

**Materials for today:**  
**<https://bit.ly/2Ti9gUM>**

# Structured populations

- i. Endless varieties most beautiful & why we care
- ii. Life cycle → Model
- iii. Basic matrix model analysis
- iv. Exercises



# Conservation

Which life stage would be best to **help**?



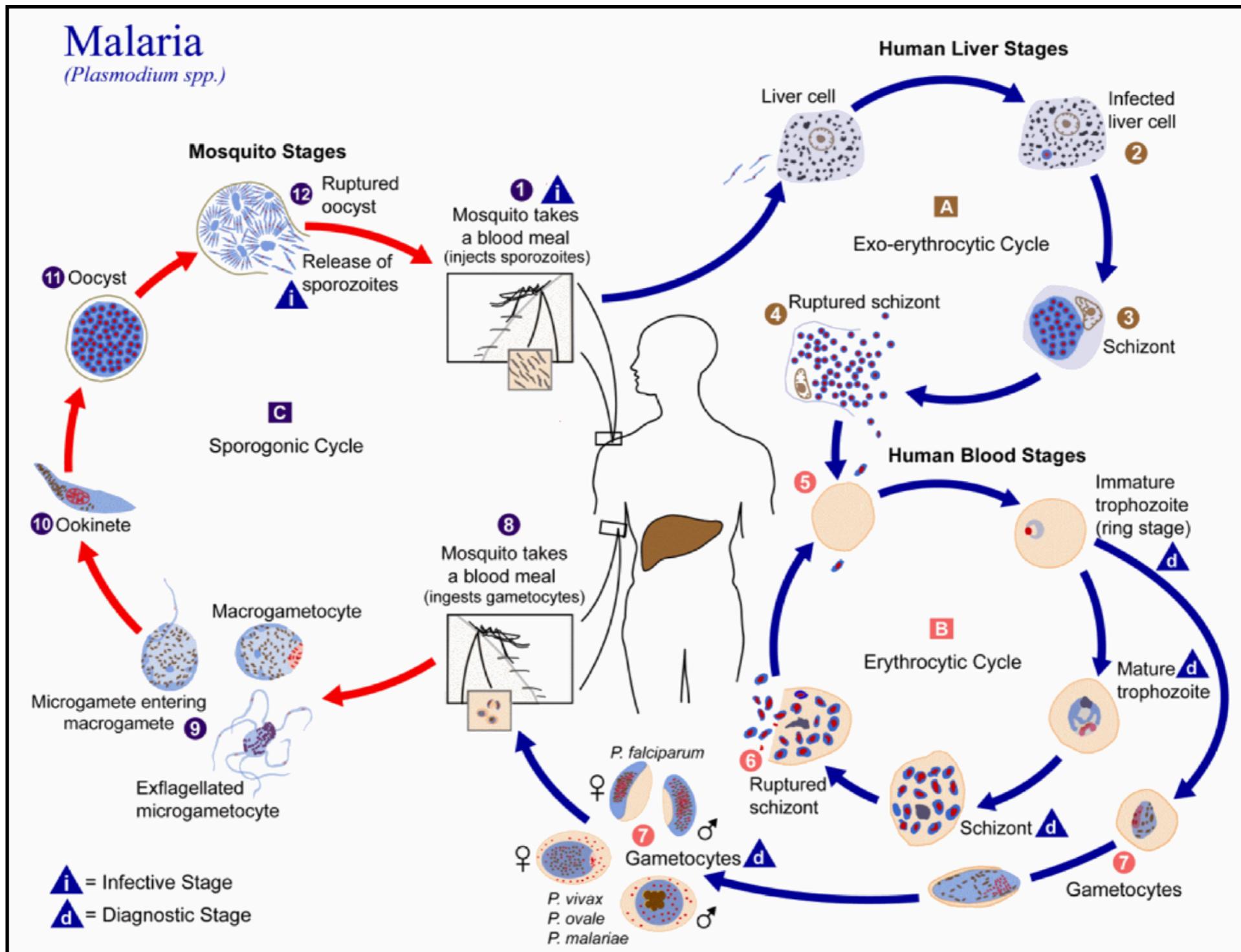
# Conservation

Which life stage would be best to **harm**?



# Disease

Where would an intervention be **most effective**?



# Structured populations

- i. Endless varieties most beautiful & why we care
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  - c. Conservation: Rare & invasive species
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# Unstructured population models

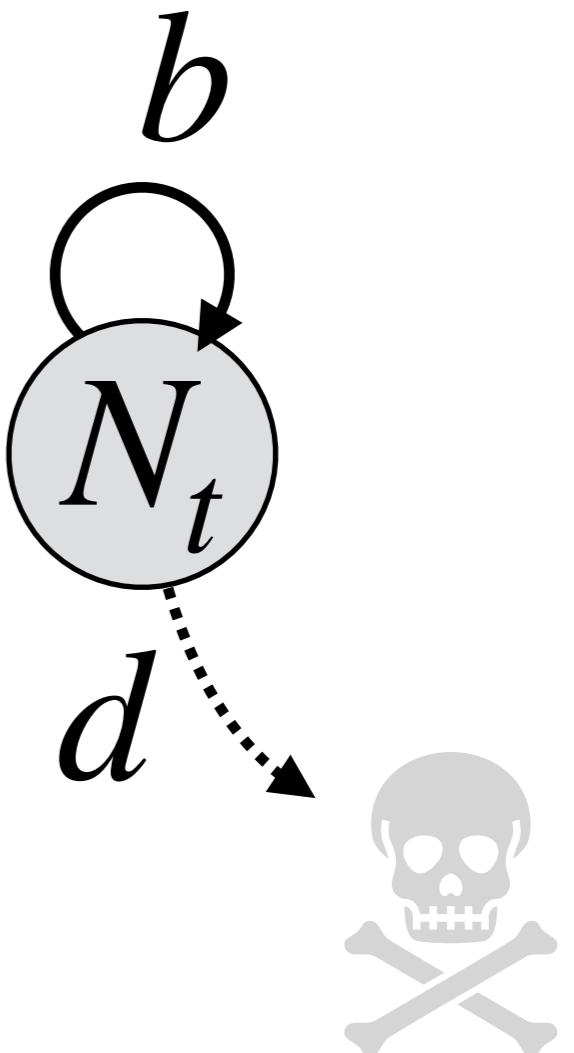
$$N_{t+1} = bN_t - dN_t$$

$$= (b - d)N_t$$

$$= \lambda N_t$$

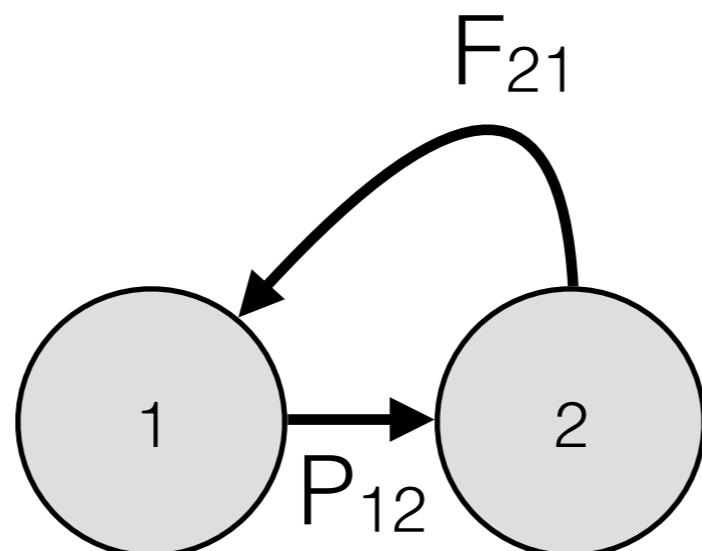


Finite rate of increase



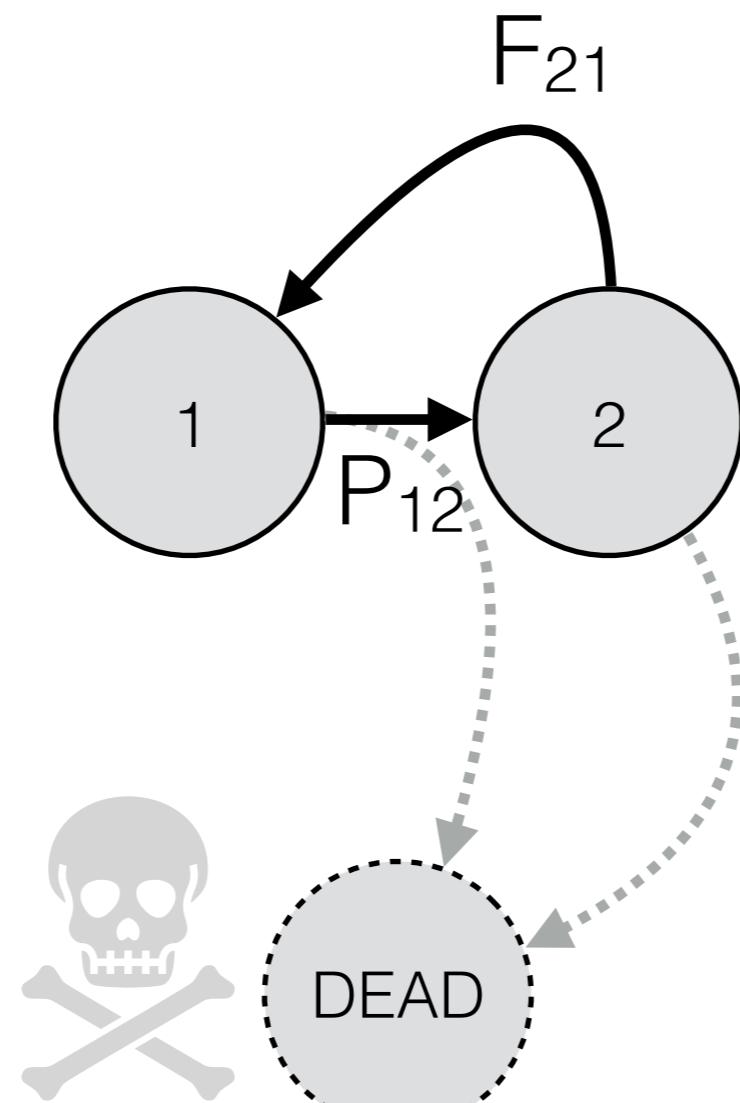
# A simple structured life cycle

*Verbascum thapsus*



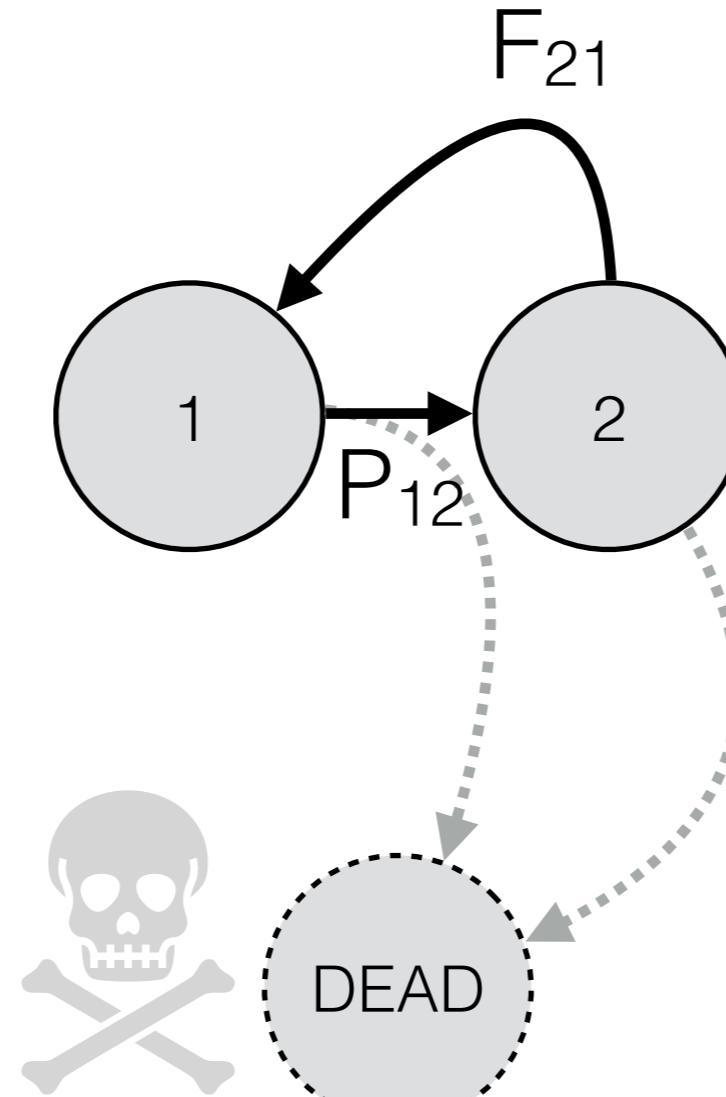
# A simple structured life cycle

*Verbascum thapsus*



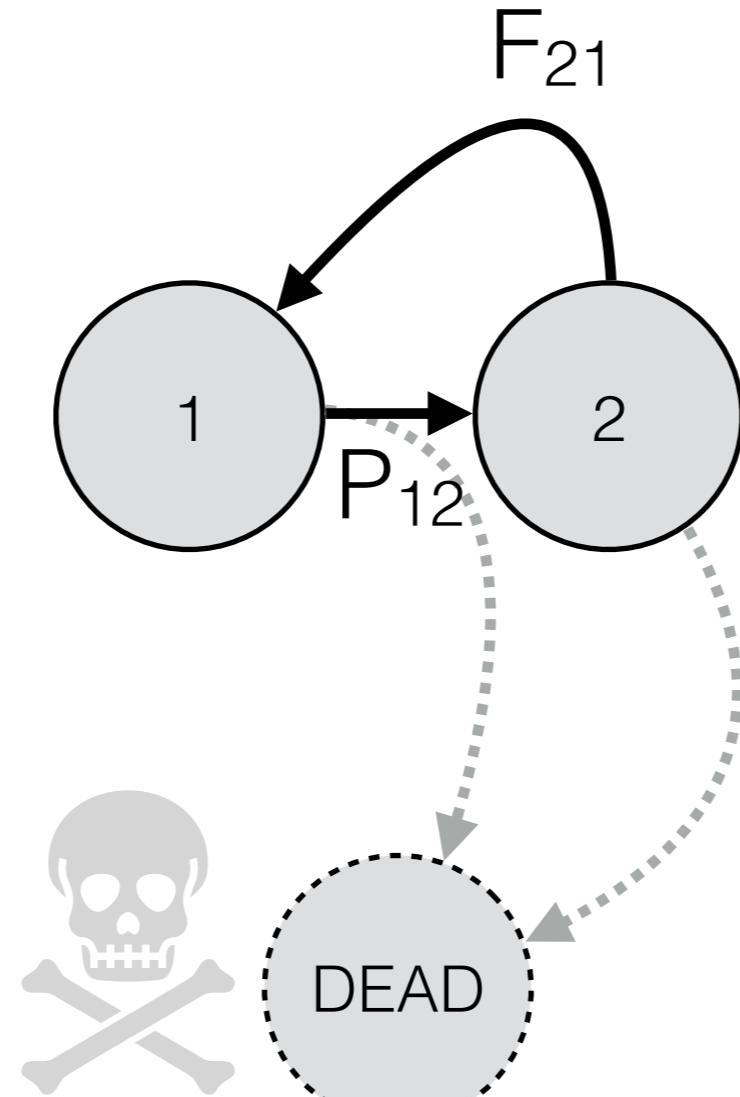
# Stage-specific equations

*Verbascum thapsus*



# Stage-specific equations

*Verbascum thapsus*



$$N_{1(t+1)} = F_{21}N_{2(t)}$$

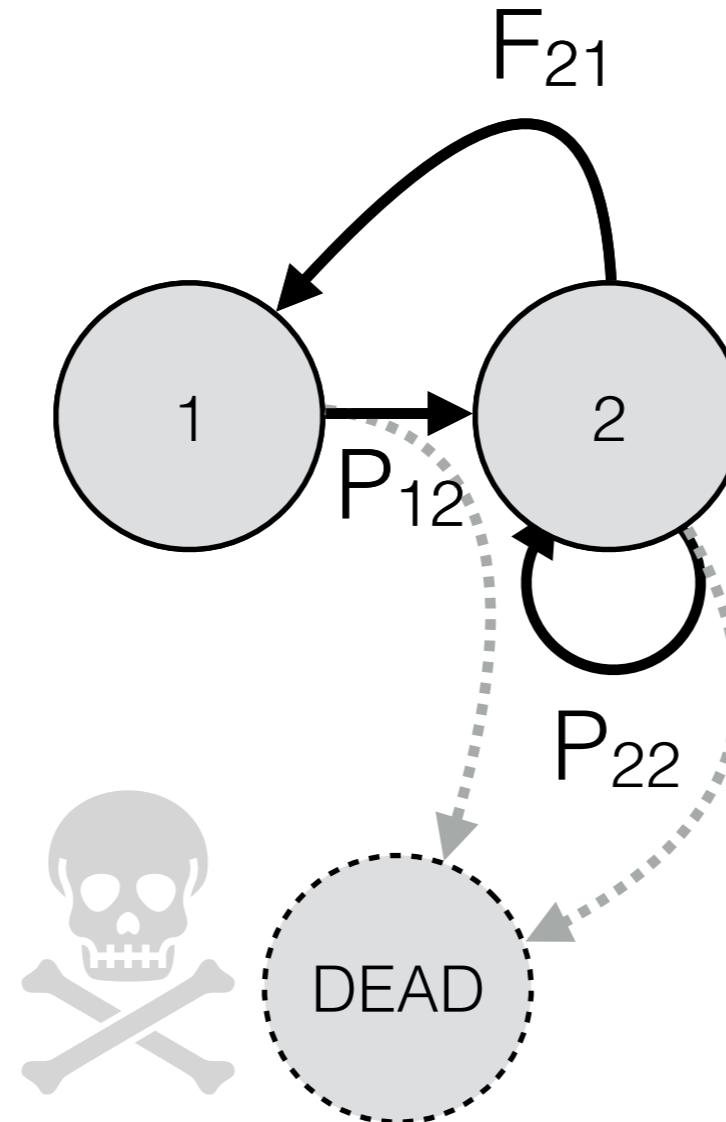
$$N_{2(t+1)} = P_{12}N_{1(t)}$$

$$N_{\text{DEAD}(t+1)} = (1 - P_{12})N_{1(t)} + N_{2(t)}$$



# Stage-specific equations

*Verbascum thapsus*



$$N_{1(t+1)} = F_{21}N_{2(t)}$$

$$N_{2(t+1)} = P_{12}N_{1(t)} + P_{22}N_{2(t)}$$

$$N_{\text{DEAD}(t+1)} = (1 - P_{12})N_{1(t)} + (1 - P_{22})N_{2(t)}$$





RESEARCH

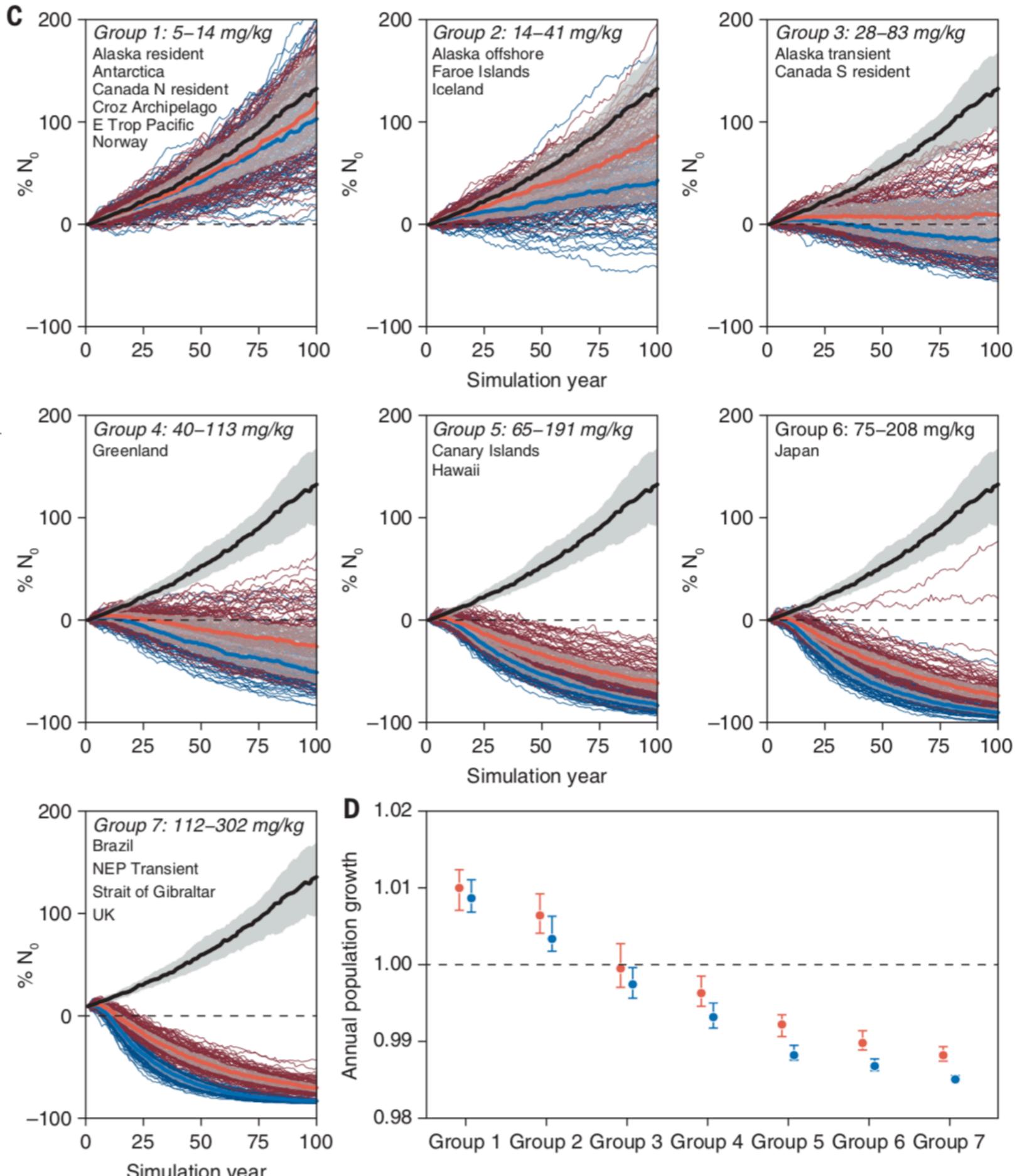
PERSISTENT CHEMICALS

## Predicting global killer whale population collapse from PCB pollution

Jean-Pierre Desforges<sup>1\*</sup>, Ailsa Hall<sup>2\*</sup>, Bernie McConnell<sup>2</sup>, Aqqalu Rosing-Asvid<sup>3</sup>, Jonathan L. Barber<sup>4</sup>, Andrew Brownlow<sup>5</sup>, Sylvain De Guise<sup>6,7</sup>, Igor Eulaers<sup>1</sup>, Paul D. Jepson<sup>8</sup>, Robert J. Letcher<sup>9</sup>, Milton Levin<sup>6</sup>, Peter S. Ross<sup>10</sup>, Filipa Samarra<sup>11</sup>, Gísli Vikingsson<sup>11</sup>, Christian Sonne<sup>1</sup>, Rune Dietz<sup>1\*</sup>

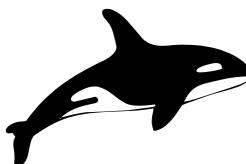
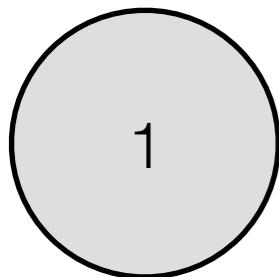
Killer whales (*Orcinus orca*) are among the most highly polychlorinated biphenyl (PCB)-contaminated mammals in the world, raising concern about the health consequences of current PCB exposures. Using an individual-based model framework and globally available data on PCB concentrations in killer whale tissues, we show that PCB-mediated effects on reproduction and immune function threaten the long-term viability of >50% of the world's killer whale populations. PCB-mediated effects over the coming 100 years predicted that killer whale populations near industrialized regions, and those feeding at high trophic levels regardless of location, are at high risk of population collapse. Despite a near-global ban of PCBs more than 30 years ago, the world's killer whales illustrate the troubling persistence of this chemical class.

No PCB  
Reproductive effects  
Reproduction + Immunity effects

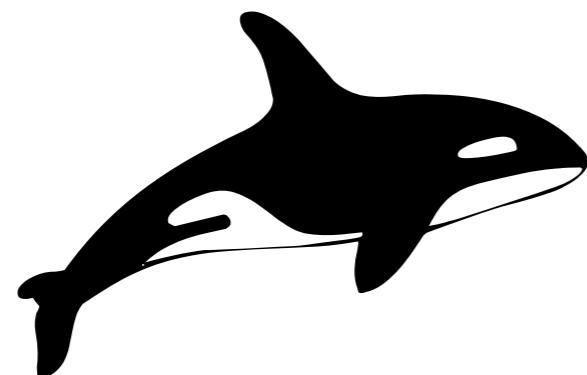
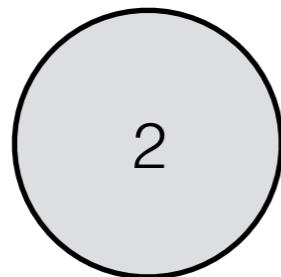


# Group exercise: Orca life cycle

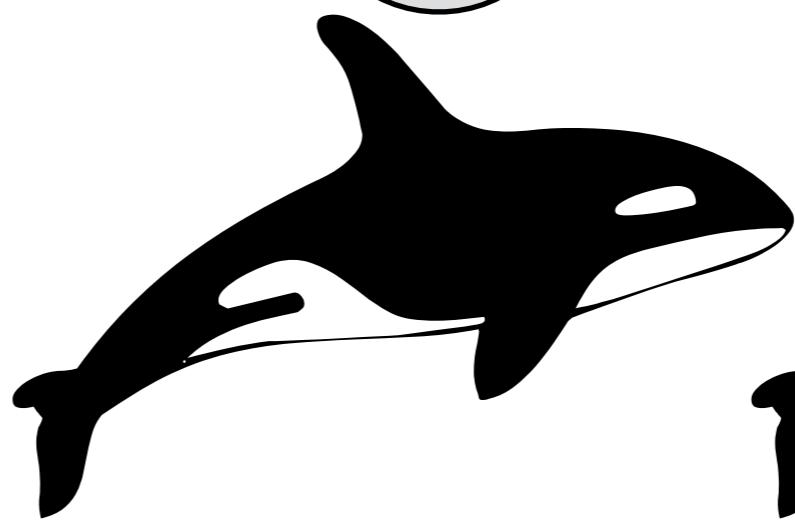
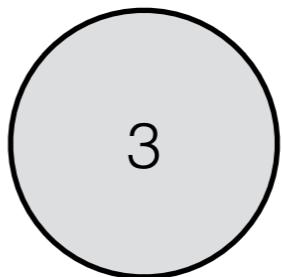
Calf



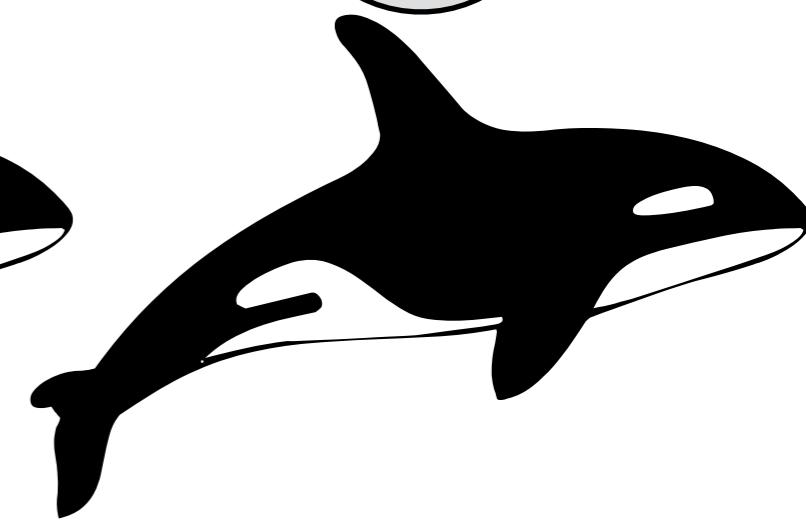
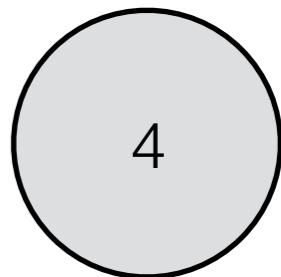
Small adult



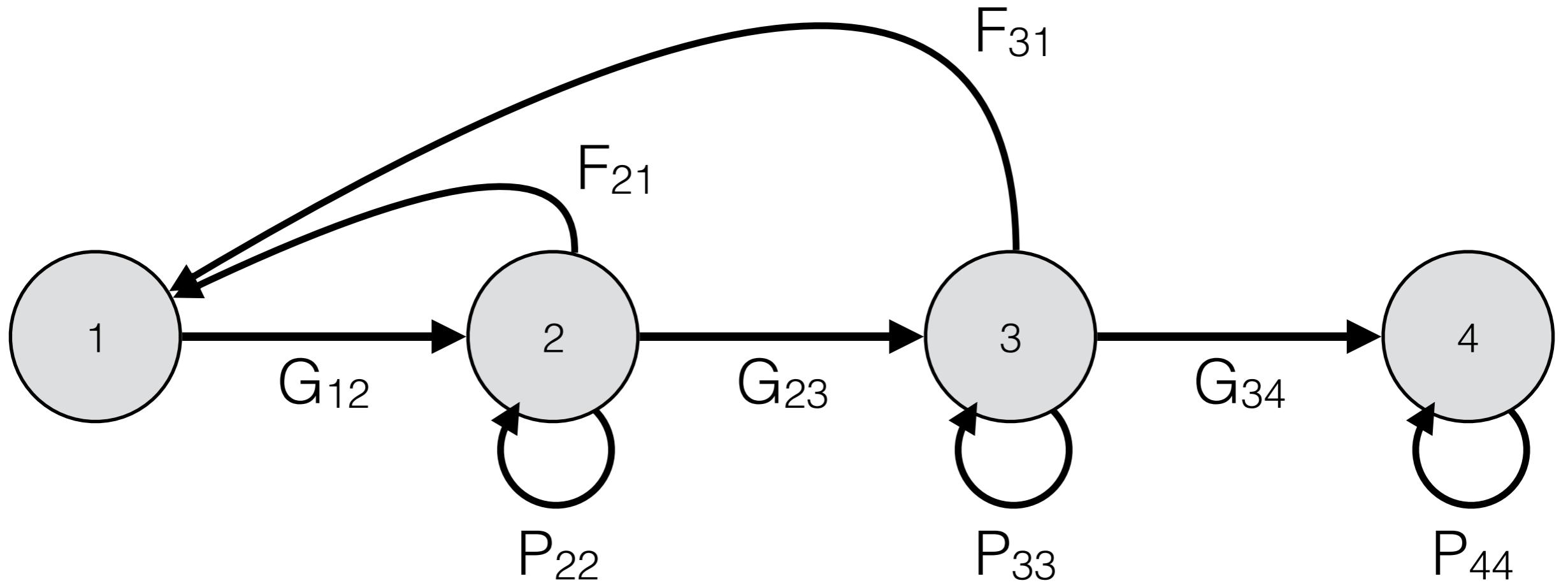
Large adult



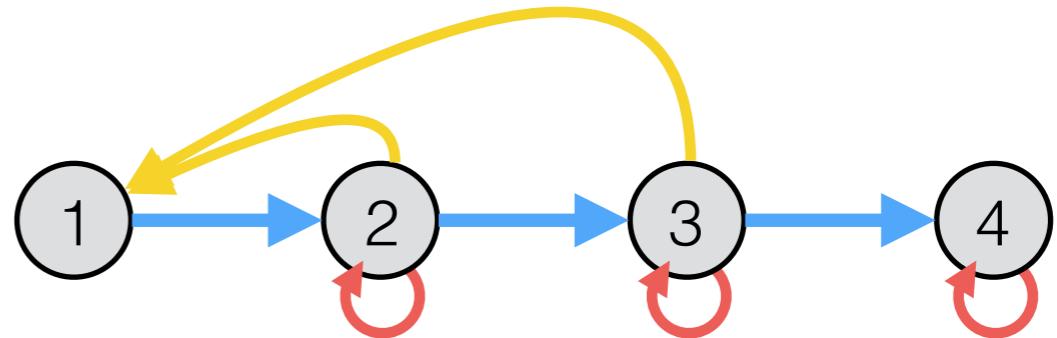
Post-reproductive  
adult



# Group exercise: Orca life cycle



# The transition matrix

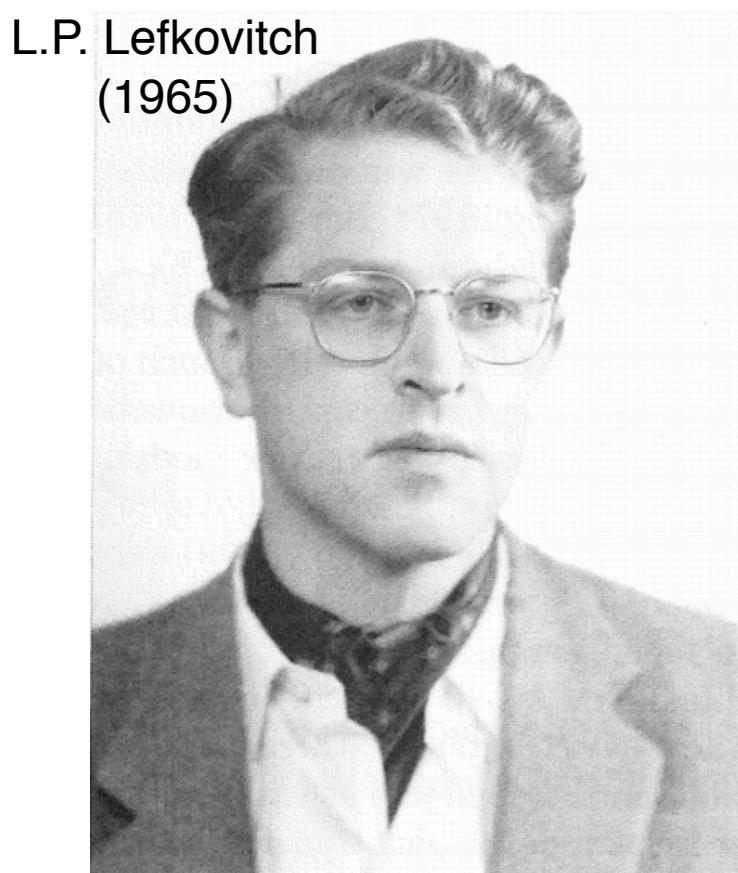


$$N_{1,t+1} = F_{21}N_{2,t} + F_{31}N_{3,t}$$

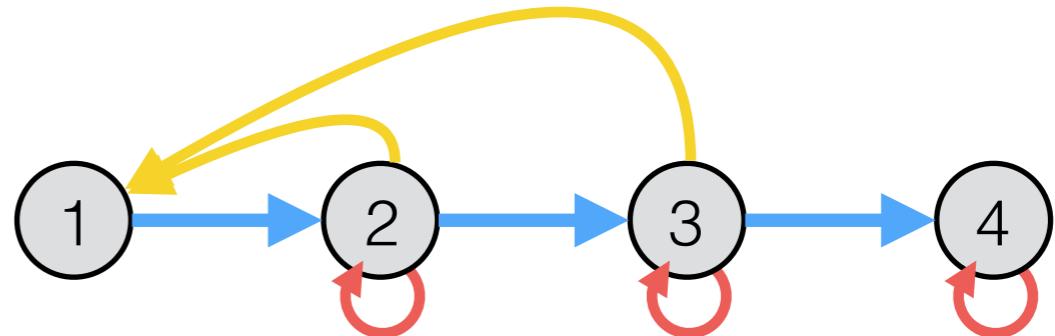
$$N_{2,t+1} = G_{12}N_{1,t} + P_{22}N_{2,t}$$

$$N_{3,t+1} = G_{23}N_{2,t} + P_{33}N_{3,t}$$

$$N_{4,t+1} = G_{34}N_{3,t} + P_{44}N_{4,t}$$



# The transition matrix

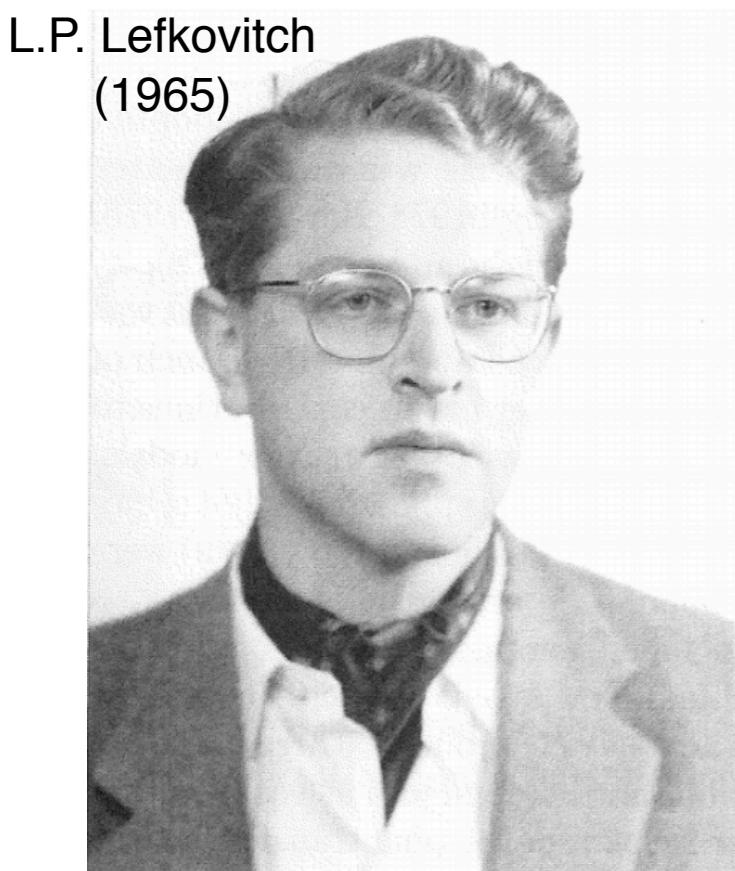


$$N_{1,t+1} = F_{21}N_{2,t} + F_{31}N_{3,t}$$

$$N_{2,t+1} = G_{12}N_{1,t} + P_{22}N_{2,t}$$

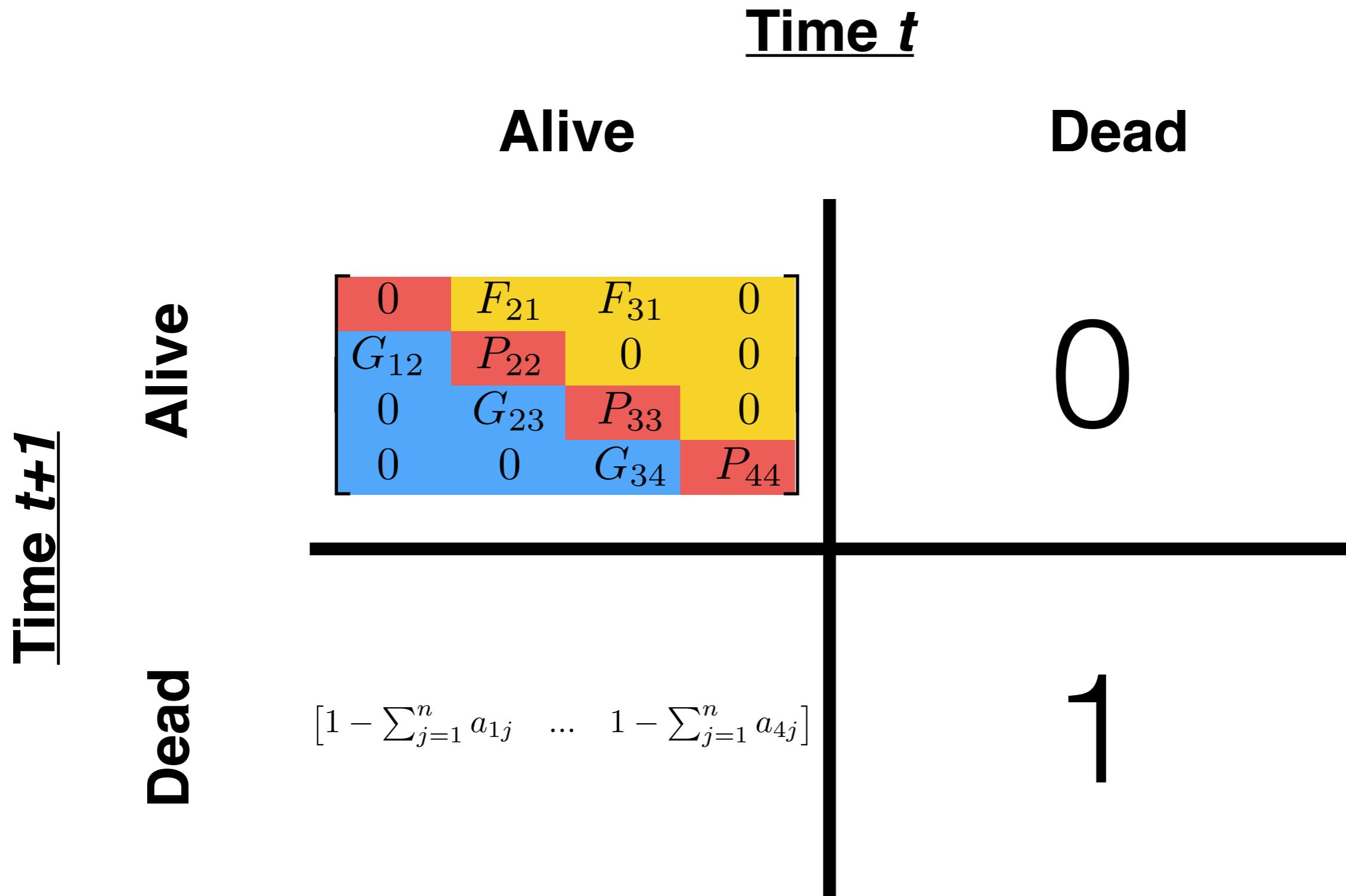
$$N_{3,t+1} = G_{23}N_{2,t} + P_{33}N_{3,t}$$

$$N_{4,t+1} = G_{34}N_{3,t} + P_{44}N_{4,t}$$



$$\begin{bmatrix} N_{1,t+1} \\ N_{2,t+1} \\ N_{3,t+1} \\ N_{4,t+1} \end{bmatrix} = \begin{bmatrix} 0 & F_{21} & F_{31} & 0 \\ G_{12} & P_{22} & 0 & 0 \\ 0 & G_{23} & P_{33} & 0 \\ 0 & 0 & G_{34} & P_{44} \end{bmatrix} \begin{bmatrix} N_{1,t} \\ N_{2,t} \\ N_{3,t} \\ N_{4,t} \end{bmatrix}$$

# Death is an absorbing state





**U = survival & growth**

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.9775 & 0.9111 & 0 & 0 \\ 0 & 0.0736 & 0.9534 & 0 \\ 0 & 0 & 0.0452 & 0.9804 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0.0043 & 0.1132 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sum_{i=1}^n a_{ij} = [0.9775 \quad 0.9847 \quad 0.9986 \quad 0.9804]$$

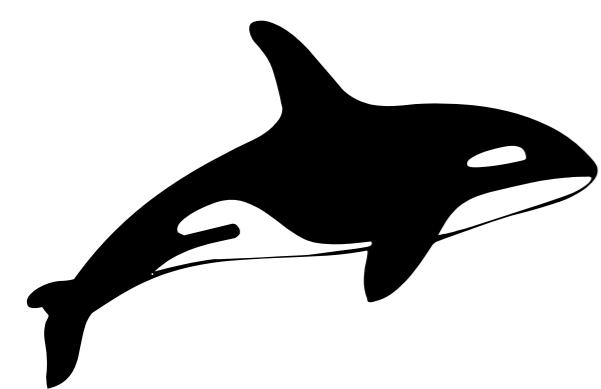
**Stage-specific  
survival rates**

**F = fecundity**

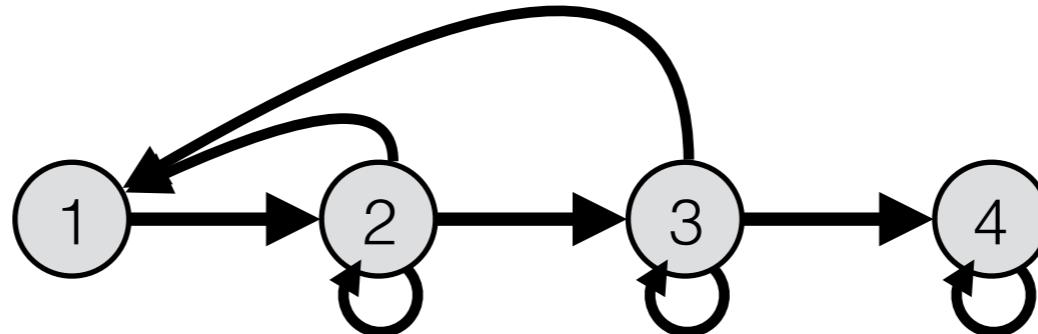
**C = clonality**

# Structured populations

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# Projection

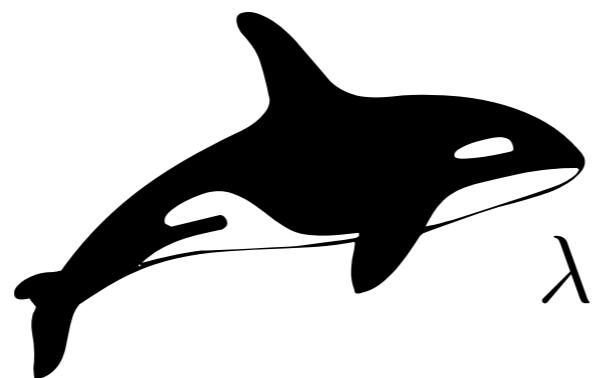


$$\begin{bmatrix} N_{1,t+1} \\ N_{2,t+1} \\ N_{3,t+1} \\ N_{4,t+1} \end{bmatrix} = \begin{bmatrix} 0 & 0.0043 & 0.1132 & 0 \\ 0.9775 & 0.9111 & 0 & 0 \\ 0 & 0.0736 & 0.9534 & 0 \\ 0 & 0 & 0.0452 & 0.9804 \end{bmatrix} \begin{bmatrix} N_{1,t} \\ N_{2,t} \\ N_{3,t} \\ N_{4,t} \end{bmatrix}$$

Matrix multiply,  
& iterate

# Equilibrium: Population growth ( $\lambda$ )

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \lambda \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$$



$$\lambda = 1.025$$



	Population size	Population decline	Population growth rate ( $\lambda$ )
Critically endangered	–	$\geq 80\%$ 10 years	$\lambda \leq (1 - 0.8)^{\frac{1}{10}}$ $\leq 0.851$
	$N < 250$	$\geq 25\%$ 3 years	$\leq 0.909$
Endangered	–	$\geq 50\%$ 10 years	$\leq 0.933$
	$N < 2500$	$\geq 20\%$ 5 years	$\leq 0.956$
Vulnerable	–	$\geq 20\%$ 10 years	$\leq 0.978$
	$N < 10000$	$\geq 10\%$ 10 years	$\leq 0.990$

# Equilibrium: Stage structure ( $w$ )

$$\begin{bmatrix} a_{11} & \dots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \lambda \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$$



$$w = \begin{bmatrix} 0.0370 \\ 0.3161 \\ 0.3229 \\ 0.3240 \end{bmatrix} = \begin{bmatrix} 3.70\% \\ 31.61\% \\ 32.29\% \\ 32.40\% \end{bmatrix}$$

# Structured vs. unstructured population dynamics

orca.R

Shiny app  
[matrixProjector.R]

**<https://rstudio.cloud/project/239821>**

Which stages have the  
biggest impact on  $\lambda$ ?

seaturtle.R

**<https://rstudio.cloud/project/239821>**



0	0	0	4.665	61.896
0.675	0.703	0	0	0
0	0.047	0.657	0	0
0	0	0.019	0.682	0
0	0	0	0.061	0.8091

No intervention:

$$\lambda = 0.952$$

Endangered

# Protect hatchlings



No intervention:

0	0	0	4.665	61.896
0.675	0.703	0	0	0
0	0.047	0.657	0	0
0	0	0.019	0.682	0
0	0	0	0.061	0.8091

Hatchling protection:

0	0	0	4.665	61.896
1.000	0.703	0	0	0
0	0.047	0.657	0	0
0	0	0.019	0.682	0
0	0	0	0.061	0.8091

$$\lambda_{\text{control}} = 0.952$$

Endangered

$$\lambda_{\text{hatching protection}} = 0.974$$

Vulnerable



# Turtle Exclusion Device

No intervention:

0	0	0	4.665	61.896
0.675	0.703	0	0	0
0	0.047	0.657	0	0
0	0	0.019	0.682	0
0	0	0	0.061	0.8091

Turtle Exclusion Device (TED):

0	0	0	4.665	61.896
0.675	0.703	0	0	0
0	0.047	0.757	0	0
0	0	0.022	0.769	0
0	0	0	0.069	0.877

0.675	0.750	0.676	0.743	0.8091
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0.675	0.750	0.77918	0.83814	0.876703
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$$\lambda_{\text{control}} = 0.952$$

Endangered

$$\lambda_{\text{TED}} = 1.027$$

Recovering

# Sensitivity

“Fixed change in  $\lambda$  for a fixed change in  $a_{ij}$ ”

$$\frac{\partial \lambda}{\partial a_{ij}} = \frac{v_i w_j}{w \cdot v}$$

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \lambda \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$$

$$\begin{bmatrix} v_1 & \dots & v_n \end{bmatrix} \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} = \lambda_L \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix}$$

# Elasticity

“Proportional change in  $\lambda$  for a proportional change in  $a_{ij}$ ”

$$e_{ij} = \frac{a_{ij}}{\lambda} \frac{\partial \lambda}{\partial a_{ij}}$$

0	0	0	4.665	61.896
0.675	0.703	0	0	0
0	0.047	0.657	0	0
0	0	0.019	0.682	0
0	0	0	0.061	0.8091

## Sensitivity



## Elasticity



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Next week:  
Noisy environments

