = \psi$. Now since $\mathrm{H}_{\text{ét}}\,\psi^* = \omega_G \otimes \mathbf{A}_{\mathrm{f}}$, the composition

$$\omega_G \otimes \mathbf{A}_{\mathrm{f}} \xrightarrow{\eta} \mathrm{H}_{\text{ét}}\, F \xrightarrow{\mathrm{H}_{\text{ét}}(\varepsilon)^{-1}} \mathrm{H}_{\text{ét}}\, \psi^*$$

defines an automorphism of $\omega_G \otimes \mathbf{A}_{\mathrm{f}}$ and hence an element $g \in G(\mathbf{A}_{\mathrm{f}})$. It is easy to verify that ε is an isomorphism from $\Phi([h,g])$ to (F,η), proving that Φ surjects onto $\mathbf{Mot}(G,X)$. ☐

Similarly to Lemma 2.3.11, we now relate the set $\mathbf{Mot}(V,\Omega)$ from Definition 2.3.2.

Lemma 2.3.15. *Let (V, b_V) be a quadratic space over $\mathbf{Q}$ of signature $(2,n)$ with $n \geq 1$, and $\omega_V\colon \mathbf{Q} \to \det V$ an isomorphism of vector spaces, and let $(\mathrm{SO}(V),\Omega)$ be the associated Shimura datum. The bijection $\Psi\colon \mathbf{Mot}(\mathrm{SO}(V)) \to \mathbf{Mot}(V)$ given in Lemma 2.3.11 maps $\mathbf{Mot}(SO(V),\Omega)$ onto $\mathbf{Mot}(V,\Omega)$.*

Proof. Given $(F,\eta) \in \mathbf{Mot}(\mathrm{SO}(V),\Omega)$ and $\eta^{\mathrm{B}}\colon \omega_{\mathrm{SO}(V)} \to \mathrm{H_B}\,F$ an isomorphism of tensor functors as in (2.8), then setting $\alpha^{\mathrm{B}} := \eta^{\mathrm{B}}_V\colon V \to F(V)_{\mathrm{B}}$ shows that $(F(V), F(b_V), F(\omega_V), \eta_V)$ is an element of $\mathbf{Mot}(V,\Omega)$.

Conversely, let $(M,b,\omega,\alpha) \in \mathbf{Mot}(V,\Omega)$, and let $\alpha^{\mathrm{B}}\colon V \to M_{\mathrm{B}}$ be an isomorphism as in (2.5). Then there exists an $h \in \Omega$ for which the diagram

$$\begin{tikzcd} \mathrm{SO}(V) \arrow[rr, "\alpha^{\mathrm{B}}"] & & \mathrm{SO}(M_{\mathrm{B}}) \\ & \mathcal{G}_{\mathrm{ab}} \arrow[ul, "\widetilde{h}"] \arrow[ur] & \end{tikzcd}$$

commutes. Moreover, if we set $g = (\alpha^{\mathrm{B}})^{-1}\alpha \in \mathrm{SO}(V)(\mathbf{A}_{\mathrm{f}})$, then α^{B} is an isomorphism from $((V,\widetilde{h}), b_V, \omega_V, g)$ to (M,b,ω,α). It follows that Ψ maps $\Phi([h,g])$ to (M,b,ω,α). ☐

Using pullback of motives as in (2.4) we can define a left $\mathrm{Aut}(\mathbf{C})$-action on $\mathbf{Mot}(G)$ by having $\sigma \in \mathrm{Aut}(\mathbf{C})$ act on a pair $(F,\eta) \in \mathbf{Mot}(G)$ as

$$\sigma(F,\eta) \quad := \quad \begin{tikzcd} G\text{-}\mathbf{Rep} \arrow[r, "F"] \arrow[dr] & \mathbf{Mot}_{\mathrm{ab},\mathbf{C}} \arrow[r, "\sigma^*"] \arrow[d] & \mathbf{Mot}_{\mathrm{ab},\mathbf{C}} \arrow[dl] \\ & \mathbf{A}_{\mathrm{f}}\text{-}\mathbf{Mod} & \end{tikzcd}$$

It is clear that the $G(\mathbf{A}_{\mathrm{f}})$-action and $\mathrm{Aut}(\mathbf{C})$-action commute.

The subset $\mathbf{Mot}(G,X)$ is not necessarily $\mathrm{Aut}(\mathbf{C})$-stable. However, in the next three subsections we will prove the following result.

Theorem 2.3.16. *Let (G, X) be a Shimura datum of Hodge or of orthogonal type with reflex field E. Then $\mathbf{Mot}(G, X)$ is an $\operatorname{Aut}(\mathbf{C}/E)$-stable subset of $\mathbf{Mot}(G)$, and the bijection $\operatorname{Sh}(G, X)(\mathbf{C}) \to \mathbf{Mot}(G, X)$ is $\operatorname{Aut}(\mathbf{C}/E)$-equivariant.*

Remark 2.3.17. Since the map $\mathbf{Mot}(\mathrm{SO}(V)) \to \mathbf{Mot}(V)$ given in Lemma 2.3.11 is $\operatorname{Aut}(\mathbf{C})$-equivariant, Theorem 2.3.3 is a corollary of Theorem 2.3.16 and Lemma 2.3.15.

2.3.2 Siegel Shimura data

Let (V, ψ_V) be a symplectic vector space over $\mathbf{Q}$, and let $(\mathrm{GSp}, \mathcal{H})$ be the associated Shimura datum as in Example 2.1.1.

Definition 2.3.18. Let R be a $\mathbf{Q}$-algebra, and consider R-modules V_1, V_2, L_1, and L_2 endowed with homomorphisms $\psi_1\colon \bigwedge^2 V_1 \to L_1$ and $\psi_2\colon \bigwedge^2 V_2 \to L_2$. A **similitude** from the tuple (V_1, L_1) to the tuple (V_2, L_2) is a pair of isomorphisms

$$(\varphi\colon V_1 \xrightarrow{\sim} V_2,\ \lambda\colon L_1 \xrightarrow{\sim} L_2)$$

for which the diagram

$$\begin{tikzcd} \bigwedge^2 V_1 \arrow[r, "\psi_1"] \arrow[d, "\varphi"'] & L_1 \arrow[d, "\lambda"] \\ \bigwedge^2 V_2 \arrow[r, "\psi_2"'] & L_2 \end{tikzcd}$$

commutes.

Definition 2.3.19. We define the set $\mathbf{Mot}(V, \psi_V)$ as the set of isomorphism classes of tuples $(M, L, \psi, \alpha, \beta)$, where

- M and L are abelian motives over $\mathbf{C}$,
- $\psi\colon \bigwedge^2 M \to L$ is a morphism of motives,
- $(\alpha, \beta)\colon (V \otimes \mathbf{A}_\mathrm{f}, \mathbf{A}_\mathrm{f}) \to (M_{\text{ét}}, L_{\text{ét}})$ is a similitude.

Here, two tuples $(M_1, L_1, \psi_1, \alpha_1, \beta_1)$ and $(M_2, L_2, \psi_2, \alpha_2, \beta_2)$ are said to be isomorphic if there exists a pair of isomorphisms $\varphi\colon M_1 \to M_2$ and $\lambda\colon L_1 \to L_2$ for which the diagrams

$$\begin{tikzcd} \bigwedge^2 M_1 \arrow[r, "\psi_1"] \arrow[d, "\varphi"'] & L_1 \arrow[d, "\lambda"] \\ \bigwedge^2 M_2 \arrow[r, "\psi_2"'] & L_2 \end{tikzcd}, \qquad \begin{tikzcd} M_{1,\text{ét}} \arrow[rr, "\varphi_{\text{ét}}"] & & M_{2,\text{ét}} \\ & V \otimes \mathbf{A}_\mathrm{f} \arrow[ul, "\alpha_1"] \arrow[ur, "\alpha_2"'] & \end{tikzcd},$$

and

$$\begin{tikzcd} L_{1,\text{ét}} \arrow[rr, "\lambda_{\text{ét}}"] & & L_{2,\text{ét}} \\ & V \otimes \mathbf{A}_{\mathrm{f}} \arrow[ul, "\beta_1"] \arrow[ur, "\beta_2"'] & \end{tikzcd}$$

commute. Pullback of motives gives a left $\mathrm{Aut}(\mathbf{C})$-action on $\mathbf{Mot}(V, \psi_V)$.

Definition 2.3.20. We define $\mathbf{AV}(V, \psi_V)$ to be the set of isogeny classes of tuples (A, λ, α), where

- A is an abelian variety over $\mathbf{C}$,
- λ is a polarization on A,
- $\alpha\colon V \otimes \mathbf{A}_{\mathrm{f}} \to \mathrm{H}^1_{\text{ét}}(A, \mathbf{A}_{\mathrm{f}})$ is a similitude, where $\mathrm{H}^1_{\text{ét}}(A, \mathbf{A}_{\mathrm{f}})$ is endowed with the symplectic form induced by λ.

Two tuples $(A_1, \lambda_1, \alpha_1)$ and $(A_2, \lambda_2, \alpha_2)$ are said to be isogenic if there is an isogeny $\varphi \in \mathrm{Hom}(A_1, A_2) \otimes \mathbf{Q}$ making the diagram

$$\begin{tikzcd} \mathrm{H}^1_{\text{ét}}(A_2, \mathbf{A}_{\mathrm{f}}) \arrow[rr, "\varphi^*"] & & \mathrm{H}^1_{\text{ét}}(A_1, \mathbf{A}_{\mathrm{f}}) \\ & V \otimes \mathbf{A}_{\mathrm{f}} \arrow[ul, "\alpha_2"] \arrow[ur, "\alpha_1"'] & \end{tikzcd},$$

commute. Pullback of schemes gives a left $\mathrm{Aut}(\mathbf{C})$-action on $\mathbf{AV}(V, \psi_V)$.

Proposition 2.3.21. *The map* $\mathrm{Sh}(\mathrm{GSp}, \mathcal{H})(\mathbf{C}) \to \mathbf{Mot}(\mathrm{GSp})$ *given in Lemma 2.3.12 is* $\mathrm{Aut}(\mathbf{C})$*-equivariant.*

Proof. In [D2, §4], Deligne defines a bijection $\mathrm{Sh}(\mathrm{GSp}, \mathcal{H})(\mathbf{C}) \to \mathbf{AV}(V, \psi_V)$, as follows. Let (h, g) be an element of $\mathrm{Sh}(\mathrm{GSp}, \mathcal{H})(\mathbf{C})$. Fix a lattice $\Lambda \subseteq V$ such that ψ_V restricts to a perfect pairing on Λ. Then (Λ, h) is a $\mathbf{Z}$-Hodge structure of type $d(1,0) + d(0,1)$, where $2d$ is the dimension of V, and $\psi_V|_\Lambda$ is a polarization on (Λ, h). This gives rise to a polarized abelian variety (A, λ) (unique up to isomorphism) such that $\mathrm{H}^1(A, \mathbf{Z})$ is isomorphic to Λ as a polarized $\mathbf{Z}$-Hodge structure. Pick such an isomorphism $f\colon \Lambda \to \mathrm{H}^1(A, \mathbf{Z})$. Now we define α to be the composition

$$V \otimes \mathbf{A}_{\mathrm{f}} \xrightarrow{g} V \otimes \mathbf{A}_{\mathrm{f}} \xrightarrow{f \otimes \mathbf{A}_{\mathrm{f}}} \mathrm{H}^1(A, \mathbf{A}_{\mathrm{f}}).$$

The pair (h, g) is mapped to (A, λ, α). As a consequence of the Shimura-Taniyama formula [D2, Théorème 4.19], this map is $\mathrm{Aut}(\mathbf{C})$-equivariant.

Next, there is a map from $\mathbf{AV}(V, \psi_V)$ to $\mathbf{Mot}(V, \psi_V)$, given by mapping (A, λ, α) to the tuple $(\mathfrak{h}^1(A), \mathbf{1}(-1), \psi_\lambda, \alpha, \beta)$, where ψ_λ is the symplectic form associated with λ, and β is the factor of similitude of α. This map is clearly $\mathrm{Aut}(\mathbf{C})$-equivariant.

We now define a map $\mathbf{Mot}(\mathrm{GSp}) \to \mathbf{Mot}(V, \psi_V)$. Denote by χ the 1-dimensional representation of GSp given by the similitude character. Then we map $(F, \eta) \in \mathbf{Mot}(\mathrm{GSp})$ to $(F(V), F(\chi), \psi_V, \eta_V, \eta_\chi)$. This map is clearly $\mathrm{Aut}(\mathbf{C})$-equivariant, and similarly to Lemma 2.3.11, one can show that this map is a bijection.

We now have a diagram

$$\begin{tikzcd}
\mathrm{Sh}(\mathrm{GSp}, \mathcal{H})(\mathbf{C}) \arrow[r, "x"] \arrow[d, "z"'] & \mathbf{Mot}(\mathrm{GSp}) \arrow[d, hook, "y"] \\
\mathbf{AV}(V, \psi_V) \arrow[r, "w"'] & \mathbf{Mot}(V, \psi_V)
\end{tikzcd}$$

From the constructions of the maps x, y, z, and w it can be seen that for $(h, g) \in \mathrm{Sh}(\mathrm{GSp}, \mathcal{H})(\mathbf{C})$, the tuples $yx(h, g)$ and $wz(h, g)$ have the same Betti realization, and are therefore isomorphic by Theorem 2.2.2. It follows that the diagram commutes. Since y, z, and w are $\mathrm{Aut}(\mathbf{C})$-equivariant, and since y is injective, we conclude that x is $\mathrm{Aut}(\mathbf{C})$-equivariant. $\square$

Remark 2.3.22. Note that this proves Theorem 2.3.16 for Siegel Shimura data.

2.3.3 Shimura data of Hodge type

Before proving Theorem 2.3.16 for Shimura data of Hodge type, we need the following lemma.

Lemma 2.3.23. *Let $\iota\colon G \to G'$ be a homomorphism of affine group schemes over $\mathbf{Q}$. If ι is a closed immersion, then the induced map $\mathbf{Mot}(G) \to \mathbf{Mot}(G')$ is injective.*

Proof. In this proof, we will identify $\mathbf{Mot}_{\mathrm{ab},\mathbf{C}}$ with $\mathcal{G}_{\mathrm{ab}}$-$\mathbf{Rep}$. That is, we think of an abelian motive M as the vector space $\mathrm{H}_{\mathrm{B}}(M)$ endowed with a $\mathcal{G}_{\mathrm{ab}}$-action.

Let $(F_1, \eta_1), (F_2, \eta_2) \in \mathbf{Mot}(G)$, and suppose they have the same image under the map $\mathbf{Mot}(G) \to \mathbf{Mot}(G')$. That is, assume we have an isomorphism $\varphi\colon F_1\iota^* \to F_2\iota^*$ of tensor functors from G'-$\mathbf{Rep}$ to $\mathbf{Mot}_{\mathrm{ab},\mathbf{C}}$ such that for every $W \in G'$-$\mathbf{Rep}$, the diagram of $\mathbf{A}_{\mathrm{f}}$-modules

$$\begin{tikzcd}
\mathrm{H}_{\text{ét}}\, F_1(W) \arrow[rr, "\mathrm{H}_{\text{ét}}(\varphi_W)"] & & \mathrm{H}_{\text{ét}}\, F_2(W) \\
& W \otimes \mathbf{A}_{\mathrm{f}} \arrow[ul, "\eta_{1,W}"] \arrow[ur, "\eta_{2,W}"'] &
\end{tikzcd}$$

commutes. We wish to show that (F_1, η_1) is isomorphic to (F_2, η_2). This amounts to showing that for $V \in G$-$\mathbf{Rep}$, the functorial isomorphism of $\mathbf{A}_{\mathrm{f}}$-modules

$$\eta_{2,V}\eta_{1,V}^{-1}\colon\ \mathbf{A}_{\mathrm{f}} \otimes F_1 V \longrightarrow \mathbf{A}_{\mathrm{f}} \otimes F_2 V$$

restricts to a $\mathcal{G}_{\mathrm{ab}}$-equivariant map $\varphi_V\colon F_1V \to F_2V$.

Since ι is a closed immersion, [DM1, Proposition 2.21] implies that there exist $W \in G'$-**Rep**, $\widetilde{V} \in G$-**Rep**, an injective G-equivariant map $\widetilde{V} \to W$, and a surjective G-equivariant map $\widetilde{V} \to V$.

We first show that $\eta_{2,V}\eta_{1,V}^{-1}(F_1V) = F_2V$. From $\eta_{2,\iota^*W}\eta_{1,\iota^*W}^{-1} = \mathbf{A}_{\mathrm{f}} \otimes \varphi_W$, it follows that $\eta_{2,\iota^*W}\eta_{1,\iota^*W}^{-1}(F_1W) = F_2W$. Therefore,

$$\eta_{2,\widetilde{V}}\eta_{1,\widetilde{V}}^{-1}(F_1\widetilde{V}) = (\mathbf{A}_{\mathrm{f}} \otimes F_2\widetilde{V}) \cap F_2W = F_2(\widetilde{V}).$$

This and the surjectivity of $F_1\widetilde{V} \to F_1V$ (which follows from Remark 2.3.7) imply that $\eta_{2,V}\eta_{1,V}^{-1}(F_1V) = F_2V$. We denote the resulting isomorphisms of vector spaces $\varphi_{\widetilde{V}}\colon F_1\widetilde{V} \to F_2\widetilde{V}$ and $\varphi_V\colon F_1V \to F_2V$.

We now show that φ_V is $\mathcal{G}_{\mathrm{ab}}$-equivariant. Consider the commutative diagram

$$\begin{tikzcd}
F_1W \arrow[d, "\varphi_W"'] & F_1\widetilde{V} \arrow[l, hook'] \arrow[r, two heads] \arrow[d, "\varphi_{\widetilde{V}}"'] & F_1V \arrow[d, "\varphi_V"'] \\
F_2W & F_2\widetilde{V} \arrow[l, hook'] \arrow[r, two heads] & F_2V
\end{tikzcd}$$

The horizontal maps are injective and surjective because F is exact, cf. Remark 2.3.7. A diagram chase combined with the fact that φ_W is $\mathcal{G}_{\mathrm{ab}}$-equivariant shows that φ_V is $\mathcal{G}_{\mathrm{ab}}$-equivariant, completing the proof. □

Proposition 2.3.24. *Let (G, X) be a Hodge type Shimura datum. The map $\mathrm{Sh}(G, X)(\mathbf{C}) \to \mathbf{Mot}(G)$ given in Lemma 2.3.12 is $\mathrm{Aut}(\mathbf{C})$-equivariant.*

Proof. Let (G, X) be a Shimura datum of Hodge type with reflex field E, and let $(G, X) \hookrightarrow (\mathrm{GSp}, \mathcal{H})$ be an embedding into a Siegel Shimura datum. Then there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Sh}(G, X)(\mathbf{C}) \arrow[r] \arrow[d] & \mathbf{Mot}(G) \arrow[d, hook] \\
\mathrm{Sh}(\mathrm{GSp}, \mathcal{H})(\mathbf{C}) \arrow[r] & \mathbf{Mot}(\mathrm{GSp})
\end{tikzcd}$$

It is clear from the definitions that the map $\mathbf{Mot}(G) \to \mathbf{Mot}(\mathrm{GSp})$ is $\mathrm{Aut}(\mathbf{C})$-equivariant, and it is injective by Lemma 2.3.23. Moreover, by Proposition 2.3.21 the map $\mathrm{Sh}(\mathrm{GSp}, \mathcal{H})(\mathbf{C}) \to \mathbf{Mot}(\mathrm{GSp})$ is $\mathrm{Aut}(\mathbf{C})$-equivariant, and $\mathrm{Sh}(G, X)(\mathbf{C}) \to \mathrm{Sh}(\mathrm{GSp}, \mathcal{H})(\mathbf{C})$ is $\mathrm{Aut}(\mathbf{C}/E)$-equivariant since $(G, X) \to (\mathrm{GSp}, \mathcal{H})$ induces a morphism $\mathrm{Sh}(G, X) \to \mathrm{Sh}(\mathrm{GSp}, \mathcal{H})_E$. It follows that the map $\mathrm{Sh}(G, X)(\mathbf{C}) \to \mathbf{Mot}(G)$ is $\mathrm{Aut}(\mathbf{C}/E)$-equivariant. □

Remark 2.3.25. Note that this proves Theorem 2.3.16 for Shimura data of Hodge type.

2.3.4 Shimura data of orthogonal type

We will need that the Picard group of $\mathbf{A}_\mathrm{f}$ is trivial, and later for Lemma 5.1.14 we will also need that the Picard group of $\mathbf{A}$ is trivial, where where $\mathbf{A}$ denotes the ring of adèles of $\mathbf{Q}$. This is unsurprising, but it is difficult to find a proof in the literature, so we include it here.

Lemma 2.3.26. *The Picard groups of $\mathbf{A}$ and $\mathbf{A}_\mathrm{f}$ are trivial.*

Proof. Note that since $\mathbf{A} = \mathbf{R} \times \mathbf{A}_\mathrm{f}$, we have $\mathrm{Pic}(\mathbf{A}) \cong \mathrm{Pic}(\mathbf{A}_\mathrm{f})$, so it suffices to show that $\mathrm{Pic}(\mathbf{A}) = 1$.

For a finite set of primes S, define

$$R_S = \mathbf{R} \times \prod_{p \in S} \mathbf{Q}_p \times \prod_{p \notin S} \mathbf{Z}_p \,.$$

By definition, $\mathbf{A}$ is the colimit $\mathrm{colim}_S\, R_S$. Therefore [T2, Tag 01ZL] says that for every line bundle $\mathcal{L}$ on $\mathrm{Spec}(\mathbf{A})$, there exists an S and a line bundle $\mathcal{L}_S$ on $\mathrm{Spec}(R_S)$ such that $\mathcal{L}$ is the pullback of $\mathcal{L}_S$ to $\mathrm{Spec}(\mathbf{A})$. It follows that it suffices to show that $\mathrm{Pic}(R_S)$ is trivial.

More generally, suppose that we are given a set of rings $\{R_i\}_{i \in I}$ with $\mathrm{Pic}(R_i) = 1$ for all i, and set $R = \prod_{i \in I} R_i$. If P is a $\mathbf{G}_m$-torsor on $\mathrm{Spec}(R)_{\mathrm{Zariski}}$, then P is affine, since affineness is Zariski-local on the target. Therefore

$$P(R) = \prod_{i \in I} P(R_i),$$

which is non-empty since $P(R_i)$ is non-empty for every $i \in I$ by $\mathrm{Pic}(R_i) = 1$. This shows that P is the trivial torsor, and hence that $\mathrm{Pic}(R) = 1$. Since the Picard groups of $\mathbf{R}$, $\mathbf{Q}_p$, and $\mathbf{Z}_p$ are trivial, this proves the lemma. $\square$

Proposition 2.3.27. *Let (SO, Ω) be a Shimura datum of orthogonal type. The map $\mathrm{Sh}(\mathrm{SO}, \Omega)(\mathbf{C}) \to \mathbf{Mot}(\mathrm{SO})$ given in Lemma 2.3.12 is $\mathrm{Aut}(\mathbf{C})$-equivariant.*

Proof. As in (2.1) , we have a Shimura datum $(\mathrm{GSpin}, \Omega')$ of Hodge type and a morphism $(\mathrm{GSpin}, \Omega') \to (\mathrm{SO}, \Omega)$. Since the map $\mathrm{GSpin} \to \mathrm{SO}$ fits in a short exact sequence

$$1 \to \mathbf{G}_m \longrightarrow \mathrm{GSpin} \longrightarrow \mathrm{SO} \to 1,$$

and since $\mathrm{H}^1(\mathbf{A}_{\mathrm{f},\text{ét}}, \mathbf{G}_m) = \mathrm{Pic}(\mathbf{A}_\mathrm{f}) = 1$ by Lemma 2.3.26, the map $\mathrm{GSpin}(\mathbf{A}_\mathrm{f}) \to \mathrm{SO}(\mathbf{A}_\mathrm{f})$ is surjective. Moreover, the map $\Omega' \to \Omega$ is a bijection, so the map $\mathrm{Sh}(\mathrm{GSpin}, \Omega')(\mathbf{C}) \to \mathrm{Sh}(\mathrm{SO}, \Omega)(\mathbf{C})$ is surjective. There is a commutative diagram

$$\begin{tikzcd} \mathrm{Sh}(\mathrm{GSpin}, \Omega')(\mathbf{C}) \arrow[r] \arrow[d, two heads] & \mathbf{Mot}(\mathrm{GSpin}) \arrow[d] \\ \mathrm{Sh}(\mathrm{SO}, \Omega)(\mathbf{C}) \arrow[r] & \mathbf{Mot}(\mathrm{SO}) \end{tikzcd}$$

Because $(\mathrm{GSpin}, \Omega')$ is of Hodge type, Proposition 2.3.24 states that the map $\mathrm{Sh}(\mathrm{GSpin}, \Omega')(\mathbf{C}) \to \mathbf{Mot}(\mathrm{GSpin})$ is $\mathrm{Aut}(\mathbf{C})$-equivariant. In addition to this, $\mathbf{Mot}(\mathrm{GSpin}) \to \mathbf{Mot}(\mathrm{SO})$ and $\mathrm{Sh}(\mathrm{GSpin}, \Omega')(\mathbf{C}) \to \mathrm{Sh}(\mathrm{SO}, \Omega)(\mathbf{C})$ are clearly $\mathrm{Aut}(\mathbf{C})$-equivariant. It follows that $\mathrm{Sh}(\mathrm{SO}, \Omega)(\mathbf{C}) \to \mathbf{Mot}(\mathrm{SO})$ is $\mathrm{Aut}(\mathbf{C})$-equivariant, wich was to be shown. $\square$

Remark 2.3.28. This concludes the proof of Theorem 2.3.16. By Remark 2.3.17, this also finishes the proof of Theorem 2.3.3.

3

Moduli of polarized hyperkähler varieties

The main result of this chapter is that the moduli stack of polarized hyperkähler varieties is a separated Deligne-Mumford stack over $\mathbf{Q}$ (Theorem 3.3.2). In a later chapter we will also see that it is smooth (Corollary 4.1.16). This result is well known to the experts. Our account closely follows that in [R4] and [H2, Chapter 5], where the same result is proved for polarized K3 surfaces (that is, for two-dimensional polarized hyperkähler varieties) in mixed characteristic.

The first section collects the basic definitions and facts about hyperkähler varieties. The second section is about polarizations on hyperkähler varieties, and Picard schemes. We also state some important results by Matsusaka and Mumford on the moduli of polarized varieties. The final section contains the main result and its proof.

3.1 Hyperkähler varieties

Definition 3.1.1. A complex scheme X is called a **hyperkähler variety** if the following conditions hold:

1. X is connected, smooth, and projective,
2. $\mathrm{H}^0(X, \Omega^2_X)$ is spanned by a nowhere degenerate 2-form,
3. $\pi_1(X) = 1$.

Remark 3.1.2. Since the Hodge–de Rham spectral sequence degenerates at the E_1-page for compact Kähler manifolds (and in particular for smooth projective complex varieties), the 2-form in the definition is automatically closed. For this reason, hyperkähler varieties are sometimes called irreducible holomorphic symplectic varieties.

Lemma 3.1.3. *Let X be a smooth projective connected complex scheme for which there exists a nowhere degenerate 2-form in $\mathrm{H}^0(X, \Omega^2_X)$. The étale fundamental group $\pi_1^{\text{ét}}(X)$ is trivial if and only if $\pi_1(X)$ is.*

Proof. By [G2, Corollaire XII.5.2], $\pi_1^{\text{ét}}(X)$ is the profinite completion of $\pi_1(X)$. In particular, if $\pi_1(X)$ is trivial, then so is $\pi_1^{\text{ét}}(X)$.

For the converse, note that the Bogomolov decomposition theorem implies that there exists an exact sequence $1 \to \mathbf{Z}^{2k} \to \pi_1(X) \to G \to 1$, with G a finite group (this is explained in the statement immediately following [B2, Théorème 1]). Since the profinite completion $\pi_1^{\text{ét}}(X)$ of $\pi_1(X)$ is trivial, the group G is trivial. Now $\pi_1(X) \cong \mathbf{Z}^{2k}$, so that $\pi_1^{\text{ét}}(X) = 1$ implies that $k = 0$, showing that $\pi_1(X) = 1$. □

Definition 3.1.4. Let K be a field of characteristic 0, and let $\overline{K}$ be an algebraic closure of K. A scheme X over K is called a **hyperkähler variety** if the following conditions hold:

1. X is geometrically connected, smooth, and projective,
2. $\mathrm{H}^0(X, \Omega^2_{X/K})$ is spanned by a nowhere degenerate 2-form,
3. $\pi_1^{\text{ét}}(X_{\overline{K}}) = 1$.

Remark 3.1.5. Since K has characteristic 0, the degeneration of the Hodge–de Rham spectral sequence at E_1 again shows that the 2-form is closed.

Remark 3.1.6. Lemma 3.1.3 shows that when $K = \mathbf{C}$, Definitions 3.1.1 and 3.1.4 agree. This allows us to apply the results of [B2], which use definition 3.1.1.

Example 3.1.7. Two-dimensional hyperkähler varieties are K3 surfaces. Conversely, every K3 surface over a field of characteristic 0 is a hyperkähler variety.

Example 3.1.8. For higher-dimensional examples, consider a K3 surface S over a field K of characteristic 0, and n an integer greater than or equal to 2. Then by [B2, Théorème 3] and the Lefschetz principle, the Hilbert scheme $S^{[n]}$ of n points on S is a $2n$-dimensional hyperkähler variety over K. Deformations of hyperkähler varieties of this type are also hyperkähler varieties, known as **$\mathbf{K3}^{[n]}$-type hyperkähler varieties**. We will return to these varieties in Section 4.7.

Example 3.1.9. The only other known examples are the so-called generalized Kummer varieties ([B2, § 7]), which are higher-dimensional analogues of Kummer K3 surfaces, and the more recent examples in dimension 6 and 10 constructed by O'Grady as symplectic desingularizations of certain moduli spaces of sheaves on abelian surfaces and K3 surfaces ([O2] and [O1]).

Proposition 3.1.10. *If X/K is a hyperkähler variety of dimension $2n$, then*

1. *for every prime ℓ, $\mathrm{H}^1_{\text{ét}}(X_{\overline{K}}, \mathbf{Z}_\ell) = 0$, and $\mathrm{H}^2_{\text{ét}}(X_{\overline{K}}, \mathbf{Z}_\ell)$ is a free $\mathbf{Z}_\ell$-module;*
2. *$\dim_K \mathrm{H}^i(X, \mathcal{O}_X) = \dim_K \mathrm{H}^0(X, \Omega^i_{X/K}) = (1 + (-1)^i)/2$, and $\chi(X, \mathcal{O}_X) = n + 1$;*
3. *$\mathrm{Pic}(X)$ is torsion-free;*
4. *the Kodaira dimension of X is 0.*

Proof. The first point follows from the étale analogues of Hurewicz' theorem and the universal coefficient theorem. Point 2 follows from [B2, Proposition 3] and the Lefschetz principle. The torsion-freeness of $\mathrm{Pic}(X)$ follows from Point 1 and the Kummer sequence. The fourth point follows immediately from the triviality of the canonical sheaf, which is due to the existence of a non-degenerate 2-form. $\square$

The following lemma states that small deformations of hyperkähler varieties are hyperkähler varieties, which will be useful in the sequel.

Lemma 3.1.11. *Let S be a scheme over $\mathbf{Q}$, and suppose $f\colon X \to S$ is a proper smooth morphism of schemes with projective and geometrically connected fibers. If $s \in S$ is a point for which X_s is a hyperkähler variety, then there exists a open neighborhood U of s in S such that all fibers of $X_U \to U$ are hyperkähler varieties.*

Proof. First assume S is reduced and locally Noetherian. Corollaire X.3.9 in [G2] shows that the fundamental group of the geometric fibers of f is constant on the connected components of S, which are open because S is locally Noetherian, and hence locally connected [T2, Tag 04MF].

Let $2n$ be the relative dimension of f. The sets $\{t \in S \mid \chi(X_t, \mathcal{O}_{X_t}) = n+1\}$ and $\{t \in S \mid h^i(X_t, \mathcal{O}_{X_t}) \leq \frac{1}{2}(1+(-1)^i), i = 0, 1, \dots, 2n\}$ are open and both contain s by Part 2 of Proposition 3.1.10. Their intersection is $\{t \in S \mid h^i(X_t, \mathcal{O}_{X_t}) = \frac{1}{2}(1+(-1)^i), i = 0, 1, \dots, 2n\}$. It follows from this and Hodge symmetry that $h^0\left(X_t, \Omega^2_{X_t/\Bbbk(t)}\right) = 1$ for all t in an open neighborhood of s.

The constancy of $t \mapsto h^0\left(X_t, \Omega^2_{X_t/\Bbbk(t)}\right)$ near s allows us to apply Grauert's direct image theorem (which uses the reducedness of S) to extend a symplectic form on X_s to nearby fibers, proving the lemma for reduced and locally Noetherian S.

For not necessarily reduced S, we obtain the result by applying the reduced case to S_{red}. A standard limit argument gets rid of the Noetherian hypothesis, see for instance [GW, Theorem 10.66]. $\square$

3.2 Polarizations on hyperkähler varieties

We need some properties of the Picard sheaf of smooth proper morphisms whose fibers are hyperkähler varieties. For an algebraic space X over a scheme S, let $\mathrm{Pic}_{X/S}$ denote the fppf sheafification of the presheaf $T \mapsto \mathrm{Pic}(X_T)$ on $(\mathbf{Sch}/S)_{\mathrm{fppf}}$.

Remark 3.2.1. When S is a scheme over $\mathbf{Q}$, and $f\colon X \to S$ a smooth proper morphism of schemes whose fibers are hyperkähler varieties, $f_* \mathcal{O}_X \cong \mathcal{O}_S$ holds universally because f is proper and has geometrically connected fibers. It follows that the étale sheafification of $T \mapsto \mathrm{Pic}(X_T)$ is equal to $\mathrm{Pic}_{X/S}$ [FGI$^+$, pg. 257]. Moreover, for every S-scheme T there is an exact sequence [BLR, Proposition 8.4]

$$0 \to \mathrm{Pic}(T) \to \mathrm{Pic}(X_T) \to \mathrm{Pic}_{X_T/T}(T).$$

So given a section $\lambda \in \mathrm{Pic}_{X/S}(S)$, we can find an étale cover $S' \to S$ such that the pullback of λ to $X_{S'}$ lies in $\mathrm{Pic}(X_{S'})/\mathrm{Pic}(S') \subseteq \mathrm{Pic}_{X_{S'}/S'}(S')$.

Following [BLR], we call a morphism $X \to S$ strongly projective if it is finitely presented and there exists a locally free sheaf E on S of constant finite rank and a closed immersion $X \to \mathbf{P}(E)$ over S. Let S be a quasi-compact scheme. For a strongly projective flat morphism $X \to S$ with geometrically integral fibers, $\mathrm{Pic}_{X/S}$ is a scheme [BLR, Theorem 8.2.5].

The following proposition is proved for families of K3 surfaces in [R4, Lemma 3.1.6]. The proof applies verbatim to families of hyperkähler varieties of higher dimension.

Proposition 3.2.2. *Let S be a quasi-compact scheme over $\mathbf{Q}$, and X/S a strongly projective smooth morphism whose fibers are hyperkähler varieties. Then multiplication by $n \in \mathbf{Z}_{>0}$ is a closed immersion $[n]\colon \mathrm{Pic}_{X/S} \to \mathrm{Pic}_{X/S}$.*

Definition 3.2.3. Let S be a $\mathbf{Q}$-scheme, and let $X \to S$ be a smooth proper morphism of algebraic spaces whose fibers are hyperkähler varieties. A **polarization** on X/S is an element $\lambda \in \mathrm{Pic}_{X/S}(S)$ such that for every geometric point $\overline{s}$ of S, the pullback $\lambda_{\overline{s}} \in \mathrm{Pic}(X_{\overline{s}})$ is ample.

Let $P \in \mathbf{Q}[t]$ be a polynomial. A polarized morphism of algebraic spaces $(X \to S, \lambda)$ is said to have Hilbert polynomial P if every geometric fiber $(X_{\overline{s}}, \lambda_{\overline{s}})$ has Hilbert polynomial P.

Remark 3.2.4. Polarizations always exist, étale locally. To see this, let X/S be as in Definition 3.2.3, let $\overline{s}$ be a geometric point of S, and pick an ample line bundle L_0 on $X_{\overline{s}}$. Then L_0 is an element of the stalk of the sheaf $\mathrm{Pic}_{X/S}$ on $S_{\text{ét}}$, and therefore extends to a section $\lambda \in \mathrm{Pic}_{X/S}(U)$, where U is an étale neighborhood of $\overline{s}$. Since ampleness is open on the base, we can find an étale neighborhood $V \to U$ of $\overline{s}$ on which λ is a polarization.

Lemma 3.2.5. *Let $P \in \mathbf{Q}[t]$ be a polynomial. There exists an integer $m \in \mathbf{Z}_{\geq 0}$ such that the following holds. For any scheme S over $\mathbf{Q}$ and any smooth proper morphism of schemes $X \to S$ whose fibers are hyperkähler varieties endowed with a polarization $\lambda \in \mathrm{Pic}_{X/S}(S)$ with Hilbert polynomial P, there exists an étale cover $U \to S$ such that*

- *$(f\colon X_U \to U, \lambda_U)$ is a polarized proper smooth morphism of schemes whose fibers are hyperkähler varieties with Hilbert polynomial P,*
- *λ_U is the image under $\mathrm{Pic}(X_U) \to \mathrm{Pic}_{X_U/U}(U)$ of a line bundle L on X_U,*
- *$f_*(L^{\otimes m})$ is free of rank $P(m)$,*
- *$L^{\otimes m}$ is relatively very ample.*

Proof. Matsusaka's big theorem [M2] gives an integer $m \in \mathbf{Z}_{\geq 0}$ such that if K is a field of characteristic 0 and $(X/K, \lambda)$ is a polarized hyperkähler variety with Hilbert polynomial P, then $m\lambda \in \mathrm{Pic}_{X/K}(K)$ is the class of a very ample line bundle on $X_{\overline{K}}$.

Let $(X \to S, \lambda)$ be a polarized smooth proper morphism of schemes whose fibers are hyperkähler varieties with Hilbert polynomial P. Using Remark 3.2.1, we find an étale cover $U \to S$ such that the first two conditions of the lemma are satisfied.

Kodaira vanishing and the triviality of the canonical sheaf of X imply that $\mathrm{H}^1(X_s, L_s^{\otimes m}) = 0$ for all $s \in U$, so from [MFK, Chapter 0, §5, a)] it follows that $f_*(L^{\otimes m})$ is locally free. The statement about the rank follows from Kodaira vanishing and the definition of the Hilbert polynomial. By further refining the cover $U \to S$ we can globally liberate $f_*(L^{\otimes m})$.

The fact that $L^{\otimes m}$ is relatively very ample follows from the choice of m. □

We will need the following results on moduli of non-ruled polarized varieties, due to Matsusaka and Mumford. See also [P2, Theorem 4.3, Proposition 4.4]. Note that ruled varieties have Kodaira dimension $-\infty$, so by Point 4 of Proposition 3.1.10, the lemmas apply to hyperkähler varieties.

Lemma 3.2.6 (Matsusaka-Mumford, [MM1, Chapter 1, Theorem 2])**.** *Let R be a dvr with fraction field K, and X_1, X_2 smooth proper R-schemes with non-ruled special fibers, equipped with relatively ample line bundles L_1 and L_2. Any isomorphism $f\colon X_{1,K} \to X_{2,K}$ with $f^*[L_{2,K}] = [L_{1,K}]$ extends uniquely to an isomorphism $X_1 \to X_2$ with $f^*[L_2] = [L_1]$.*

Lemma 3.2.7 (Matsusaka-Mumford, [MM1, Chapter 1, Corollary 2])**.** *Let X be a non-ruled variety over an algebraically closed field K with $\mathrm{H}^0(X, \Omega^1_{X/K}) = 0$, equipped with an ample line bundle L. Then the number of automorphisms of X preserving the class of L in $\mathrm{Pic}(X)$ is finite.*

3.3 The moduli stack of polarized hyperkähler varieties

In this section we define the stack of polarized hyperkähler varieties, and prove some of its properties. We closely follow [R4] and [H2, Chapter 5].

Definition 3.3.1. The **moduli stack of polarized hyperkähler varieties** is defined as the groupoid fibration $\mathbf{HK} \to \mathbf{Sch}/\mathbf{Q}$ whose objects are pairs $(X \to S, \lambda \in \mathrm{Pic}_{X/S}(S))$ where S is a $\mathbf{Q}$-scheme, $X \to S$ is a smooth proper morphisms of algebraic spaces whose fibers are hyperkähler varieties, and λ is a polarization on X. Morphisms $(X' \to S', \lambda') \to (X \to S, \lambda)$ are those cartesian squares

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

for which $f^*\lambda = \lambda'$ in $\mathrm{Pic}_{X'/S'}(S')$. The functor $\mathbf{HK} \to \mathbf{Sch}/\mathbf{Q}$ maps $(X \to S, \lambda)$ to S, and a cartesian square as above to the morphism $S' \to S$.

The following is the main theorem of this chapter. See also Lemma 3.3.9, Lemma 3.3.10, and Corollary 4.1.16 in the next chapter.

Theorem 3.3.2. *The groupoid fibration* $\mathbf{HK}$ *is a smooth separated Deligne-Mumford stack over* $\mathbf{Q}$. *Its dimension at a* $\mathbf{C}$*-point* (X, λ) *is equal to* $b_2(X) - 3$.

Remark 3.3.3. In this section we will only prove that **HK** is a separated Deligne-Mumford stack, locally of finite type over $\mathbf{Q}$. The assertion about the smoothness and dimension of **HK** will be proved in a later chapter and is a consequence of the local Torelli theorem for complex hyperkähler varieties [B2, Théorème 5]. See Corollary 4.1.16.

Lemma 3.3.4. *The groupoid fibration* **HK** *is a stack on* $(\mathbf{Sch}\,/\,\mathbf{Q})_{\text{ét}}$.

Proof. This follows immediately from the fact that the groupoid fibration of algebraic spaces is a stack [T2, Tag 04UA]. $\square$

Fix a polynomial $P \in \mathbf{Q}[t]$ and an integer $m, \in \mathbf{Z}_{\geq 0}$, and define $N := P(m)-1$. Denote by Hilb the Hilbert scheme $\mathrm{Hilb}_{\mathbf{P}^N}^{P(mt)}$, which parametrizes closed subschemes Z of $\mathbf{P}^N$ such that $\mathcal{O}(1)|_Z$ has Hilbert polynomial $P(mt)$. Let $\mathcal{Z} \subseteq \mathbf{P}^N \times \mathrm{Hilb}$ be the universal family.

Let $H_{m,P}\colon (\mathbf{Sch}/\mathbf{Q})^{\mathrm{opp}} \to \mathbf{Set}$ be the subfunctor of Hilb sending a $\mathbf{Q}$-scheme S to the set of those $Z \in \mathrm{Hilb}(S)$ satisfying

1. $Z \to S$ is smooth, and its fibers are hyperkähler varieties,
2. there exists an $L \in \mathrm{Pic}(Z)$ such that $\mathcal{O}(1)|_Z = mL$ in $\mathrm{Pic}(Z)/\,\mathrm{Pic}(S)$,
3. for any geometric point $\overline{s}$ of S, the restriction map
$$\mathrm{H}^0(\mathbf{P}^N_{\overline{s}}, \mathcal{O}(1)) \to \mathrm{H}^0(Z_{\overline{s}}, L_{\overline{s}}^{\otimes m})$$
is an isomorphism.

Lemma 3.3.5. *The functor* $H_{m,P}$ *is representable by a scheme, and the inclusion* $H_{m,P} \to \mathrm{Hilb}$ *is an immersion.*

Proof. We need to show that the locus in Hilb over which $\mathcal{Z}$ satisfies the given properties is a locally closed subscheme.

Let H_1 be the set of $s \in \mathrm{Hilb}$ such that $\mathcal{Z}_s$ is a hyperkähler variety over $\Bbbk(s)$. This is an open set by the fact that smoothness is an open condition and Lemma 3.1.11.

Consider the cartesian square

$$\begin{tikzcd} \mathrm{Pic}_{\mathcal{Z}_{H_1}/H_1} \arrow[r, "{[m]}"] & \mathrm{Pic}_{\mathcal{Z}_{H_1}/H_1} \\ H_2 \arrow[u] \arrow[r] & H_1 \arrow[u, "i^*\,\mathcal{O}(1)"'] \end{tikzcd}.$$

Then $H_2 \to H_1$ is a closed immersion by Proposition 3.2.2. Since $\mathrm{Pic}_{\mathcal{Z}_{H_2}/H_2}$ is a scheme by Theorem 8.2.5 in [BLR], there exists a Poincaré bundle on $\mathrm{Pic}_{\mathcal{Z}_{H_2}/H_2} \times_{H_2} \mathcal{Z}_{H_2}$. This, combined with the fact that by construction $\mathcal{O}(1)|_{\mathcal{Z}_{H_2}} \in m\,\mathrm{Pic}_{\mathcal{Z}_{H_2}/H_2}(H_2)$, shows that Property 2 holds over H_2.

That Property 3 defines a locally closed subscheme $H_{m,P}$ of H_2 is proved exactly as in the final part of the proof of Proposition 5.1 in [MFK]. $\square$

Now pick $m \in \mathbf{Z}_{\geq 0}$ associated with P as in Lemma 3.2.5, and let $\mathbf{HK}^P \subseteq \mathbf{HK}$ be the open substack of pairs $(X \to S, \lambda)$ with Hilbert polynomial P. Note that the action of the group scheme $\mathrm{PGL} := \underline{\mathrm{Aut}}(\mathbf{P}^N_{\mathbf{Q}})$ on Hilb restricts to an action on $H_{m,P}$.

Lemma 3.3.6. *The universal family $\mathcal{Z}_{H_{m,P}} \to H_{m,P}$ yields a* PGL*-invariant morphism $H_{m,P} \to \mathbf{HK}^P$, which in turn induces an equivalence $[H_{m,P}/\mathrm{PGL}] \xrightarrow{\sim} \mathbf{HK}^P$. In particular, $\mathbf{HK}^P$ is an algebraic stack.*

Proof. To simplify the notation, we define $H = H_{m,P}$.

The natural relatively very ample line bundle on $\mathcal{Z}_H \to H$ is of the form $m\lambda$ for exactly one $\lambda \in \mathrm{Pic}_{\mathcal{Z}_H/H}(H)$ by the definition of H and Proposition 3.2.2. This defines the morphism $(\mathcal{Z}_H, \lambda)\colon H \to \mathbf{HK}^P$.

To establish the required equivalence, we use [LMB, Proposition 3.8]. This proposition states that it suffices to show

1. for every $\mathbf{Q}$-scheme U and every morphism $\xi\colon U \to \mathbf{HK}^P$ there exists an étale cover $f\colon V \to U$ such that $f^*\xi$ is in the essential image of $H \to \mathbf{HK}^P$;
2. $H \times_{\mathbf{HK}^P} H$ is equivalent to $H \times \mathrm{PGL}$ as an $(H \times H)$-stack, where $H \times \mathrm{PGL} \to H \times H$ is given by $(h, g) \mapsto (h, hg)$.

For Point 1, let $(X, \lambda)\colon U \to \mathbf{HK}^P$, and take an étale cover $V \to U$ as in Lemma 3.2.5. That is, there exists a line bundle L on X_V such that λ_V is the class of L in $\mathrm{Pic}_{X/U}(V)$, the line bundle $L^{\otimes m}$ is relatively very ample, and f_*L is free of rank $N+1$. It follows that $L^{\otimes m}$ gives rise to a closed immersion $X_V \to \mathbf{P}\, f_*(L^{\otimes m})$ satisfying the Conditions 1, 2, and 3 preceding Lemma 3.3.5, so that we have a morphism $V \to H$. In particular, (X_V, λ_V) is in the essential image of $H \to \mathbf{HK}^P$.

To prove Point 2, define $\Phi\colon H \times \mathrm{PGL} \to H \times_{\mathbf{HK}^P} H$ by $(h, g) \mapsto (h, hg, g^{-1}|_h)$. This is a morphism of $(H \times H)$-stacks which is clearly fully faithful. We want to see that it is an equivalence.

To see the essential surjectivity of Φ, consider $Z_1, Z_2 \in H(U)$ and an isomorphism $\phi\colon (Z_1, L_1) \to (Z_1, L_2)$, where L_1 and L_2 are as in the second point of Lemma 3.3.5. There is a commutative diagram

$$\begin{tikzcd} & \mathbf{P}(f_{1,*}L_1^{\otimes m}) \arrow[r] & \mathbf{P}(f_{2,*}L_2^{\otimes m}) \arrow[dr] & \\ \mathbf{P}^N_U \arrow[ur] & & & \mathbf{P}^N_U \\ & Z_1 \arrow[ul] \arrow[uu] \arrow[r, "\phi"'] & Z_2 \arrow[uu] \arrow[ur] & \end{tikzcd}.$$

The top arrow is the isomorphism induced by the fact that $\phi^*(L_2) \cong f_1^*(M) \otimes L_1$ for some $M \in \mathrm{Pic}(U)$, and the morphisms $\mathbf{P}(f_{i,*}L_i^{\otimes m}) \longleftrightarrow \mathbf{P}^N_U$ are the isomorphisms induced by Point 3 of Lemma 3.3.5. All other morphisms are the obvious ones. The composition of the top arrows is now an element $g \in \mathrm{PGL}(U) = \mathrm{Aut}_U(\mathbf{P}^N_U)$ with $Z_1 g^{-1} = Z_2$, proving the essential surjectivity of Φ. $\square$

Lemma 3.3.7. *The stack* $\mathbf{HK}^P$ *is of finite type over* $\mathbf{Q}$.

Proof. Since the Hilbert scheme is of finite type $\mathbf{Q}$, it follows that H is of finite type over $\mathbf{Q}$. Using [T2, Tags 06U8, 050X], we find that $\mathbf{HK}^P$ is of finite type over $\mathbf{Q}$. □

Lemma 3.3.8. *The stack* $\mathbf{HK}$ *is a Deligne-Mumford stack, locally of finite type over* $\mathbf{Q}$.

Proof. Since $\mathbf{HK}$ is the disjoint union of all $\mathbf{HK}^P$, where P ranges over all polynomials $P \in \mathbf{Q}[t]$, Lemmas 3.3.6 and 3.3.7 show that $\mathbf{HK}^P$ is an algebraic stack, locally of finite type over $\mathbf{Q}$.

To show that $\mathbf{HK}$ is a Deligne-Mumford stack, it suffices to show that the diagonal $\Delta\colon \mathbf{HK} \to \mathbf{HK} \times_{\mathbf{Q}} \mathbf{HK}$ is of finite type and that the geometric points of $\mathbf{HK}$ have finite and reduced automorphism groups [O3, Remark 8.3.4].

By Lemma 3.2.7, and because group schemes over a field of characteristic 0 are reduced [G1, Corollaire $\mathrm{VI_B}$.1.6.1] (see also [P1, Corollaire 4.2.8]), the automorphism group of a geometric point of $\mathbf{HK}$ is finite and reduced.

By [T2, Tag 04XS], the diagonal Δ is locally of finite type. Since $\mathbf{HK}$ is the disjoint union of the Noetherian stacks $\mathbf{HK}^P$, and since morphisms between Noetherian stacks are quasi-compact, Δ is also compact, and hence of finite type. □

Lemma 3.3.9. *The stack* $\mathbf{HK}$ *is a Deligne-Mumford stack, locally of finite type over* $\mathbf{Q}$.

Proof. This follows immediately from the fact that $\mathbf{HK}$ is the disjoint union of all $\mathbf{HK}^P$, where P ranges over all of $\mathbf{Q}[t]$, Lemma 3.3.8, and Lemma 3.3.7. □

Lemma 3.3.10. *The Deligne-Mumford stack* $\mathbf{HK}$ *is separated over* $\mathbf{Q}$.

Proof. Since $\mathbf{HK}$ is locally of finite type over $\mathbf{Q}$ by Lemma 3.3.9, we can apply the valuative criterion for separatedness of morphisms of locally Noetherian stacks [LMB, Proposition 7.8]. That is, we need to show that for a complete dvr R over $\mathbf{Q}$ with fraction field K and algebraically closed residue field, and two points (X_1, λ_1), $(X_2, \lambda_2) \in \mathbf{HK}(R)$, any isomorphism $f\colon (X_1, \lambda_1)_K \to (X_2, \lambda_2)_K$ over K extends uniquely to an isomorphism $(X_1, \lambda_1) \to (X_2, \lambda_2)$ over R.

Since R is complete and has algebraically closed residue field, the étale covers of R are all trivial. Therefore, by Remark 3.2.1, the λ_i are the classes of relatively ample line bundles on X_1 and X_2, respectively. This allows us to apply Lemma 3.2.6, which says that the isomorphism f extends uniquely to an isomorphism $(X_1, \lambda_1) \to (X_2, \lambda_2)$, proving the lemma. □

4
Period maps for hyperkähler varieties

It is well-known that the canonical model of a Siegel Shimura stack is the moduli stack of principally polarized abelian varieties over $\mathbf{Q}$ (see [D2]). This follows almost immediately from Deligne's definition of canonical models. An analogue of this result for polarized K3 surfaces over $\mathbf{Q}$ was initially proved by Rizov in [R3], then via a different argument by Madapusi-Pera in [MP1], and finally a slightly stronger version was proved by Taelman in [T1]. In this chapter, we extend this result to higher-dimensional polarized hyperkähler varieties over $\mathbf{Q}$. More precisely, we will construct a degree 2 étale cover $\mathbf{HK}_{\mathrm{or}}$ of the moduli stack $\mathbf{HK}$ of polarized hyperkähler varieties over $\mathbf{Q}$, and then give an étale morphism from $\mathbf{HK}_{\mathrm{or}}$ to an orthogonal Shimura stack, known as the period map.

In the first section we collect some important results from the literature on hyperkähler varieties over $\mathbf{C}$. In particular, we recall some basic facts about a quadratic form on the second cohomology of a hyperkähler variety known as the Beauville-Bogomolov-Fujiki form (the BBF form), and state the global Torelli theorem of Verbitsky, which roughly says that the geometry of a hyperkähler variety is largely determined by its second cohomology endowed with the BBF form (Theorem 4.1.12). We also show that $\mathbf{HK}$ is smooth (Corollary 4.1.16).

In the next section, we define a BBF form for hyperkähler varieties over non-closed fields of characteristic 0. The most important result in this section is the étale monodromy invariance of this form (Theorem 4.2.4). The third section introduces the notion of an orientation on a hyperkähler variety, which yields the degree 2 étale cover $\mathbf{HK}_{\mathrm{or}}$ of $\mathbf{HK}$ on which we will construct the period map.

Section 4.4 is an introduction to Shimura stacks, following [T1]. It is also shown that, over $\mathbf{C}$, orthogonal Shimura stacks are moduli stacks of Hodge structures endowed with a bilinear pairing and a trivialization of the determinant, known as an orientation. Then, in Section 4.5, we use this modular interpretation to give a morphism from $\mathbf{HK}_{\mathrm{or},\mathbf{C}}$ to an orthogonal Shimura stack, mapping a hyperkähler variety endowed with an orientation to its second cohomology, endowed with the BBF form and the orientation. We then use the results in Chapter 2 to prove the main theorem of this chapter, which states that this morphism descends to $\mathbf{Q}$ (Theorem 4.5.2).

The final two sections give stronger versions of this result for specific examples of hyperkähler varieties. Following [T1], Section 4.6 shows that we can in fact obtain a period map on the moduli stack of polarized K3 surfaces, rather than on the stack of oriented polarized K3 surfaces (Theorem 4.6.4). In Section 4.7, we consider hyperkähler varieties deformation equivalent to the Hilbert scheme of points on a K3 surface, known as $\mathrm{K3}^{[n]}$-type hyperkähler varieties. We extend a result of Markman on the monodromy of $\mathrm{K3}^{[n]}$-type varieties to $\mathrm{K3}^{[n]}$-type varieties over non-closed fields of characteristic 0 (Theorem 4.7.12), and combine this with Verbitsky's Torelli theorem to give a period map for such varieties over $\mathbf{Q}$ which is actually an open immersion (Theorem 4.7.18).

4.1 The global Torelli theorem

In this section, we recall the definition of a quadratic form on the second cohomology of a complex hyperkähler variety, known as the Beauville-Bogomolov-Fujiki form. Endowed with this form and its natural Hodge structure, the second cohomology captures much of the geometry of a hyperkähler variety. This is Verbitsky's global Torelli theorem, which we state in the first subsection, see Theorem 4.1.12. In the second subsection we show that the moduli stack of polarized hyperkähler varieties is smooth, see Corollary 4.1.16.

4.1.1 The global Torelli theorem

In this subsection we define the Beauville-Bogomolov-Fujiki form, state some of its properties, and state the global Torelli theorem. We will call a complex Kähler manifold X a **hyperkähler manifold** if it is compact, simply connected, and if $\mathrm{H}^0(X, \Omega^1_X)$ is spanned by a nowhere degenerate 2-form.

Theorem 4.1.1. *Let X be a hyperkähler manifold. Then there exists a unique primitive quadratic form $q\colon \mathrm{H}^2(X, \mathbf{Z}(1)) \to \mathbf{Z}$ such that*

1. *q is a $\mathbf{Q}$-multiple of the quadratic form*

$$Q\colon \alpha \longmapsto \int_X \sqrt{\mathrm{td}_X}\, \alpha^2$$

   on $\mathrm{H}^2(X, \mathbf{Q}(1))$,

2. *there exists a Kähler class $\omega \in \mathrm{H}^2(X, \mathbf{R}(1))$ with $q(\omega) > 0$.*

Proof. In [B2, Théorème 5], Beauville defines a primitive quadratic form q_X on $\mathrm{H}^2(X, \mathbf{Z}(1))$ which is positive on all Kähler classes. Moreover, it is shown in [F, Remark 4.12] that q_X is a multiple of Q.

Now suppose q is another primitive quadratic form on $\mathrm{H}^2(X, \mathbf{Z}(1))$ satisfying the two conditions, and let $\omega \in \mathrm{H}^2(X, \mathbf{R}(1))$ be a Kähler class on which q is positive. Then by the first condition, q is equal to cq_X, where $c \in \mathbf{Q}^\times$. Since q is primitive, $c \in \{\pm 1\}$. Moreover, $c = q_X(\omega)/q(\omega)$ is positive because $q_X(\omega)$ is positive on all Kähler classes. It follows that $q = q_X$. ☐

Definition 4.1.2. Let X be a hyperkähler manifold. The quadratic form on $\mathrm{H}^2(X, \mathbf{Z}(1))$ given in Theorem 4.1.1 is called the **Beauville-Bogomolov-Fujiki form** or **BBF form** of X, which we denote q_X.

Occasionally it will be more convenient to work with the **BBF pairing**

$$b_X\colon \operatorname{Sym}^2 \mathrm{H}^2(X, \mathbf{Z}(1)) \longrightarrow \mathbf{Z},$$

which is defined by

$$b_X(v, w) := q_X(v + w) - q_X(v) - q_X(w).$$

Remark 4.1.3. As is noted in the proof of Theorem 4.1.1, there holds $q_X(\omega) > 0$ for every Kähler class. In particular, if L is an ample line bundle on X, we have $q_X(c_1(L)) \in \mathbf{Z}_{>0}$.

Example 4.1.4. When S is a complex K3 surface, the BBF form on $\mathrm{H}^2(S, \mathbf{Z}(1))$ is simply the quadratic form induced by the cup product. In particular, it is an even self-dual $\mathbf{Z}$-lattice of signature $(3, 19)$. These properties determine the isometry class of $\mathrm{H}^2(S, \mathbf{Z}(1))$ by [S3, Chapter V, Theorem 5].

In general, the BBF form is not necessarily self-dual, as the following example shows.

Example 4.1.5. Let S be a K3 surface, $X = S^{[n]}$ be the Hilbert scheme of n points on S. In [B2], Beauville gives the following description of the BBF form on X. Let $S^{(n)}$ be the nth symmetric product of S. That is, $S^{(n)}$ is the quotient of S^n by the action of S_n given by permuting the coordinates. Then there is a natural map $S^{[n]} \to S^{(n)}$, and the inverse image of the singular locus of $S^{(n)}$ is a divisor E on $S^{[n]}$. There exists a $\delta \in \mathrm{H}^2(X, \mathbf{Z}(1))$ with $2\delta = E$, and such that

$$\mathrm{H}^2(S^{[n]}, \mathbf{Z}(1)) \cong \mathrm{H}^2(S, \mathbf{Z}(1)) \oplus \mathbf{Z}\,\delta$$

as quadratic spaces. There holds $q(\delta) = 2 - 2n$. In particular, the discriminant

$$\Delta(\mathrm{H}^2(X, \mathbf{Z}(1))) := \mathrm{H}^2(X, \mathbf{Z}(1))^\vee / \mathrm{H}^2(X, \mathbf{Z}(1))$$

is isomorphic to $\mathbf{Z}/(2n-2)\,\mathbf{Z}$, and is generated by δ.

Example 4.1.6. For the remaining known examples of complex hyperkähler varieties as in Example 3.1.9, the BBF form has also been computed. They can all be found in the table in [R1].

Let $(\Lambda, b\colon \operatorname{Sym}^2 \Lambda \to \mathbf{Z})$ be a $\mathbf{Z}$-lattice of signature $(3, n)$, and suppose that Λ is endowed with a $\mathbf{Z}$-Hodge structure. Then (Λ, b) is called a **Hodge lattice of K3 type** if the pairing $b\colon \operatorname{Sym}^2 \Lambda \to \mathbf{Z}(0)$ is a morphism of Hodge structures, Λ has type $(-1, 1), (0, 0), (1, -1)$, the spaces $\Lambda^{1,-1}$ and $\Lambda^{-1,1}$ are one-dimensional and orthogonal to $\Lambda^{0,0}$, and the space $(\Lambda \otimes \mathbf{R}) \cap (\Lambda^{1,-1} \oplus \Lambda^{-1,1})$ is positive-definite.

Proposition 4.1.7 ([GHJ, Section 22.3])**.** *Let X be a hyperkähler manifold. Then the BBF form has signature $(3, b_2(X) - 3)$, and it endows $\mathrm{H}^2(X, \mathbf{Z}(1))$ with the structure of a Hodge lattice of K3 type.*

The following proposition is a consequence of Point 2 in Theorem 4.1.1.

Proposition 4.1.8. *Let X/S be a proper smooth map of complex analytic spaces whose fibers are hyperkähler manifolds. Then there exists a unique quadratic form*

$$q_{X/S} \colon \mathrm{R}^2 f_* \mathbf{Z}(1) \longrightarrow \mathbf{Z}$$

such that for every $s \in S$, the form $q_{X/S}$ restricts to the BBF form on $\mathrm{H}^2(X_s, \mathbf{Z}(1))$.

Definition 4.1.9. The quadratic form $q_{X/S}$ in Proposition 4.1.8 is called the **BBF form** of X/S. The associated morphism of variations of Hodge structures

$$b_{X/S} \colon \operatorname{Sym}^2 \mathrm{R}^2 f_* \mathbf{Z}(1) \longrightarrow \mathbf{Z}(0)$$

given by

$$b_{X/S}(v, w) = q_{X/S}(v + w) - q_{X/S}(v) - q_{X/S}(w).$$

is known as the **BBF pairing** of X/S.

Before we can state the global Torelli theorem, we need the notion of a parallel transport operator.

Definition 4.1.10. Let X_0 and X_1 be hyperkähler manifolds. Suppose we have

- a smooth proper morphism of complex analytic spaces $f \colon \mathfrak{X} \to T$,
- $0, 1 \in T(\mathbf{C})$,
- a path γ in T from 0 to 1,
- isomorphisms $\psi_0 \colon \mathfrak{X}_0 \to X_0$ and $\psi_1 \colon X_1 \to \mathfrak{X}_1$, where $\mathfrak{X}_0$ and $\mathfrak{X}_1$ are the fibers of f over 0 and 1, respectively.

Then the induced homomorphism

$$\mathrm{H}^2(X_0, \mathbf{Z}(1)) \xrightarrow{\psi_0^*} \mathrm{H}^2(\mathfrak{X}_0, \mathbf{Z}(1)) \xrightarrow{\gamma} \mathrm{H}^2(\mathfrak{X}_1, \mathbf{Z}(1)) \xrightarrow{\psi_1^*} \mathrm{H}^2(X_1, \mathbf{Z}(1))$$

is called a **parallel transport operator**.

Remark 4.1.11. It is easy to verify that the composition of parallel transport operators is again a parallel transport operator. By Proposition 4.1.8, parallel transport operators preserve the BBF form.

The following is known as the global Torelli theorem for hyperkähler manifolds. It was originally proved by Verbitsky in [V]. See also [M1] and [H1].

Theorem 4.1.12. *Let X_0 and X_1 be hyperkähler manifolds, and $\varphi \colon \mathrm{H}^2(X_0, \mathbf{Z}(1)) \to \mathrm{H}^2(X_1, \mathbf{Z}(1))$ a homomorphism of abelian groups. Then there exists an isomorphism $f \colon X_1 \to X_0$ with $\varphi = f^*$ if an only if φ is a morphism of Hodge structures, an isometry, a parallel transport operator, and there exists a Kähler class ω on X_0 such that $\varphi(\omega)$ is a Kähler class on X_1.*

Remark 4.1.13. When X_0 and X_1 are complex K3 surfaces, the isomorphism f in Theorem 4.1.12 is unique by [H2, Proposition 15.2.1]. In general, the isomorphism f is not unique. For example, if X is a complex generalized Kummer variety of dimension $2n-2$ with $n \geq 2$, then the kernel of $\mathrm{Aut}(X) \to \mathrm{O}(\mathrm{H}^2(X, \mathbf{Z}(1)))$ is isomorphic to a semidirect product of $\mathbf{Z}/2\mathbf{Z}$ with $(\mathbf{Z}/n\mathbf{Z})^{\oplus 4}$, as is shown in [BNS, Corollary 3.3].

Corollary 4.1.14. *Let (X_0, λ_0) and (X_1, λ_1) be polarized hyperkähler varieties, and $\varphi\colon \mathrm{H}^2(X_1, \mathbf{Z}(1)) \to \mathrm{H}^2(X_0, \mathbf{Z}(1))$ a Hodge isometry mapping $c_1(\lambda_1)$ to $c_1(\lambda_0)$. If φ is a parallel transport operator, then there exists an isomorphism $f\colon (X_0, \lambda_0) \to (X_1, \lambda_1)$ inducing φ.*

Proof. This follows immediately from Theorem 4.1.12 and the fact that if λ is an ample line bundle, then $c_1(\lambda)$ is a Kähler class. $\square$

4.1.2 Deformations of polarized hyperkähler varieties

Let (X_0, λ_0) be a polarized complex hyperkähler variety. We are interested in the deformation theory of the pair (X_0, λ_0). Let $\mathbf{Art}_{\mathbf{C}}$ be the category of local Artinian $\mathbf{C}$-algebras. We define a functor $\mathrm{Def}(X_0, \lambda_0)\colon \mathbf{Art}_{\mathbf{C}} \to \mathbf{Set}$ by mapping a local Artinian $\mathbf{C}$-algebra A with maximal ideal $\mathfrak{m}$ to the set of equivalence classes of tuples

$$(f\colon X \longrightarrow \mathrm{Spec}(A),\ \lambda \in \mathrm{Pic}_{X/A}(A),\ \varphi\colon X_0 \longrightarrow X_{\mathfrak{m}}).$$

Here, f is a smooth proper morphism of algebraic spaces whose fibers are hyperkähler varieties, $\lambda \in \mathrm{Pic}_{X/A}(A)$ is a polarization, and φ is an isomorphism of schemes mapping λ_0 to $\lambda_{\mathfrak{m}}$. Two such tuples $(f\colon X \to \mathrm{Spec}(A), \lambda, \varphi)$ and $(f'\colon X' \to \mathrm{Spec}(A), \lambda', \varphi')$ are said to be equivalent if there exists an isomorphism of algebraic spaces $X \to X'$ over A mapping λ to λ', and such that the diagram

$$\begin{tikzcd} X_{\mathfrak{m}} \arrow[rr] \arrow[dr, "\varphi"'] & & X'_{\mathfrak{m}} \arrow[dl, "\varphi'"] \\ & X_0 & \end{tikzcd}$$

commutes.

Theorem 4.1.15. *Let (X_0, λ_0) be a polarized complex hyperkähler variety with second Betti number b_2. Then the functor* $\mathrm{Def}(X_0, \lambda_0)\colon \mathbf{Art}_{\mathbf{C}} \to \mathbf{Set}$ *is prorepresented by the formal power series ring $\mathbf{C}[\![t_1, \ldots, t_{b_2-3}]\!]$.*

Proof. Let **Germs** be the category of germs of complex analytic spaces. Its objects are pairs (S, s), with S a complex analytic space, and $s \in S$. A morphism $(S, s) \to (S', s')$ in **Germs** consists of an open neighborhood U of s, and a morphism $\varphi\colon U \to S'$ mapping s to s'. Define a functor $\mathrm{Def}_{\mathrm{an}}(X_0)\colon \mathbf{Germs}^{\mathrm{opp}} \to \mathbf{Set}$ by mapping a germ (S, s) to equivalence classes of pairs $(f\colon X \to U, \varphi\colon X_0 \to X_s)$, where U is a neighborhood of s, f is a proper smooth map of complex analytic spaces, and φ is an isomorphism. Two pairs $(f\colon X \to U, \varphi)$ and $(f'\colon X' \to U', \varphi')$ are equivalent if there exists a neighborhood $V \subseteq U \cap U'$ of s and an isomorphism of complex spaces $X_V \to X'_V$ making the diagram

$$\begin{tikzcd} X_V \arrow[rr] \arrow[dr, "\varphi"'] & & X'_V \arrow[dl, "\varphi'"] \\ & X_0 & \end{tikzcd}$$

commute. The Bogomolov-Tian-Todorov theorem states that since X_0 is a compact Kähler manifold with trivial canonical bundle, the functor $\mathrm{Def}_{\mathrm{an}}(X_0)$ is represented by the germ $(\mathbf{C}^n, 0)$, for some $n \in \mathbf{Z}_{\geq 0}$.

Let $\Omega^{\pm}_{\mathrm{H}^2(X_0,\mathbf{R}(1))}$ be the complex manifold parametrizing Hodge structures of K3 type on $\mathrm{H}^2(X_0, \mathbf{R}(1))$, and let $\mathfrak{X} \to \mathrm{Def}_{\mathrm{an}}(X_0)$ be the universal deformation of X_0. Since $\mathrm{Def}_{\mathrm{an}}(X_0)$ is simply connected, we can canonically identify $\mathrm{H}^2(\mathfrak{X}_s, \mathbf{Z}(1))$ with $\mathrm{H}^2(X_0, \mathbf{Z}(1))$ for each $s \in \mathrm{Def}_{\mathrm{an}}(X_0)$. This gives rise to a morphism $p\colon \mathrm{Def}_{\mathrm{an}}(X_0) \to \Omega^{\pm}_{\mathrm{H}^2(X_0,\mathbf{Z}(1))}$. The local Torelli theorem [B2, Théorème 5] states that p is a local isomorphism. This implies that $n = b_2 - 2$.

Let $\mathrm{Def}_{\mathrm{an}}(X_0, \lambda_0)\colon \mathbf{Germs}^{\mathrm{opp}} \to \mathbf{Set}$ be the functor parametrizing polarized deformations of (X_0, λ_0), defined similarly to $\mathrm{Def}_{\mathrm{an}}(X_0)$ and $\mathrm{Def}(X_0, \lambda_0)$. Then $\mathrm{Def}_{\mathrm{an}}(X_0, \lambda_0)$ is a subfunctor of $\mathrm{Def}_{\mathrm{an}}(X_0)$, and is in fact the inverse image of $\Omega^{\pm}_{\mathrm{H}^2(X_0,\mathbf{R}(1))\cap\lambda_0^{\perp}}$ under p. In particular, $\mathrm{Def}_{\mathrm{an}}(X_0, \lambda_0)$ is represented by the germ $(\mathbf{C}^{b_2-3}, 0)$.

Let $\mathbf{A}$ be the category of $\mathbf{C}$-algebras which are isomorphic to quotients of $\mathcal{O}^{\mathrm{an}}_{\mathbf{C}^n,0}$ for some $n \in \mathbf{Z}_{\geq 0}$. Then $(S, s) \mapsto \mathcal{O}^{\mathrm{an}}_{S,s}$ defines an equivalence $\mathbf{Germs}^{\mathrm{opp}} \to \mathbf{A}$, where $\mathcal{O}^{\mathrm{an}}_S$ denotes the structure sheaf of S. Since the analytification of a finite $\mathbf{C}$-scheme is still finite, the category $\mathbf{A}$ contains $\mathbf{Art}_{\mathbf{C}}$ as a full subcategory, and there is a 2-commutative diagram

$$\begin{tikzcd} \mathbf{Art}_{\mathbf{C}} \arrow[rr] \arrow[dr, "\mathrm{Def}(X_0,\lambda_0)"'] & & \mathbf{A} \arrow[dl, "\mathrm{Def}_{\mathrm{an}}(X_0,\lambda_0)"] \\ & \mathbf{Set} & \end{tikzcd}$$

In particular, since the completion of $\mathcal{O}^{\mathrm{an}}_{\mathbf{C}^{b_2-3},0}$ is $\mathbf{C}[\![t_1, \ldots, t_{b_2-3}]\!]$, and since Artinian algebras are complete, we have, for any $A \in \mathbf{Art}_{\mathbf{C}}$

$$\mathrm{Def}(X_0, \lambda_0)(A) = \mathrm{Hom}(\mathcal{O}^{\mathrm{an}}_{\mathbf{C}^n,0}, A) = \mathrm{Hom}(\mathbf{C}[\![t_1, \ldots, t_{b_2-3}]\!], A).$$

$\square$

The following corollary finishes the proof of Theorem 3.3.2.

Corollary 4.1.16. *The moduli stack* $\mathbf{HK}$ *of polarized hyperkähler varieties over* $\mathbf{Q}$ *is smooth. Its dimension at a* $\mathbf{C}$*-point* (X, λ) *is equal to* $b_2(X) - 3$*.*

Proof. For the smoothness assertion, it suffices to prove that $\mathbf{HK}_{\mathbf{C}}$ is smooth over $\mathbf{C}$. We already know that $\mathbf{HK}_{\mathbf{C}}$ is locally of finite type. From [T2, Tag 02HX] it follows that we need to check that if A is an Artinian $\mathbf{C}$-algebra, and $I \subseteq A$ an ideal with $I^2 = 0$, then for any morphism $\operatorname{Spec}(A/I) \to \mathbf{HK}$, there exists a 2-commutative diagram

$$\begin{tikzcd} \operatorname{Spec}(A/I) \arrow[r] \arrow[d] & \mathbf{HK} \\ \operatorname{Spec}(A) \arrow[ur] & \end{tikzcd}$$

More concretely, this means that given a smooth proper morphism $X \to \operatorname{Spec}(A/I)$ whose fibers are hyperkähler varieties and a polarization $\lambda \in \operatorname{Pic}_{X/\operatorname{Spec}(A/I)}(A/I)$, we want to find a smooth proper morphism $X' \to \operatorname{Spec}(A)$ whose fibers are hyperkähler varieties and a polarization $\lambda' \in \operatorname{Pic}_{X/\operatorname{Spec}(A)}(A)$ such that the pullback of the pair (X', λ') to A/I is isomorphic to (X, λ). This follows immediately from Theorem 4.1.15.

Let (X, λ) be a $\mathbf{C}$-point of $\mathbf{HK}$, and let $\mathbf{C}[\varepsilon]$ be the ring of dual numbers over $\mathbf{C}$. Then $\operatorname{Def}(X, \lambda)(\mathbf{C}[\varepsilon])$ is a finite-dimensional $\mathbf{C}$-vector space, and the dimension of $\mathbf{HK}$ at the point (X, λ) is equal to that of $\operatorname{Def}(X, \lambda)(\mathbf{C}[\varepsilon])$. It therefore follows from Theorem 4.1.15 that the dimension of $\mathbf{HK}$ at (X, λ) is $b_2(X) - 3$. □

4.2 The BBF form on étale cohomology

In this section we extend the notion of BBF form to hyperkähler varieties over arbitrary fields of characteristic 0. The main result is the étale monodromy invariance of the BBF form, Theorem 4.2.4.

Lemma 4.2.1. *Let* X *be a hyperkähler variety over a field* K *of characteristic* 0*. Then there exists a unique primitive quadratic form* $q\colon \mathrm{H}^2_{\text{ét}}(X_{\overline{K}}, \widehat{\mathbf{Z}}(1)) \to \widehat{\mathbf{Z}}$ *such that*

1. q *is a* $\mathbf{Q}$*-multiple of the quadratic form*
$$Q_X : \alpha \longmapsto \int_X \sqrt{\mathrm{td}_{X_{\overline{K}}}}\,\alpha^2$$
on $\mathrm{H}^2_{\text{ét}}(X_{\overline{K}}, \mathbf{A}_{\mathrm{f}}(1))$*,*
2. *there exists an ample line bundle* L *on* $X_{\overline{K}}$ *for which* $q(c_1(L)) \in \mathbf{Z}_{>0}$*.*

Proof. By a spreading out argument, we may assume that K is of finite type over $\mathbf{Q}$. We choose an embedding of K into $\mathbf{C}$. Now using Artin's comparison isomorphism and Theorem 4.1.1, we obtain a primitive quadratic form on $\mathrm{H}^2_{\text{ét}}(X_{\overline{K}}, \widehat{\mathbf{Z}}(1))$

satisfying the conditions of the lemma. In fact, by Remark 4.1.3, we obtain a quadratic form $q\colon \mathrm{H}^2_{\text{ét}}(X_{\overline{K}}, \widehat{\mathbf{Z}}(1)) \to \widehat{\mathbf{Z}}$ satisfying the stronger condition that *for every* ample line bundle $L \in \mathrm{Pic}(X_{\overline{K}})$ there holds $q(c_1(L)) \in \mathbf{Z}_{>0}$.

Now suppose q' is another primitive quadratic form on $\mathrm{H}^2_{\text{ét}}(X_{\overline{K}}, \widehat{\mathbf{Z}}(1))$ satisfying the conditions of the lemma. Let L be an ample line bundle on $X_{\overline{K}}$ such that $q'(c_1(L)) \in \mathbf{Z}_{>0}$. Since q and q' both satisfy condition 1, there exists a $c \in \mathbf{Q}^\times$ with $q = cq'$. Because q and q' are primitive, we have $c \in \widehat{\mathbf{Z}}^\times \cap \mathbf{Q}^\times = \{\pm 1\}$. It now follows from the fact that $q'(c_1(L))$ and $q(c_1(L))$ are positive integers that $c = 1$, proving the uniqueness. ☐

Definition 4.2.2. Let X be a hyperkähler variety over a field K of characteristic 0. The quadratic form on $\mathrm{H}^2_{\text{ét}}(X_{\overline{K}}, \widehat{\mathbf{Z}}(1))$ given in Lemma 4.2.1 is called the **Beauville-Bogomolov-Fujiki form** or **BBF form** of X, and is denoted q_X. The bilinear pairing $b_X\colon \mathrm{Sym}^2 \mathrm{H}^2_{\text{ét}}(X_{\overline{K}}, \widehat{\mathbf{Z}}(1)) \to \widehat{\mathbf{Z}}$ associated with q_X is called the **BBF pairing**.

Remark 4.2.3. The proof of Lemma 4.2.1 shows that if X is a hyperkähler variety over $\mathbf{C}$, then the Artin comparison isomorphism between singular and étale cohomology gives an isometry from $\mathrm{H}^2(X, \mathbf{Z}(1)) \otimes \widehat{\mathbf{Z}}$ endowed with the BBF form from Definition 4.1.2 to $\mathrm{H}^2_{\text{ét}}(X, \widehat{\mathbf{Z}}(1))$ endowed with the BBF form from Definition 4.2.2.

Theorem 4.2.4. *Let S be a $\mathbf{Q}$-scheme, $f\colon X \to S$ a proper smooth morphism of algebraic spaces whose fibers are hyperkähler varieties. Then there exists a unique quadratic form*

$$q_{X/S}\colon \mathrm{R}^2_{\text{ét}} f_* \widehat{\mathbf{Z}}(1) \longrightarrow \widehat{\mathbf{Z}}(0)$$

such that for every geometric point $\overline{s}$ of S, the form $q_{X/S}$ restricts to the BBF form on $\mathrm{H}^2_{\text{ét}}(X_{\overline{s}}, \widehat{\mathbf{Z}}(1))$.

Proof. The uniqueness is clear, so we go on to prove the existence of the form.

First, we prove the existence for those $f\colon X \to S$ which admit a polarization $\lambda \in \mathrm{Pic}_{X/S}(S)$. We assume without loss of generality that S is connected.

Let $\overline{s}$ and $\overline{s}'$ be geometric points of S, and γ a path in $S_{\text{ét}}$ from $\overline{s}$ to $\overline{s}'$. Then γ induces an isomorphism of $\widehat{\mathbf{Z}}$-modules $\gamma_*\colon \mathrm{H}^2_{\text{ét}}(X_{\overline{s}}, \widehat{\mathbf{Z}}(1)) \to \mathrm{H}^2_{\text{ét}}(X_{\overline{s}'}, \widehat{\mathbf{Z}}(1))$. Let q and q' be the BBF forms on $\mathrm{H}^2_{\text{ét}}(X_{\overline{s}}, \widehat{\mathbf{Z}}(1))$ and $\mathrm{H}^2_{\text{ét}}(X_{\overline{s}'}, \widehat{\mathbf{Z}}(1))$, respectively. It suffices to show that $q\gamma_* = q'$.

For the forms $Q := Q_{X_{\overline{s}}}$ and $Q' := Q_{X_{\overline{s}'}}$ from Lemma 4.2.1 it is clear that $Q\gamma_* = Q'$. Since q is a primitive $\mathbf{Q}$-multiple of Q, it follows that the form $q\gamma_*$ is a primitive $\mathbf{Q}$-multiple of Q'. Moreover, because $\lambda_{\overline{s}}$ extends to a section λ of $\mathrm{Pic}_{X/S}$ over S, we have $(q\gamma_*)(c_1(\lambda_{\overline{s}})) = q(c_1(\lambda_{\overline{s}}))$, which is an element of $\mathbf{Z}_{>0}$ by part 2 of Lemma 4.2.1. Since the BBF form is uniquely determined by the two conditions in Lemma 4.2.1, it follows that $q\gamma_* = q'$, proving the theorem for polarizable families of hyperkähler varieties.

Now let $f\colon X \to S$ be as in the statement of the theorem. By Remark 3.2.4, there exists an étale cover $U \to S$ and a polarization $\lambda \in \mathrm{Pic}_{X/S}(U)$. Let $f_U\colon X_U \to U$ be the pullback of X to U. Then the first part of this proof shows that there exists a quadratic form

$$q_U\colon \mathrm{R}^2_{\text{ét}} f_{U,*} \widehat{\mathbf{Z}}(1) \longrightarrow \widehat{\mathbf{Z}}(0)$$

in $U_{\text{ét}}$ which restricts to the étale BBF form on geometric fibers. It suffices to show that q_U descends to S.

Let pr_1 and pr_2 denote the projections $U \times_S U \to U$, and let X_1 and X_2 be the pullbacks of X_U along pr_1 and pr_2, respectively. Then stalk-wise $\mathrm{pr}_1^* q_U$ and $\mathrm{pr}_2^* q_U$ are the BBF forms of the geometric fibers of X_1 and X_2, respectively. From $X_1 \cong X_2$ it follows that $\mathrm{pr}_1^* q_U = \mathrm{pr}_2^* q_U$, since isomorphisms of hyperkähler varieties preserve the BBF form. In particular, q_U descends to S, proving the theorem. $\square$

Remark 4.2.5. Let S be a $\mathbf{Q}$-scheme, and let $f\colon X \to S$ be a proper smooth morphism of algebraic spaces whose fibers are hyperkähler varieties. The quadratic form

$$q_{X/S}\colon \mathrm{R}^2_{\text{ét}} f_* \widehat{\mathbf{Z}}(1) \longrightarrow \widehat{\mathbf{Z}}$$

given in Theorem 4.2.4 is called the **BBF form** of X/S. The associated bilinear pairing $b_{X/S}\colon \mathrm{Sym}^2 \mathrm{R}^2_{\text{ét}} f_* \widehat{\mathbf{Z}}(1) \to \widehat{\mathbf{Z}}$ is known as the **BBF pairing** of X/S.

This quadratic form is preserved under base change in the following sense. Suppose we are given a morphism $\varphi\colon S' \to S$ of $\mathbf{Q}$-schemes. Define $f'\colon X' \to S'$ by the cartesian square

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "\varphi"'] & S \end{tikzcd}$$

Then f' is a smooth proper morphism of algebraic spaces whose fibers are hyperkähler varieties. Smooth and proper base change for étale cohomology give an isomorphism

$$\varphi^*\left(\mathrm{R}^2_{\text{ét}} f_* \widehat{\mathbf{Z}}(1)\right) \longrightarrow \mathrm{R}^2_{\text{ét}} f'_* \widehat{\mathbf{Z}}(1)$$

of local systems on $S'_{\text{ét}}$ which is compatible with the BBF forms.

Remark 4.2.6. Let X be a complex hyperkähler variety, and $\sigma \in \mathrm{Aut}(\mathbf{C})$. It is not clear whether X and $\sigma^* X$ have isometric BBF forms on their singular cohomology. However, it can be shown that they have the same genus, as follows. Remark 4.2.5 shows that $\mathrm{H}^2(X, \mathbf{Z}(1)) \otimes \widehat{\mathbf{Z}}$ and $\mathrm{H}^2(\sigma^* X, \mathbf{Z}(1)) \otimes \widehat{\mathbf{Z}}$ are isometric. In addition, by Proposition 4.1.7 they have the same signature, so they have the same genus.

For all known examples of complex hyperkähler varieties (see Examples 3.1.7 through 3.1.9), the BBF form Λ satisfies the inequality $\mathrm{rk}(\Lambda) \geq \mathrm{length}(\Delta(\Lambda)) + 2$, as can be seen in the table in [R1]. Here, $\mathrm{length}(\Delta(\Lambda))$ denotes the minimal number of elements required to generate $\Delta(\Lambda)$. By [N1, Theorem 1.14.2], this inequality and the indefiniteness of Λ imply that the genus of Λ contains exactly one isometry class. In particular, if X is one of the known examples of complex hyperkähler varieties, then X and $\sigma^* X$ have isometric BBF forms on their singular cohomology.

4.3 Orientations on hyperkähler varieties

This section serves to introduce the moduli stack $\mathbf{HK}_{\mathrm{or}}$ of oriented polarized hyperkähler varieties. The main result is the rather technical Theorem 4.3.6, which we need to construct a morphism from a connected component of $\mathbf{HK}_{\mathrm{or}}$ to an orthogonal Shimura stack.

We will need to work with motives over a finitely generated field k of characteristic 0. Similarly to the algebraically closed case discussed in Section 2.2, they form a semisimple Tannakian category $\mathbf{Mot}_k$.

Let $\overline{k}$ be an algebraic closure of k. Then the composition of the pullback functor $\mathbf{Mot}_k \to \mathbf{Mot}_{\overline{k}}$ with the fiber functor $\mathrm{H}_{\text{ét}} \colon \mathbf{Mot}_{\overline{k}} \to \mathbf{A}_{\mathrm{f}}\text{-}\mathbf{Mod}$ yields a fiber functor on $\mathbf{Mot}_k$, which we denote by $\mathrm{H}_{\overline{k},\text{ét}}$. Equation (2.3) shows that $\mathrm{H}_{\overline{k},\text{ét}}$ gives rise to a functor $\mathbf{Mot}_k \to \mathrm{Gal}_k\text{-}\mathbf{Rep}_{\mathbf{A}_{\mathrm{f}}}$, where $\mathrm{Gal}_k\text{-}\mathbf{Rep}_{\mathbf{A}_{\mathrm{f}}}$ denotes the category of $\mathbf{A}_{\mathrm{f}}$-modules endowed with a continuous $\mathbf{A}_{\mathrm{f}}$-linear Gal_k-action. We will abusively denote this functor with $\mathrm{H}_{\overline{k},\text{ét}}$ as well. Similarly, an embedding $\iota \colon k \to \mathbf{C}$ and the Betti realization functor give rise to a fiber functor $\mathrm{H}_\iota \colon \mathbf{Mot}_k \to \mathbf{Q}\text{-}\mathbf{HS}$. For a motive $M \in \mathbf{Mot}_k$, we denote by $M_{\text{ét}}$ and M_ι the images of M under $\mathrm{H}_{\overline{k},\text{ét}}$ and H_ι, respectively.

Lemma 4.3.1. *Let k be a finitely generated field of characteristic 0, let X be a hyperkähler variety over k with $b_2(X) > 3$, and let $\omega_{[4]} \colon \mathbf{Z}/4\mathbf{Z} \to \det \mathrm{H}^2_{\text{ét}}(X_{\overline{k}}, \mu_4)$ be a Gal_k-equivariant isomorphism. Then there exists an isomorphism of motives over k*

$$\omega \colon \mathbf{1} \longrightarrow \det(\mathfrak{h}^2(X)(1))$$

such that $\omega_{\text{ét}} \colon \mathbf{A}_{\mathrm{f}} \to \det \mathrm{H}^2_{\text{ét}}(X_{\overline{k}}, \mathbf{A}_{\mathrm{f}}(1))$ of ω restricts to an isomorphism $\widehat{\mathbf{Z}} \to \det \mathrm{H}^2_{\text{ét}}(X_{\overline{k}}, \widehat{\mathbf{Z}}(1))$ satisfying $\omega_{\text{ét}}|_{\widehat{\mathbf{Z}}} \otimes \mathbf{Z}/4\mathbf{Z} = \omega_{[4]}$. Moreover, for every embedding $\iota \colon k \to \mathbf{C}$, the map $\omega_\iota \colon \mathbf{Q} \to \det \mathrm{H}^2(X \times_{k,\iota} \mathbf{C}, \mathbf{Q}(1))$ restricts to an isomorphism $\mathbf{Z} \to \det \mathrm{H}^2(\iota^* X, \mathbf{Z}(1))$.

Proof. We first show that $\det(\mathfrak{h}^2(X)(1))$ is isomorphic to $\mathbf{1}$. Since $b_2(X) > 3$, [A1, Theorem 1.5.1] shows that $\det(\mathfrak{h}^2(X)(1))$ is an abelian motive over k of rank 1 and weight 0. It follows that $\det(\mathfrak{h}^2(X)(1))$ is an Artin motive. In particular, since its rank is 1, it follows that $\det(\mathfrak{h}^2(X)(1))$ is the motive associated with a quadratic character $\chi \colon \mathrm{Gal}_k \to \{\pm 1\}$. For any prime number ℓ, the character χ agrees with the composition

$$\mathrm{Gal}_k \longrightarrow \mathrm{O}(\mathrm{H}^2_{\text{ét}}(X_{\overline{k}}, \mathbf{Z}_\ell(1))) \xrightarrow{\det} \{\pm 1\}.$$

The commutative diagram

$$\begin{tikzcd}
 & \mathrm{O}(\mathrm{H}^2_{\text{ét}}(X_{\overline{k}}, \mathbf{Z}_2(1))) \arrow[d] \arrow[dr, "\det"] & \\
\mathrm{Gal}_k \arrow[ur] \arrow[dr] & & \{\pm 1\} \\
 & \mathrm{GL}(\mathrm{H}^2_{\text{ét}}(X_{\overline{k}}, \mu_4)) \arrow[ur, "\det"'] &
\end{tikzcd}$$

and the existence of $\omega_{[4]}$ show that χ is trivial, and hence that there exists an isomorphism $\omega\colon \mathbf{1} \cong \det(\mathfrak{h}^2(X)(1))$.

For $\iota\colon k \hookrightarrow \mathbf{C}$, we endow $\det \mathrm{H}^2(X_{\iota,\mathbf{C}}, \mathbf{Z}(1))$ with the quadratic form induced by the BBF form. Similarly, $\det \mathrm{H}^2_{\text{ét}}(X_{\overline{k}}, \widehat{\mathbf{Z}}(1))$ is also endowed with the quadratic form induced by the BBF form. Then, by Theorem 4.2.4, $\det \mathrm{H}^2(X_{\iota,\mathbf{C}}, \mathbf{Z}(1)) \otimes \widehat{\mathbf{Z}}$ and $\det \mathrm{H}^2_{\text{ét}}(X_{\overline{k}}, \widehat{\mathbf{Z}}(1))$ are isometric. In particular, since the genus of a rank 1 lattice contains only one isometry class, the discriminant d of $\det \mathrm{H}^2(X_{\iota,\mathbf{C}}, \mathbf{Z}(1))$ is independent of the choice of ι.

Endow $\mathbf{1}$ with the unique quadratic form q such that for any $\iota\colon k \hookrightarrow \mathbf{C}$, the quadratic form q_ι on $\mathbf{Q}$ restricts to a quadratic form on $\mathbf{Z}$ with discriminant d. By rescaling the isomorphism $\omega\colon \mathbf{1} \to \det(\mathfrak{h}^2(X)(1))$, we may assume that it is an isometry with respect to q and the BBF form. Let $\iota\colon k \hookrightarrow \mathbf{C}$. Then, by construction, the two sublattices $\omega_\iota(\mathbf{Z})$ and $\det \mathrm{H}^2(X_{\iota,\mathbf{C}}, \mathbf{Z}(1))$ of $\det \mathrm{H}^2(X_{\iota,\mathbf{C}}, \mathbf{Q}(1))$ have the same discriminant, and are therefore equal. It follows that ω_ι restricts to an isomorphism $\mathbf{Z} \to \det \mathrm{H}^2(X_{\iota,\mathbf{C}}, \mathbf{Z}(1))$. From the Artin comparison isomorphisms it now follows that $\omega_{\text{ét}}$ restricts to an isomorphism $\widehat{\mathbf{Z}} \to \det \mathrm{H}^2_{\text{ét}}(X_{\overline{k}}, \widehat{\mathbf{Z}}(1))$.

Multiplying ω with -1 if necessary guarantees that $\omega_{\text{ét}} \otimes \mathbf{Z}/4\mathbf{Z} = \omega_{[4]}$. □

Lemma 4.3.2. *Let S be a normal $\mathbf{Q}$-scheme of finite type, $f\colon X \to S$ a smooth proper morphism of algebraic spaces whose fibers are hyperkähler varieties satisfying $b_2 > 3$, and let $\omega_{[4]}\colon \mathbf{Z}/4\mathbf{Z} \to \det \mathrm{R}^2_{\text{ét}} f_*\mu_4$ be an isomorphism of local systems on $S_{\text{ét}}$. Then there are unique isomorphisms of local systems*

$$\omega_{\text{ét}}\colon \widehat{\mathbf{Z}} \longrightarrow \det \mathrm{R}^2_{\text{ét}} f_* \widehat{\mathbf{Z}}(1)$$

on $S_{\text{ét}}$ and

$$\omega_{\text{an}}\colon \mathbf{Z} \longrightarrow \det \mathrm{R}^2 f_{\mathbf{C},*} \mathbf{Z}(1)$$

on $S_{\mathbf{C}}$ satisfying $\omega_{\text{ét}}|_{S_{\mathbf{C}}} = \omega_{\text{an}} \otimes \widehat{\mathbf{Z}}$ and $\omega_{[4]} = \omega_{\text{ét}} \otimes \mathbf{Z}/4\mathbf{Z}$.

Proof. The uniqueness of the isomorphisms is clear, so we go on to prove existence.

Without loss of generality we may assume that S is connected. Let η be the generic point of S, and $\overline{\eta}$ an algebraic closure of η. Lemma 4.3.1 shows that the restriction of the local system $\det \mathrm{R}^2_{\text{ét}} f_* \widehat{\mathbf{Z}}(1)$ to η is constant. Since $\pi_1^{\text{ét}}(\eta, \overline{\eta}) \to \pi_1^{\text{ét}}(S, \overline{\eta})$ is surjective by [G2, Proposition V.8.2], we conclude that $\det \mathrm{R}^2_{\text{ét}} f_* \widehat{\mathbf{Z}}(1)$ is constant.

Let s be a $\mathbf{C}$-point of S. Then we have a commutative diagram

$$\begin{tikzcd} & \mathrm{GL}(\mathrm{H}^2(X_s, \mathbf{Z}(1))) \arrow[dd] \arrow[dr, "\det"] & \\ \pi_1(S_{\mathbf{C}}, s) \arrow[ur] \arrow[dr] & & \{\pm 1\} \\ & \mathrm{GL}(\mathrm{H}^2(X_s, \mu_4)) \arrow[ur, "\det"'] & \end{tikzcd} \tag{4.1}$$

By the existence of $\omega_{[4]}$, the composition $\pi_1(S_{\mathbf{C}}, s) \to \mathrm{GL}(\mathrm{H}^2(X_s, \mu_4)) \to \{\pm 1\}$ is trivial, so by (4.1) we conclude that $\pi_1(S_{\mathbf{C}}, s) \to \mathrm{GL}(\mathrm{H}^2(X_s, \mathbf{Z}(1))) \to \{\pm 1\}$ is trivial as well. It follows that the local system $\det \mathrm{R}^2 f_{\mathbf{C},*}\, \mathbf{Z}(1)$ is constant on $S_{\mathbf{C}}$.

Since the local systems $\det \mathrm{R}^2_{\text{ét}} f_*\, \widehat{\mathbf{Z}}(1)$ and $\det \mathrm{R}^2 f_{\mathbf{C},*}\, \mathbf{Z}(1)$ are constant, the isomorphisms given in Lemma 4.3.1, applied to η, give the desired isomorphisms of local systems $\omega_{\text{ét}}$ and ω_{an}. ☐

Definition 4.3.3. Let S be a $\mathbf{Q}$-scheme, and let $f\colon X \to S$ be a smooth proper morphism of algebraic spaces whose fibers are hyperkähler varieties. An **orientation** on X/S is an isomorphism of sheaves of finite abelian groups

$$\omega\colon\ \mathbf{Z}/4\,\mathbf{Z} \longrightarrow \det \mathrm{R}^2_{\text{ét}} f_* \mu_4$$

on $S_{\text{ét}}$.

The **moduli stack of oriented polarized hyperkähler varieties $\mathbf{HK}_{\text{or}}$** is defined to be the stack parameterizing tuples $(X/S, \lambda, \omega)$, where $(X/S, \lambda) \in \mathbf{HK}$ is an element such that the fibers of X/S have second Betti number greater than 3, and ω is an orientation on X/S. Let $f\colon \mathfrak{X} \to \mathbf{HK}_{\text{or}}$ be the universal hyperkähler variety. We denote by $\omega_{[4]}$ the universal orientation $\mathbf{Z}/4\,\mathbf{Z} \to \det \mathrm{R}^2_{\text{ét}} f_* \mu_4$.

Remark 4.3.4. Since $\mathbf{HK}_{\text{or}}$ is a degree 2 étale cover of $\mathbf{HK}$, it is itself a smooth separated Deligne-Mumford stack over $\mathbf{Q}$.

Remark 4.3.5. The condition on the second Betti number of the hyperkähler varieties parameterized by $\mathbf{HK}_{\text{or}}$ is there to ensure that their motives are abelian, cf. Remark 2.2.1. This will allow us to apply Lemma 4.3.2.

Theorem 4.3.6. *There are unique isomorphisms of local systems*

$$\omega_{\text{ét}}\colon\ \widehat{\mathbf{Z}} \longrightarrow \det \mathrm{R}^2_{\text{ét}} f_*\, \widehat{\mathbf{Z}}(1)$$

on $\mathbf{HK}_{\text{or},\text{ét}}$ and

$$\omega_{\text{an}}\colon\ \mathbf{Z} \longrightarrow \det \mathrm{R}^2 f_{\mathbf{C},*}\, \mathbf{Z}(1)$$

on $\mathbf{HK}_{\text{or},\mathbf{C}}$ such that $\omega_{\text{ét}}|_{\mathbf{HK}_{\text{or},\mathbf{C}}} = \omega_{\text{an}} \otimes \widehat{\mathbf{Z}}$ and such that $\omega_{[4]} = \omega_{\text{ét}} \otimes \mathbf{Z}/4\,\mathbf{Z}$.

Proof. By Corollary 4.1.16, $\mathbf{HK}_{\text{or}}$ is smooth, and in particular normal and of finite type over $\mathbf{Q}$. It follows that we can apply Lemma 4.3.2 to conclude the proof. ☐

The following lemma will be useful in our treatment of the moduli stack of polarized K3 surfaces.

Lemma 4.3.7. *There is a rank 1 local $\mathbf{Z}$-system D on $\mathbf{HK}_{\text{ét}}$, endowed with an injective morphism of sheaves $D \to \det \mathrm{R}^2_{\text{ét}} f_*\, \widehat{\mathbf{Z}}(1)$ on $\mathbf{HK}_{\text{ét}}$ and an isomorphism of sheaves $D|_{\mathbf{HK}_{\mathbf{C}}} \to \det \mathrm{R}^2 f_{\mathbf{C},*}\, \mathbf{Z}(1)$ on $\mathbf{HK}_{\mathbf{C}}$ such that the diagram*

$$\begin{tikzcd}
 & \det \mathrm{R}^2 f_{\mathbf{C},*}\, \mathbf{Z}(1) \arrow[dd, "\otimes \widehat{\mathbf{Z}}"] \\
D|_{\mathbf{HK}_{\mathbf{C}}} \arrow[ur] \arrow[dr] & \\
 & \det \mathrm{R}^2_{\text{ét}} f_{\mathbf{C},*}\, \widehat{\mathbf{Z}}(1)
\end{tikzcd} \tag{4.2}$$

commutes.

Proof. Being a degree 2 étale cover of $\mathbf{HK}$, the stack $\mathbf{HK}_{\mathrm{or}}$ comes with a natural $\{\pm1\}$-action making it a $\{\pm1\}$-torsor on $\mathbf{HK}_{\text{ét}}$. In addition to this, we have a $\mathbf{Z}^\times$-torsor $\mathrm{Isom}(\mathbf{Z}, \det \mathrm{R}^2 f_{\mathbf{C},*}\, \mathbf{Z}(1))$ on $\mathbf{HK}_{\mathbf{C}}$, and a $\widehat{\mathbf{Z}}^\times$-torsor $\mathrm{Isom}(\widehat{\mathbf{Z}}, \det \mathrm{R}^2_{\text{ét}}\, f_*\, \widehat{\mathbf{Z}}(1))$. Theorem 4.3.6 gives morphisms of sheaves

$$\omega_{\text{ét}}\colon\ \mathbf{HK}_{\mathrm{or}} \longrightarrow \mathrm{Isom}\Big(\widehat{\mathbf{Z}}, \det \mathrm{R}^2_{\text{ét}}\, f_*\, \widehat{\mathbf{Z}}(1)\Big) \tag{4.3}$$

on $\mathbf{HK}_{\text{ét}}$ and

$$\omega_{\mathrm{an}}\colon\ \mathbf{HK}_{\mathrm{or},\mathbf{C}} \longrightarrow \mathrm{Isom}\big(\mathbf{Z}, \det \mathrm{R}^2 f_{\mathbf{C},*}\, \mathbf{Z}(1)\big) \tag{4.4}$$

on $\mathbf{HK}_{\mathbf{C}}$ such that the diagram

$$\begin{tikzcd} & \mathrm{Isom}(\mathbf{Z}, \det \mathrm{R}^2 f_{\mathbf{C},*}\, \mathbf{Z}(1)) \arrow[d, "\otimes \widehat{\mathbf{Z}}"] \\ \mathbf{HK}_{\mathrm{or},\mathbf{C}} \arrow[ur] \arrow[dr] & \\ & \mathrm{Isom}(\widehat{\mathbf{Z}}, \det \mathrm{R}^2_{\text{ét}}\, f_*\, \widehat{\mathbf{Z}}(1))|_{\mathbf{HK}_{\mathbf{C}}} \end{tikzcd} \tag{4.5}$$

commutes. It is easily verified that the maps $\omega_{\text{ét}}$ and ω_{an} are $\{\pm1\}$-equivariant. In particular, ω_{an} is an isomorphism, since its source and target are torsors under the same group $\mathbf{Z}^\times = \{\pm1\}$.

There is an equivalence from the groupoid of rank 1 local $\mathbf{Z}$-systems on $\mathbf{HK}_{\text{ét}}$ (respectively $\mathbf{HK}_{\mathbf{C}}$) to the groupoid of $\{\pm1\}$-torsors on $\mathbf{HK}_{\text{ét}}$ (respectively $\mathbf{HK}_{\mathbf{C}}$) given by mapping a local system L to $\mathrm{Isom}(\mathbf{Z}, L)$. Similarly, $\mathrm{Isom}(\widehat{\mathbf{Z}}, -)$ gives an equivalence from the groupoid of rank 1 local $\widehat{\mathbf{Z}}$-systems on $\mathbf{HK}_{\text{ét}}$ to the groupoid of $\widehat{\mathbf{Z}}^\times$-torsors on $\mathbf{HK}_{\text{ét}}$.

It follows that $\mathbf{HK}_{\mathrm{or}}$ gives rise to a rank 1 local $\mathbf{Z}$-system on $\mathbf{HK}_{\text{ét}}$. Equations (4.3) and (4.4) yield injective morphisms of sheaves $D \to \det \mathrm{R}^2_{\text{ét}}\, f_*\, \widehat{\mathbf{Z}}(1)$ and $D|_{\mathbf{HK}_{\mathbf{C}}} \to \det \mathrm{R}^2 f_{\mathbf{C},*}\, \mathbf{Z}(1)$. The commutativity of the diagram in (4.5) shows that the diagram in (4.2) commutes, proving the lemma. □

Corollary 4.3.8. *Let S be a $\mathbf{Q}$-scheme, $\overline{s}$ a geometric point of S, and X/S a smooth proper morphism of schemes whose fibers are hyperkähler varieties with $b_2 > 3$, endowed with a polarization $\lambda \in \mathrm{Pic}_{X/S}(S)$. For every prime number ℓ, consider the monodromy representation $\rho_\ell\colon \pi_1^{\text{ét}}(S, \overline{s}) \to \mathrm{O}(\mathrm{H}^2_{\text{ét}}(X_{\overline{s}}, \mathbf{Z}_\ell(1)))$. Then the composition $\det \rho_\ell\colon \pi_1^{\text{ét}}(S, \overline{s}) \to \{\pm1\}$ is independent of ℓ.*

Remark 4.3.9. When the base scheme is normal, and the fibers of f are K3 surfaces, this result holds in mixed characteristic, and without the existence of a polarization. We sketch a proof. By first spreading out and using Chebotarev density we reduce to the case of a K3 surface X over a finite field. Over a finite field, the Weil conjectures [D3, Théorème 1.3] imply that the determinant of the Frobenius on H^2 can be expressed in terms of the zeta function of X, which is independent of ℓ.

Saito uses this argument to prove an analogous result for the middle cohomology of any even-dimensional proper smooth variety, see [S2, Lemma 3.2].

4.4 Shimura stacks

In this section, we introduce Shimura stacks (following [T1]), and we give a modular interpretation of orthogonal Shimura stacks over $\mathbf{C}$ in terms of variations of $\mathbf{Z}$-Hodge structures.

4.4.1 General Shimura stacks

Let (G, X) be a Shimura datum with reflex field E. As in Chapter 2, we assume that $Z(\mathbf{Q})$ is discrete in $G(\mathbf{A}_\mathrm{f})$, where Z denotes the center of G. Let $\mathcal{K}$ be a profinite group, and let $i\colon \mathcal{K} \to G(\mathbf{A}_\mathrm{f})$ be a continuous homomorphism with finite kernel and open image (for example, $\mathcal{K} \subseteq G(\mathbf{A}_\mathrm{f})$ a compact open subgroup).

We define the **Shimura stack** $\mathrm{Sh}_\mathcal{K}[G, X]$ as follows. Let $\mathcal{K}' \subseteq \mathcal{K}$ be an open normal subgroup such that $i|_{\mathcal{K}'}\colon \mathcal{K}' \to G(\mathbf{A}_\mathrm{f})$ is injective and has neat image (see [M4] for the definition of a neat compact open subgroup of $G(\mathbf{A}_\mathrm{f})$). Then the Shimura *variety* $\mathrm{Sh}_{i(\mathcal{K}')}(G, X)$ is smooth and defined over E. Moreover, the finite group $\mathcal{K} / \mathcal{K}'$ acts on $\mathrm{Sh}_{i(\mathcal{K}')}(G, X)$ via right multiplication, and we let $\mathrm{Sh}_\mathcal{K}[G, X]$ be the quotient stack

$$\mathrm{Sh}_\mathcal{K}[G, X] := \left[\mathrm{Sh}_{i(\mathcal{K}')}(G, X)/(\mathcal{K} / \mathcal{K}')\right].$$

Now $\mathrm{Sh}_\mathcal{K}[G, X]$ is a smooth separated Deligne-Mumford stack over E, whose coarse moduli space $\mathrm{Sh}_\mathcal{K}(G, X)$ is isomorphic to the Shimura variety $\mathrm{Sh}_{i(\mathcal{K})}(G, X)$. The $G(\mathbf{A}_\mathrm{f})$-action on $\mathrm{Sh}(G, X)$ endows it with the structure of a $\mathcal{K}$-torsor on $\mathrm{Sh}_\mathcal{K}[G, X]_\text{ét}$.

Example 4.4.1. Let (G, X) be the Siegel Shimura datum associated with a symplectic $\mathbf{Q}$-vector space of dimension 2, that is, $(G, X) = (\mathrm{GL}_2, \mathcal{H})$. For $\mathcal{K} = \mathrm{GL}_2(\widehat{\mathbf{Z}})$, the stack $\mathrm{Sh}_\mathcal{K}[G, X]$ is equivalent to the moduli stack of elliptic curves over $\mathbf{Q}$. More generally, for $(G, X) = (\mathrm{GSp}_{2g}, \mathcal{H})$ and $\mathcal{K} = \mathrm{GSp}_{2g}(\widehat{\mathbf{Z}})$, the stack $\mathrm{Sh}_\mathcal{K}[G, X]$ is equivalent to the moduli stack of principally polarized abelian varieties of dimension g over $\mathbf{Q}$.

Lemma 4.4.2. *Let S be a smooth separated $\mathbf{C}$-scheme. Then the functor*

$$\mathrm{Hom}(S, \mathrm{Sh}_\mathcal{K}[G, X]_\mathbf{C}) \longrightarrow \mathrm{Hom}(S^\mathrm{an}, \mathrm{Sh}_\mathcal{K}[G, X]^\mathrm{an}_\mathbf{C})$$

given by analytification is an equivalence of groupoids.

Proof. First suppose $\mathcal{K}$ is a neat compact open subgroup of $G(\mathbf{A}_\mathrm{f})$, so that $\mathrm{Sh}_\mathcal{K}[G, X] = \mathrm{Sh}_\mathcal{K}(G, X)$. Then the lemma is well known and a consequence of [M4, Lemma 5.13] and Borel's theorem [M4, Theorem 3.14].

For more general $i\colon \mathcal{K} \to G(\mathbf{A}_\mathrm{f})$, let $\mathcal{K}'$ be a open normal subgroup of $\mathcal{K}$ such that $i|_{\mathcal{K}'}$ is injective, and such that $i(\mathcal{K}')$ is neat. Then $\mathrm{Sh}_\mathcal{K}[G, X]^\mathrm{an}_\mathbf{C}$ is the quotient stack

$$\left[\mathrm{Sh}_{i(\mathcal{K}')}(G, X)^\mathrm{an}_\mathbf{C}/(\mathcal{K} / \mathcal{K}')\right],$$

so a morphism $\psi\colon S^\mathrm{an} \to \mathrm{Sh}_\mathcal{K}[G, X]^\mathrm{an}_\mathbf{C}$ corresponds to a $(\mathcal{K} / \mathcal{K}')$-torsor P on S^an and a $(\mathcal{K} / \mathcal{K}')$-equivariant holomorphic map $\varphi\colon P \to \mathrm{Sh}_{i(\mathcal{K}')}(G, X)^\mathrm{an}_\mathbf{C}$. By [G2, Corollaire XII.5.2], the torsor P is the analytification of a $(\mathcal{K}/\mathcal{K}')$-torsor P_{alg} on S. Moreover, by the case of the lemma for neat compact open subgroups of $G(\mathbf{A}_{\mathrm{f}})$, the map φ is the analytification of a $(\mathcal{K}/\mathcal{K}')$-equivariant morphism $\varphi_{\mathrm{alg}}\colon P \to \mathrm{Sh}_{i(\mathcal{K}')}(G,X)_{\mathbf{C}}$. It follows that ψ is the analytification of a morphism $S \to \mathrm{Sh}_{\mathcal{K}}[G,X]_{\mathbf{C}}$. □

The analytification of $\mathrm{Sh}_{\mathcal{K}}[G,X]_{\mathbf{C}}$ can be identified with the quotient stack

$$[G(\mathbf{Q})\backslash X \times G(\mathbf{A}_{\mathrm{f}})/\mathcal{K}]$$

Its groupoid of $\mathbf{C}$-points has as objects pairs (h,g), with $h \in X$ and $g \in G(\mathbf{A}_{\mathrm{f}})$. A morphism $(h,g) \to (h',g')$ consists of $\gamma \in G(\mathbf{Q})$ and $k \in \mathcal{K}$ with $\gamma h = h'$ and $\gamma g i(k) = g'$.

Let V be a finite-dimensional $\mathbf{Q}$-vector space, endowed with a homomorphism $\rho\colon G \to \mathrm{GL}(V)$ and a continuous linear right $\mathcal{K}$-action, and assume that these two actions commute. Then the quotient stack

$$\Big[G(\mathbf{Q})\backslash X \times V \times G(\mathbf{A}_{\mathrm{f}})/\mathcal{K}\Big]$$

is a variation of $\mathbf{Q}$-Hodge structures on $\mathrm{Sh}_{\mathcal{K}}[G,X]^{\mathrm{an}}_{\mathbf{C}}$. Its fiber over a point $(h,g) \in \mathrm{Sh}_{\mathcal{K}}[G,X]^{\mathrm{an}}_{\mathbf{C}}$ is V, endowed with the Hodge structure ρh.

Now consider a full $\widehat{\mathbf{Z}}$-lattice $L \subseteq V \otimes \mathbf{A}_{\mathrm{f}}$ such that for all $v \in L$ and $k \in \mathcal{K}$ we have $i(k)vk^{-1} \in L$. Then the quotient

$$\Big[G(\mathbf{Q})\backslash X \times \{(v,g) \in V \times G(\mathbf{A}_{\mathrm{f}}) \mid v \in g(L) \cap V\}/\mathcal{K}\Big] \tag{4.6}$$

is a variation of $\mathbf{Z}$-Hodge structures on $\mathrm{Sh}_{\mathcal{K}}[G,X]^{\mathrm{an}}_{\mathbf{C}}$. The stalk over a point (h,g) of $\mathrm{Sh}_{\mathcal{K}}[G,X]^{\mathrm{an}}_{\mathbf{C}}$ is the finitely generated free abelian group $g(L)\cap V$, endowed with the Hodge structure given by ρh.

4.4.2 Orthogonal Shimura stacks

We now apply the constructions of Section 4.4.1 to give a modular interpretation of orthogonal Shimura stacks in terms of variations of $\mathbf{Z}$-Hodge structures. We will do this for various choices of $\mathcal{K}$, which arise naturally from the moduli stacks of hyperkähler varieties that we consider in later sections.

Let (Λ_0, b_0) be a $\mathbf{Z}$-lattice of signature $(3,n)$ with $n \geq 1$, let $\lambda_0 \in \Lambda_0$ be an element with $b_0(\lambda_0,\lambda_0) > 0$, and let $\omega_0\colon \mathbf{Z} \to \det \Lambda_0$ be an isomorphism of abelian groups. Define V to be the signature $(2,n)$ quadratic space $(\mathbf{Q}\,\lambda_0)^\perp \subseteq \Lambda_0 \otimes \mathbf{Q}$, and let (SO,Ω) be the Shimura datum associated with V as in Section 2.3. Define $\mathcal{K}_0$ to be the profinite group

$$\mathcal{K}_0 := \big\{g \in \mathrm{SO}(\Lambda_0)(\widehat{\mathbf{Z}}) \mid g(\lambda_0) = \lambda_0\big\},$$

which we endow with the injective map $i\colon \mathcal{K}_0 \to \mathrm{SO}(\mathbf{A}_{\mathrm{f}})$ sending g to the restriction of $g \otimes \mathbf{A}_{\mathrm{f}}$ to $V \otimes \mathbf{A}_{\mathrm{f}} \subseteq \Lambda_0 \otimes \mathbf{A}_{\mathrm{f}}$. The image $i(\mathcal{K}_0)$ is a compact open subgroup of $\mathrm{SO}(\mathbf{A}_{\mathrm{f}})$, so we have a Shimura stack $\mathrm{Sh}_{\mathcal{K}_0}[\mathrm{SO},\Omega]$.

Let SO act on $\Lambda_0 \otimes \mathbf{Q} = V \oplus \mathbf{Q}\,\lambda_0$ by having g act as $g \oplus \mathrm{id}$, and let $\mathcal{K}_0$ act trivially on $\Lambda_0 \otimes \mathbf{Q}$. Then $\Lambda_0 \otimes \mathbf{Q}$ and the $\widehat{\mathbf{Z}}$-sublattice $\Lambda_0 \otimes \widehat{\mathbf{Z}}$ of $\Lambda_0 \otimes \mathbf{A}_\mathrm{f}$ induce a variation of $\mathbf{Z}$-Hodge structures $\mathbb{A}$ on $\mathrm{Sh}_{\mathcal{K}_0}[\mathrm{SO}, \Omega]^{\mathrm{an}}_{\mathbf{C}}$, as in (4.6). The stalk of $\mathbb{A}$ over a point $(h, g) \in \mathrm{Sh}_{\mathcal{K}_0}[\mathrm{SO}, \Omega]^{\mathrm{an}}_{\mathbf{C}}$ is the finitely generated free abelian group $g(\Lambda_0 \otimes \widehat{\mathbf{Z}}) \cap \Lambda_0 \otimes \mathbf{Q}$, endowed with the Hodge structure given by h.

The pairing on Λ_0, the element $\lambda_0 \in \Lambda_0$, and the isomorphism $\omega_0 \colon \mathbf{Z} \to \det \Lambda_0$ induce the following morphisms of $\mathbf{Z}$-VHS:

- $b\colon \mathrm{Sym}^2 \mathbb{A} \longrightarrow \mathbf{Z}(0)$,
- $\lambda\colon \mathbf{Z}(0) \longrightarrow \mathbb{A}$,
- $\omega\colon \mathbf{Z}(0) \longrightarrow \det \mathbb{A}$.

The following lemma gives a universal property for the tuple $(\mathbb{A}, b, \lambda, \omega)$.

Lemma 4.4.3. *Let S be a complex analytic space. Pulling back the tuple $(\mathbb{A}, b, \lambda, \omega)$ induces an equivalence of groupoids from* $\mathrm{Hom}(S, \mathrm{Sh}_{\mathcal{K}_0}[\mathrm{SO}, \Omega]^{\mathrm{an}}_{\mathbf{C}})$ *to the groupoid of tuples $(\Lambda, b, \lambda, \omega)$ where*

- *Λ is a $\mathbf{Z}$-VHS on S,*
- *$b\colon \mathrm{Sym}^2 \Lambda \to \mathbf{Z}(0)$ is a morphism of $\mathbf{Z}$-VHS making the stalks of Λ K3-type Hodge lattices of signature $(3, n)$,*
- *λ is a positive global section of Λ of type $(0, 0)$,*
- *$\omega\colon \mathbf{Z}(0) \to \det \Lambda$ is an isomorphism of $\mathbf{Z}$-VHS,*

such that for every $s \in S$, there exists an isometry $\Lambda_s \otimes \widehat{\mathbf{Z}} \to \Lambda_0 \otimes \widehat{\mathbf{Z}}$ mapping λ_s and ω_s to λ_0 and ω_0, respectively.

Proof. We construct a quasi-inverse of the natural functor from $\mathrm{Hom}(S, \mathrm{Sh}_{\mathcal{K}_0}[\mathrm{SO}, \Omega])$ to the groupoid of tuples $(\Lambda, b, \lambda, \omega)$. Without loss of generality we may assume that S is connected.

We first show that the fibers of $\Lambda \otimes \mathbf{Q}$ are isomorphic to the tuple $(\Lambda_0 \otimes \mathbf{Q}, b_0, \lambda_0, \omega_0)$. Let $s \in S$, and let $\psi_s \colon \Lambda_0 \otimes \widehat{\mathbf{Z}} \to \Lambda_s \otimes \widehat{\mathbf{Z}}$ be an isometry mapping λ_0 to λ_s and ω_0 to ω_s. The quadratic spaces $\Lambda_s \otimes \mathbf{Q}$ and $\Lambda_0 \otimes \mathbf{Q}$ have the same signature, and ψ_s induces an isometry $\Lambda_s \otimes \mathbf{A}_\mathrm{f} \cong \Lambda_0 \otimes \mathbf{A}_\mathrm{f}$, so by the Hasse-Minkowski theorem [S3, Chapter IV, Theorem 9], there is an isometry $\varphi_s \colon \Lambda_s \otimes \mathbf{Q} \to \Lambda_0 \otimes \mathbf{Q}$. By an argument similar to that in the proof of Lemma 2.3.11, we can use the existence of ψ_s to modify φ_s so as to ensure that it maps λ_s to λ_0 and ω_s to ω_0.

Let $\mathcal{K} \subseteq \mathcal{K}_0$ be a neat open normal subgroup, so that $H := \mathcal{K}_0 \,/\, \mathcal{K}$ is a finite group, and $\mathrm{Sh}_{\mathcal{K}}[\mathrm{SO}, \Omega]_{\mathbf{C}}$ is equal to the Shimura variety $\mathrm{Sh}_{\mathcal{K}}(\mathrm{SO}, \Omega)_{\mathbf{C}}$. Define S' to be the quotient sheaf

$$\mathrm{Isom}\Big((\Lambda_0 \otimes \widehat{\mathbf{Z}}, b_0, \lambda_0, \omega_0), (\Lambda \otimes \widehat{\mathbf{Z}}, b, \lambda, \omega)\Big) \,/\, \mathcal{K}$$

on S. Then S' is an H-torsor on S. We will construct an H-equivariant map $S' \to \mathrm{Sh}_{\mathcal{K}}(\mathrm{SO}, \Omega)_{\mathbf{C}}$. Since $\mathrm{Sh}_{\mathcal{K}_0}[\mathrm{SO}, \Omega]$ is by definition the quotient stack $[\mathrm{Sh}_{\mathcal{K}}(\mathrm{SO}, \Omega)/H]$, this induces a morphism $S \to \mathrm{Sh}_{\mathcal{K}_0}[\mathrm{SO}, \Omega]_{\mathbf{C}}$.

Let $U \subseteq S'$ be a connected open set on which the local system underlying Λ is constant, and let $\varphi\colon \Lambda_U \otimes \mathbf{Q} \to \Lambda_0 \otimes \mathbf{Q}$ be an isometry mapping λ to λ_0 and ω to ω_0. By the constancy of Λ, we can also find an isometry $\psi\colon \Lambda_0 \otimes \widehat{\mathbf{Z}} \to \Lambda \otimes \widehat{\mathbf{Z}}$ representing the universal section of

$$\mathrm{Isom}\Big((\Lambda_0 \otimes \widehat{\mathbf{Z}}, b_0, \lambda_0, \omega_0), (\mathbb{A} \otimes \widehat{\mathbf{Z}}, b, \lambda, \omega)\Big)/\,\mathcal{K}$$

over S'.

By definition of variations of Hodge structures, there is a holomorphic map $f\colon U \to X$ mapping $s \in U$ to the image under φ_s of the Hodge structure on $\Lambda_s \otimes \mathbf{Q}$. Note that for any $s \in U$, the composition $\varphi_s \circ \psi_s$ defines an element of $\mathrm{SO}(\mathbf{A}_{\mathrm{f}})$. The constancy of Λ_U and the connectedness of U imply that this element does not depend on the choice of s, and we will denote it with g. Now define $U \to \mathrm{Sh}_{\mathcal{K}}(\mathrm{SO}, \Omega)_{\mathbf{C}}$ by mapping a point $s \in U$ to $(f(s), g)$. Since

$$\mathrm{Sh}_{\mathcal{K}}(\mathrm{SO}, \Omega)_{\mathbf{C}} = \mathrm{SO}(\mathbf{Q})\backslash X \times \mathrm{SO}(\mathbf{A}_{\mathrm{f}})/\,\mathcal{K},$$

this map does not depend on the choice of φ and ψ, and it is clearly H-equivariant. Moreover, [M4, Lemma 5.13] gives a decomposition of $\mathrm{Sh}_{\mathcal{K}}(\mathrm{SO}, \Omega)_{\mathbf{C}}$ into quotients of the form $\Gamma\backslash X$, with Γ a discrete group acting properly discontinuously on X. This decomposition can be used to show that $U \to \mathrm{Sh}_{\mathbf{C}}(\mathrm{SO}, \Omega)_{\mathbf{C}}$ is holomorphic.

Applying this construction to an open cover of S' on which the local system underlying Λ is constant gives an H-equivariant holomorphic map $S' \to \mathrm{Sh}_{\mathcal{K}}[\mathrm{SO}, \Omega]_{\mathbf{C}}$, and hence a morphism $S \to \mathrm{Sh}_{\mathcal{K}_0}[\mathrm{SO}, \Omega]_{\mathbf{C}}$. ☐

Consider an open subgroup $\mathcal{K} \subseteq \mathcal{K}_0$. We will generalize Lemma 4.4.3 to give a modular interpretation of $\mathrm{Sh}_{\mathcal{K}}[\mathrm{SO}, \Omega]^{\mathrm{an}}_{\mathbf{C}}$.

The group $\mathcal{K}$ acts from the right on the isomorphism sheaf

$$I := \mathrm{Isom}\left((\Lambda_0 \otimes \widehat{\mathbf{Z}}, b_0, \lambda_0, \omega_0), (\mathbb{A} \otimes \widehat{\mathbf{Z}}, b, \lambda, \omega)\right)$$

on $\mathrm{Sh}_{\mathcal{K}_0}[\mathrm{SO}, \Omega]^{\mathrm{an}}_{\mathbf{C}}$. In the proof of the following lemma we will construct a section α of the quotient sheaf $I/\,\mathcal{K}$ over $\mathrm{Sh}_{\mathcal{K}}[\mathrm{SO}, \Omega]^{\mathrm{an}}_{\mathbf{C}}$. We will refer to α as the **universal level-$\mathcal{K}$ structure** on $\mathbb{A}$ over $\mathrm{Sh}_{\mathcal{K}}[\mathrm{SO}, \Omega]^{\mathrm{an}}_{\mathbf{C}}$. The following lemma says that $\mathrm{Sh}_{\mathcal{K}}[\mathrm{SO}, \Omega]^{\mathrm{an}}_{\mathbf{C}}$ is a moduli stack for $\mathbf{Z}$-VHS endowed with a level $\mathcal{K}$-structure.

Lemma 4.4.4. *Let S be a complex analytic space. There exists a section α of the quotient sheaf $I/\,\mathcal{K}$ over $\mathrm{Sh}_{\mathcal{K}}[\mathrm{SO}, \Omega]^{\mathrm{an}}_{\mathbf{C}}$ such that pulling back the tuple $(\mathbb{A}, b, \lambda, \omega, \alpha)$ induces an equivalence of groupoids from $\mathrm{Hom}(S, \mathrm{Sh}_{\mathcal{K}}[\mathrm{SO}, \Omega]^{\mathrm{an}}_{\mathbf{C}})$ to the groupoid of tuples $(\Lambda, b, \lambda, \omega, \alpha)$ where $(\Lambda, b, \lambda, \omega)$ is as in Lemma 4.4.3, and α is a global section of the quotient sheaf*

$$\mathrm{Isom}\left((\Lambda_0 \otimes \widehat{\mathbf{Z}}, b_0, \lambda_0, \omega_0), (\Lambda \otimes \widehat{\mathbf{Z}}, b, \lambda, \omega)\right)/\,\mathcal{K}\,.$$

Proof. We will only construct the universal level $\mathcal{K}$-structure on the restriction of $\mathbb{A}$ to $\mathrm{Sh}_{\mathcal{K}}[\mathrm{SO}, \Omega]^{\mathrm{an}}_{\mathbf{C}}$. The rest of the proof is similar to that of Lemma 4.4.3, and therefore omitted.

Let Ω^+ be a connected component of Ω, and let $\mathrm{SO}(\mathbf{Q})_+ \subseteq \mathrm{SO}(\mathbf{Q})$ be the stabilizer of this component with respect to the action of $\mathrm{SO}(\mathbf{Q})$ on $\pi_0(\Omega)$. Let $\mathcal{C}$ be a set of representatives of the quotient set $\mathrm{SO}(\mathbf{Q})_+ \backslash \mathrm{SO}(\mathbf{A}_\mathrm{f}) / \mathcal{K}$, and for $g \in \mathrm{SO}(\mathbf{A}_\mathrm{f})$, let Γ_g be the group $\mathrm{SO}(\mathbf{Q})_+ \cap g\,\mathcal{K}\,g^{-1}$. Then the stack $\mathrm{Sh}_\mathcal{K}[\mathrm{SO},\Omega]^{\mathrm{an}}_\mathbf{C}$ is equivalent to the disjoint union

$$\coprod_{g\in\mathcal{C}} \left[\Gamma_g \backslash \Omega^+\right], \tag{4.7}$$

as can be seen in [M4, Lemma 5.13]. Since Ω^+ is simply connected, the analytic stack $[\Gamma_g\backslash\Omega^+]$ is connected, and its fundamental group is Γ_g.

For $g \in \mathcal{C}$, let Λ_g be the $\mathbf{Z}$-lattice $g(\Lambda_0 \otimes \widehat{\mathbf{Z}}) \cap \Lambda_0 \otimes \mathbf{Q}$. Then $\lambda_0 \in \Lambda_g$, and ω_0 induces an isomorphism $\omega_g\colon \mathbf{Z} \to \det\Lambda_g$. The pullback of $\mathbb{A}$ to $[\Gamma_g\backslash\Omega^+]$ is $[\Gamma_g\backslash(\Omega^+ \times \Lambda_g)]$. To give a section of $I/\mathcal{K}$ over $[\Gamma_g\backslash\Omega^+]$ is therefore equivalent to giving an isometry $\psi\colon \Lambda_0 \otimes \widehat{\mathbf{Z}} \to \Lambda_g \otimes \widehat{\mathbf{Z}}$ preserving λ_0 and mapping ω_0 to ω_g such that for every $\gamma \in \Gamma_g$, there exists a $k \in \mathcal{K}$ with $\gamma\psi = \psi k$. Since $\Lambda_g \otimes \widehat{\mathbf{Z}} = g(\Lambda_0 \otimes \widehat{\mathbf{Z}})$, the choice $\psi = g$ gives an isometry satisfying these conditions.

It can be checked that this defines a section

$$\alpha\colon \mathrm{Sh}_\mathcal{K}[\mathrm{SO},\Omega]_\mathbf{C} \longrightarrow I/\mathcal{K},$$

completing the proof of the lemma. □

We now introduce a compact open subgroup $\mathcal{K}_F$ of $\mathrm{SO}(\mathbf{A}_\mathrm{f})$ that will play an important role in our treatment of the moduli stack of polarized hyperkähler varieties which are deformation equivalent to a Hilbert scheme of points on a K3 surface. The group $\mathcal{K}_0$ acts on the discriminant group $\Delta(\Lambda_0) := \Lambda_0^\vee/\Lambda_0$. This induces an action on the quotient set $F(\Lambda_0) := \Delta(\Lambda_0)/\{\pm 1\}$, where -1 acts as $-\mathrm{id}_{\Delta(\Lambda_0)}$. We define the group $\mathcal{K}_F \subseteq \mathcal{K}_0$ as

$$\mathcal{K}_F := \left\{ g \in \mathcal{K}_0 \mid \Delta(g) = \pm\,\mathrm{id}_{\Delta(\Lambda_0)} \right\}.$$

The universal level-$\mathcal{K}_F$ structure α on the restriction of $\mathbb{A}$ to $\mathrm{Sh}_{\mathcal{K}_F}[\mathrm{SO},\Omega]^{\mathrm{an}}_\mathbf{C}$ induces an isomorphism of sheaves of finite sets $\overline{\alpha}\colon F(\Lambda_0) \to F(\mathbb{A})$ on $\mathrm{Sh}_{\mathcal{K}_F}[\mathrm{SO},\Omega]_\mathbf{C}$.

The next lemma follows almost immediately from Lemma 4.4.4.

Lemma 4.4.5. *Let S be a complex analytic space. Pulling back the tuple $(\mathbb{A}, b, \lambda, \omega, \overline{\alpha})$ induces an equivalence of groupoids from $\mathrm{Hom}(S, \mathrm{Sh}_{\mathcal{K}_F}[\mathrm{SO},\Omega]^{\mathrm{an}}_\mathbf{C})$ to the groupoid of tuples $(\Lambda, b, \lambda, \omega, \overline{\alpha})$ where $(\Lambda, b, \lambda, \omega)$ is as in Lemma 4.4.3, and $\overline{\alpha}\colon F(\Lambda_0) \to F(\Lambda)$ is an isomorphism of sheaves of sets on S such that for every $s \in S$, there exists an isometry $\psi\colon \Lambda_0 \otimes \widehat{\mathbf{Z}} \to \Lambda_s \otimes \widehat{\mathbf{Z}}$ mapping λ_0 and ω_0 to λ and ω, and such that $\overline{\alpha}_s$ is induced by ψ.*

Now suppose that the rank of Λ_0 is even. Taelman showed in [T1] that we can find a Shimura stack parametrizing Hodge lattices of K3 type in the same genus as Λ_0, without the need for adding an orientation.

Let $\mathcal{K}$ be the profinite group

$$\left\{ g \in \mathrm{O}(\Lambda_0)(\widehat{\mathbf{Z}}) \mid g(\lambda_0) = \lambda_0 \ \text{ and } \ \det(g) \in \{\pm 1\} \right\}. \tag{4.8}$$

The requirement that $\det g \in \{\pm 1\}$ says that for every prime p the determinant $\det g_p \in \{\pm 1\} \subseteq \mathbf{Z}_p^\times$ is the same. That is, $\det g_p$ is either 1 for all p or -1 for all p. Consider the continuous homomorphism

$$i\colon \mathcal{K} \longrightarrow \mathrm{SO}(\mathbf{A}_\mathrm{f}), \quad g \longmapsto \det(g)g|_{V\otimes \mathbf{A}_\mathrm{f}},$$

where as before V denotes the orthogonal complement of λ_0 in $\Lambda_0 \otimes \mathbf{Q}$. Note that the determinant of $\det(g)g|_{V\otimes\mathbf{A}_\mathrm{f}}$ is 1 by the evenness of $\mathrm{rk}\,\Lambda_0$, so that i indeed lands in $\mathrm{SO}(\mathbf{A}_\mathrm{f})$. Moreover, i has open image and a finite kernel (of order ≤ 2), so this gives rise to a Shimura stack $\mathrm{Sh}_\mathcal{K}[\mathrm{SO},\Omega]$.

We construct a universal $\mathbf{Z}$-VHS on $\mathrm{Sh}_\mathcal{K}[\mathrm{SO},\Omega]_\mathbf{C}$. As above, let $g \in \mathrm{SO}$ act on $\Lambda_0 \otimes \mathbf{Q}$ as $g \oplus \mathrm{id}$, and let $\mathcal{K}$ act from the right on $\Lambda_0 \otimes \mathbf{Q}$ via the determinant on V, and as the identity on $\mathbf{Q}\,\lambda_0$. Then these actions and the $\widehat{\mathbf{Z}}$-lattice $\Lambda_0 \otimes \widehat{\mathbf{Z}}$ in $\Lambda_0 \otimes \mathbf{A}_\mathrm{f}$ give rise to a $\mathbf{Z}$-VHS $\mathbb{A}$ as in (4.6). As before, the pairing $b_0\colon \mathrm{Sym}^2\,\Lambda_0 \to \mathbf{Z}$ and the element λ_0 give rise to

- a morphism of $\mathbf{Z}$-VHS $b_0\colon \mathrm{Sym}^2\,\mathbb{A} \to \mathbf{Z}(0)$,
- a global section λ of $\mathbb{A}$ of type $(0,0)$ satisfying $b(\lambda,\lambda) > 0$.

The following lemma states that the tuple $(\mathbb{A}, b, \lambda)$ is universal.

Lemma 4.4.6. *Let S be a complex analytic space. Pulling back the tuple $(\mathbb{A}, b, \lambda)$ induces an equivalence of groupoids from $\mathrm{Hom}(S, \mathrm{Sh}_\mathcal{K}[\mathrm{SO},\Omega]_\mathbf{C}^{\mathrm{an}})$ to the groupoid of tuples (Λ, b, λ) where*

- *Λ is a $\mathbf{Z}$-VHS on S,*
- *$b\colon \mathrm{Sym}^2\,\Lambda \to \mathbf{Z}(0)$ is a morphism of $\mathbf{Z}$-VHS making the stalks of Λ K3-type Hodge lattices of signature $(3,n)$,*
- *λ is a global section of Λ of type $(0,0)$ satisfying $b(\lambda,\lambda) > 0$,*

such that for every $s \in S$, there exists an isometry $\psi\colon \Lambda_s \otimes \widehat{\mathbf{Z}} \to \Lambda_0 \otimes \widehat{\mathbf{Z}}$ mapping λ_s to λ_0, and mapping the rank 1 free abelian group $\det(\Lambda_s)$ to $\det(\Lambda_0)$.

Proof. We prove the lemma for the case $S = \mathrm{Spec}(\mathbf{C})$. The additional details needed to prove the lemma for more general complex analytic spaces are similar to the proof of Lemma 4.4.3, and are therefore omitted.

The statement of the lemma gives a functor F from the groupoid

$$\mathrm{Sh}_\mathcal{K}[\mathrm{SO},\Omega](\mathbf{C}) = [\mathrm{SO}(\mathbf{Q})\backslash\Omega \times \mathrm{SO}(\mathbf{A}_\mathrm{f})/\,\mathcal{K}]$$

to the groupoid of tuples (Λ, b, λ) on $\mathrm{Spec}(\mathbf{C})$. On objects, F is defined by mapping an object (h,g) to the $\mathbf{Z}$-lattice $\Lambda_g := g(\Lambda_0 \otimes \widehat{\mathbf{Z}}) \cap \Lambda_0 \otimes \mathbf{Q}$ endowed with the Hodge structure h, and the positive type $(0,0)$ element λ_0. A morphism from (h,g) to (h',g') in consists of $\gamma \in \mathrm{SO}(\mathbf{Q})$ and $k \in \mathcal{K}$ with $\gamma h = h'$ and $\gamma g k = g'$. Then F maps (γ, k) to the restriction of $(\det(k)\gamma) \oplus \mathrm{id}_{\mathbf{Q}\,\lambda_0}$ to Λ_g.

To see that F is faithful, suppose we have (γ_0, k_0) and (γ_1, k_1) in $\mathrm{SO}(\mathbf{Q}) \times \mathcal{K}$ giving morphisms from (h,g) to (h',g'). If $F(\gamma_0,k_0) = F(\gamma_1,k_1)$, then in particular

$\det(k_0)\gamma_0 = \det(k_1)\gamma_1$. Since $\det(\gamma_0) = \det(\gamma_1) = 1$, this implies that $\gamma_0 = \gamma_1$, and $\det(k_0) = \det(k_1)$. Moreover, we have $\gamma_0 g i(k_0) = g' = \gamma_1 g i(k_1)$, so that $i(k_0) = i(k_1)$. Since

$$k_0 \otimes \mathbf{A}_\mathrm{f} = (\det(k_0)i(k_0)) \oplus \mathrm{id}_{\mathbf{A}_\mathrm{f}\,\lambda_0} = (\det(k_1)i(k_1)) \oplus \mathrm{id}_{\mathbf{A}_\mathrm{f}\,\lambda_0} = k_1 \otimes \mathbf{A}_\mathrm{f},$$

we obtain $k_0 = k_1$.

For the fullness of F, let (h, g) and (h', g') be points of $\mathrm{Sh}_\mathcal{K}[\mathrm{SO}, \Omega]^{\mathrm{an}}_\mathbf{C}$, and suppose we have a Hodge isometry $\varphi\colon \Lambda_g \to \Lambda_{g'}$ preserving λ_0. Since $\Lambda_g \otimes \mathbf{Q}$ and $\Lambda_{g'} \otimes \mathbf{Q}$ are equal to $\Lambda_0 \otimes \mathbf{Q}$, the isometry $\gamma := \det(\varphi)\varphi|_{\mathbf{Q}\,\lambda_0^\perp}$ is an element of $\mathrm{SO}(\mathbf{Q})$. Note that $\Lambda_g \otimes \widehat{\mathbf{Z}} = g(\Lambda_0 \otimes \widehat{\mathbf{Z}})$, and $\Lambda_{g'} = g'(\Lambda_0 \otimes \widehat{\mathbf{Z}})$, so we define k to be the composition

$$\Lambda_0 \otimes \widehat{\mathbf{Z}} \xrightarrow{g} g\Big(\Lambda_0 \otimes \widehat{\mathbf{Z}}\Big) \xrightarrow{\varphi\otimes\widehat{\mathbf{Z}}} g'\Big(\Lambda_0 \otimes \widehat{\mathbf{Z}}\Big) \xrightarrow{(g')^{-1}} \Lambda_0 \otimes \widehat{\mathbf{Z}}\,.$$

Then k satisfies $\det(k) = \det(\varphi) \in \{\pm 1\}$ so that $k \in \mathcal{K}$. It can be verified that (γ, k) is a morphism from (h, g) to (h', g') with $\varphi = F(\gamma, k)$.

To prove the essential surjectivity, consider a tuple (Λ, b, λ) on $\mathrm{Spec}(\mathbf{C})$. Let $\psi\colon \Lambda_0 \otimes \widehat{\mathbf{Z}} \to \Lambda \otimes \widehat{\mathbf{Z}}$ be an isometry mapping λ_0 to λ and $\det(\Lambda_0)$ to $\det(\Lambda)$. It follows that if we endow $\det(\Lambda_0)$ and $\det(\Lambda)$ with the pairings induced by b_0 and b, then the lattices $\det(\Lambda_0)$ and $\det(\Lambda)$ have the same discriminant $d \in \mathbf{Z}$.

By the existence of ψ and the fact that Λ and Λ_0 have the same signature, the Hasse-Minkowski theorem [S3, Chapter IV, Theorem 9] gives an isometry $\varphi\colon \Lambda \otimes \mathbf{Q} \to \Lambda_0\,\mathbf{Q}$ mapping λ to λ_0. The sublattices $\det(\Lambda_0)$ and $\varphi\det(\Lambda)$ have the same discriminant d, so it follows that φ maps $\det(\Lambda)$ to $\det(\Lambda_0)$.

Define $h \in \Omega$ as the image under φ of the Hodge structure on $\mathbf{Q}\,\lambda^\perp \subseteq \Lambda \otimes \mathbf{Q}$. Consider the composition $g := \varphi|_{\mathbf{A}_\mathrm{f}\,\lambda^\perp}\psi|_{V\otimes\mathbf{A}_\mathrm{f}}\colon V\otimes\mathbf{A}_\mathrm{f} \to V\otimes\mathbf{A}_\mathrm{f}$, where V denotes the orthogonal complement of λ_0 in $\Lambda_0 \otimes \mathbf{Q}$. Since φ and ψ map the $\mathbf{Z}$-lattices $\det(\Lambda_0)$ and $\det(\Lambda)$ into each other, $\det g$ is an element of $\{\pm 1\}$. By composing φ with $-\mathrm{id}_V \oplus \mathrm{id}_{\mathbf{Q}\,\lambda_0}$ if necessary, we ensure that $\det(g) = 1$, so that $g \in \mathrm{SO}(\mathbf{A}_\mathrm{f})$. Note that even after this modification, $\varphi\colon \Lambda\otimes\mathbf{Q} \to \Lambda_0\otimes\mathbf{Q}$ is still a Hodge isometry mapping λ to λ_0.

With these choices it can be checked that φ induces a Hodge isometry $\Lambda \to \Lambda_g$ mapping λ to λ_0, where Λ_g is endowed with the Hodge structure given by h. This shows that F is essentially surjective, completing the proof of the lemma. □

4.5 Period maps

Let $\mathbf{M}\,/\,\mathbf{Q}$ be a connected component of the moduli stack $\mathbf{HK}_\mathrm{or}$ of oriented polarized hyperkähler varieties over $\mathbf{Q}$. In this section, we will associate to $\mathbf{M}$ an orthogonal Shimura stack $\mathrm{Sh}_{\mathcal{K}_0}[\mathrm{SO}, \Omega]$ over $\mathbf{Q}$, and we will construct an étale morphism $\mathbf{M}_\mathbf{C} \to \mathrm{Sh}_{\mathcal{K}_0}[\mathrm{SO}, \Omega]_\mathbf{C}$, called the **period map**. The main result of this section is that this period map descends to a morphism $\mathbf{M} \to \mathrm{Sh}_{\mathcal{K}_0}[\mathrm{SO}, \Omega]$, defined over $\mathbf{Q}$. This is a generalization of a result of Rizov ([R3, Theorem 3.16]). Our treatment closely follows the proof of that result given by Madapusi-Pera in [MP1].

Let $f\colon \mathfrak{X} \to \mathbf{M}$ be the universal hyperkähler variety, λ the universal polarization on $\mathfrak{X}$, and $\omega_{[4]}\colon \mathbf{Z}/4\mathbf{Z} \to \det \mathrm{R}^2_{\text{ét}} f_* \mu_4$ the universal orientation. As we saw in Theorem 4.3.6, $\omega_{[4]}$ gives rise to isomorphisms of local systems

$$\omega_{\text{ét}}\colon \widehat{\mathbf{Z}} \longrightarrow \det \mathrm{R}^2_{\text{ét}} f_* \widehat{\mathbf{Z}}(1)$$

on $\mathbf{M}_{\text{or},\text{ét}}$

$$\omega_{\text{an}}\colon \mathbf{Z} \longrightarrow \det \mathrm{R}^2 f_{\mathbf{C},*} \mathbf{Z}(1)$$

on $\mathbf{M}_{\text{or},\mathbf{C}}$.

Let $x_0 := (X_0, \lambda_{x_0}, \omega_{[4],x_0})$ be a $\mathbf{C}$-point of $\mathbf{M}$. Then $\Lambda_0 := \mathrm{H}^2(X_0, \mathbf{Z}(1))$ endowed with the BBF pairing b_0 is a $\mathbf{Z}$-lattice of signature $(3,n)$, for some $n \geq 1$, and $\lambda_0 := c_1(\lambda_{x_0})$ is a positive element of Λ_0. Moreover, $\omega_0 := \omega_{\text{an},x_0}$ is an isomorphism $\mathbf{Z} \to \det \Lambda_0$. Let V be the orthogonal complement of $\mathbf{Q}\,\lambda_0$ in $\Lambda_0 \otimes \mathbf{Q}$. Then as in Section 4.4.2, the tuple $(\Lambda_0, \lambda_0, b_0, \omega_0)$ gives rise to an orthogonal Shimura datum

$$(\mathrm{SO}, \Omega) := (\mathrm{SO}(V), \Omega_V),$$

and a compact open subgroup $\mathcal{K}_0 \subseteq \mathrm{SO}(\mathbf{A}_\mathrm{f})$ defined as

$$\mathcal{K}_0 := \left\{ g \in \mathrm{SO}(\Lambda_0)(\widehat{\mathbf{Z}}) \mid g(\lambda_0) = \lambda_0 \right\}.$$

Lemma 4.5.1. *Let $x = (X, \lambda_x, \omega_{[4],x})$ be a $\mathbf{C}$-point of $\mathbf{M}$. Then the tuple*

$$\left(\mathrm{H}^2(X, \mathbf{Z}(1)) \otimes \widehat{\mathbf{Z}},\, b_X,\, c_1(\lambda_x),\, \omega_{\text{an},x} \right)$$

is isomorphic to $(\Lambda_0 \otimes \widehat{\mathbf{Z}}, b_0, \lambda_0, \omega_0)$.

Proof. This is a consequence of the connectedness of $\mathbf{M}$ and the fact that the BBF pairing on $\mathrm{R}^2 f_{\mathbf{C},*} \mathbf{Z}(1)$ and the isomorphism of sheaves ω_{an} on $\mathbf{M}_{\mathbf{C}}$ extend to morphisms of local systems on $\mathbf{M}_{\text{ét}}$ over $\mathbf{Q}$, which is the content of Theorems 4.2.4 and 4.3.6. $\square$

On $\mathbf{M}_{\mathbf{C}}$, we now have the following Hodge-theoretic data:

- a $\mathbf{Z}$-VHS $\mathrm{R}^2 f_{\mathbf{C},*} \mathbf{Z}(1)$,
- the BBF pairing $b_{\text{an}}\colon \mathrm{Sym}^2 \mathrm{R}^2 f_{\mathbf{C},*} \mathbf{Z}(1) \to \mathbf{Z}(0)$, making the stalks of $\mathrm{R}^2 f_{\mathbf{C},*} \mathbf{Z}(1)$ K3-type Hodge lattices of signature $(3,n)$, by Propositions 4.1.7 and 4.1.8,
- $\lambda_{\text{an}} := c_1(\lambda|_{\mathbf{M}_{\mathbf{C}}})$ is a positive global section of $\mathrm{R}^2 f_{\mathbf{C},*} \mathbf{Z}(1)$ of type $(0,0)$ by Remark 4.1.3,
- the orientation $\omega_{\text{an}}\colon \mathbf{Z}(0) \to \det \mathrm{R}^2 f_{\mathbf{C},*} \mathbf{Z}(1)$ from Theorem 4.3.6, which is an isomorphism of $\mathbf{Z}$-VHS.

By Lemma 4.4.2, Lemma 4.4.3, and Lemma 4.5.1, these data gives rise to a morphism of complex Deligne-Mumford stacks

$$\mathbf{M}_{\mathbf{C}} \longrightarrow \mathrm{Sh}_{\mathcal{K}_0}[\mathrm{SO}, \Omega]_{\mathbf{C}},$$

known as the **period map**. Note that since the reflex field of (SO, Ω) is $\mathbf{Q}$ by Lemma 2.1.3, $\mathrm{Sh}_{\mathcal{K}_0}[\mathrm{SO}, \Omega]$ is a stack over $\mathbf{Q}$. We will show that the period map descends to a morphism defined over $\mathbf{Q}$.

First note that the tuple $(\mathrm{R}^2 f_{\mathbf{C},*} \widehat{\mathbf{Z}}(1),\, b_{\mathrm{an}},\, \lambda_{\mathrm{an}},\, \omega_{\mathrm{an}})$ on $\mathbf{M}_{\mathbf{C},\text{ét}}$ extends uniquely to a tuple on $\mathbf{M}_{\text{ét}}$ over $\mathbf{Q}$ consisting of

- the local $\widehat{\mathbf{Z}}$-system $\mathrm{R}^2_{\text{ét}} f_* \widehat{\mathbf{Z}}(1)$,
- the étale BBF form $b_{\text{ét}} \colon \operatorname{Sym}^2 \mathrm{R}^2_{\text{ét}} f_* \widehat{\mathbf{Z}}(1) \to \widehat{\mathbf{Z}}$ from Theorem 4.2.4,
- the polarization $\lambda_{\text{ét}} := c_1(\lambda) \in \mathrm{H}^0(\mathbf{M}, \mathrm{R}^2_{\text{ét}} f_* \widehat{\mathbf{Z}}(1))$,
- the orientation $\omega_{\text{ét}} \colon \widehat{\mathbf{Z}} \to \det \mathrm{R}^2 f_* \widehat{\mathbf{Z}}(1)$ from Theorem 4.3.6.

This gives rise to a $\mathcal{K}_0$-torsor

$$\operatorname{Isom}\Big((\Lambda_0 \otimes \widehat{\mathbf{Z}},\, b_0,\, \lambda_0,\, \omega_0), (\mathrm{R}^2_{\text{ét}} f_* \widehat{\mathbf{Z}}(1),\, b_{\text{ét}},\, \lambda_{\text{ét}},\, \omega_{\text{ét}})\Big)$$

on $\mathbf{M}_{\text{ét}}$. Similarly, the $\mathrm{SO}(\mathbf{A}_{\mathrm{f}})$-action on $\mathrm{Sh}(\mathrm{SO}, \Omega)$ endows it with the structure of a $\mathcal{K}_0$-torsor on $\mathrm{Sh}_{\mathcal{K}_0}[\mathrm{SO}, \Omega]_{\text{ét}}$.

Theorem 4.5.2. *The period map $\mathbf{M}_{\mathbf{C}} \to \mathrm{Sh}_{\mathcal{K}_0}[\mathrm{SO}, \Omega]_{\mathbf{C}}$ defined by the tuple $(\mathrm{R}^2 f_{\mathbf{C},*} \mathbf{Z}(1),\, b_{\mathrm{an}},\, \lambda_{\mathrm{an}},\, \omega_{\mathrm{an}})$ descends to a morphism $\mathbf{M} \to \mathrm{Sh}_{\mathcal{K}_0}[\mathrm{SO}, \Omega]$ defined over $\mathbf{Q}$. This morphism is étale, and it pulls the $\mathcal{K}_0$-torsor $\mathrm{Sh}(\mathrm{SO}, \Omega)$ on $\mathrm{Sh}_{\mathcal{K}_0}[\mathrm{SO}, \Omega]_{\text{ét}}$ back to the $\mathcal{K}_0$-torsor*

$$\operatorname{Isom}\Big(\Big(\Lambda_0 \otimes \widehat{\mathbf{Z}},\, b_0,\, \lambda_0,\, \omega_0\Big), \Big(\mathrm{R}^2_{\text{ét}} f_* \widehat{\mathbf{Z}}(1),\, b_{\text{ét}},\, \lambda_{\text{ét}},\, \omega_{\text{ét}}\Big)\Big)$$

on $\mathbf{M}_{\text{ét}}$.

Proof. Let $\mathcal{K} \subseteq \mathcal{K}_0$ be a normal neat open subgroup, so that $H = \mathcal{K}_0 \,/\, \mathcal{K}$ is a finite group, $\mathrm{Sh}_{\mathcal{K}}[\mathrm{SO}, \Omega] = \mathrm{Sh}_{\mathcal{K}}(\mathrm{SO}, \Omega)$, and

$$\mathrm{Sh}_{\mathcal{K}_0}[\mathrm{SO}, \Omega] = [\mathrm{Sh}_{\mathcal{K}}(\mathrm{SO}, \Omega)/H]\,.$$

Now define $\mathbf{M}_{\mathcal{K}}$ to be the stack of objects of $\mathbf{M}$ endowed with a level $\mathcal{K}$-structure. That is, $\mathbf{M}_{\mathcal{K}}$ is the H-torsor

$$\operatorname{Isom}\Big((\Lambda_0 \otimes \widehat{\mathbf{Z}}, b_0, \lambda_0, \omega_0), (\mathrm{R}^2_{\text{ét}} f_* \widehat{\mathbf{Z}}(1), b_{\text{ét}}, \lambda_{\text{ét}}, \omega_{\text{ét}})\Big) \,/\, \mathcal{K} \tag{4.9}$$

on $\mathbf{M}_{\text{ét}}$. Then $\mathbf{M}_{\mathcal{K}}$ is a smooth separated Deligne-Mumford stack. By Lemma 4.4.4 and Lemma 4.4.2, the tuple $(\mathrm{R}^2 f_{\mathbf{C},*},\, b_{\mathrm{an}},\, \lambda_{\mathrm{an}},\, \omega_{\mathrm{an}})$ restricted to $\mathbf{M}_{\mathcal{K},\mathbf{C}}$, endowed with the canonical level-$\mathcal{K}$ structure, yields an H-equivariant morphism of complex Deligne-Mumford stacks $\mathbf{M}_{\mathcal{K},\mathbf{C}} \to \mathrm{Sh}_{\mathcal{K}}(\mathrm{SO}, \Omega)_{\mathbf{C}}$ for which the diagram

$$\begin{tikzcd} \mathbf{M}_{\mathcal{K},\mathbf{C}} \arrow[r] \arrow[d] & \mathrm{Sh}_{\mathcal{K}}(\mathrm{SO}, \Omega)_{\mathbf{C}} \arrow[d] \\ \mathbf{M}_{\mathbf{C}} \arrow[r] & \mathrm{Sh}_{\mathcal{K}_0}[\mathrm{SO}, \Omega]_{\mathbf{C}} \end{tikzcd} \tag{4.10}$$

commutes.

We will show that the morphism $f\colon \mathbf{M}_{\mathcal{K},\mathbf{C}} \to \mathrm{Sh}_{\mathcal{K}}(\mathrm{SO},\Omega)_{\mathbf{C}}$ descends to $\mathbf{Q}$. The commutative diagram (4.10) will then imply that $\mathbf{M}_{\mathbf{C}} \to \mathrm{Sh}_{\mathcal{K}_0}[\mathrm{SO},\Omega]_{\mathbf{C}}$ descends to $\mathbf{Q}$ as well.

Let P be the groupoid of tuples $(X,\lambda,\omega_{[4]},\alpha)$, where $x := (X,\lambda,\omega_{[4]})$ is a complex point of $\mathbf{M}$, and $\alpha\colon \Lambda_0 \otimes \widehat{\mathbf{Z}} \to \mathrm{H}^2(X,\widehat{\mathbf{Z}}(1))$ is an isometry mapping λ_0 to $c_1(\lambda)$, and ω_0 to $\omega_{\mathrm{an},x}$. This groupoid comes with a forgetful functor $P \to \mathbf{M}_{\mathcal{K}}(\mathbf{C})$ mapping $(X,\lambda,\omega_{[4]},\alpha)$ to $(X,\lambda,\omega_{[4]},\alpha \mod \mathcal{K})$.

Next, we construct a functor $f'\colon P \to \mathrm{Sh}(\mathrm{SO},\Omega)(\mathbf{C})$. Let $(X,\lambda,\omega_{[4]},\alpha) \in P$. By an argument similar to that in the proofs of Lemma 2.3.11 and Lemma 4.4.3, there exists an isometry $\varphi\colon \mathrm{H}^2(X,\mathbf{Q}(1)) \to \Lambda_0\otimes\mathbf{Q}$ mapping $c_1(\lambda)$ to λ_0 and ω_{an} to ω_0. Let h be the image under φ of the Hodge structure on $\mathbf{Q}\,c_1(\lambda)^\perp \subseteq \mathrm{H}^2(X,\mathbf{Q}(1))$, and define g to be the composition $\varphi|_{\mathbf{A}_\mathrm{f}\,c_1(\lambda)^\perp} \circ \alpha|_{V\otimes\mathbf{A}_\mathrm{f}}$. Then $h \in X$ and $g \in \mathrm{SO}(\mathbf{A}_\mathrm{f})$, so this yields an element of $\mathrm{Sh}(\mathrm{SO},\Omega)(\mathbf{C})$ which does not depend on the choice of φ. It can be checked that this gives a functor $f'\colon P \to \mathrm{Sh}(\mathrm{SO},\Omega)(\mathbf{C})$.

Now we have a commutative diagram of groupoids

$$\begin{tikzcd}
P \arrow[r, "f'"] \arrow[d] & \mathrm{Sh}(\mathrm{SO},\Omega)(\mathbf{C}) \arrow[d] \\
\mathbf{M}_{\mathcal{K}}(\mathbf{C}) \arrow[r, "f(\mathbf{C})"'] & \mathrm{Sh}_{\mathcal{K}}(\mathrm{SO},\Omega)(\mathbf{C})
\end{tikzcd}$$

If we show that f' is $\mathrm{Aut}(\mathbf{C})$-equivariant in the sense that for every $\sigma \in \mathrm{Aut}(\mathbf{C})$ and every $(X,\lambda,\omega_{[4]},\alpha)$ in P there holds

$$f'\left(\sigma^*X,\,\sigma^*\lambda,\,\sigma^*\omega_{[4]},\,\sigma^*\alpha\right) = \sigma f\left(X,\,\lambda,\,\omega_{[4]},\,\alpha\right),$$

then it follows that $f(\mathbf{C})$ is $\mathrm{Aut}(\mathbf{C})$-equivariant in a similar sense, which implies that f descends to a map $\mathbf{M}_{\mathcal{K}} \to \mathrm{Sh}_{\mathcal{K}}(\mathrm{SO},\Omega)$.

Corollary 2.3.5 gives an $\mathrm{Aut}(\mathbf{C})$-equivariant map $\Phi\colon\ \mathrm{Sh}(\mathrm{SO},\Omega)(\mathbf{C}) \to \mathbf{Mot}(\Lambda_0\otimes\mathbf{Q},\lambda_0)$ where $\mathbf{Mot}(\Lambda_0\otimes\mathbf{Q},\lambda_0)$ is as in Definition 2.3.4. Theorem 2.2.2 and Example 2.2.1 imply that the composition $P \to \mathrm{Sh}(\mathrm{SO},\Omega)(\mathbf{C}) \to \mathbf{Mot}(\Lambda_0\otimes\mathbf{Q},\lambda_0)$ is given by mapping $(X,\lambda,\omega_{[4]},\alpha)$ to the tuple

$$\left(\mathfrak{h}^2(X),\,b_{\mathrm{an},x}\otimes\mathbf{Q},\,\lambda_{\mathrm{an},x}\otimes\mathbf{Q},\,\omega_{\mathrm{an},x}\otimes\mathbf{Q},\,\alpha\otimes\mathbf{A}_\mathrm{f}\right),$$

where x denotes the $\mathbf{C}$-point of $\mathbf{M}$ corresponding to $(X,\lambda,\omega_{[4]})$. The required $\mathrm{Aut}(\mathbf{C})$-equivariance of f' now follows from the fact that $\sigma^*\,\mathfrak{h}^2(X) = \mathfrak{h}^2(\sigma^*X)$, and the existence of $b_{\text{ét}}$, $\lambda_{\text{ét}}$, and $\omega_{\text{ét}}$.

We now show that the period map pulls the $\mathcal{K}_0$-torsor $\mathrm{Sh}(\mathrm{SO},\Omega)$ on $\mathrm{Sh}_{\mathcal{K}_0}[\mathrm{SO},\Omega]_{\text{ét}}$ back to the isomorphism sheaf

$$I := \mathrm{Isom}\Big(\left(\Lambda_0\otimes\widehat{\mathbf{Z}},\,b_0,\,\lambda_0,\,\omega_0\right),\,\left(\mathrm{R}^2_{\text{ét}}\,f_*\,\widehat{\mathbf{Z}}(1),\,b_{\text{ét}},\,\lambda_{\text{ét}},\,\omega_{\text{ét}}\right)\Big)$$

on $\mathbf{M}_{\text{ét}}$. It follows from (4.9) that I is the inverse limit $\lim_{\mathcal{K}} \mathbf{M}_{\mathcal{K}}$, where $\mathcal{K}$ ranges over all normal neat open subgroups of $\mathcal{K}_0$. Since $\mathrm{Sh}(\mathrm{SO}, \Omega)$ is equal to the limit $\lim_{\mathcal{K}} \mathrm{Sh}_{\mathcal{K}}(\mathrm{SO}, \Omega)$, it follows from the cartesian squares

$$\begin{array}{ccc} \mathbf{M}_{\mathcal{K}} & \longrightarrow & \mathrm{Sh}_{\mathcal{K}}(\mathrm{SO}, \Omega) \\ \downarrow & & \downarrow \\ \mathbf{M} & \longrightarrow & \mathrm{Sh}_{\mathcal{K}_0}[\mathrm{SO}, \Omega] \end{array}$$

that the period map pulls $\mathrm{Sh}(\mathrm{SO}, \Omega)$ back to I.

To see that the period map $\mathbf{M} \to \mathrm{Sh}_{\mathcal{K}_0}[\mathrm{SO}, \Omega]$ is étale, it suffices to show that $\mathbf{M}_{\mathbf{C}} \to \mathrm{Sh}_{\mathcal{K}_0}[\mathrm{SO}, \Omega]_{\mathbf{C}}$ is étale. This is an immediate consequence of the local Torelli theorem (see [A1, Proposition 3.3.1] and [B2, Théorème 5]) and the decomposition of $\mathrm{Sh}_{\mathcal{K}_0}[\mathrm{SO}, \Omega]_{\mathbf{C}}$ given in (4.7) and [M4, Lemma 5.13]. □

Remark 4.5.3. The period map in Theorem 4.5.2 is not in general an open immersion. In order for it to be an open immersion, it is necessary that it is fully faithful on $\mathbf{C}$-points. An example where the period map is neither full nor faithful is when $\mathbf{M}$ is a moduli stack of polarized oriented generalized Kummer varieties (see Example 3.1.9). The fact that the period map is not faithful in this case follows from [BNS, Corollary 3.3]. The failure of fullness follows from the global Torelli theorem (Corollary 4.1.14) and the fact that not every Hodge isometry between generalized Kummers is a parallel transport operator by [M5, Theorem 4.3].

One result pertaining to the failure of faithfulness of the period map is the result of Hassett and Tschinkel which states that when X is a complex hyperkähler variety, then the kernel of $\mathrm{Aut}(X) \to \mathrm{O}(\mathrm{H}^2(X, \mathbf{Z}(1)))$ is a deformation invariant of X. See [HT, Theorem 2.1] for a more precise statement. It implies that the relative inertia stack of $\mathbf{M} \to \mathrm{Sh}_{\mathcal{K}_0}[\mathrm{SO}, \Omega]$ is a local system of finite groups on $\mathbf{M}$.

In contrast with the period map for abelian varieties, the period map for hyperkähler is not surjective. See [D1] for a description of the image of the period map when $\mathbf{M}$ is a moduli stack of hyperkähler varieties deformation equivalent to a Hilbert scheme of points on a K3 surface.

4.6 K3 surfaces

In this section we consider moduli stacks of polarized hyperkähler varieties whose second Betti number is even (for example K3 surfaces). For such moduli stacks we can use Lemma 4.4.6 to eliminate the orientations occurring in Theorem 4.5.2. This results in a period map from a connected component of $\mathbf{HK}$ to an orthogonal Shimura stack, see Theorem 4.6.2. This section follows work on moduli stacks of K3 surfaces of Taelman in [T1].

Let $\mathbf{M}$ be a connected component of $\mathbf{HK}$ for which the hyperkähler varieties parameterized by $\mathbf{M}$ have even second Betti number, let $f\colon \mathfrak{X} \to \mathbf{M}$ be the universal hyperkähler variety, and let λ be the universal polarization on $\mathfrak{X}$.

Let $x_0 = (X_0, \lambda_{x_0})$ be a $\mathbf{C}$-point of $\mathbf{M}$. Then $\Lambda_0 := \mathrm{H}^2(X_0, \mathbf{Z}(1))$ endowed with the BBF pairing b_0 is a $\mathbf{Z}$-lattice of signature $(3, n)$, with n an odd and positive, and $\lambda_0 := c_1(\lambda_{x_0})$ is a positive element of Λ_0. Let V be the orthogonal complement of λ_0 in $\Lambda_0 \otimes \mathbf{Q}$. Then as in Section 4.4.2, the tuple $(\Lambda_0, b_0, \lambda_0)$ gives rise to an orthogonal Shimura datum

$$(\mathrm{SO}, \Omega) := (\mathrm{SO}(V), \Omega_V),$$

and a profinite group $\mathcal{K}$ defined as in (4.8), namely

$$\mathcal{K} := \left\{ g \in \mathrm{O}(\Lambda_0)(\widehat{\mathbf{Z}}) \mid g(\lambda_0) = \lambda_0 \ \text{ and } \ \det(g) \in \{\pm 1\} \right\}.$$

We endow $\mathcal{K}$ with the continuous homomorphism $i\colon \mathcal{K} \to \mathrm{SO}(\mathbf{A}_{\mathrm{f}})$ given by mapping $g \in \mathcal{K}$ to $\det(g) g|_{V \otimes \mathbf{A}_{\mathrm{f}}}$. Since the reflex field of (SO, Ω) is $\mathbf{Q}$ by Lemma 2.1.3, this yields a Shimura stack $\mathrm{Sh}_{\mathcal{K}}[\mathrm{SO}, \Omega]$ over $\mathbf{Q}$.

We will construct a $\mathcal{K}$-torsor on $\mathbf{M}$ as follows. On $\mathbf{M}_{\text{ét}}$, we have

- the local $\widehat{\mathbf{Z}}$-system $\mathrm{R}^2_{\text{ét}} f_* \widehat{\mathbf{Z}}(1)$,
- the étale BBF form $b_{\text{ét}}\colon \mathrm{Sym}^2 \mathrm{R}^2_{\text{ét}} f_* \widehat{\mathbf{Z}}(1) \to \widehat{\mathbf{Z}}$,
- the polarization $\lambda_{\text{ét}} := c_1(\lambda) \in \mathrm{H}^0(\mathbf{M}, \mathrm{R}^2_{\text{ét}} f_* \widehat{\mathbf{Z}}(1))$.

In addition to this, Lemma 4.3.7 gives a rank 1 local $\mathbf{Z}$-system D on $\mathbf{M}_{\text{ét}}$ endowed with an injective morphism of sheaves $D \to \det \mathrm{R}^2_{\text{ét}} f_* \widehat{\mathbf{Z}}(1)$ and an isomorphism of sheaves $D|_{\mathbf{M}_\mathbf{C}} \to \det \mathrm{R}^2 f_{\mathbf{C},*} \mathbf{Z}(1)$ for which the diagram

$$\begin{tikzcd} & \det \mathrm{R}^2 f_{\mathbf{C},*} \mathbf{Z}(1) \arrow[d, "\otimes \widehat{\mathbf{Z}}"] \\ D|_{\mathbf{M}_\mathbf{C}} \arrow[ur] \arrow[dr] & \\ & \det \mathrm{R}^2_{\text{ét}} f_{\mathbf{C},*} \widehat{\mathbf{Z}}(1) \end{tikzcd}$$

commutes. Now the isomorphism sheaf

$$\mathrm{Isom}\Big((\Lambda_0 \otimes \widehat{\mathbf{Z}},\, b_0,\, \lambda_0,\, \det \Lambda_0), (\mathrm{R}^2 f_* \widehat{\mathbf{Z}}(1),\, b_{\text{ét}},\, \lambda_{\text{ét}},\, D) \Big)$$

is a $\mathcal{K}$-torsor on $\mathbf{M}_{\text{ét}}$.

Lemma 4.6.1. *Let $x = (X, \lambda_x)$ be a $\mathbf{C}$-point of $\mathcal{M}$. Then there exists an isometry $\psi\colon \mathrm{H}^2(X, \mathbf{Z}(1)) \otimes \widehat{\mathbf{Z}} \to \Lambda_0 \otimes \widehat{\mathbf{Z}}$ mapping $c_1(\lambda_x)$ to λ_0, such that ψ maps $\det \mathrm{H}^2(X, \mathbf{Z}(1))$ to $\det \Lambda_0$.*

Proof. This follows from the connectedness of the stack $\mathbf{M}$ over $\mathbf{Q}$, and the existence of $b_{\text{ét}}$, $\lambda_{\text{ét}}$ and D.

Let Λ be the BBF lattice $\mathrm{H}^2(X, \mathbf{Z}(1))$, and let γ be a path from x to x_0 in $\mathbf{M}_{\text{ét}}$. Then the existence of the local $\widehat{\mathbf{Z}}$-system $\mathrm{R}^2_{\text{ét}} f_* \widehat{\mathbf{Z}}(1)$ implies that the path

γ induces an isomorphism $\psi\colon \Lambda \otimes \widehat{\mathbf{Z}} \to \Lambda_0 \otimes \widehat{\mathbf{Z}}$. It follows from the existence of $b_{\text{ét}}$ and $\lambda_{\text{ét}}$ that ψ is an isometry, and that it maps $c_1(\lambda_x)$ to λ_0.

To see that ψ maps $\det \Lambda$ to $\det \Lambda_0$, we consider the local $\mathbf{Z}$-system D on $\mathbf{M}_{\text{ét}}$. The equivalence between rank 1 local $\mathbf{Z}$-systems on $\mathbf{M}_{\text{ét}}$ and $\mathbf{Z}^\times$-torsors on $\mathbf{M}_{\text{ét}}$ can be used to show that γ induces a functorial isomorphism $\gamma_D\colon D_x \to D_{x_0}$. In particular, since ψ is induced by γ, we have a commutative diagram

$$\begin{tikzcd} D_x \arrow[r, "\sim"] \arrow[d, "\gamma_D"'] & \det \Lambda \arrow[r, hook] & \det \Lambda \otimes \widehat{\mathbf{Z}} \arrow[d, "\det \psi"] \\ D_{x_0} \arrow[r, "\sim"'] & \det \Lambda_0 \arrow[r, hook] & \det \Lambda_0 \otimes \widehat{\mathbf{Z}} \end{tikzcd}$$

It follows from this diagram that ψ maps $\det \Lambda$ into $\det \Lambda_0$, which was to be shown. $\square$

On $\mathcal{M}_{\mathbf{C}}$, we have the following Hodge-theoretic data:

- a variation of $\mathbf{Z}$-Hodge structures $\mathrm{R}^2 f_{\mathbf{C},*}\, \mathbf{Z}(1)$,
- the BBF pairing $b_{\text{an}}\colon\ \mathrm{Sym}^2 \mathrm{R}^2 f_{\mathbf{C},*}\, \mathbf{Z}(1) \to \mathbf{Z}(0)$, making the stalks of $\mathrm{R}^2 f_{\mathbf{C},*}\, \mathbf{Z}(1)$ K3-type Hodge lattices of signature $(3,n)$,
- a positive type $(0,0)$ global section $\lambda_{\text{an}} := c_1(\lambda|_{\mathbf{M}_{\mathbf{C}}})$ of $\mathrm{R}^2 f_{\mathbf{C},*}\, \mathbf{Z}(1)$.

It follows from Lemma 4.6.1, Lemma 4.4.6, and Lemma 4.4.2 that the tuple

$$\big(\mathrm{R}^2 f_{\mathbf{C},*}\, \mathbf{Z}(1),\, b_{\text{an}},\, \lambda_{\text{an}}\big)$$

gives rise to a morphism $\mathbf{M}_{\mathbf{C}} \to \mathrm{Sh}_{\mathcal{K}}[\mathrm{SO},\Omega]_{\mathbf{C}}$, known as the **period map**.

The following theorem states that the period map descends to $\mathbf{Q}$. Its proof is similar to that of Theorem 4.5.2, and is therefore omitted.

Theorem 4.6.2. *The period map* $\mathbf{M}_{\mathbf{C}} \to \mathrm{Sh}_{\mathcal{K}}[\mathrm{SO},\Omega]_{\mathbf{C}}$ *defined by the tuple*

$$\big(\mathrm{R}^2 f_{\mathbf{C},*}\, \mathbf{Z}(1),\, b_{\text{an}},\, \lambda_{\text{an}}\big)$$

descends to a morphism $\mathbf{M} \to \mathrm{Sh}_{\mathcal{K}}[\mathrm{SO},\Omega]$ *defined over* $\mathbf{Q}$. *This morphism is étale, and it pulls the* $\mathcal{K}$*-torsor* $\mathrm{Sh}(\mathrm{SO},\Omega)$ *on* $\mathrm{Sh}_{\mathcal{K}}[\mathrm{SO},\Omega]_{\text{ét}}$ *back to the* $\mathcal{K}$*-torsor*

$$\mathrm{Isom}\Big((\Lambda_0 \otimes \widehat{\mathbf{Z}},\, b_0,\, \lambda_0,\, \det \Lambda_0), (\mathrm{R}^2 f_*\, \widehat{\mathbf{Z}}(1),\, b_{\text{ét}},\, \lambda_{\text{ét}},\, D)\Big)$$

on $\mathbf{M}_{\text{ét}}$.

We now apply Theorem 4.6.2 and the global Torelli theorem (Corollary 4.1.14) to K3 surfaces to show that the stack of primitively polarized K3 surfaces is an open substack of a Shimura stack. Let $\mathbf{K3}_{2d}$ be the moduli stack over $\mathbf{Q}$ of primitively polarized K3 surfaces of degree $2d$, and $f\colon \mathfrak{X} \to \mathbf{K3}_{2d}$ the universal K3 surface. Then $\mathbf{K3}_{2d}$ is a connected component of $\mathbf{HK}$ parametrizing hyperkähler varieties with even second Betti number. We maintain the notations from earlier in the section. That is, Λ_0 is the BBF lattice of a $\mathbf{C}$-point of $\mathbf{K3}_{2d}$, we denote by (SO, Ω) the associated orthogonal Shimura datum, and so on.

In this case, the lattice Λ_0 is the K3 lattice Λ_{K3}, which is the unique even self-dual lattice of signature $(3, 19)$, and λ_0 is a primitive element of Λ_{K3} of length $2d$.

For polarized K3 surfaces, the global Torelli theorem (see Corollary 4.1.14) has the following form.

Theorem 4.6.3 ([H2, Theorem 5.3 and Proposition 15.2.1])**.** *Let (X_0, λ_0) and (X_1, λ_1) be polarized K3 surfaces, and*

$$\varphi\colon\ \mathrm{H}^2(X_1, \mathbf{Z}(1)) \longrightarrow \mathrm{H}^2(X_0, \mathbf{Z}(1))$$

a Hodge isometry mapping $c_1(\lambda_1)$ to $c_1(\lambda_0)$. Then there exists a unique isomorphism $f\colon (X_0, \lambda_0) \to (X_1, \lambda_1)$ inducing φ.

Combining this with Theorem 4.6.2 yields the following theorem, which states that $\mathbf{K3}_{2d}$ is an open substack of an orthogonal Shimura stack.

Theorem 4.6.4. *The period map $\mathbf{K3}_{2d,\mathbf{C}} \to \mathrm{Sh}_{\mathcal{K}}[\mathrm{SO}, \Omega]_{\mathbf{C}}$ defined by the tuple*

$$\big(\mathrm{R}^2 f_{\mathbf{C},*}\, \mathbf{Z}(1),\, b_{\mathrm{an}},\, \lambda_{\mathrm{an}}\big)$$

descends to a morphism $\mathbf{K3}_{2d} \to \mathrm{Sh}_{\mathcal{K}}[\mathrm{SO}, \Omega]$ defined over $\mathbf{Q}$. This morphism is an open immersion, and it pulls the $\mathcal{K}$-torsor $\mathrm{Sh}(\mathrm{SO}, \Omega)$ on $\mathrm{Sh}_{\mathcal{K}}[\mathrm{SO}, \Omega]_{\text{ét}}$ back to the $\mathcal{K}$-torsor

$$\mathrm{Isom}\Big((\Lambda_0 \otimes \widehat{\mathbf{Z}},\, b_0,\, \lambda_0,\, \det \Lambda_0), (\mathrm{R}^2 f_* \,\widehat{\mathbf{Z}}(1),\, b_{\text{ét}},\, \lambda_{\text{ét}},\, D)\Big)$$

on $\mathbf{K3}_{2d,\text{ét}}$.

Remark 4.6.5. The only other known examples of complex hyperkähler varieties with even second Betti number are O'Grady's examples, see Example 3.1.9. Let OG_6 and OG_{10} be O'Grady's examples of dimension 6 and 10, respectively.

By [MW, Theorem 5.2], the kernel of $\mathrm{Aut}(\mathrm{OG}_6) \to \mathrm{O}(\mathrm{H}^2(\mathrm{OG}_6, \mathbf{Z}(1)))$ is isomorphic to $(\mathbf{Z}/2\,\mathbf{Z})^{\oplus 8}$. It follows that the period map is not faithful. Since open immersions of stacks are representable, it follows that Theorem 4.6.4 does not extend to the moduli space of polarized hyperkähler varieties deformation equivalent to OG_6. However, the period map is full on $\mathbf{C}$-points in this case by [MR, Theorem 5.4].

For OG_{10} the map $\mathrm{Aut}(\mathrm{OG}_{10}) \to \mathrm{O}(\mathrm{H}^2(\mathrm{OG}_{10}, \mathbf{Z}(1)))$ is injective, by [MW, Theorem 3.1], so the period map in Theorem 4.6.2 is faithful. However, by [M5, Theorem 5.3], the analogue of Theorem 4.6.3 with K3 surfaces replaced with varieties deformation equivalent to OG_{10} does not hold, which shows that the period map is not full, and hence not an open immersion.

4.7 The K3$^{[n]}$ deformation type

Throughout this section, n is an integer greater than or equal to 2. In this section we consider hyperkähler varieties that are deformation equivalent to the Hilbert scheme of n points on a K3 surface, known as $\mathbf{K3}^{[n]}$**-type** hyperkähler varieties. In the first subsection we extend a theorem of Markman on the monodromy of such varieties over $\mathbf{C}$ to such varieties over non-closed fields of characteristic 0, see Theorem 4.7.12. In the second subsection, we use this to give an open immersion from a connected component of the moduli stack of polarized oriented $\mathbf{K3}^{[n]}$-type hyperkähler varieties to an orthogonal Shimura stack.

4.7.1 $\mathrm{K3}^{[n]}$-type hyperkähler varieties and their monodromy

Definition 4.7.1. A hyperkähler variety X over $\mathbf{C}$ is said to be of $\mathbf{K3}^{[n]}$ **type** if there exists a proper smooth morphism $\mathfrak{X} \to T$ of complex analytic spaces, with T connected, such that one of the fibers of f is isomorphic to X, and another is isomorphic to $S^{[n]}$ for some projective K3 surface S over $\mathbf{C}$.

By [H2, Theorem 7.1.1], any two complex K3 surfaces are deformation equivalent in a complex analytic sense. The following lemma is an algebraic analogue of this result for projective complex K3 surfaces. The lemma is well known, but it is difficult to find an argument in the literature, so we sketch a proof.

Lemma 4.7.2. *Let S_1 and S_2 be two projective complex K3 surfaces. Then there exists a smooth proper morphism of complex schemes $f\colon \mathfrak{S} \to T$ whose fibers are K3 surfaces, with T connected, such that one fiber of f is isomorphic to S_1, and another is isomorphic to S_2.*

Proof sketch. Pick primitive polarizations on S_1 and S_2 of degrees $2d$ and $2e$, respectively.

Let $N \subseteq \Lambda_{\mathrm{K3}}$ be a signature $(1,1)$ lattice with primitive elements λ and μ of length $2d$ and $2e$, respectively. Moreover, assume that the orthogonal complements of λ and μ in N do not contain any δ with $\delta^2 = -2$. Now consider the period domain

$$\Omega := \{[z] \in \Omega_{\Lambda_{\mathrm{K3}}} \mid zN = 0\}.$$

It parametrizes K3-type Hodge structures on Λ_{K3} for which all of N is of type $(0,0)$. Let Ω^0 be the open subset

$$\Omega \Big\backslash \bigcup_{\substack{\delta \in \Lambda_{\mathrm{K3}} \\ \delta^2=-2,\, \delta N = 0}} W_\delta,$$

where W_δ denotes the subset of Ω orthogonal to δ. Then Ω^0 is non-empty, because it is the complement of countably many hyperplanes. The surjectivity of the period map [H2, Theorem 6.3.1] implies that there exists a complex K3 surface S with $N \subseteq \mathrm{Pic}(S)$ such that if $\delta \in \mathrm{Pic}(S)$ with $\delta^2 = -2$, then $\delta\lambda \neq 0$ and $\delta\mu \neq 0$.

Let $\mathcal{C}$ be the subset of $\{x \in \mathrm{Pic}(S) \otimes \mathbf{R} \mid x^2 > 0\}$ containing the ample cone of S. By [H2, Corollary 8.1.6], the ample cone of S is a connected component of

$$Y := \mathcal{C} \Big\backslash \bigcup_{\substack{\delta \in \mathrm{Pic}(S) \\ \delta^2=0}} H_\delta$$

where $H_\delta \subseteq \mathrm{Pic}(S) \otimes \mathbf{R}$ denotes the orthogonal complement in $\mathrm{Pic}(S) \otimes \mathbf{R}$ of δ. By construction, λ and μ are elements of Y. By [H2, Proposition 8.2.6] there is a subgroup of $\mathrm{O}(\mathrm{Pic}(S))$ that acts transitively on the set of connected components of Y. It follows that there exist $\varphi, \psi \in \mathrm{O}(\mathrm{Pic}(S))$ such that $\varphi(\lambda)$ and $\psi(\mu)$ are ample. It follows that S has primitive polarizations of degree $2d$ and $2e$.

Let $\mathbf{K3}_{2d,\mathbf{C}}$ and $\mathbf{K3}_{2e,\mathbf{C}}$ be the moduli stacks of primitively polarized complex K3 surfaces of degree $2d$ and $2e$, respectively. These are irreducible Deligne-Mumford stacks of finite type over $\mathbf{C}$. By [DM2, Proposition 4.14], there exist connected schemes T_1 and T_2 and surjective morphisms $T_1 \to \mathbf{K3}_{2d,\mathbf{C}}$ and $T_2 \to \mathbf{K3}_{2e,\mathbf{C}}$.

The polarizations on S of degree $2d$ and $2e$ give rise to $\mathbf{C}$-points x_1 and x_2 of T_1 and T_2. Let t_1 and t_2 be $\mathbf{C}$-points of T_1 and T_2 lifting x_1 and x_2, respectively. Gluing T_1 and T_2 at the points t_1 and t_2 we obtain a scheme T. By pulling back the universal K3 surfaces on $\mathbf{K3}_{2d,\mathbf{C}}$ and $\mathbf{K3}_{2e,\mathbf{C}}$ to T_1 and T_2, and gluing them along the fibers over the points $f_1(t)$ and $f_2(t)$, we obtain the desired morphism $\mathfrak{S} \to T$ deforming S_1 to S_2. $\square$

The following lemma shows that in the definition of $\mathrm{K3}^{[n]}$-type hyperkähler varieties, one can replace the complex analytic spaces by algebraic spaces over $\mathbf{C}$.

Lemma 4.7.3. *Let X be a hyperkähler variety over $\mathbf{C}$ of $K3^{[n]}$ type, and S a projective complex K3 surface. Then there exists a smooth proper morphism of complex algebraic spaces $f\colon \mathfrak{X} \to T$ whose fibers are hyperkähler varieties, with T connected, such that one fiber of f is isomorphic to X, and another is isomorphic to $S^{[n]}$.*

Proof. In [MP3, Corollary 1.2], Mongardi and Pacienza show that varieties *birational* to the Hilbert scheme of points on a projective complex K3 surface are dense in the moduli space of polarized complex hyperkähler varieties of $\mathrm{K3}^{[n]}$ type. It follows that there exists a projective complex K3 surface S', a complex hyperkähler variety X' birational to $(S')^{[n]}$, and a smooth proper morphism of complex algebraic spaces $\mathfrak{X}_1 \to T_1$, one of whose fibers is isomorphic to X, and another is isomorphic to X'.

Since X' and $(S')^{[n]}$ are birational complex hyperkähler varieties, [R2, Proposition 2.1] says that there exists a smooth proper morphism of complex algebraic spaces $\mathfrak{X}_2 \to T_2$ whose fibers are hyperkähler varieties, one of whose fibers is isomorphic to X', and another isomorphic to $(S')^{[n]}$.

By Lemma 4.7.2, there exists a smooth proper morphism of complex algebraic spaces $\mathfrak{S} \to T_3$ whose fibers are K3 surfaces, one of which is S', and another is S. It follows that there is a smooth proper morphism of complex algebraic spaces $\mathfrak{X}_3 \to T_3$ whose fibers are the Hilbert schemes of n points on the fibers of $\mathfrak{S} \to T_3$.

By gluing $\mathfrak{X}_1 \to T_1$, $\mathfrak{X}_2 \to T_2$, and $\mathfrak{X}_3 \to T_3$ together along appropriate points, we obtain a smooth proper morphism of complex algebraic spaces whose fibers are hyperkähler varieties, one of which is isomorphic to X, and another to $S^{[n]}$. $\square$

Lemma 4.7.4. *Let X be a complex hyperkähler variety of $K3^{[n]}$ type, and let $\sigma \in \mathrm{Aut}(\mathbf{C})$. Then the pullback σ^*X of X along σ is a hyperkähler variety of $K3^{[n]}$ type. Moreover, the BBF forms on $\mathrm{H}^2(X, \mathbf{Z}(1))$ and $\mathrm{H}^2(\sigma^*X, \mathbf{Z}(1))$ are isometric.*

Proof. Let S, T, and $f\colon \mathfrak{X} \to T$ be as in Lemma 4.7.3. Then $\sigma^* f\colon \sigma^* \mathfrak{X} \to \sigma^* T$ is a smooth proper morphism of algebraic spaces such that one of the fibers is isomorphic to $\sigma^* X$, and another is isomorphic to $\sigma^*(S^{[n]})$. Since $\sigma^*(S^{[n]}) = (\sigma^* S)^{[n]}$, and since $\sigma^* S$ is a K3 surface, it follows that $\sigma^* X$ is of $\mathrm{K3}^{[n]}$ type. In particular, X and $\sigma^* X$ have isometric BBF forms on singular cohomology by Proposition 4.1.8. ☐

Definition 4.7.5. A hyperkähler variety X over a field k of characteristic 0 is said to be of **$\mathbf{K3}^{[n]}$ type** if X descends to a hyperkähler variety X_K over a subfield K of $\mathbf{C}$ such that $X_{\mathbf{C}}$ is of $\mathrm{K3}^{[n]}$ type.

Remark 4.7.6. Lemma 4.7.4 shows that when $k = \mathbf{C}$, Definition 4.7.5 is equivalent to Definition 4.7.1.

Lemma 4.7.7. *Let X be a hyperkähler variety of $K3^{[n]}$ type over an algebraically closed field k of characteristic 0, and S a K3 surface over k. Then there exists a smooth proper morphism of algebraic spaces $f\colon \mathfrak{X} \to T$ over k whose fibers are hyperkähler varieties, with T connected, such that one fiber of f is isomorphic to X, and another is isomorphic to $S^{[n]}$.*

Proof. This follows from Lemma 4.7.3 by a spreading out argument. ☐

Lemma 4.7.8. *Let S be a connected finite type $\mathbf{Q}$-scheme, and X/S a smooth proper morphism of algebraic spaces whose fibers are hyperkähler varieties. If there exists a $\mathbf{C}$-point s_0 of S such that X_{s_0} is of $K3^{[n]}$ type, then every fiber of X/S is of $K3^{[n]}$ type.*

Proof. Let s be a $\mathbf{C}$-point of S. We will show that the fiber X_s is of $\mathrm{K3}^{[n]}$ type. By the connectedness of S, we can find a $\sigma \in \mathrm{Aut}(\mathbf{C})$ such that σs_0 is in the same component of $S^{\mathrm{an}}_{\mathbf{C}}$ as s. By Lemma 4.7.4, $X_{\sigma s_0} = \sigma^* X_{s_0}$ is of $\mathrm{K3}^{[n]}$ type, so it follows that X_s is of $\mathrm{K3}^{[n]}$ type.

Now let k be a field of characteristic 0, and $s \in S(k)$. Since S is of finite type, we can find a finitely generated subfield $k' \subseteq k$ and a point $s' \in S(k')$ such that s factors through s'. Next, we choose an embedding $k' \subseteq \mathbf{C}$. By the above, $X_{s',\mathbf{C}}$ is of $\mathrm{K3}^{[n]}$ type, so by definition X_s is of $\mathrm{K3}^{[n]}$ type. ☐

Let S be a $\mathbf{Q}$-scheme, and $f\colon X \to S$ be a smooth proper morphism of algebraic spaces whose fibers are hyperkähler varieties of $\mathrm{K3}^{[n]}$ type. Consider the discriminant

$$\Delta(X/S) := \Delta\Big(\mathrm{R}^2_{\text{ét}}\, f_*\, \widehat{\mathbf{Z}}(1)\Big)\,,$$

which is a local system of finite abelian groups on $S_{\text{ét}}$, with fibers isomorphic to $\mathbf{Z}/(2n-2)\,\mathbf{Z}$, see Example 4.1.5. Now let $F(X/S)$ be the local system of finite sets on $S_{\text{ét}}$ defined as the quotient of $\Delta(X/S)$ by the action of $\{\pm 1\}$, with -1 acting as $-\mathrm{id}_{\Delta(X/S)}$.

Theorem 4.7.9 (Markman)**.** *Let S be a scheme over an algebraically closed field of characteristic 0, and let X/S be a smooth proper morphism of algebraic spaces whose fibers are hyperkähler varieties of $K3^{[n]}$ type. Then the local system $F(X/S)$ on $S_{\text{ét}}$ is constant.*

Proof. For S of finite type over $\mathbf{C}$, this follows from [M1, Lemma 9.2] and [G2, Corollaire XII.5.2]. A spreading out argument shows that the result holds for any scheme S over $\mathbf{C}$. We obtain the result for general algebraically closed ground fields of characteristic 0 via the Lefschetz principle. □

Lemma 4.7.10. *Let S be a K3 surface over a field k of characteristic* 0. *Then the local system $F(S^{[n]}/k)$ on $k_{\text{ét}}$ is constant.*

Proof. We will show that the local system $\Delta(S^{[n]}/k)$ is constant. A fortiori its quotient $F(S^{[n]}/k)$ is then also constant.

As in Example 4.1.5, there is a natural morphism $S^{[n]} \to S^{(n)}$, and the inverse image of the singular locus defines a divisor E on $S^{[n]}$. Then $\Delta(S^{[n]}/k)$ is generated by $\delta \in \mathrm{H}^2_{\text{ét}}(S^{[n]}_{\overline{k}}, \widehat{\mathbf{Z}}(1))$ satisfying $2\delta = E$. Since E is Gal_k-invariant, and since $\mathrm{H}^2_{\text{ét}}(S^{[n]}_{\overline{k}}, \widehat{\mathbf{Z}}(1))$ is a free $\widehat{\mathbf{Z}}$-module, it follows that δ is Gal_k-invariant. This shows that Gal_k acts trivially on $\Delta(S^{[n]}/k)$. □

Lemma 4.7.11. *Let k be a perfect field, let $\overline{k}$ be an algebraic closure of k, let $\mathcal{M}$ be a stack on $(\mathbf{Sch}/k)_{\text{ét}}$, and let $\mathcal{F}$ be a local system of finite sets on the big étale site $\mathcal{M}_{\text{ét}}$ of $\mathcal{M}$. If*

1. *for any algebraically closed extension Ω of $\overline{k}$ and $y, z \in \mathcal{M}_{\overline{k}}(\Omega)$ there exists a connected algebraic space T over Ω and a morphism $T \to \mathcal{M}_{\overline{k}}$ which has y and z in its image,*

2. *the restriction of $\mathcal{F}$ to $\mathcal{M}_{\overline{k}}$ is constant, and*

3. *there exists a point $x \in \mathcal{M}(k)$ such that $x^*\mathcal{F}$ is constant,*

then $\mathcal{F}$ is constant.

Proof. Let $\mathsf{LS}(\mathcal{M}_{\overline{k}})$ be the category of local systems of finite sets on $\mathcal{M}_{\overline{k},\text{ét}}$. For an algebraically closed extension Ω of $\overline{k}$, a geometric point $x_0 \in \mathcal{M}_{\overline{k}}(\Omega)$ induces a functor x_0^* from $\mathsf{LS}(\mathcal{M}_{\overline{k}})$ to the category $\mathbf{fSet}$ of finite sets, via pullback. For two geometric points x_0 and x_1 of $\mathcal{M}_{\overline{k}}$, a path from x_0 to x_1 in $\mathcal{M}_{\overline{k}}$ consists of an isomorphism of functors $x_0^* \to x_1^*$. Assumption 1 implies that we can find a path between any two geometric points of $\mathcal{M}_{\overline{k}}$.

Let $\overline{x}$ be the $\overline{k}$-point of $\mathcal{M}_{\overline{k}}$ corresponding to the k-point x of $\mathcal{M}$ in assumption 3. We define F_0 to be the finite set $\overline{x}^*\mathcal{F}$. Let $\mathcal{F}_0 \in \mathsf{LS}(\mathcal{M})$ be the constant sheaf of finite sets on $\mathcal{M}_{\text{ét}}$ associated with F_0. We will show that $\mathcal{F}$ is isomorphic to $\mathcal{F}_0$.

By assumption 2, the sheaf $\mathcal{F}|_{\mathcal{M}_{\overline{k}}}$ is constant, so there exists an isomorphism $\beta\colon \mathcal{F}|_{\mathcal{M}_{\overline{k}}} \to \mathcal{F}_0|_{\mathcal{M}_{\overline{k}}}$ which satisfies $\overline{x}^*\beta = \mathrm{id}_{\overline{x}^*\mathcal{F}}$.

We claim that the condition that $\overline{x}^*\beta = \mathrm{id}_{\overline{x}^*\mathcal{F}}$ determines β uniquely. To see this, first note that the big étale site of $\mathcal{M}_{\overline{k}}$ has enough points by [T2, Tag 06W4], so β is determined by the morphisms $y^*\beta\colon y^*\mathcal{F} \to y^*\mathcal{F}_0$, where y ranges over all geometric points of $\mathcal{M}_{\overline{k}}$. For a geometric point y of $\mathcal{M}_{\overline{k}}$, let γ be a path from $\overline{x}$

to y. Then by the functoriality of γ there is a commutative diagram of bijections

$$\begin{tikzcd} \overline{x}^*\mathcal{F} \arrow[r, "\overline{x}^*\beta"] \arrow[d, "\gamma_{\mathcal{F}}"'] & \overline{x}^*\mathcal{F}_0 \arrow[d, "\gamma_{\mathcal{F}_0}"] \\ y^*\mathcal{F} \arrow[r, "y^*\beta"'] & y^*\mathcal{F}_0 \end{tikzcd}$$

This shows that $y^*\beta$ is determined by $\overline{x}^*\beta$, so that the condition $\overline{x}^*\beta = \mathrm{id}_{\overline{x}^*\mathcal{F}}$ uniquely determines β.

Let $\sigma \in \mathrm{Gal}_k$. Then σ acts trivially on $\overline{x}^*\mathcal{F}_0$ because $\mathcal{F}_0$ is constant, and it acts trivially on $\overline{x}^*\mathcal{F}$ by assumption 3. It follows that $\sigma\beta\colon \mathcal{F}|_{\mathcal{M}_{\overline{k}}} \to \mathcal{F}_0|_{\mathcal{M}_{\overline{k}}}$ satisfies $\overline{x}^*(\sigma\beta) = \mathrm{id}$, so that $\sigma\beta = \beta$. It follows that β induces an isomorphism $\mathcal{F} \to \mathcal{F}_0$, showing that $\mathcal{F}$ is constant on $\mathcal{M}_{\text{ét}}$. □

Theorem 4.7.12. *Let S be a scheme over $\mathbf{Q}$, and let X/S be a smooth proper morphism of algebraic spaces whose fibers are hyperkähler varieties of $K3^{[n]}$ type. Then the local system $F(X/S)$ on $S_{\text{ét}}$ is constant.*

Proof. Let $\mathbf{K3}^{[n]}$ be the groupoid fibration on $\mathbf{Sch}/\mathbf{Q}$ whose objects are proper smooth morphisms of algebraic spaces $f\colon Y \to S$, where S is a $\mathbf{Q}$-scheme, such that all fibers of f are hyperkähler varieties of $\mathrm{K3}^{[n]}$ type. Then $\mathbf{K3}^{[n]}$ is a stack for the étale topology. The assignment

$$Y/S \longmapsto F(Y/S)(S)$$

defines a local system $\mathcal{F}$ of finite sets on the big étale site $\mathbf{K3}^{[n]}_{\text{ét}}$ of $\mathbf{K3}^{[n]}$. The theorem is equivalent to $\mathcal{F}$ being constant. Lemma 4.7.7, Theorem 4.7.9, and Lemma 4.7.10 show that $\mathbf{K3}^{[n]}$ and $\mathcal{F}$ satisfy the hypotheses of Lemma 4.7.11, so $\mathcal{F}$ is constant. □

Remark 4.7.13. The results proved in this section have analogues for generalized Kummer varieties (see Example 3.1.9).

The main results used in proving that being of $\mathrm{K3}^{[n]}$ type is an algebraic condition (see Lemma 4.7.3 and Lemma 4.7.4) are [MP3, Corollary 1.2] and [R2, Proposition 2.1]. These results hold for generalized Kummer varieties, and the arguments given here for $\mathrm{K3}^{[n]}$-type varieties carry over almost verbatim to such varieties (with the complex K3 surface and its Hilbert scheme of points in Lemma 4.7.3 replaced by a complex abelian surface and the associated generalized Kummer variety).

Let S be a $\mathbf{Q}$-scheme. For a smooth proper morphism $X \to S$ whose fibers are generalized Kummer varieties, we denote by $F(X/S)$ the quotient by $\{\pm 1\}$ of the sheaf of finite abelian groups

$$\Delta(\mathrm{R}^2_{\text{ét}}\, f_*\, \widehat{\mathbf{Z}}(1))$$

on $S_{\text{ét}}$. It follows from [M5, Theorem 4.3] that if S is a scheme over an algebraically closed field of characteristic 0, and if X/S admits an orientation, then the local system of finite sets $F(X/S)$ is constant, giving an analogue of Theorem 4.7.9.

Suppose X is the generalized Kummer variety associated with an abelian surface over a field k of characteristic 0. Then the description given in [B2] of the BBF lattice $\mathrm{H}^2_{\text{ét}}(X_{\overline{k}}, \widehat{\mathbf{Z}}(1))$ shows that its discriminant is generated by an algebraic cycle on X, which allows one to prove an analogue of Lemma 4.7.10. Ultimately this leads to an analogue of Theorem 4.7.12 for *oriented* generalized Kummer varieties.

4.7.2 Period maps for K3$^{[n]}$-type hyperkähler varieties

We will now apply the results of the preceding subsection to obtain a period map over $\mathbf{Q}$ for oriented polarized hyperkähler varieties of K3$^{[n]}$-type which is an open immersion.

Let $\mathbf{M}$ be a connected component of $\mathbf{HK}_{\text{or}}$ such that one of the points of $\mathbf{M}$ is a K3$^{[n]}$-type hyperkähler variety. We use the notation from Section 4.5. In particular, we let $f\colon \mathfrak{X} \to \mathbf{M}$ be the universal hyperkähler variety, λ the universal polarization, and $\omega_{[4]}$ the universal orientation on $\mathfrak{X}$. By Lemma 4.7.8, every fiber of $\mathfrak{X} / \mathbf{M}$ is of K3$^{[n]}$ type.

The constructions in Section 4.5 yield a $\mathbf{Z}$-VHS $\mathrm{R}^2 f_{\mathbf{C},*}\,\mathbf{Z}(1)$ on $\mathbf{M}_{\mathbf{C}}$, endowed with data b_{an}, λ_{an}, and ω_{an}, arising from the BBF form, the polarization, and the orientation on $\mathfrak{X}$, respectively. They also yield a local $\widehat{\mathbf{Z}}$-system $\mathrm{R}^2_{\text{ét}} f_*\,\widehat{\mathbf{Z}}(1)$ on $\mathbf{M}_{\text{ét}}$, endowed with data $b_{\text{ét}}$, $\lambda_{\text{ét}}$, and $\omega_{\text{ét}}$. Moreover, we pick a $\mathbf{C}$-point $x_0 = (X_0, \lambda_{x_0}, \omega_{[4],x_0})$, which gives rise to $\Lambda_0 := \mathrm{H}^2(X_0, \mathbf{Z}(1))$, endowed with the BBF form b_0, the polarization $\lambda_0 := \lambda_{\text{an},x_0}$, and the orientation $\omega_0 := \omega_{\text{an},x_0}$.

Theorem 4.7.12 gives rise to extra structure on $\mathrm{R}^2 f_{\mathbf{C},*}\,\mathbf{Z}(1)$ and $\mathrm{R}^2_{\text{ét}} f_*\,\widehat{\mathbf{Z}}(1)$, as follows. As in the proof of Theorem 4.7.12, let $\mathbf{K3}^{[n]}$ be the stack over $\mathbf{Q}_{\text{ét}}$ whose objects are smooth proper morphisms of algebraic spaces $f\colon X \to S$, where S is a $\mathbf{Q}$-scheme, such that all fibers of f afe hyperkähler varieties of K3$^{[n]}$ type. The assignment

$$Y/S \longmapsto F(Y/S)(S)$$

defines a local system $\mathcal{F}$ of finite sets on the big étale site of $\mathbf{K3}^{[n]}$. Let F_0 be the constant sheaf of finite sets associated with $F(X_0)$. Then Theorem 4.7.12 shows that we can pick an isomorphism of sheaves

$$\overline{\beta}\colon F_0 \longrightarrow \mathcal{F} \tag{4.11}$$

such that $\overline{\beta}$ is the identity over the point X_0 of $\mathbf{K3}^{[n]}$. Now $\mathfrak{X} / \mathbf{M}$ yields a morphism $\mathbf{M} \to \mathbf{K3}^{[n]}$, which allows us to pull the isomorphism $\overline{\beta}$ back to give an isomorphism

$$\overline{\alpha}\colon F_0 \longrightarrow F(\mathfrak{X} / \mathbf{M})$$

of sheaves of finite sets on $\mathbf{M}_{\text{ét}}$. Note that $\overline{\alpha}_{x_0}$ is the identity.

Since $F(\mathfrak{X} / \mathbf{M})$ is a constant local system of finite sets, there is an isomorphism of sheaves $\overline{\alpha}\colon F_0 \to F(\mathfrak{X} / \mathbf{M})$ such that $\overline{\alpha}_{x_0}$ is the identity.

The proof of the following lemma is similar to that of Lemma 4.5.1, and hence omitted.

Lemma 4.7.14. *Let $x = (X, \lambda, \omega_{[4]})$ be a $\mathbf{C}$-point of $\mathbf{M}$. Then there exists an isometry $\psi\colon \Lambda_0 \otimes \widehat{\mathbf{Z}} \to \mathrm{H}^2(X, \mathbf{Z}(1)) \otimes \widehat{\mathbf{Z}}$ mapping λ_0 and ω_0 to $\lambda_{\mathrm{an},x}$ and $\omega_{\mathrm{an},x}$, and such that ψ induces $\overline{\alpha}_x$.*

Let (SO, Ω) be the orthogonal Shimura datum associated with $(\Lambda_0, b_0, \lambda_0, \omega_0)$ as in Section 4.5. Moreover, let $\mathcal{K}_F$ be the profinite group

$$\left\{g \in \mathcal{K}_0 \mid \Delta(g) = \pm\,\mathrm{id}_{\Delta(\Lambda_0)}\right\},$$

viewed as a compact open subgroup of $\mathrm{SO}(\mathbf{A}_\mathrm{f})$ by mapping $g \in \mathcal{K}_F$ to $g|_{\mathbf{A}_\mathrm{f}\,\lambda_0^\perp}$. Now Lemma 4.7.14, Lemma 4.4.5, and Lemma 4.4.2 show that the tuple

$$(\mathrm{R}^2\, f_{\mathbf{C},*}\, \mathbf{Z}(1),\, b_{\mathrm{an}},\, \lambda_{\mathrm{an}},\, \omega_{\mathrm{an}},\, \alpha)$$

gives rise to a morphism of complex Deligne-Mumford stacks $\mathbf{M}_\mathbf{C} \to \mathrm{Sh}_{\mathcal{K}_F}[\mathrm{SO}, \Omega]_\mathbf{C}$.

Note that the isomorphism sheaf

$$\mathrm{Isom}\left((\Lambda_0 \otimes \widehat{\mathbf{Z}},\, b_0,\, \lambda_0,\, \omega_0,\, \mathrm{id}_{F(\Lambda_0)}), (\mathrm{R}^2_{\text{ét}}\, f_*\, \widehat{\mathbf{Z}}(1),\, b_{\text{ét}},\ \lambda_{\text{ét}},\, \omega_{\text{ét}})\right)$$

is a $\mathcal{K}_F$-torsor on $\mathbf{M}_{\text{ét}}$. Moreover, $\mathrm{Sh}(\mathrm{SO}, \Omega))$ is a $\mathcal{K}_F$-torsor on $\mathrm{Sh}[\mathrm{SO}, \Omega]_{\text{ét}}$. The following theorem states that the period map $\mathbf{M}_\mathbf{C} \to \mathrm{Sh}_{\mathcal{K}_F}[\mathrm{SO}, \Omega]_\mathbf{C}$ descends to $\mathbf{Q}$. Its proof is similar to that of Theorem 4.5.2.

Theorem 4.7.15. *The period map $\mathbf{M}_\mathbf{C} \to \mathrm{Sh}_{\mathcal{K}_F}[\mathrm{SO}, \Omega]_\mathbf{C}$ defined by the tuple*

$$(\mathrm{R}^2\, f_{\mathbf{C},*}\, \mathbf{Z}(1),\, b_{\mathrm{an}},\, \lambda_{\mathrm{an}},\, \omega_{\mathrm{an}},\, \alpha)$$

descends to a morphism $\mathbf{M} \to \mathrm{Sh}_{\mathcal{K}_F}[\mathrm{SO}, \Omega]$ defined over $\mathbf{Q}$. This morphism is étale, and pulls the $\mathcal{K}_F$-torsor $\mathrm{Sh}(\mathrm{SO}, \Omega)$ on $\mathrm{Sh}_{\mathcal{K}_F}[\mathrm{SO}, \Omega]_{\text{ét}}$ back to the $\mathcal{K}_F$-torsor.

$$\mathrm{Isom}\left((\Lambda_0 \otimes \widehat{\mathbf{Z}},\, b_0,\, \lambda_0,\, \omega_0,\, \mathrm{id}_{F(\Lambda_0)}), (\mathrm{R}^2_{\text{ét}}\, f_*\, \widehat{\mathbf{Z}}(1),\, b_{\text{ét}},\ \lambda_{\text{ét}},\, \omega_{\text{ét}})\right)$$

on $\mathbf{M}_{\text{ét}}$.

We wish to show that this period map is an open immersion. For this, we need the following two lemmas. The first implies that the period map is faithful. The second is a consequence of characterization of parallel transport operators for $\mathrm{K3}^{[n]}$-type hyperkähler varieties, due to Markman. In conjunction with Verbitsky's global Torelli theorem (Corollary 4.1.14), this will allow us to show that the period map is full.

Lemma 4.7.16 ([B1, Proposition 10] and [HT, Theorem 2.1])**.** *Let X be a complex hyperkähler variety of $K3^{[n]}$ type. Then the natural homomorphism $\mathrm{Aut}(X) \to \mathrm{O}(\mathrm{H}^2(X, \mathbf{Z}(1)))$ is injective.*

Lemma 4.7.17. *Let $x = (X, \lambda, \omega_{[4]})$ and $x' = (X', \lambda', \omega'_{[4]})$ be $\mathbf{C}$-points of $\mathbf{M}$, and let $\varphi\colon \mathrm{H}^2(X', \mathbf{Z}(1)) \to \mathrm{H}^2(X, \mathbf{Z}(1))$ be a Hodge isometry mapping $\lambda_{\mathrm{an},x'}$, $\omega_{\mathrm{an},x'}$, and $\overline{\alpha}_{x'}$ to $\lambda_{\mathrm{an},x}$, $\omega_{\mathrm{an},x'}$, and $\overline{\alpha}_x$. Then φ is induced by an isomorphism $x \to x'$ in $\mathbf{M}$.*

Proof. By the global Torelli theorem for polarized hyperkähler manifolds, Corollary 4.1.14, it suffices to show that φ is a parallel transport operator in the sense of Definition 4.1.10.

By applying Lemma 4.7.3 to both X and X', we can find a smooth proper morphism of algebraic spaces $f\colon \mathfrak{Y} \to T$ whose fibers are hyperkähler varieties, with T connected, such that one fiber of f (over $t \in T$, say) is isomorphic to X, and another (over $t' \in T$) is isomorphic to X'.

Pick a path γ in T^{an} from t to t'. Then γ induces a parallel transport operator $\psi\colon \mathrm{H}^2(X, \mathbf{Z}(1)) \to \mathrm{H}^2(X', \mathbf{Z}(1))$. Since f defines a morphism $T \to \mathbf{K3}^{[n]}$, it follows that $\psi\overline{\beta}_X = \overline{\beta}_{X'}$, where $\overline{\beta}$ is the isomorphism of sheaves on $\mathbf{K3}^{[n]}_{\text{ét}}$ given in (4.11).

Now the composition $\varphi\psi$ is an element of $\mathrm{O}(\mathrm{H}^2(X, \mathbf{Z}(1)))$. Since $\overline{\alpha}_x = \overline{\beta}_X$ by definition of $\overline{\alpha}$, it follows that $\varphi\psi(\overline{\alpha}_x) = \overline{\alpha}_x$, so that $\varphi\psi$ acts as $\pm\,\mathrm{id}$ on $\Delta(\mathrm{H}^2(X, \mathbf{Z}(1)))$. By [M1, Lemma 9.2], this implies that $\varphi\psi$ is a parallel transport operator. Since ψ is a parallel transport operator, and since the composition of parallel transport operators is a parallel transport operator, it follows that φ is a parallel transport operator. $\square$

The following theorem is an immediate consequence of Lemma 4.7.16 and Lemma 4.7.17.

Theorem 4.7.18. *The period map $\mathbf{M} \to \mathrm{Sh}_{\mathcal{K}_F}[\mathrm{SO}, \Omega]$ from Theorem 4.7.18 is an open immersion.*

Remark 4.7.19. By Remark 4.7.13, we can prove a statement similar to Theorem 4.7.18 for moduli spaces of oriented polarized hyperkähler varieties deformation equivalent to a generalized Kummer variety. The resulting period map is full by [M5, Theorem 4.3] and Corollary 4.1.14. However it is not an open immersion, since it is not faithful by [BNS, Corollary 3.3].

5

The spinor norm of monodromy operators

In this chapter, we will compute the spinor norm of monodromy operators on K3 surfaces.

In the first section, we recall the definition and basic facts about the spinor norm. In the second section, we state the main result, and compare it to known results. The proof of the result makes use of a theorem of Deligne on the connected components of Shimura varieties, which is stated in the third section. The proof of the main result is given in the fourth section. In the final two sections, we apply the result to sharpen a theorem of Elsenhans and Jahnel on the zeta function of K3 surfaces over finite fields, and to give a necessary condition for a lattice to be the Néron-Severi lattice of a K3 surface over a non-closed field.

5.1 The spinor norm

5.1.1 Generalities

In this section we recall the definition of the spinor norm, and list some results which we will need in later sections. None of the results in this section are original, and proofs for most of them can be found in [C, Appendix C], [K], and [MM2]. We provide proofs for the results which are harder to find in the literature.

Throughout this section, (V, q) will be a quadratic form of over a commutative ring R. That is, V is a locally free R-module of constant finite rank, and q is a map $V \to R$ such that $q(\lambda v) = \lambda^2 q(v)$ for all $\lambda \in R$ and $v \in V$, and such that the map $b_q \colon V \times V \to R$ given by

$$(v, w) \longmapsto q(v + w) - q(v) - q(w)$$

is a bilinear form. The map b_q is known as the bilinear form associated with q.

Moreover, in this section, we assume that V is self-dual in the sense that b_q induces an isomorphism $V \to V^\vee$. In case 2 is *not* invertible in R, we additionally assume that V has even dimension.

The reason for restricting our attention to quadratic forms satisfying these conditions is that for such quadratic forms, the group scheme $\mathrm{O}(V)$ admits a natural central extension in the fppf topology, namely

$$1 \longrightarrow \mu_2 \longrightarrow \mathrm{Pin}(V) \longrightarrow \mathrm{O}(V) \longrightarrow 1. \tag{5.1}$$

The group $\mathrm{Pin}(V)$ is known as the **Pin group of** V, and is constructed using the Clifford algebra of V. See [C, Appendix C.5].

Definition 5.1.1. The connecting homomorphism $\mathrm{O}(V)(R) \to \mathrm{H}^1(R_{\mathrm{fppf}}, \mu_2)$ coming from (5.1) is known as the **spinor norm**, and will be denoted ν_V.

Remark 5.1.2. We denote by $-V$ the quadratic form $(V, -q)$. Note that $\mathrm{O}(V) = \mathrm{O}(-V)$. In general, ν_{-V} does *not* coincide with ν_V, see Lemma 5.1.9. Some authors refer to ν_{-V} as the spinor norm, notably [H2].

Remark 5.1.3. In working with the spinor norm, we will frequently make use of the exact sequence

$$1 \longrightarrow R^\times/2 \longrightarrow \mathrm{H}^1(R_{\mathrm{fppf}}, \mu_2) \longrightarrow \mathrm{Pic}(R)[2] \longrightarrow 1$$

coming from the Kummer sequence. In particular, when $\mathrm{Pic}(R)$ is trivial, we will identify the spinor norm with a map $\mathrm{O}(V)(R) \to R^\times/2$.

Example 5.1.4. In later sections, we will frequently consider self-dual *even* $\mathbf{Z}$-lattices Λ. Such lattices arise as the associated bilinear form of a self-dual quadratic form over $\mathbf{Z}$. Moreover, these lattices have even rank by [H2, Theorem 14.1.1], so we have a spinor norm $\nu_\Lambda \colon \mathrm{O}(\Lambda)(\mathbf{Z}) \to \mathbf{Z}^\times/2 = \{\pm 1\}$.

The following lemmas collect some basic identities for the spinor norm.

Lemma 5.1.5. *Let V and W be quadratic spaces over R. For $g \in \mathrm{O}(V)(R)$ and $h \in \mathrm{O}(W)(R)$, the direct sum $g \oplus h$ is an orthogonal transformation of $V \oplus W$, and*

$$\nu_{V \oplus W}(g \oplus h) = \nu_V(g)\nu_W(h).$$

Lemma 5.1.6. *Let V be a quadratic space over R, and $R \to R'$ a ring homomorphism. Then the diagram*

$$\begin{array}{ccc} \mathrm{O}(V)(R) & \longrightarrow & \mathrm{O}(V)(R') \\ \nu \downarrow & & \downarrow \nu \\ \mathrm{H}^1(R_{\mathrm{fppf}}, \mu_2) & \longrightarrow & \mathrm{H}^1(R'_{\mathrm{fppf}}, \mu_2) \end{array}$$

commutes.

Similarly to (5.1), the group $\mathrm{O}(V)$ also admits a natural central extension $\mathrm{GPin}(V)$ by $\mathbf{G}_m$, constructed using the Clifford algebra of V, see [C, Appendix C.4]. This group scheme comes with a morphism $N \colon \mathrm{GPin}(V) \to \mathbf{G}_m$ known as the **Clifford norm**, the kernel of which is $\mathrm{Pin}(V)$. The facts we need about $\mathrm{GPin}(V)$

and $\mathrm{Pin}(V)$ are summarized by the following commutative diagram:

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
1 \arrow[r] & \mu_2 \arrow[r] \arrow[d] & \mathrm{Pin}(V) \arrow[r] \arrow[d] & \mathrm{O}(V) \arrow[r] \arrow[d, "\mathrm{id}"] & 1 \\
1 \arrow[r] & \mathbf{G}_m \arrow[r] \arrow[d, "2"'] & \mathrm{GPin}(V) \arrow[r] \arrow[d, "N"] & \mathrm{O}(V) \arrow[r] & 1 \\
 & \mathbf{G}_m \arrow[r, "\mathrm{id}"'] \arrow[d] & \mathbf{G}_m \arrow[d] & & \\
 & 1 & 1 & &
\end{tikzcd}.$$

The two columns on the left are fppf exact sequences, and the two top rows are central extensions. The top left square is cartesian.

The group scheme $\mathrm{GPin}(V)$ is related to the group scheme $\mathrm{GSpin}(V)$ of Section 2.3 by the cartesian square

$$\begin{tikzcd}
\mathrm{GSpin}(V) \arrow[r] \arrow[d] & \mathrm{GPin}(V) \arrow[d] \\
\mathrm{SO}(V) \arrow[r] & \mathrm{O}(V)
\end{tikzcd}.$$

The following lemma relates the spinor norm to the Clifford norm. This is useful because the Clifford norm is a morphism of group schemes, whereas the spinor norm is not.

Lemma 5.1.7. *Let V be a quadratic space over R. Then the diagram*

$$\begin{tikzcd}
\mathrm{GPin}(V)(R) \arrow[r, "N"] \arrow[d] & R^\times \arrow[d] \\
\mathrm{O}(V)(R) \arrow[r, "\nu"'] & \mathrm{H}^1(R_{\mathrm{fppf}}, \mu_2)
\end{tikzcd},$$

commutes, where the map $R^\times \to \mathrm{H}^1(R_{\mathrm{fppf}}, \mu_2)$ is the connecting homomorphism coming from the Kummer sequence.

Proof. The short exact sequence

$$1 \to \mu_2 \longrightarrow \mathbf{G}_m \times \mathrm{Pin}(V) \longrightarrow \mathrm{GPin}(V) \to 1,$$

gives rise to a connecting homomorphism $\delta\colon \mathrm{GPin}(V)(R) \to \mathrm{H}^1(R_{\mathrm{fppf}}, \mu_2)$. The commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \mu_2 \arrow[r] \arrow[d] & \mathbf{G}_m \times \mathrm{Pin}(V) \arrow[r] \arrow[d] & \mathrm{GPin}(V) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mu_2 \arrow[r] & \mathrm{Pin}(V) \arrow[r] & \mathrm{O}(V) \arrow[r] & 1
\end{tikzcd}$$

shows that δ coincides with the composition $\mathrm{GPin}(V)(R) \to \mathrm{O}(V)(R) \xrightarrow{\nu} \mathrm{H}^1(R_{\mathrm{fppf}}, \mu_2)$. Similarly, the commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \mu_2 \arrow[r] \arrow[d] & \mathbf{G}_m \times \mathrm{Pin}(V) \arrow[r] \arrow[d] & \mathrm{GPin}(V) \arrow[r] \arrow[d, "N"] & 1 \\
1 \arrow[r] & \mu_2 \arrow[r] & \mathbf{G}_m \arrow[r, "2"'] & \mathbf{G}_m \arrow[r] & 1
\end{tikzcd}$$

shows that δ coincides with the composition $\mathrm{GPin}(V)(R) \xrightarrow{N} R^\times \to \mathrm{H}^1(R_{\mathrm{fppf}}, \mu_2)$. From this we conclude the lemma. $\square$

Suppose $v \in V$ with $q(v) \in R^\times$. Then

$$w \longmapsto w - \frac{b_q(v, w)}{q(v)} v$$

defines an element $r_v \in \mathrm{O}(V)(R)$, called the reflection through v. The following lemma computes the spinor norm on reflections.

Lemma 5.1.8. *Let $v \in V$ be such that $q(v) \in R^\times$. Then $\nu(r_v)$ is the image of $q(v)$ under the map $R^\times/2 \to \mathrm{H}^1(R_{\mathrm{fppf}}, \mu_2)$ coming from the Kummer sequence.*

5.1.2 Quadratic forms over fields of characteristic $\neq 2$

In this subsection, we take R to be a field k of characteristic $\neq 2$.

The Cartan-Dieudonné theorem says that $\mathrm{O}(V)(k)$ is generated by reflections, so Lemma 5.1.8 allows us to compute the spinor norm of any orthogonal transformation. We can also use this to see how ν_V relates to ν_{-V}.

Lemma 5.1.9. *Let V be a quadratic form over a field k of characteristic $\neq 2$. Then for all $g \in \mathrm{O}(V)(k)$,*

$$\nu_V(g) = \det(g)\nu_{-V}(g)$$

in $k^\times/2$.

Proof. By the Cartan-Dieudonné theorem it suffices to check this on reflections, which can be done using Lemma 5.1.8 and the fact that the determinant of a reflection is -1. $\square$

Remark 5.1.10. This result holds for more general base rings. For instance, suppose Λ is an even self-dual lattice. Then the injectivity of $\mathbf{Z}^\times/2 \to \mathbf{R}^\times/2$ and Lemma 5.1.9 applied to $\Lambda \otimes \mathbf{R}$ show that $\nu_\Lambda = \det \cdot \nu_{-\Lambda}$.

Over fields of characteristic $\neq 2$ there is another convenient way to compute spinor norms, given by the Zassenhaus formula.

Lemma 5.1.11 (Zassenhaus formula, [C, Theorem C.5.7])**.** *Let V be a quadratic space over a field k of characteristic $\neq 2$. For $g \in \mathrm{O}(V)(k)$, let $V_0 \subseteq V$ be the maximal subspace on which $1+g$ is nilpotent, and let V_1 be its orthogonal complement. Then*

$$\nu(g) = \mathrm{disc}(V_0) \det \left(\frac{1+g}{2} \bigg|_{V_1} \right) \quad \text{in } k^\times/2,$$

where $\mathrm{disc}(V_0)$ is defined to be the determinant of the Gram matrix of V_0 with respect to any basis of V_0.

By applying the Zassenhaus formula to $-\mathrm{id}_V$, we immediately obtain the following identity.

Lemma 5.1.12. *If V is a quadratic space over a field k of characteristic $\neq 2$, then*

$$\nu(-\mathrm{id}_V) = \mathrm{disc}(V)$$

holds in $k^\times/2$.

The Zassenhaus formula also has the following consequence, which we will use in Section 5.5.

Lemma 5.1.13. *Let k be a field of characteristic $\neq 2$, let V be a quadratic form over k, and $g \in \mathrm{O}(V)(k)$. If g does not have -1 as an eigenvalue, then $\det(g)$ is a square in $k^\times$.*

Proof. Lemma 5.1.11, applied to both V and $-V$, says that

$$\nu_V(g) = \det\left(\frac{1+g}{2}\right) = \nu_{-V}(g)$$

in $k^\times/2$. Combining this with Lemma 5.1.9, which states that $\nu_{-V}(g) = \det(g)\nu_V(g)$, yields the result. ☐

5.1.3 The image of the spinor norm over arithmetically interesting rings

In this subsection we collect some results on the image of the spinor norm.

The first says that the spinor norm is surjective on adelic points and $\mathbf{Q}$-points for indefinite quadratic spaces of rank ≥ 3 over $\mathbf{Q}$.

Lemma 5.1.14. *Let V be an indefinite quadratic space over $\mathbf{Q}$ of rank ≥ 3. The Clifford norms $N_{\mathbf{A}} \colon \mathrm{GSpin}(V)(\mathbf{A}) \to \mathbf{A}^\times$ and $N_{\mathbf{Q}} \colon \mathrm{GSpin}(V)(\mathbf{Q}) \to \mathbf{Q}^\times$ are surjective. The spinor norms $\nu_{\mathbf{A}} \colon \mathrm{SO}(V)(\mathbf{A}) \to \mathbf{A}^\times/2$ and $\nu_{\mathbf{Q}} \colon \mathrm{SO}(V)(\mathbf{Q}) \to \mathbf{Q}^\times/2$ are surjective.*

Proof. We will use Spin, GSpin, and SO to denote $\mathrm{Spin}(V)$, $\mathrm{GSpin}(V)$, and $\mathrm{SO}(V)$, respectively.

If R is a $\mathbf{Q}$-algebra with $\mathrm{Pic}(R) = 1$, then the map $\mathrm{GSpin}(R) \to \mathrm{SO}(R)$ is surjective. Therefore we can conclude from Lemma 5.1.7 that if N_R is surjective, then ν_R is also surjective. As such, since $\mathbf{A}$ and $\mathbf{Q}$ have trivial Picard groups (see Lemma 2.3.26), we only have to show the surjectivity of $N_{\mathbf{A}}$ and $N_{\mathbf{Q}}$.

By [C, Lemma C.4.1, Proposition C.4.10] Spin is simply connected, from which it follows that $\mathrm{H}^1(\mathbf{Q}_{p,\text{ét}}, \mathrm{Spin}) = 1$ for every prime p ([PR, Theorem 6.4]), and hence that $N_{\mathbf{Q}_p}$ is surjective for all p. Let $\Lambda \subseteq V$ be a full $\mathbf{Z}$-lattice. This yields integral models $\mathrm{GSpin}(\Lambda)$ and $\mathrm{Spin}(\Lambda)$ of the group schemes GSpin and Spin. Given the surjectivity of $N_{\mathbf{Q}_p}$ for all p, to show that $N_{\mathbf{A}_\mathrm{f}}$ is surjective, it suffices to show that $N_{\mathbf{Z}_p} \colon \mathrm{GSpin}(\Lambda)(\mathbf{Z}_p) \to \mathbf{Z}_p^\times$ is surjective for all but finitely many p.

Let p be an odd prime number coprime to the discriminant of Λ, so that $\mathrm{Spin}(\Lambda \otimes \mathbf{Z}_p)$ is a smooth connected group scheme, which follows from [C, Theorem C.1.5], the smoothness of μ_2 over $\mathbf{Z}_p$, and the short exact sequence

$$1 \to \mu_2 \longrightarrow \mathrm{Spin}(\Lambda \otimes \mathbf{Z}_p) \longrightarrow \mathrm{SO}(\Lambda \otimes \mathbf{Z}_p) \to 1.$$

For $\lambda \in \mathbf{Z}_p^\times$, we wish to find an element of $\mathrm{GSpin}(\Lambda)(\mathbf{Z}_p)$ lifting λ. That is, we want to show the existence of the diagonal dashed arrow in the diagram

$$\begin{tikzcd}
\mathrm{GSpin}(\Lambda \otimes \mathbf{Z}_p) \arrow[r, "N"] & \mathbf{G}_m \\
\mathrm{Spec}(\mathbf{F}_p) \arrow[r] \arrow[u, dashed] & \mathrm{Spec}(\mathbf{Z}_p) \arrow[u, "\lambda"'] \arrow[ul, dashed]
\end{tikzcd}$$

Since the kernel $\mathrm{Spin}(\Lambda \otimes \mathbf{F}_p)$ of N over $\mathbf{F}_p$ is connected, Lang's theorem [PR, Theorem 6.1] shows the existence of the dashed arrow on the left. The smoothness of the kernel $\mathrm{Spin}(\Lambda \otimes \mathbf{Z}_p)$ of N implies the smoothness of N itself [EvdGM, Corollary 4.33]. This allows us to apply Hensel's lemma [DG, Théorème 18.5.17] to show the existence of the diagonal dashed arrow.

Let $e_1, e_2 \in V_{\mathbf{R}}$ be orthogonal elements with $e_1^2 = -1$ and $e_2^2 = 1$, which exist because V is indefinite. Then for $\lambda \in \mathbf{R}$ we have $N_{\mathbf{R}}(\lambda e_1 e_2) = -\lambda^2$ and $N_{\mathbf{R}}(\lambda) = \lambda^2$, proving the surjectivity of $N_{\mathbf{R}}$, which, combined with the surjectivity of $N_{\mathbf{A}_\mathrm{f}}$, implies the surjectivity of $N_{\mathbf{A}}$.

To see that $N_{\mathbf{Q}}$ is surjective, it suffices to show the triviality of the connecting homomorphism $\delta \colon \mathbf{Q}^\times \to \mathrm{H}^1(\mathbf{Q}_{\text{ét}}, \mathrm{Spin})$ derived from the short exact sequence

$$1 \to \mathrm{Spin} \longrightarrow \mathrm{GSpin} \longrightarrow \mathbf{G}_m \to 1.$$

From the surjectivity of $N_{\mathbf{R}}$ it follows that $\mathbf{R}^\times \to \mathrm{H}^1(\mathbf{R}_{\text{ét}}, \mathrm{Spin})$ is trivial, and in particular that the composition $\mathbf{Q}^\times \to \mathrm{H}^1(\mathbf{Q}_{\text{ét}}, \mathrm{Spin}) \to \mathrm{H}^1(\mathbf{R}_{\text{ét}}, \mathrm{Spin})$ is trivial. From the Hasse principle for simply connected groups ([PR, Theorem 6.6]), which says that $\mathrm{H}^1(\mathbf{Q}_{\text{ét}}, \mathrm{Spin}) \to \mathrm{H}^1(\mathbf{R}_{\text{ét}}, \mathrm{Spin})$ is a bijection, we obtain that δ is trivial. $\square$

Let ℓ be a prime number, and Λ a $\mathbf{Z}_\ell$-lattice. When $\ell = 2$ we require Λ to be even, to ensure that it has an associated quadratic form. Note that Λ is automatically even when ℓ is odd. We denote by $\Delta(\Lambda)$ the discriminant form of Λ, i.e., the group $\Lambda^\vee/\Lambda$ endowed with the natural quadratic form $\Lambda^\vee/\Lambda \to \mathbf{Q}/\mathbf{Z}$ induced by the extension of the bilinear form on Λ to $\Lambda^\vee$. Note that $\mathrm{O}(\Lambda)$ acts on $\Delta(\Lambda)$. We denote by $\widetilde{\mathrm{O}}(\Lambda)$ the group

$$\left\{ g \in \mathrm{O}(\Lambda) \mid g|_{\Delta(\Lambda)} = \mathrm{id}_{\Delta(\Lambda)} \right\}.$$

On this group, we can define a $\mathbf{Z}_\ell^\times/2$-valued spinor norm ν_Λ, as the following lemma shows.

Lemma 5.1.15. *Let Λ be an even $\mathbf{Z}_\ell$-lattice. There is a unique homomorphism $\nu_\Lambda \colon \widetilde{\mathrm{O}}(\Lambda) \to \mathbf{Z}_\ell^\times/2$ for which the square*

$$\begin{tikzcd}
\widetilde{\mathrm{O}}(\Lambda) \arrow[r, "\nu_\Lambda"] \arrow[d] & \mathbf{Z}_\ell^\times/2 \arrow[d] \\
\mathrm{O}(\Lambda \otimes \mathbf{Q}_\ell) \arrow[r, "\nu_{\Lambda \otimes \mathbf{Q}_\ell}"'] & \mathbf{Q}_\ell^\times/2
\end{tikzcd}.$$

commutes.

Proof. The uniqueness of ν_Λ follows from the injectivity of $\mathbf{Z}_\ell^\times/2 \to \mathbf{Q}_\ell^\times/2$.

Note that even though Λ need not be self-dual, $\Lambda \otimes \mathbf{Q}_\ell$ is self-dual, so we have a spinor norm $\nu_{\Lambda \otimes \mathbf{Q}_\ell} \colon \mathrm{O}(\Lambda \otimes \mathbf{Q}_\ell) \to \mathbf{Q}_\ell^\times/2$. We need to prove that the image of the composition

$$\widetilde{\mathrm{O}}(\Lambda) \longrightarrow \mathrm{O}(\Lambda \otimes \mathbf{Q}_\ell) \longrightarrow \mathbf{Q}_\ell^\times/2$$

is contained in $\mathbf{Z}_\ell^\times/2$.

Let Λ' be a self-dual even $\mathbf{Z}_\ell$-lattice into which Λ embeds. Then Definition 5.1.1 gives a spinor norm $\nu \colon \mathrm{O}(\Lambda') \to \mathbf{Z}_\ell^\times/2$. Moreover, the map $\mathrm{O}(\Lambda \otimes \mathbf{Q}_\ell) \to \mathrm{O}(\Lambda' \otimes \mathbf{Q}_\ell)$ given by $g \mapsto g \oplus \mathrm{id}_{\Lambda^\perp}$ restricts to an injective map $\widetilde{\mathrm{O}}(\Lambda) \to \mathrm{O}(\Lambda')$. Now the diagram

$$\begin{tikzcd}
\widetilde{\mathrm{O}}(\Lambda) \arrow[r] \arrow[d] & \mathrm{O}(\Lambda') \arrow[d] \arrow[dr, "\nu"] & \\
\mathrm{O}(\Lambda \otimes \mathbf{Q}_\ell)) \arrow[r] \arrow[dr, "\nu"'] & \mathrm{O}(\Lambda' \otimes \mathbf{Q}_\ell) \arrow[d, "\nu"] & \mathbf{Z}_\ell^\times/2 \arrow[dl] \\
& \mathbf{Q}_\ell^\times/2 &
\end{tikzcd}$$

which commutes by Lemma 5.1.5 and Lemma 5.1.6, shows that ν_Λ lands in $\mathbf{Z}_\ell^\times/2$. □

Remark 5.1.16. When confusion is unlikely to arise, we will denote the map ν_Λ by ν.

We are interested in the image of $(\det, \nu)\colon \widetilde{\mathrm{O}}(\Lambda) \to \{\pm 1\} \times \mathbf{Z}_\ell^\times/2$. We now define some invariants of Λ in terms of which completely determine the image of $(\det, \nu)$.

For a finite abelian group A, we denote by $\mathrm{length}(A)$ the minimal number of elements needed to generate A. Note that $\mathrm{rk}\,\Lambda \geq \mathrm{length}(\Delta(\Lambda))$.

Let ℓ be an odd prime number, and Λ an even $\mathbf{Z}_\ell$-lattice. Then by [N1, Theorem 1.9.1], there exists a unique (up to isomorphism) $\mathbf{Z}_\ell$-lattice Λ_1 of rank $\mathrm{length}(\Delta(\Lambda))$ whose discriminant form is isomorphic to $\Delta(\Lambda)$. It is clear that $\mathrm{disc}(\Lambda_1) \in \mathbf{Z}_\ell^\times/2$ only depends on the discriminant form $\Delta(\Lambda)$.

Definition 5.1.17. For Λ and Λ_1 as above, we denote the invariant $\mathrm{disc}(\Lambda_1) \in \mathbf{Z}_\ell^\times/2$ of $\Delta(\Lambda)$ with $\mathrm{disc}(\Delta(\Lambda))$.

Theorem 5.1.18 ([MM2, Theorem VII.12.1])**.** *Let ℓ be an odd prime number, and Λ an even $\mathbf{Z}_\ell$-lattice. Then*

$$(\det, \nu)\, \widetilde{\mathrm{O}}(\Lambda) = \begin{cases} \{(1,1)\} & \textit{if } \mathrm{rk}\,\Lambda = \mathrm{length}(\Delta(\Lambda)) \\ \{(1,1), (-1, 2\,\mathrm{disc}(\Delta(\Lambda)))\} & \textit{if } \mathrm{rk}\,\Lambda = \mathrm{length}(\Delta(\Lambda)) + 1 \\ \{\pm 1\} \times \mathbf{Z}_\ell^\times/2 & \textit{otherwise,} \end{cases}$$

as a subgroup of $\{\pm 1\} \times \mathbf{Z}_\ell^\times/2$.

Remark 5.1.19. For $\ell = 2$, the image of $(\det, \nu)$ is also completely determined by $\Delta(\Lambda)$ and $\mathrm{rk}\,\Lambda$, but the result is much more complicated than for odd ℓ. The interested reader is referred to [MM2, Theorems VII.12.2, VII.12.3, VII.12.4] for the full statement.

For an even self-dual $\mathbf{Z}_\ell$-lattice Λ' and a sublattice Λ of Λ', we denote by $\mathrm{O}(\Lambda', \Lambda)$ the group

$$\mathrm{O}(\Lambda', \Lambda) = \{g \in \mathrm{O}(\Lambda') \mid g|_\Lambda = \mathrm{id}_\Lambda\}\,. \tag{5.2}$$

Consider the product $\det \cdot \nu\colon \mathrm{O}(\Lambda') \to \mathbf{Z}_\ell^\times/2$ of the spinor norm $\nu\colon \mathrm{O}(\Lambda') \to \mathbf{Z}_\ell^\times/2$ with the composition

$$\mathrm{O}(\Lambda') \xrightarrow{\det} \mu_2(\mathbf{Z}_\ell) \longrightarrow \mathbf{Z}_\ell^\times/2.$$

In Section 5.6, it will be useful to know when the image of $\mathrm{O}(\Lambda', \Lambda)$ under $\det \cdot \nu$ is trivial. The following corollary of Theorem 5.1.18 gives a necessary and sufficient criterion in terms of the ranks of Λ' and Λ, the invariant $\mathrm{disc}(\Delta(\Lambda))$, and $\mathrm{length}(\Delta(\Lambda))$.

Corollary 5.1.20. *Let ℓ be an odd prime number, Λ' a self-dual even $\mathbf{Z}_\ell$-lattice, and Λ a primitive sublattice of Λ'. Then* $\det \cdot \nu(\mathrm{O}(\Lambda', \Lambda)) = 1$ *if and only if*

$$\mathrm{rk}\,\Lambda + \mathrm{length}(\Delta(\Lambda)) = \mathrm{rk}\,\Lambda' - 1$$

and the product

$$(-1)^{\operatorname{rk}\Lambda} 2 \operatorname{disc}(\Delta(\Lambda))$$

is equal to 1 *in* $\mathbf{Z}_\ell^\times/2$*, or if*

$$\operatorname{rk}\Lambda + \operatorname{length}(\Delta(\Lambda)) = \operatorname{rk}\Lambda'.$$

Proof. Let $\Lambda^\perp$ be the orthogonal complement of Λ in Λ'. Since Λ' is self-dual, there is an isomorphism $\mathrm{O}(\Lambda', \Lambda) \to \widetilde{\mathrm{O}}(\Lambda^\perp)$ mapping g to its restriction to $\Lambda^\perp$. Similarly to the proof of Lemma 5.1.15, there is a commutative diagram

$$\begin{tikzcd} \mathrm{O}(\Lambda', \Lambda) \arrow[r, "\sim"] \arrow[dr, "\det \cdot \nu"'] & \widetilde{\mathrm{O}}(\Lambda^\perp) \arrow[d, "\det \cdot \nu"] \\ & \mathbf{Z}_\ell^\times/2 \end{tikzcd},$$

so that $\det \cdot \nu(\mathrm{O}(\Lambda', \Lambda)) = \det \cdot \nu(\widetilde{\mathrm{O}}(\Lambda^\perp))$. It now follows from Theorem 5.1.18 applied to $\Lambda^\perp$ that $\det \cdot \nu(\mathrm{O}(\Lambda, \Lambda'))$ is trivial if and only if

$$\operatorname{rk}\Lambda^\perp = \operatorname{length}(\Delta(\Lambda^\perp)) + 1$$

and $-2\operatorname{disc}(\Delta(\Lambda^\perp))$ is a square in $\mathbf{Z}_\ell^\times$, or if $\operatorname{rk}\Lambda^\perp = \operatorname{length}(\Delta(\Lambda^\perp))$.

We will now restate these conditions in terms of invariants of Λ. Let Γ be the unique even $\mathbf{Z}_\ell$-lattice of rank equal to $\operatorname{length}(\Delta(\Lambda))$ whose discriminant form is isomorphic to $\Delta(\Lambda)$ (see [N1, Theorem 1.9.1]). Then $\operatorname{disc}(\Delta(\Lambda))$ is equal to $\operatorname{disc}(\Gamma)$. Moreover, since $\Lambda^\perp$ is the orthogonal complement of Λ in the self-dual lattice Λ', we have $\Delta(\Lambda^\perp) \cong -\Delta(\Lambda)$. It follows that

$$\operatorname{disc}(\Delta(\Lambda^\perp)) = \operatorname{disc}(-\Gamma) = (-1)^{\operatorname{rk}\Gamma}\operatorname{disc}(\Gamma) = (-1)^{\operatorname{length}(\Delta(\Lambda))}\operatorname{disc}(\Delta(\Lambda)).$$

Moreover, $\operatorname{length}(\Delta(\Lambda^\perp)) = \operatorname{length}(\Delta(\Lambda))$, and $\operatorname{rk}\Lambda^\perp = \operatorname{rk}\Lambda' - \operatorname{rk}\Lambda$. This finishes the proof of the corollary. □

5.2 Statement of the result

In this section we state the main result of this chapter, which computes the spinor norm of monodromy operators on K3 surfaces. We also give some corollaries of the main theorem which are proved in later sections.

Before stating the main theorem, we introduce some notation.

Recall that the Kronecker-Weber theorem identifies $\mathrm{Gal}_{\mathbf{Q}}^{\mathrm{ab}}$ with $\widehat{\mathbf{Z}}^\times$. Let ℓ be a prime number. The surjection $\widehat{\mathbf{Z}}^\times \to \mathbf{Z}_\ell^\times/2$ gives rise to a number field K. For $\ell = 2$, the ring of integers $\mathcal{O}_K$ is $\mathbf{Z}[\zeta_8]$, where ζ_8 is a primitive 8th root of unity, and for odd ℓ the ring of integers is $\mathbf{Z}\left[\frac{1+\sqrt{\ell^*}}{2}\right]$, where $\ell^* = (-1)^{\frac{\ell-1}{2}}\ell$. Since K is unramified away from ℓ, the ring $\mathcal{O}_K\left[\frac{1}{\ell}\right]$ is étale over $\mathbf{Z}[\frac{1}{\ell}]$. Moreover, the action of $\mathrm{Gal}(K/\mathbf{Q}) \cong \mathbf{Z}_\ell^\times/2$ on K extends to an action on $\mathcal{O}_K\left[\frac{1}{\ell}\right]$, making

$T_\ell := \mathrm{Spec}(\mathcal{O}_K[\frac{1}{\ell}])$ a $\mathbf{Z}_\ell^\times/2$-torsor on $\mathbf{Z}[\frac{1}{\ell}]_{\text{ét}}$. In particular, T_ℓ has degree 2 over $\mathbf{Z}[\frac{1}{\ell}]$ when ℓ is odd, and degree 4 when $\ell = 2$.

For a $\mathbf{Z}[\frac{1}{\ell}]$-scheme S, we denote by $T_{\ell,S}$ the $(\mathbf{Z}_\ell^\times/2)$-torsor on $S_{\text{ét}}$ defined by the cartesian diagram

$$\begin{tikzcd} T_{\ell,S} \arrow[r] \arrow[d] & T_\ell \arrow[d] \\ S \arrow[r] & \mathrm{Spec}\big(\mathbf{Z}\big[\frac{1}{\ell}\big]\big) \end{tikzcd} \quad . \tag{5.3}$$

Given a geometric point $\overline{s}$ of S, we denote the homomorphism $\pi_1^{\text{ét}}(S,\overline{s}) \to \mathbf{Z}_\ell^\times/2$ associated with $T_{\ell,S}$ by χ_ℓ.

The following is the main result of this chapter. It is proved in Section 5.4.

Theorem 5.2.1. *Let ℓ be a prime number, $d \in \mathbf{Z}_{>0}$, S a scheme over $\mathbf{Z}\big[\frac{1}{2d\ell}\big]$, and $\overline{s}$ a geometric point of S. For a projective K3 surface $f\colon X \to S$ of degree $2d$, the following diagram commutes:*

$$\begin{tikzcd} \pi_1^{\text{ét}}(S,\overline{s}) \arrow[r] \arrow[dr, "\chi_\ell"'] & \mathrm{O}(\mathrm{H}^2_{\text{ét}}(X_{\overline{s}}, \mathbf{Z}_\ell(1))) \arrow[d, "\det \cdot \nu"] \\ & \mathbf{Z}_\ell^\times/2 \end{tikzcd} \quad ,$$

where ν denotes the spinor norm.

Using the triviality of χ_ℓ when $S = \mathrm{Spec}(F)$, with F an algebraically closed field, we immediately obtain the following corollary.

Corollary 5.2.2. *Let S be a scheme over an algebraically closed field F of characteristic p, let ℓ be a prime number distinct from p, $s \in S(F)$, and X a projective K3 surface over S, of degree coprime to p. Then the composition $\pi_1^{\text{ét}}(S,s) \to \mathrm{O}(\mathrm{H}^2_{\text{ét}}(X_s, \mathbf{Z}_\ell(1))) \xrightarrow{\det\cdot\nu} \mathbf{Z}_\ell^\times/2$ is trivial.*

As a corollary, we obtain [H2, Proposition 7.5.5], which states the same result for complex K3 surfaces. Note that if X is a complex K3 surface, [H2] works with $\nu_{-\mathrm{H}^2(X,\mathbf{Z}(1))}$ instead of $\nu_{\mathrm{H}^2(X,\mathbf{Z}(1))}$. As is shown in remark 5.1.10,

$$\nu_{-\mathrm{H}^2(X,\mathbf{Z}(1))} = \det \cdot \nu_{\mathrm{H}^2(X,\mathbf{Z}(1))}.$$

Corollary 5.2.3. *Let S be a scheme over $\mathbf{C}$, $s \in S(\mathbf{C})$, and X a projective K3 surface over S. Then the composition $\pi_1(S,s) \to \mathrm{O}(\mathrm{H}^2(X_s, \mathbf{Z}(1))) \xrightarrow{\det\cdot\nu} \mathbf{Z}^\times/2$ is trivial.*

Proof. This follows by applying Corollary 5.2.2 and using the injectivity of $\mathbf{Z}^\times/2 \to \mathbf{Z}_\ell^\times/2$ for $\ell \equiv 3(4)$. ☐

The result also allows us to compute the spinor norm of the Frobenius operator acting on the second cohomology of a K3 surface over a finite field. The proof is contained in Section 5.5. The corollary also gives rise to a restriction on the zeta function of a K3 surface over a finite field, see Corollary 5.5.3

Corollary 5.2.4. *Let $\mathbf{F}_q$ be a finite field, X a K3 surface over $\mathbf{F}_q$ of degree coprime to q, and ℓ a prime number coprime to q. Then*

$$\nu\left(\mathrm{Frob}_q\big|_{\mathrm{H}^2_{\text{ét}}\left(X_{\overline{\mathbf{F}}_q}, \mathbf{Z}_\ell(1)\right)}\right) = q \cdot \det\left(\mathrm{Frob}_q\big|_{\mathrm{H}^2_{\text{ét}}\left(X_{\overline{\mathbf{F}}_q}, \mathbf{Z}_\ell(1)\right)}\right)$$

in $\mathbf{Z}_\ell^\times/2$.

The theorem also gives rise to a necessary condition for the realizability of sublattices of the K3 lattice as Néron-Severi groups of K3 surfaces over a given field, as the following corollary shows. See Theorem 5.6.1 for a slightly stronger statement.

Corollary 5.2.5. *Let k be a field, let ℓ be an odd prime number, and let X/k be a K3 surface of degree coprime to the characteristic of k. If*

$$\mathrm{rk}(\mathrm{Pic}(X)) + \mathrm{length}(\Delta(\mathrm{Pic}(X) \otimes \mathbf{Z}_\ell)) = 22$$

then ℓ^* is a square in k.

5.3 The reciprocity law for Shimura stacks

This section contains some Shimura-theoretic preliminaries necessary for the proof of Theorem 5.2.1. The first subsection is about Deligne's reciprocity law for the connected components of a Shimura variety. In the second subsection we apply Deligne's reciporicity law to orthogonal Shimura stacks.

5.3.1 The reciprocity law for Shimura stacks

In this subsection we recall a result of Deligne on the structure of the set of connected components of a Shimura varieties and the Galois action on it, known as Deligne's reciprocity law. All of these results can be found in [D4]. We work in the slightly more general setting of Shimura stacks, but the results carry over to our setting with minimal modifications.

If G is a reductive group over a number field E, we denote by $\widetilde{G}$ the universal covering of the derived subgroup of G. We then define $\pi(G)$ to be the quotient set

$$\pi(G) = G(\mathbf{A}_E)/G(E)\widetilde{G}(\mathbf{A}_E).$$

Lemma 5.3.1 ([D4, Corollaire 2.0.8, (2.4.0.1)])**.** *Let E be a number field, and G a reductive group over E. Then $G(E)\widetilde{G}(\mathbf{A}_E) \subseteq G(\mathbf{A}_E)$ is a normal subgroup, and the quotient $\pi(G)$ is a locally compact Hausdorff abelian group. This construction defines a functor*

$$\pi\colon \Big(\text{reductive } E\text{-groups}\Big) \longrightarrow \begin{pmatrix}\text{locally compact Hausdorff} \\ \text{abelian groups}\end{pmatrix}$$

Remark 5.3.2. If E'/E is a finite extension, and G_E a reductive E-group, Deligne constructs a natural homomorphism

$$N_{E'/E}\colon \pi(G_{E'}) \longrightarrow \pi(G_E),$$

called the norm, see [D4, (2.4.0.1)]. This homomorphism is needed to state Deligne's reciprocity law in full generality. In all Shimura data we will deal with outside this section the reflex field is $\mathbf{Q}$, so all norms we encounter are the identity.

Example 5.3.3. If $G = \mathrm{GL}_2$ over $\mathbf{Q}$, then $\widetilde{G} = \mathrm{SL}_2$, so the determinant yields an isomorphism $\pi(G) \cong \mathbf{Q}^\times \setminus \mathbf{A}^\times$. In the next subsection we will see that when $G = \mathrm{SO}(V)$, where V is a quadratic space over $\mathbf{Q}$ of signature $(2, n)$ with $n \geq 1$, then the spinor norm yields an isomorphism $\pi(\mathrm{SO}(V)) \cong \mathbf{Q}^\times \setminus \mathbf{A}^\times/2$.

Example 5.3.4. If E is a number field, and $G = \mathbf{G}_{m,E}$, then $\pi(G) = E^\times \setminus \mathbf{A}_E^\times$. Artin's reciprocity law is a homomorphism $\pi(\mathbf{G}_{m,E}) \to \mathrm{Gal}_E^{\mathrm{ab}}$ inducing an isomorphism $\pi_0\pi(\mathbf{G}_{m,E}) \to \mathrm{Gal}_E^{\mathrm{ab}}$. We will denote its reciprocal by

$$\mathrm{art}_E\colon \pi_0\pi(\mathbf{G}_{m,E}) \longrightarrow \mathrm{Gal}_E^{\mathrm{ab}}.$$

We now restrict our attention to reductive $\mathbf{Q}$-groups G. Let G^{ad} be the adjoint group of G. We use $G(\mathbf{R})_+$ to denote the inverse image of the identity component of $G^{\mathrm{ad}}(\mathbf{R})$ under $G(\mathbf{R}) \to G^{\mathrm{ad}}(\mathbf{R})$. By $\overline{\pi}_0\pi(G)$ we denote the quotient group

$$\overline{\pi}_0\pi(G) = \big(\pi_0\pi(G)\big)/\pi_0(G(\mathbf{R})_+).$$

This construction is relevant to us because of the following result.

Lemma 5.3.5 ([D4, Proposition 1.2.7, Résumé 2.1.16])**.** *Let (G, X) be a Shimura datum with reflex field E, and $\pi_0 \operatorname{Sh}(G, X)$ the E-scheme of connected components of $\operatorname{Sh}(G, X)$. Then $\overline{\pi}_0\pi(G)$ is profinite, and the $G(\mathbf{A}_\mathrm{f})$-action on $\pi_0 \operatorname{Sh}(G, X)$ factors through $G(\mathbf{A}_\mathrm{f}) \to \overline{\pi}_0\pi(G)$, endowing $\pi_0 \operatorname{Sh}(G, X)$ with the structure of a $\overline{\pi}_0\pi(G)$-torsor on E. Moreover, $\pi_0(X)$ is a $G(\mathbf{R})/G(\mathbf{R})_+$-torsor.*

Example 5.3.6. Consider the Siegel Shimura datum (G, X) associated with a symplectic $\mathbf{Q}$-vector space of dimension 2, as in Example 2.1.1. That is, $(G, X) = (\mathrm{GL}_2, \mathcal{H})$, where $\mathcal{H}$ is the double half plane, which parametrizes Hodge structures on $\mathbf{R}^2$ of type $(0, 1) + (1, 0)$. The reflex field of (G, X) is $\mathbf{Q}$. It is easy to see that $\mathrm{GL}_2(\mathbf{R})_+$ is connected, so that $\overline{\pi}_0\pi(G) = \pi_0\pi(G)$. The determinant and Artin reciprocity therefore yield an isomorphism

$$\overline{\pi}_0\pi(G) \xrightarrow{\det} \pi_0(\mathbf{Q}^\times \setminus \mathbf{A}^\times) \xrightarrow{\mathrm{art}_\mathbf{Q}} \mathrm{Gal}_\mathbf{Q}^{\mathrm{ab}}. \tag{5.4}$$

Lemma 5.3.5 gives a $\mathrm{Gal}_\mathbf{Q}^{\mathrm{ab}}$-action on the source, and $\mathrm{Gal}_\mathbf{Q}^{\mathrm{ab}}$ acts on the target by translation. We will see in Example 5.3.8 that Deligne's reciprocity law implies that the isomorphism above is $\mathrm{Gal}_\mathbf{Q}^{\mathrm{ab}}$-equivariant.

For a commutative $\mathbf{C}$-algebra A, the map $A \otimes_\mathbf{R} \mathbf{C} \to A \times A$, $a \otimes z \mapsto (za, \overline{z}a)$ is an isomorphism of rings. This yields an isomorphism $\mathbf{G}_{m,\mathbf{C}} \times \mathbf{G}_{m,\mathbf{C}} \to \mathbf{S}_\mathbf{C}$, which we will use to identify these two group schemes. Let (G, X) be a Shimura datum. For $h \in X$, we define $\mu_h \colon \mathbf{G}_{m,\mathbf{C}} \to G_{\mathbf{C}}$ as $z \mapsto h_{\mathbf{C}}(z, 1)$. The reflex field E of (G, X) is by definition the unique smallest subfield of $\mathbf{C}$ such that the $G(\mathbf{C})$-conjugacy class of μ_h is defined over E. Then E is a number field. It can be shown that μ_h induces a continuous group homomorphism $\pi(\mathbf{G}_{m,E}) \to \pi(G_E)$, which we denote with $\pi\mu_h$, see [D4, §2.4].

We now define a continuous homomorphism

$$r_{(G,X)} \colon \operatorname{Gal}_E \longrightarrow \overline{\pi}_0\pi(G), \tag{5.5}$$

as the following composition

$$\operatorname{Gal}_E \to \operatorname{Gal}_E^{\mathrm{ab}} \xrightarrow{\operatorname{art}_E^{-1}} \pi_0\pi(\mathbf{G}_{m,E}) \xrightarrow{\pi_0\pi(\mu_h)} \pi_0\pi(G_E) \xrightarrow{\pi_0 N_{E/\mathbf{Q}}} \pi_0\pi(G) \to \overline{\pi}_0\pi(G).$$

We now have two $\overline{\pi}_0\pi(G)$-torsors on E, namely $\pi_0 \operatorname{Sh}(G, X)$, and the one defined by (5.5). The following theorem of Deligne says that these two torsors are isomorphic.

Theorem 5.3.7 (Deligne's reciprocity law, [D4, Théorème 2.6.3])**.** *Let (G, X) be a Shimura datum with reflex field E. Then the $\overline{\pi}_0\pi(G)$-torsor $\pi_0 \operatorname{Sh}(G, X)$ on E is isomorphic to the one defined by* (5.5).

Example 5.3.8. We again consider the Shimura datum $(G, X) = (\mathrm{GL}_2, \mathcal{H})$, as in Example 5.3.6. Let $h \colon \mathbf{S} \to G_{\mathbf{R}}$ be an element of $\mathcal{H}$. Then the composition $\det h$ corresponds to the Tate Hodge structure $\mathbf{Q}(-1)$, so that $\det \mu_h \colon \mathbf{G}_{m,\mathbf{C}} \to \mathbf{G}_{m,\mathbf{C}}$ is the identity. From Deligne's reciprocity law it now follows that (5.4) is $\operatorname{Gal}_{\mathbf{Q}}$-equivariant.

We will rephrase 5.3.7 in a way which is more convenient for our purposes. Let (G, X) be a Shimura datum with reflex field E, and $\mathcal{K}$ a profinite group endowed with a continuous homomorphism $\mathcal{K} \to G(\mathbf{A}_{\mathrm{f}})$ with finite kernel and open image. The $G(\mathbf{A}_{\mathrm{f}})$-action on $\operatorname{Sh}(G, X)$ turns $\operatorname{Sh}(G, X)$ into a $\mathcal{K}$-torsor on $\operatorname{Sh}_{\mathcal{K}}[G, X]_{\text{ét}}$. For a geometric point $\overline{s}$ of $\operatorname{Sh}_{\mathcal{K}}[G, X]$, and a geometric point $\widetilde{s}$ of $\operatorname{Sh}(G, X)$ lying over $\overline{s}$, this $\mathcal{K}$-torsor gives rise to a homomorphism $\pi_1^{\text{ét}}(\operatorname{Sh}_{\mathcal{K}}[G, X], \overline{s}) \to \mathcal{K}$.

Lemma 5.3.9. *Let $\widetilde{s}$ be a geometric point of $\operatorname{Sh}(G, X)$, $\overline{s}$ its image in $\operatorname{Sh}_{\mathcal{K}}[G, X]$, and $\rho \colon \pi_1^{\text{ét}}(\operatorname{Sh}_{\mathcal{K}}[G, X], \overline{s}) \to \mathcal{K}$ the resulting homomorphism. Then the diagram*

$$\begin{tikzcd}
\pi_1^{\text{ét}}(\operatorname{Sh}_{\mathcal{K}}[G, X], \overline{s}) \arrow[r, "\rho"] \arrow[dd] & \mathcal{K} \arrow[d] \\
 & G(\mathbf{A}_{\mathrm{f}}) \arrow[d] \\
\operatorname{Gal}_E \arrow[r, "r_{(G,X)}"'] & \overline{\pi}_0\pi(G)
\end{tikzcd}$$

commutes, where $r_{(G,X)}$ is defined by (5.5).

Proof. Let θ be the composition $\mathcal{K} \to G(\mathbf{A}_{\mathrm{f}}) \to \overline{\pi}_0\pi(G)$. Using θ to change the structure group of the $\mathcal{K}$-torsor $\mathrm{Sh}(G, X)$ gives rise to a $\overline{\pi}_0\pi(G)$-torsor on $\mathrm{Sh}_{\mathcal{K}}[G, X]_{\text{ét}}$, which we denote $\theta_* \mathrm{Sh}(G, X)$. By Theorem 5.3.7, it suffices to show that $\theta_* \mathrm{Sh}(G, X)$ is isomorphic to the pullback of $\pi_0 \mathrm{Sh}(G, X)$ to $\mathrm{Sh}_{\mathcal{K}}[G, X]$.

Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Sh}(G, X) \arrow[r] \arrow[d] & \pi_0(\mathrm{Sh}(G, X)) \arrow[d] \\
\mathrm{Sh}_{\mathcal{K}}[G, X] \arrow[r] & \mathrm{Spec}(E)
\end{tikzcd}.$$

Here, $\mathcal{K}$ acts on $\pi_0(\mathrm{Sh}(G, X))$ via θ, and the map $\mathrm{Sh}(G, X) \to \pi_0(\mathrm{Sh}(G, X))$ is $\mathcal{K}$-equivariant. This proves the lemma. $\square$

Example 5.3.10. As in Example 5.3.8, consider the Shimura datum $(\mathrm{GL}_2, \mathcal{H})$. Let $\mathcal{K}$ be the compact open subgroup $\mathrm{GL}_2(\widehat{\mathbf{Z}})$ of $\mathrm{GL}_2(\mathbf{A}_{\mathrm{f}})$, and $\overline{s}$ a geometric point of $\mathrm{Sh}_{\mathcal{K}}[\mathrm{GL}_2, \mathcal{H}]$. Then Lemma 5.3.9 and Example 5.3.8 can be used to show that the diagram

$$\begin{tikzcd}
\pi_1^{\text{ét}}(\mathrm{Sh}_{\mathcal{K}}[\mathrm{GL}_2, \mathcal{H}], \overline{s}) \arrow[r] \arrow[d] & \mathcal{K} \arrow[d, "\det"] \\
\mathrm{Gal}_{\mathbf{Q}} \arrow[r, "\chi"'] & \widehat{\mathbf{Z}}^{\times}
\end{tikzcd}$$

commutes. Here, $\chi\colon \mathrm{Gal}_{\mathbf{Q}} \to \widehat{\mathbf{Z}}^{\times}$ is the cyclotomic character, that is, it is the composition of $\mathrm{Gal}_{\mathbf{Q}} \to \mathrm{Gal}_{\mathbf{Q}}^{\mathrm{ab}}$ with the isomorphism $\mathrm{Gal}_{\mathbf{Q}}^{\mathrm{ab}} \to \widehat{\mathbf{Z}}^{\times}$ given by the Kronecker-Weber theorem. Note that $\mathrm{Sh}_{\mathcal{K}}[\mathrm{GL}_2, \mathcal{H}]$ is the moduli stack of elliptic curves over $\mathbf{Q}$. This can be used to show that for any scheme S over $\mathbf{Q}$ with geometric point $\overline{s}$, and any family E of elliptic curves over S, the diagram

$$\begin{tikzcd}
\pi_1^{\text{ét}}(S, \overline{s}) \arrow[r] \arrow[d] & \mathrm{GL}(\mathrm{H}^1(E_{\overline{s}}, \widehat{\mathbf{Z}})) \arrow[d, "\det"] \\
\mathrm{Gal}_{\mathbf{Q}} \arrow[r, "\chi^{-1}"'] & \widehat{\mathbf{Z}}^{\times}
\end{tikzcd}$$

commutes.

5.3.2 Orthogonal Shimura stacks

In this section, we apply Deligne's reciprocity law to orthogonal Shimura stacks. Before stating the main result, we need to introduce some notation.

Throughout this section, V is a quadratic space over $\mathbf{Q}$ of signature $(2, n)$, with $n \geq 1$. We denote by (SO, Ω) the associated Shimura datum as in Section 2.3. That is, $\mathrm{SO} = \mathrm{SO}(V)$ is the special orthogonal group, and $\Omega = \Omega_V$ is the period domain of Hodge structures of K3 type on $V \otimes_{\mathbf{Q}} \mathbf{R}$. By Lemma 2.1.3, the reflex field of (SO, Ω) is $\mathbf{Q}$. In addition, we let $\mathcal{K}$ be a profinite group endowed with a continuous homomorphism $\mathcal{K} \to \mathrm{SO}(\mathbf{A}_{\mathrm{f}})$ with open image and finite kernel. As we saw in Section 4.4, this gives rise to a Shimura stack $\mathrm{Sh}_{\mathcal{K}}[\mathrm{SO}, \Omega]$, which is a smooth separated Deligne-Mumford stack over $\mathbf{Q}$.

Remark 5.3.11. Artin reciprocity yields a map

$$\mathrm{Gal}_{\mathbf{Q}} \longrightarrow \mathrm{Gal}_{\mathbf{Q}}^{\mathrm{ab}}/2 \longrightarrow \pi_0(\mathbf{Q}^\times\backslash\mathbf{A}^\times/2) = \mathbf{Q}^\times\backslash\mathbf{A}^\times/2$$

which we denote CFT. Moreover, note that since $\mathbf{Q}^\times\backslash\mathbf{A}^\times/2$ is 2-torsion, it does not matter whether we use Artin's reciprocity law or its reciprocal $\mathrm{art}_{\mathbf{Q}}$ to define this map.

The following is the main result of this section.

Proposition 5.3.12. *Let $\widetilde{s}$ be a geometric point of $\mathrm{Sh}(\mathrm{SO}, \Omega)$, and $\overline{s}$ its image in $\mathrm{Sh}_{\mathcal{K}}[\mathrm{SO}, \Omega]$. Define $\xi_{\mathcal{K}} \colon \mathcal{K} \to \mathbf{Q}^\times\backslash\mathbf{A}^\times/2$ to be the composition*

$$\mathcal{K} \longrightarrow \mathrm{SO}(\mathbf{A}_{\mathrm{f}}) \xrightarrow{\nu} \mathbf{A}_{\mathrm{f}}^\times/2 \longrightarrow \mathbf{Q}^\times\backslash\mathbf{A}^\times/2,$$

where ν denotes the spinor norm, see Section 5.1. Then the diagram

$$\begin{tikzcd}
\pi_1^{\text{ét}}(\mathrm{Sh}_{\mathcal{K}}[\mathrm{SO}, \Omega], \overline{s}) \arrow[r] \arrow[d] & \mathcal{K} \arrow[d, "\xi_{\mathcal{K}}"] \\
\mathrm{Gal}_{\mathbf{Q}} \arrow[r, "\text{CFT}"'] & \mathbf{Q}^\times\backslash\mathbf{A}^\times/2
\end{tikzcd}$$

commutes.

Remark 5.3.13. Let $\mathrm{GSpin} = \mathrm{GSpin}(V)$ be the Clifford group of V, and $N \colon \mathrm{GSpin} \to \mathbf{G}_m$ the Clifford norm (see Section 2.3). The proof will show that a similar statement involving $\mathrm{GSpin}(\mathbf{A}_{\mathrm{f}}) \xrightarrow{N} \mathbf{A}^\times \to \mathbf{Q}^\times\backslash\,\mathbf{A}^\times$ and the composition

$$\mathrm{Gal}_{\mathbf{Q}} \longrightarrow \mathrm{Gal}_{\mathbf{Q}}^{\mathrm{ab}} \xrightarrow{\mathrm{art}_{\mathbf{Q}}^{-1}} \pi_0\left(\mathbf{Q}^\times\backslash\,\mathbf{A}^\times\right)$$

holds for the Shimura datum (GSpin, Ω)

The proof will make use of the morphisms of Shimura data from (2.1), namely

$$(\mathrm{SO}, \Omega) \longleftarrow (\mathrm{GSpin}, \Omega) \xrightarrow{N} (\mathbf{G}_m, \{\mathbf{Q}(-1)\}),$$

and the relation of the spinor norm to the Clifford norm given by Lemma 5.1.7. Note that by Lemma 2.1.3, the reflex fields of each of these Shimura data is $\mathbf{Q}$.

Lemma 5.3.14. *Let V be a quadratic space over $\mathbf{R}$ of signature $(2, n)$, with $n \geq 1$. Both $\mathrm{SO}(\mathbf{R})_+$ and $\mathrm{GSpin}(\mathbf{R})_+$ are connected.*

Proof. It is well known that $\mathrm{SO}(\mathbf{R})$ has two connected components. Moreover, by the last part of Lemma 5.3.5, $[\mathrm{SO}(\mathbf{R}) : \mathrm{SO}(\mathbf{R})_+] = |\pi_0(\Omega^\pm)| = 2$, proving the first assertion.
For the second assertion, note that $[\mathrm{GSpin}(\mathbf{R}) : \mathrm{GSpin}(\mathbf{R})_+] = 2$ by Lemma 5.3.5, so it suffices to show that $\pi_0(\mathrm{GSpin}(\mathbf{R})) \cong \{\pm 1\}$. For this, we use that if G is a Lie group with closed subgroup H, then the sequence

$$\pi_0(H) \to \pi_0(G) \to \pi_0(G/H) \to 1$$

is exact and functorial in (G, H). We apply this to the exact sequences $1 \to \{\pm 1\} \to \mathrm{Spin}(\mathbf{R}) \to \mathrm{SO}(\mathbf{R})_+ \to 1$ and $1 \to \mathbf{R}^\times \to \mathrm{GSpin}(\mathbf{R}) \to \mathrm{SO}(\mathbf{R}) \to 1$, and use the connectedness of $\mathrm{Spin}(\mathbf{R})$ ([PR, Proposition 7.6]) to conclude that $\pi_0(\mathrm{GSpin}(\mathbf{R})) \cong \pi_0(\mathrm{SO}(\mathbf{R})) \cong \{\pm 1\}$. □

Corollary 5.3.15. *Let V be a quadratic space over $\mathbf{Q}$ of signature $(2, n)$ with $n \geq 1$. Then $\overline{\pi}_0\pi\,\mathrm{SO} = \pi_0\pi\,\mathrm{SO}$ and $\overline{\pi}_0\pi\,\mathrm{GSpin} = \pi_0\pi\,\mathrm{GSpin}$.*

Note that $\mathrm{SO}(\mathbf{A}_\mathrm{f}) \xrightarrow{\nu} \mathbf{A}_\mathrm{f}^\times/2 \to \mathbf{Q}^\times\backslash\,\mathbf{A}^\times/2$ factors through $\pi_0\pi(\mathrm{SO})$, yielding a morphism $\pi_0\pi(\mathrm{SO}) \to \mathbf{Q}^\times\backslash\,\mathbf{A}^\times/2$ which we also denote with ν. Moreover, Corollary 5.3.15 identifies $\overline{\pi}_0\pi(\mathrm{SO})$ with $\pi_0\pi(\mathrm{SO})$, so Deligne's reciprocity law results in a homomorphism

$$\mathrm{Gal}_\mathbf{Q} \xrightarrow{r_{(\mathrm{SO},\Omega)}} \overline{\pi}_0\pi(\mathrm{SO}) = \pi_0\pi(\mathrm{SO}) \xrightarrow{\nu} \mathbf{Q}^\times\backslash\,\mathbf{A}^\times/2.$$

The following lemma states that this homomorphism coincides with the one coming from class field theory.

Proposition 5.3.16. *Let V be a quadratic space over $\mathbf{Q}$ of signature $(2, n)$ with $n \geq 1$. The diagram*

$$\begin{tikzcd} \mathrm{Gal}_\mathbf{Q} \arrow[r, "r_{(\mathrm{SO},\Omega)}"] \arrow[dr, "\mathrm{CFT}"'] & \pi_0\pi(\mathrm{SO}) \arrow[d, "\nu"] \\ & \mathbf{Q}^\times\backslash\,\mathbf{A}^\times/2 \end{tikzcd}$$

commutes.

Proof. Recall that Lemma 5.1.7 gives a commutative diagram relating the spinor norm to the Clifford norm. If we apply $\pi_0\pi$ to this commutative diagram, and factor out $\mathbf{Q}^\times$ in the bottom right corner, we obtain

$$\begin{tikzcd} \pi_0\pi(\mathrm{GSpin}) \arrow[r, "N"] \arrow[d] & \pi_0\pi(\mathbf{G}_m) \arrow[d] \\ \pi_0\pi(\mathrm{SO}) \arrow[r, "\nu"'] & \mathbf{Q}^\times\backslash\mathbf{A}^\times/2 \end{tikzcd} \tag{5.6}$$

On the other hand, let $h\colon \mathbf{S} \to \mathrm{SO}_{\mathbf{R}}$ be an element of Ω, and $\widetilde{h}\colon \mathbf{S} \to \mathrm{GSpin}$ the unique lift of h to GSpin for which $N \circ \widetilde{h}\colon \mathbf{S} \to \mathbf{G}_{m,\mathbf{R}}$ corresponds to the Lefschetz Hodge structure $\mathbf{Q}(-1)$, cf [D3, 4.2]. Since the reflex field of (SO, Ω), (GSpin, Ω), and $(\mathbf{G}_m, \{\mathbf{Q}(-1)\})$ is $\mathbf{Q}$, we obtain a commutative diagram

$$
\begin{tikzcd}
 & \pi_0\pi(\mathrm{SO}) \\
\pi_0\pi(\mathbf{G}_m) \arrow[ur, "\pi_0\pi(\mu_h)"] \arrow[r, "\pi_0\pi(\mu_{\widetilde{h}})"'] \arrow[dr, "\pi_0\pi(\mu_{N\widetilde{h}})"'] & \pi_0\pi(\mathrm{GSpin}) \arrow[u] \arrow[d, "\pi_0\pi(N)"] \\
 & \pi_0\pi(\mathbf{G}_m)
\end{tikzcd}
\tag{5.7}
$$

Note that since $N\widetilde{h}$ corresponds to $\mathbf{Q}(-1)$, there holds $\mu_{N\widetilde{h}} = \mathrm{id}_{\mathbf{G}_{m,\mathbf{Q}}}$, so the bottom map in the commutative diagram is the identity.

By combining (5.7) and (5.6) with the definition of $r_{(\mathrm{SO},\Omega)}$, we find that

$$
\begin{tikzcd}
\mathrm{Gal}_{\mathbf{Q}} \arrow[r, "\mathrm{CFT}"] \arrow[ddrr, bend right, "r_{(\mathrm{SO},\Omega)}"'] & \pi_0\pi(\mathbf{G}_m) \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr] \arrow[dr, phantom] \arrow[ddr, bend right, "\mu_h"] & & \\
 & & \pi_0\pi(\mathrm{GSpin}) \arrow[r, "N"'] \arrow[d] & \pi_0\pi(\mathbf{G}_m) \arrow[d] \\
 & & \pi_0\pi(\mathrm{SO}) \arrow[r, "\nu"'] & \mathbf{Q}^\times \backslash \mathbf{A}^\times/2
\end{tikzcd}
$$

commutes. Since the map $\mathrm{Gal}_{\mathbf{Q}} \to \mathbf{Q}^\times\backslash\mathbf{A}^\times/2$ given by composing the maps along the top of the diagram is precisely the one coming from class field theory, we obtain the desired result. $\square$

Finally, we are able to prove Proposition 5.3.12.

Proof of Proposition 5.3.12. Note that $\overline{\pi}_0\pi(\mathrm{SO}) = \pi_0\pi(\mathrm{SO})$ by Lemma 5.3.14. Therefore Lemma 5.3.9 and Proposition 5.3.16 yield a commutative diagram

$$
\begin{tikzcd}
\pi_1^{\text{ét}}(\mathrm{Sh}_{\mathcal{K}}[\mathrm{SO}, \Omega], \overline{s}) \arrow[r] \arrow[d] & \mathcal{K} \arrow[d] \\
\mathrm{Gal}_{\mathbf{Q}} \arrow[r, "r_{(\mathrm{SO},\Omega)}"] \arrow[dr, "\mathrm{CFT}"'] & \pi_0\pi(\mathrm{SO}) \arrow[d, "\nu"] \\
 & \mathbf{Q}^\times\backslash\mathbf{A}^\times/2
\end{tikzcd}
$$

proving the proposition. □

Remark 5.3.17. Lemma 5.1.14 shows that the spinor norm induces an isomorphism $\pi_0\pi(\mathrm{SO}) \to \mathbf{Q}^\times \backslash \mathbf{A}^\times/2$. Combining this with Proposition 5.3.16 and the fact that the scheme of connected components of $\mathrm{Sh}(\mathrm{SO}, \Omega)$ is a $\pi_0\pi(\mathrm{SO})$-torsor on $\mathbf{Q}_{\text{ét}}$ shows that $\pi_0(\mathrm{Sh}(\mathrm{SO}, \Omega)) \cong \mathrm{Spec}(\mathbf{Q}^{\text{quad}})$. Similarly, $\pi_0(\mathrm{Sh}(\mathrm{GSpin}, \Omega)) \cong \mathrm{Spec}(\mathbf{Q}^{\text{ab}})$, where $\mathbf{Q}^{\text{ab}}$ is the maximal abelian extension of $\mathbf{Q}$.

We end this section by making Proposition 5.3.12 more explicit in the case that is most relevant to our purposes.

Let Λ be a self-dual even $\mathbf{Z}$-lattice of signature $(3, n)$, with n odd, and $\lambda \in \Lambda$ a primitive vector of positive length. By setting $V = \lambda^\perp \otimes \mathbf{Q}$, this gives rise to an orthogonal Shimura datum $(\mathrm{SO}, \Omega) := (\mathrm{SO}(V), \Omega_V)$. Let $\mathcal{K}$ be the profinite group defined in (4.8). That is,

$$\mathcal{K} = \left\{ g \in \mathrm{O}(\Lambda \otimes \widehat{\mathbf{Z}}) \,\middle|\, g\lambda = \lambda \text{ and } \det g \in \mu_2(\mathbf{Z}) \subsetneq \mu_2(\widehat{\mathbf{Z}}) \right\}.$$

We endow $\mathcal{K}$ with the homomorphism $i\colon \mathcal{K} \to \mathrm{SO}(\mathbf{A}_{\mathrm{f}})$, $g \mapsto \det(g) g|_{V_{\mathbf{A}_{\mathrm{f}}}}$, yielding a Shimura stack $\mathrm{Sh}_{\mathcal{K}}[\mathrm{SO}, \Omega]$ over $\mathbf{Q}$.

Note that the fact that Λ is self-dual and of even rank implies that we have a spinor norm $\nu\colon \mathrm{O}(\Lambda \otimes \widehat{\mathbf{Z}}) \to \widehat{\mathbf{Z}}^\times/2$, see Definition 5.1.1. Combining this with the determinant $\det\colon \mathrm{O}(\Lambda \otimes \widehat{\mathbf{Z}}) \to \mu_2(\widehat{\mathbf{Z}})$, we obtain a map $\det \cdot \nu\colon \mathcal{K} \to \widehat{\mathbf{Z}}^\times/2$, sending $g \in \mathcal{K}$ to $\det(g)\nu(g)$. We can use this map to change the structure group of the $\mathcal{K}$-torsor $\mathrm{Sh}(\mathrm{SO}, \Omega)$ on $\mathrm{Sh}_{\mathcal{K}}[\mathrm{SO}, \Omega]_{\text{ét}}$, yielding a $\widehat{\mathbf{Z}}^\times/2$-torsor which we denote $(\det \cdot \nu)_* \mathrm{Sh}(\mathrm{SO}, \Omega)$. Aside from this we have another $\widehat{\mathbf{Z}}^\times/2$-torsor, namely $\mathrm{Spec}(\mathbf{Q}^{\text{quad}}) \times \mathrm{Sh}_{\mathcal{K}}[\mathrm{SO}, \Omega]$.

Proposition 5.3.18. *Let $(\mathrm{SO}, \Omega, \mathcal{K})$ and ν be as above. Then*

$$(\det \cdot \nu)_* \mathrm{Sh}(\mathrm{SO}, \Omega) \cong \mathrm{Spec}(\mathbf{Q}^{\text{quad}}) \times \mathrm{Sh}_{\mathcal{K}}[\mathrm{SO}, \Omega]$$

as $\widehat{\mathbf{Z}}^\times/2$-torsors on $\mathrm{Sh}_{\mathcal{K}}[\mathrm{SO}, \Omega]_{\text{ét}}$.

Proof. Define a map $\psi\colon \mathbf{Q}^\times \backslash \mathbf{A}^\times/2 \to \widehat{\mathbf{Z}}^\times/2$ to be the composition

$$\mathbf{Q}^\times \backslash \mathbf{A}^\times/2 \xrightarrow{\mathrm{art}_{\mathbf{Q}}} \mathrm{Gal}(\mathbf{Q}^{\text{quad}} / \mathbf{Q}) \longrightarrow \widehat{\mathbf{Z}}^\times/2,$$

where the second map is the isomorphism given by the Kronecker-Weber theorem. It can be shown that ψ is given explicitly by

$$(a_v)_v \longmapsto \left(a_p \,\mathrm{sgn}(a_\infty) \prod_q q^{\mathrm{ord}_q(a_q)} \right)_p.$$

In particular,

$$\psi(1_{\mathbf{R}}, -1, -1, \ldots) = -1, \tag{5.8}$$

and

$$\psi((a_v)_v) = (a_v)_v, \quad \text{for } (a_v)_v \in \widehat{\mathbf{Z}}^\times/2 \subseteq \mathbf{A}^\times/2. \tag{5.9}$$

The composition $\mathrm{Gal}_{\mathbf{Q}} \xrightarrow{\text{CFT}} \mathbf{Q}^\times \backslash \mathbf{A}^\times/2 \xrightarrow{\psi} \widehat{\mathbf{Z}}^\times/2$ corresponds to the $\widehat{\mathbf{Z}}^\times/2$-torsor $\mathrm{Spec}(\mathbf{Q}^{\mathrm{quad}})$ on $\mathbf{Q}_{\text{ét}}$. It now follows from Proposition 5.3.12 that it suffices to show that

$$\mathcal{K} \xrightarrow{i} \mathrm{SO}(\mathbf{A}_{\mathrm{f}}) \xrightarrow{\nu} \mathbf{Q}^\times \backslash \mathbf{A}^\times/2 \xrightarrow{\psi} \widehat{\mathbf{Z}}^\times/2$$

is equal to $\det \cdot \nu$.

Let $g \in \mathcal{K}$. First, note that by applying the identities in Lemma 5.1.5 and Lemma 5.1.12 to $V_{\mathbf{Q}_p}$ and $V_{\mathbf{Q}_p} \oplus \mathbf{Q}_p\,\lambda = \Lambda_{\mathbf{Q}_p}$ for all p, and using that $\det(v) \in \{\pm 1\}$, we obtain

$$\nu i(g) = \nu(\det(g) g|_{V_{\mathbf{A}_{\mathrm{f}}}}) = \mathrm{disc}(V_{\mathbf{A}_{\mathrm{f}}})^{\frac{1-\det(g)}{2}} \nu(g_{\mathbf{A}_{\mathrm{f}}}).$$

Note that since Λ is self-dual and of even rank, we have $\nu(g_{\mathbf{A}_{\mathrm{f}}}) = \nu(g) \in \widehat{\mathbf{Z}}^\times/2$. Write $\lambda^2 = 2d$, with $d \in \mathbf{Z}_{>0}$, so that $\mathrm{disc}(V_{\mathbf{A}_{\mathrm{f}}}) = -2d < 0$. Combining this with $\nu(g) \in \widehat{\mathbf{Z}}^\times$ and (5.8) and (5.9), we find that applying ψ to the equation above yields $\psi\nu i(g) = \det(g)\nu(g)$, which was to be shown. ☐

The following remark relates the proposition to the $\mathbf{Z}_\ell^\times/2$-torsors T_ℓ defined in (5.3), and hence to the maps χ_ℓ occurring in Theorem 5.2.1.

Remark 5.3.19. Note that $\mathbf{Q}^{\mathrm{quad}}$ is a Galois extension of $\mathbf{Q}$ with Galois group $\widehat{\mathbf{Z}}^\times/2$, so that $\mathrm{Spec}(\mathbf{Q}^{\mathrm{quad}})$ is a $(\widehat{\mathbf{Z}}^\times/2)$-torsor on $\mathbf{Q}_{\text{ét}}$. We denote by ζ_8 a primitive 8th root of unity, and for an odd prime ℓ, we define $\ell^* = (-1)^{\frac{\ell-1}{2}}\ell$. The diagram

$$\widehat{\mathbf{Z}}^\times/2 \left\{ \begin{tikzcd} & \mathbf{Q}^{\mathrm{quad}} \arrow[dl, dash] \arrow[dr, dash] & \\ \mathbf{Q}(\zeta_8) \arrow[dr, dash, "\mathbf{Z}_2^\times/2"'] & & \mathbf{Q}(\sqrt{\ell^*}) \arrow[dl, dash, "\mathbf{Z}_\ell^\times/2"] \\ & \mathbf{Q} & \end{tikzcd} \right.$$

shows that $T_{\ell,\mathbf{Q}}$ is obtained by changing the structure group of $\mathrm{Spec}(\mathbf{Q}^{\mathrm{quad}})$ to $\mathbf{Z}_\ell^\times/2$.

5.4 Proof of Theorem 5.2.1

We use the notation of Theorem 5.2.1. In particular, S is a $\mathbf{Z}[\frac{1}{2d\ell}]$-scheme, and $f\colon X \to S$ a projective K3 surface of degree $2d$. Recall the $\mathbf{Z}_\ell^\times/2$-torsor $T_{\ell,S}$ on $S_{\text{ét}}$ defined in (5.3). The K3 surface $f\colon X \to S$ gives rise to an $\mathrm{O}(\Lambda_{\mathrm{K3}} \otimes \mathbf{Z}_\ell)$-torsor on $S_{\text{ét}}$, namely

$$\mathrm{Isom}_S\left(\underline{\Lambda_{\mathrm{K3}}} \otimes \mathbf{Z}_\ell, \mathrm{R}^2_{\text{ét}} f_*\, \mathbf{Z}_\ell(1)\right),$$

where Λ_{K3} denotes an even self-dual lattice over $\mathbf{Z}$ of signature $(3, 19)$, and $\mathrm{R}^2_{\text{ét}}\, f_*\, \mathbf{Z}_\ell(1)$ is endowed with the cup product pairing. By changing the structure group of this torsor using the map $\det\cdot\nu\colon\ \mathrm{O}(\Lambda_{\mathrm{K3}}\otimes\mathbf{Z}_\ell)\to\mathbf{Z}_\ell^\times/2$, we obtain a $\mathbf{Z}_\ell^\times/2$-torsor on $S_{\text{ét}}$, which we denote $(\det\cdot\nu)_{\ell,X/S}$. Now Theorem 5.2.1 is equivalent to $(\det\cdot\nu)_{\ell,X/S}$ and $T_{\ell,S}$ being isomorphic as torsors on $S_{\text{ét}}$. Note that these torsors are stable under base change along morphisms $S'\to S$.

Let $\mathbf{K3}_{2d}$ be the moduli stack over $\mathbf{Z}[1/2d\ell]$ of polarized K3 surfaces of degree $2d$, and $f\colon\ \mathfrak{X}\to\mathbf{K3}_{2d}$ the universal K3 surface. Then it suffices to show that $(\det\cdot\nu)_{\ell,\mathfrak{X}\,/\,\mathbf{K3}_{2d}}$ and $T_{\ell,\mathbf{K3}_{2d}}$ are isomorphic. We will obtain the characteristic 0 case using Shimura-theoretic methods, and later deduce the general case.

Let $\lambda\in\Lambda_{\mathrm{K3}}$ be a primitive element of length $2d$, and define V to be the orthogonal complement of λ in $\Lambda_{\mathrm{K3}}\otimes\mathbf{Q}$. Let (SO,Ω) be the Shimura datum associated with V as in Section 2.3, and let $\mathcal{K}$ be the profinite group defined in (4.8), namely

$$\mathcal{K}:=\left\{g\in\mathrm{O}(\Lambda_{\mathrm{K3}})(\widehat{\mathbf{Z}})\mid g(\lambda)=\lambda\ \text{ and }\ \det(g)\in\{\pm1\}\right\}.$$

We endow $\mathcal{K}$ with the continuous homomorphism $i\colon\ \mathcal{K}\to\mathrm{SO}(\mathbf{A}_\mathrm{f})$ given by mapping $g\in\mathcal{K}$ to $\det(g)g|_{V\otimes\mathbf{A}_\mathrm{f}}$, yielding a Shimura stack $\mathrm{Sh}_{\mathcal{K}}[\mathrm{SO},\Omega]$ over $\mathbf{Q}$. Theorem 4.6.4 gives a morphism $P\colon\ \mathbf{K3}_{2d,\mathbf{Q}}\to\mathrm{Sh}_{\mathcal{K}}[\mathrm{SO},\Omega]$, defined over $\mathbf{Q}$.

Lemma 4.3.7 gives a rank 1 local $\mathbf{Z}$-system D on $\mathbf{K3}_{2d,\mathbf{Q}}$, endowed with an injective map $D\to\det\mathrm{R}^2_{\text{ét}}\,f_{\mathbf{Q},*}\,\widehat{\mathbf{Z}}(1)$. This gives rise to a $\mathcal{K}$-torsor

$$I:=\mathrm{Isom}\left((\Lambda_{\mathrm{K3}},\lambda,\det\Lambda_{\mathrm{K3}}),\left(\mathrm{R}^2_{\text{ét}}\,f_{\mathbf{Q},*}\,\widehat{\mathbf{Z}}(1),b,\lambda,D\right)\right)$$

on $\mathbf{K3}_{2d,\mathbf{Q},\text{ét}}$, where b is the pairing coming from the cup product. Consider the composition

$$\mathcal{K}\xrightarrow{\det\cdot\nu}\widehat{\mathbf{Z}}^\times/2\longrightarrow\mathbf{Z}_\ell^\times/2.$$

Then changing the structure group of I using this homomorphism is precisely the restriction to $\mathbf{K3}_{2d,\mathbf{Q},\text{ét}}$ of the torsor $(\det\cdot\nu)_{\ell,\mathfrak{X}\,/\,\mathbf{K3}_{2d}}$. Theorem 4.6.4 says that P pulls the $\mathcal{K}$-torsor $\mathrm{Sh}(\mathrm{SO},\Omega)$ on $\mathrm{Sh}_{\mathcal{K}}[\mathrm{SO},\Omega]_{\text{ét}}$ back to I, so combining this with Proposition 5.3.18 and Remark 5.3.19 proves that

$$T_{\ell,\mathbf{K3}_{2d,\mathbf{Q}}}\cong(\det\cdot\nu)_{\ell,\mathfrak{X}\,/\,\mathbf{K3}_{2d,\mathbf{Q}}}.\tag{5.10}$$

The following lemma will allow us to deduce the general case from the characteristic 0 case.

Lemma 5.4.1. *Let S be a normal locally Noetherian scheme, and G a finite group acting on S. If x is a geometric point of $[S/G]_\mathbf{Q}$, then*

$$\pi_1^{\text{ét}}([S/G]_\mathbf{Q},x)\longrightarrow\pi_1^{\text{ét}}([S/G],x)$$

is surjective.

Proof. Without loss of generality, $[S/G]$ is connected. Let T be a connected component of S, and H the subgroup $\{g\in G\mid gT\subseteq T\}$ of G. Then T is a connected H-torsor on $[S/G]$, and the resulting map $\pi_1^{\text{ét}}([S/G],x)\to H$ is surjective. Now let η be the generic point of T, and y a geometric point of $T_\mathbf{Q}$ lying over x. Then by [G2, Proposition V.8.2], $\pi_1^{\text{ét}}(\eta, y) \to \pi_1^{\text{ét}}(T, y)$ is surjective. Moreover, $\eta \to T$ factors through $T_{\mathbf{Q}}$, so

$$\pi_1^{\text{ét}}(T_{\mathbf{Q}}, y) \longrightarrow \pi_1^{\text{ét}}(T, y)$$

is surjective. The diagram

$$\begin{tikzcd}
1 \arrow[r] & \pi_1^{\text{ét}}(T, y) \arrow[r] & \pi_1^{\text{ét}}([S/G], x) \arrow[r] & H \arrow[r] & 1 \\
1 \arrow[r] & \pi_1^{\text{ét}}(T_{\mathbf{Q}}, y) \arrow[r] \arrow[u] & \pi_1^{\text{ét}}([S/G]_{\mathbf{Q}}, x) \arrow[r] \arrow[u] & H \arrow[r] \arrow[u, "\text{id}"'] & 1
\end{tikzcd}$$

now shows the surjectivity of $\pi_1^{\text{ét}}([S/G]_{\mathbf{Q}}, x) \to \pi_1^{\text{ét}}([S/G], x)$. □

Lemma 5.4.2. *Let S be a quasi-separated normal Noetherian algebraic space, and G a finite group acting on S. If x is a geometric point of $[S/G]_{\mathbf{Q}}$, then*

$$\pi_1^{\text{ét}}([S/G]_{\mathbf{Q}}, x) \longrightarrow \pi_1^{\text{ét}}([S/G], x)$$

is surjective.

Proof. Let y be a geometric point of $S_{\mathbf{Q}}$ lying over x. We will show that the map

$$\pi_1^{\text{ét}}(S_{\mathbf{Q}}, y) \longrightarrow \pi_1^{\text{ét}}(S, y)$$

is surjective. Given the surjectivity of this map, the rest of the proof proceeds exactly as the proof of Lemma 5.4.1, and is therefore omitted.

Since S is quasi-separated, normal, and Noetherian, [LMB, Corollaire 16.6.2] shows that there exists a normal scheme S' and a finite group G' acting on S' such that S is isomorphic to the quotient S'/G'. Note that S is the coarse moduli space of $[S'/G']$, so we have a canonical morphism $[S'/G'] \to S$. Let z be a geometric point of $[S'/G']_{\mathbf{Q}}$ lying over y, and consider the commutative diagram

$$\begin{tikzcd}
\pi_1^{\text{ét}}([S'/G']_{\mathbf{Q}}, z) \arrow[r] \arrow[d] & \pi_1^{\text{ét}}(S_{\mathbf{Q}}, y) \arrow[d] \\
\pi_1^{\text{ét}}([S'/G'], z) \arrow[r] & \pi_1^{\text{ét}}(S, y)
\end{tikzcd}$$

By Lemma 5.4.1, the map $\pi_1^{\text{ét}}([S'/G']_{\mathbf{Q}}, z) \to \pi_1^{\text{ét}}([S'/G'], z)$ is surjective, and by [N2, Theorem 7.11], the horizontal maps in the diagram are surjective. It follows that $\pi_1^{\text{ét}}(S_{\mathbf{Q}}, y) \to \pi_1^{\text{ét}}(S, y)$ is surjective. □

We now finish the proof of Theorem 5.2.1. Consider the sheaf of isometries

$$\mathbf{K3}_{2d,4} := \operatorname{Isom}\big((\Lambda_{\text{K3}} \otimes \mathbf{Z}/4\,\mathbf{Z}, \lambda), (\mathrm{R}^2_{\text{ét}}\, f_*\mu_4, b, \lambda)\big)$$

on $\mathbf{K3}_{2d,\text{ét}}$, where b is the cup product pairing. Since $\mathbf{K3}_{2d}$ is a smooth separated Deligne-Mumford stack over $\mathbf{Z}[1/2d]$ by [R4, Theorem 4.3.3, Proposition 4.3.11], it follows that $\mathbf{K3}_{2d,4}$ is a smooth separated Deligne-Mumford stack over $\mathbf{Z}[1/2d]$. An argument similar to that in the proof of [R4, Lemma 6.1.3] shows that the automorphism groups in $\mathbf{K3}_{2d,4}$ are trivial, so that $\mathbf{K3}_{2d,4}$ is an algebraic space.

The finite group $\mathrm{O}(\Lambda_{\mathrm{K3}} \otimes \mathbf{Z}/4\mathbf{Z})$ acts on $\mathbf{K3}_{2d,4}$. It is clear that $\mathbf{K3}_{2d}$ is isomorphic to the quotient stack $[\mathbf{K3}_{2d,4} / \mathrm{O}(\Lambda_{\mathrm{K3}} \otimes \mathbf{Z}/4\mathbf{Z})]$. It now follows from Lemma 5.4.2 and (5.10) that

$$T_{\ell,\mathbf{K3}_{2d}} \cong (\det \cdot \nu)_{\ell,\mathfrak{X}/\mathbf{K3}_{2d}},$$

which was to be shown. □

5.5 The spinor norm of the Frobenius

In this section we apply our result to K3 surfaces over a finite field $\mathbf{F}_q$ of characteristic p to compute the spinor norm of the Frobenius acting on the second cohomology. As a corollary, we obtain a special case of a theorem of Elsenhans and Jahnel.

We first compute the value of χ_ℓ on Frob_q, where $\chi_\ell \colon \mathrm{Gal}_{\mathbf{F}_q} \to \mathbf{Z}_\ell^\times/2$ is defined by (5.3).

Lemma 5.5.1. *Assume that p is an odd prime, and let ℓ be a prime distinct from p. Then $\chi_\ell(\mathrm{Frob}_q) = q$ in $\mathbf{Z}_\ell^\times/2$.*

Proof. Let $r \in \mathbf{Z}_{>0}$ be such that $q = p^r$. If r is even, then $q = 1$ in $\mathbf{Z}_\ell^\times/2$. Moreover, all elements of $\mathbf{F}_p$ are squares in $\mathbf{F}_q$. In particular, ℓ^*, -1, and 2 are all squares in $\mathbf{F}_q$, so that $\chi_\ell(\mathrm{Frob}_q) = 1$. If r is odd, then $q = p$ in $\mathbf{Z}_\ell^\times/2$. Moreover, every element of $\mathbf{F}_p$ is a square in $\mathbf{F}_p$ if and only if it is a square in $\mathbf{F}_q$, so we may assume $r = 1$.

If ℓ is odd, then quadratic reciprocity states that ℓ^* has a square root in $\mathbf{F}_p$ if and only if p has a square root in $\mathbf{F}_\ell$. By Hensel's lemma the latter statement is equivalent to p having a square root in $\mathbf{Z}_\ell$. We conclude that $\chi_\ell(\mathrm{Frob}_p) = p$.

If $\ell = 2$, then it follows from $\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}$ and $\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$ that we need to show that if $p = m \mod 8$, then $p = m \in \mathbf{Z}_2^\times/2$, where $m \in \{\pm 1, \pm 5\}$. This follows immediately from the fact that reduction modulo 8 induces an isomorphism $\mathbf{Z}_2^\times/2 \to (\mathbf{Z}/8\mathbf{Z})^\times$. □

We are now able to prove Corollary 5.2.4, which gives an expression for the spinor norm of Frob_q acting on the second cohomology of a K3 surface. For the remainder of the section, X denotes a K3 surface over $\mathbf{F}_q$.

Proof of Corollary 5.2.4. Assume that X admits a polarization of degree coprime to p, and let ℓ be a prime distinct from p. By Theorem 5.2.1 and Lemma 5.5.1,

$$\det\left(\mathrm{Frob}_q|_{\mathrm{H}^2_{\text{ét}}\left(X_{\overline{\mathbf{F}}_q},\mathbf{Z}_\ell(1)\right)}\right) \cdot \nu\left(\mathrm{Frob}_q|_{\mathrm{H}^2_{\text{ét}}\left(X_{\overline{\mathbf{F}}_q},\mathbf{Z}_\ell(1)\right)}\right) = \chi_\ell\left(\mathrm{Frob}_q\right) = q \in \mathbf{Z}_\ell^\times/2.$$

□

Let ℓ be a prime distinct from p, and Φ the characteristic polynomial of Frob_q acting on $\mathrm{H}^2_{\text{ét}}(X_{\overline{\mathbf{F}_q}}, \mathbf{Q}_\ell(1))$. Then Φ has coefficients in $\mathbf{Q}$, and does not depend on ℓ. We will use Corollary 5.2.4 to compute $\Phi(-1)$ up to squares in case $\Phi(-1) \neq 0$. First, we need the following lemma on K3 surfaces satisfying $\Phi(-1) \neq 0$.

Lemma 5.5.2. *Let ℓ be a prime distinct from p. If $\Phi(-1) \neq 0$, then*

$$\det \mathrm{H}^2_{\text{ét}}(X_{\overline{\mathbf{F}_q}}, \mathbf{Q}_\ell(1)) \cong \mathbf{Q}_\ell(0)$$

as $\mathrm{Gal}_{\mathbf{F}_q}$-representations.

Proof. We need to show that $\det \mathrm{Frob}_q = 1$. If $\ell \equiv 3(4)$, then $\det \mathrm{Frob}_q = 1$ follows from Lemma 5.1.13. For other ℓ, we use that $\det \mathrm{Frob}_q$ is ℓ-independent (this follows from the Weil conjectures, proved for K3 surfaces in [D3]) to reduce to the case where $\ell \equiv 3(4)$. □

Corollary 5.5.3. *Let X be a K3 surface over $\mathbf{F}_q$ of degree coprime to q, let ℓ be a prime coprime to q, and assume that $\Phi(-1) \neq 0$. Then*

$$\Phi(-1) = q$$

holds in $\mathbf{Q}_\ell^\times/2$.

Proof. Let F denote Frob_q acting on $\mathrm{H}^2_{\text{ét}}(X_{\overline{\mathbf{F}_q}}, \mathbf{Q}_\ell(1))$. Combining Lemma 5.5.2 and Corollary 5.2.4 yields $\nu(F) = q$ as elements of $\mathbf{Q}_\ell^\times/2$. Since $\Phi(-1) \neq 0$, the Zassenhaus formula (Lemma 5.1.11) says that

$$\nu(F) = \det\left(\frac{1+F}{2}\right) = (-2)^{22}\det(-1-F)$$

which is equal to $\Phi(-1)$ by the definition of Φ and the fact that $(-2)^{22}$ is a square. This proves the corollary. □

Remark 5.5.4. In [EJ, Proposition 3.11], Elsenhans and Jahnel obtain the same result using different methods.

5.6 Néron-Severi lattices over non-closed fields

In this section we apply Theorem 5.2.1 to give a necessary condition for a lattice to be the Néron-Severi lattice of a K3 surface over a non-closed field.

For a finite abelian group A and a prime number ℓ, we denote by $A[\ell^\infty]$ the subgroup of those $a \in A$ for which there exists an $r \in \mathbf{Z}_{\geq 0}$ with $\ell^r a = 0$. For a $\mathbf{Z}$-lattice Λ we have $\Delta(\Lambda)[\ell^\infty] = \Delta(\Lambda \otimes \mathbf{Z}_\ell)$. In particular, Definition 5.1.17 gives an invariant

$$\mathrm{disc}\left(\Delta(\Lambda)[\ell^\infty]\right) \in \mathbf{Z}_\ell^\times/2$$

of Λ.

The following theorem is the main result of this section.

Theorem 5.6.1. *Let k be a field, let ℓ be an odd prime number, and let X/k be a K3 surface of degree coprime to the characteristic of k. We denote by $\rho(X)$ the rank of* $\mathrm{Pic}(X)$*, and*

$$r_\ell(X) := \operatorname{length}\Big(\Delta\big(\operatorname{Pic}(X)\big)[\ell^\infty]\Big).$$

If $r_\ell(X) + \rho(X) = 21$ and the product

$$(-1)^{r_\ell(X)+1} 2 \operatorname{disc}\Big(\Delta\Big(\operatorname{Pic}(X)[\ell^\infty]\Big)\Big)$$

is equal to 1 in $\mathbf{Z}_\ell^\times/2$, or if $r_\ell(X) + \rho(X) = 22$, then ℓ^* is a square in k.

Proof. If ℓ is equal to the characteristic of k, then ℓ^* is trivially a square in k.

Suppose that ℓ is coprime to the characteristic of k. Then ℓ^* is a square in k if and only if the image of $\chi_\ell \colon \mathrm{Gal}_k \to \mathbf{Z}_\ell^\times/2$ is trivial, where χ_ℓ is the quadratic character defined by (5.3). Therefore we need to prove that when the conditions of the theorem are satisfied, then χ_ℓ has trivial image.

By Theorem 5.2.1, χ_ℓ is equal to the composition

$$\mathrm{Gal}_k \longrightarrow \mathrm{O}(\mathrm{H}_{\text{ét}}(X_{\overline{k}}, \mathbf{Z}_\ell(1))) \xrightarrow{\det\cdot\nu} \mathbf{Z}_\ell^\times/2.$$

The image of $\mathrm{Pic}(X)$ under c_1 in $\mathrm{H}_{\text{ét}}(X_{\overline{k}}, \mathbf{Z}_\ell(1))$ is invariant under the Gal_k-action. That is, each of its elements is Gal_k-stable. It follows that the image of χ_ℓ is contained in

$$\det\cdot\nu\,\big(\mathrm{O}\big(\mathrm{H}_{\text{ét}}(X_{\overline{k}}, \mathbf{Z}_\ell(1)),\, c_1\mathrm{Pic}(X)\big)\big)$$

where we use the notation from (5.2). Corollary 5.1.20 states that this is trivial if and only if the conditions of the theorem are satisfied, completing the proof. □

Remark 5.6.2. The main lattice-theoretic input into Theorem 5.6.1 is Theorem 5.1.18. As was noted in Remark 5.1.19, this theorem has a more complicated analogue for $\ell = 2$. One can use this to prove a version of Theorem 5.6.1 for even primes.

Example 5.6.3. Let ℓ be an odd prime, and Λ a lattice of signature $(1, 10)$ whose discriminant form is ℓ-primary and has length 11. By [M6, Remark 2.11], there exists a complex projective K3 surface whose Picard lattice is isomorphic to Λ. However, since $\operatorname{rk}\Lambda + \operatorname{length}(\Delta(\Lambda)) = 22$, it follows from Theorem 5.6.1 that there is no K3 surface over $\mathbf{Q}$ whose Picard lattice is isomorphic to Λ (or indeed over any field of characteristic 0 not containing $\sqrt{\ell^*}$).

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Summary

In this thesis I study to what extent the moduli stacks of polarized hyperkähler varieties (for example, K3 surfaces) are related to Shimura stacks. I focus in particular on hyperkähler varieties defined over non-closed fields, and the ramifications of Deligne's reciprocity law for such varieties. Chapters 1 and 2 serve as introductions to Shimura stacks and moduli of polarized hyperkähler varieties, respectively, and can be read independently of each other. From Chapter 3 on, every chapter depends on all chapters preceding it.

Chapter 1 is an introductory chapter, and gives a detailed global overview of the thesis and the main results.

The second chapter is an expository chapter about Shimura varieties and motives. The main result is that the canoncial model of a Shimura variety of abelian type (for example, an orthogonal Shimura variety) is a moduli space of abelian motives. This result is due to Deligne and Milne, and this chapter is a more Tannakian exposition of their work.

Chapter 3 gives an introduction to polarized hyperkähler varieties and their moduli. The main result, which is well known to the experts, is that the moduli stack of polarized hyperkähler varieties is a separated Deligne-Mumford stack over $\mathbf{Q}$.

In Chapter 4 I study the period map for hyperkähler varieties. This is a morphism from a degree 2 étale covering of the moduli stack of complex polarized hyperkähler varieties to an orthogonal Shimura stack. One of the main results of this chapter is that this map descends to a morphism over $\mathbf{Q}$. This is proved by combining the results of Chapter 2 and 3 with André's result that the motive of a hyperkähler variety is abelian. Furthermore, the chapter contains stronger versions of this main result for two specific families of hyperkähler varieties, namely K3 surfaces and hyperkähler varieties of type $\mathrm{K3}^{[n]}$.

The final chapter applies the results of Chapter 4 to obtain more concrete results about K3 surfaces, namely a computation of the spinor norm of monodromy operators on the second cohomology. The proof makes use of a result Deligne on the connected components of the canonical model of a Shimura variety, of which the chapter contains an exposition. This is then used to sharpen a result of Elsenhans and Jahnel on K3 surfaces over finite fields, and to give a non-trivial necessary condition for a lattice to be the Néron-Severi lattice of a K3 surface over a non-closed field.

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