Topologically billiard-like paths in triangles

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The *billiard flow* on a triangle Δ is the usual Euclidean geodesic flow on the interior of Δ, but on the edges the flow sends the angle of incidence to the angle of reflection. The billiard flow is not defined on the vertices. A *periodic billiard path* is a closed trajectory of the billiard flow. [Tab95] and [MT02] survey results on billiards.

We mark the edges of our triangles $1, 2, 3$. The *symbolic dynamics* $s_{\widehat{\gamma}}$ of a periodic billiard path $\widehat{\gamma}$ is the repeating sequence of integers marking the edges it hits. Denote the space of all marked triangles up to similarity by $\mathcal{T}$. Given $s_{\widehat{\gamma}}$, it useful to know $tile(s_{\widehat{\gamma}}) \subset \mathcal{T}$, the set of all triangles which have periodic billiard paths that realize $s_{\widehat{\gamma}}$ as their symbolic dynamics. We will say that a billiard path $\widehat{\gamma}$ on Δ is *stable* if $tile(s_{\widehat{\gamma}})$ contains an open neighborhood of Δ. Recent progress has been made on the open question, "Does every triangle have a periodic billiard path?" by Schwartz who analyzes many tiles of stable periodic billiard paths (see [Sch06a] and [Sch06b]).

We prove a new elementary result about tiles of stable periodic billiard paths:

Theorem 1 *If $\widehat{\gamma}$ is a stable periodic billiard path, then $tile(s_{\widehat{\gamma}})$ contains only acute or only obtuse triangles. Further, if $tile(s)$ contains obtuse triangles then all of the longest sides of triangles in $tile(s)$ have the same marking.*

A rephrasing of this theorem is given by figure 1.

This theorem is proved using a technique to confine $tile(s)$. Coordinatize the space of marked triangles up to similarity by the angles of the triangles as in figure 1. In section 6, we define a set of triangles $UF(s) \subset \mathcal{T}$ which consists of a finite union of lines in $\mathcal{T}$. Topological techniques demonstrate that $UF(s)$ and $tile(s)$ are disjoint. Section 7 proves that $tile(s)$ lies in only one component of $\mathcal{T} \smallsetminus UF(s)$. The final section is devoted to proving that

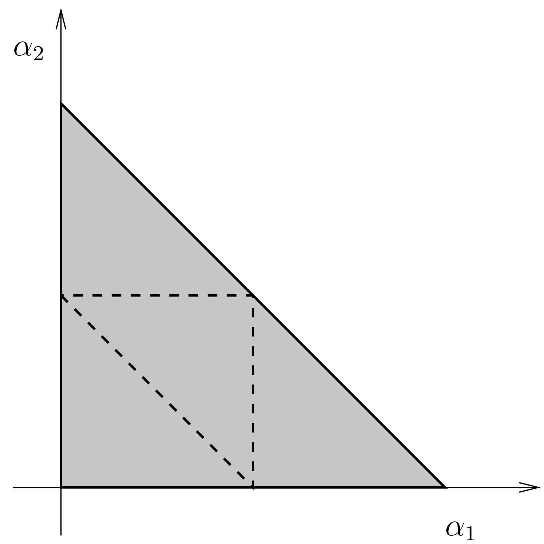


Figure 1: Points in $\mathcal{T}$ can be identified by two of their angles, α_1 and α_2. The right triangles split the parameter space of marked triangles into four components. Theorem 1 states that if $\widehat{\gamma}$ is stable, then *tile*$(s_{\widehat{\gamma}})$ is stable then it is contained in one of these four components.

the right triangles are contained in $UF(s)$ for any choice of s. This directly implies theorem 1.

Remark 2 *Heuristically, when trying to understand tile*(s)*, first we would hope to be able to localize the problem by restricting analysis to a small subset of the parameter space of triangles. In fact, the component of* $\mathcal{T} \setminus UF(s)$ *that contains tile*(s) *can be computed explicitly (see definition 28) This algorithm is implemented as part of* McBilliards[1]*, a computer program written Rich Schwartz and I to investigate billiards in triangles.* McBilliards *is capable of rigorously computing approximations of tile*(s)*. The algorithm that computes the relevant component of* $\mathcal{T} \setminus UF(s)$ *operates quite quickly and is used to localize the computations that yield approximations to tile*(s)*.*

[1]McBilliards is freely available online at `http://www.math.sunysb.edu/~pat/` or `http://www.math.brown.edu/~res/`.

1 The thrice punctured sphere

Given a triangle $\Delta \in \mathcal{T}$, we can construct a locally Euclidean surface, Σ_Δ, by gluing two copies of Δ together and removing the vertices. See figure 2.

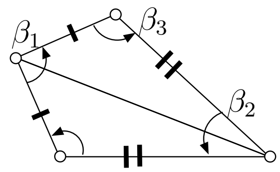


Figure 2: Σ_Δ is constructed from two copied of the triangle Δ. The β_i are curves which travel around the punctures.

Think of Σ_Δ as a triangular pillow case. There is a canonical folding map $\Sigma_\Delta \to \Delta$ which is $2-1$ except on the edges of Δ. The billiard flow on Δ can be pulled back to the Euclidean geodesic flow on Σ_Δ. Thus, a periodic billiard path $\widehat{\gamma}$ in Δ has a pull back to a closed Euclidean geodesic γ in Σ_Δ. Based on this observation, we will forget entirely about looking for billiard paths, and look for geodesics on this locally Euclidean surface instead.

The symbolic dynamics, $s_{\widehat{\gamma}}$, of the periodic billiard path $\widehat{\gamma}$ is the sequence of edges hit. The geodesic loop γ also hits the same sequence of edges, $s_\gamma = s_{\widehat{\gamma}}$, of the triangulation of Σ_Δ. We denote the set of all loops in Σ_Δ that hit edges of the triangulation in this sequence by $[\gamma]_s$. We call this collection of loops a *symbolics class*. Note that every loop in $[\gamma]_s$ lies in the same homotopy class, which we specify as $[\gamma]$.

Conversely, a repeating sequence s in $\{1, 2, 3\}$ determine a symbolics class. This corresponding symbolics class is the set of all curves which hit the edges of Σ_Δ in the sequence specified by the shortest even period of s.

The surfaces Σ_Δ are all canonically homeomorphic under a homeomorphism which preserves the triangulation. We will use Σ to denote the topological triangulated 3-punctured sphere, when we do not care about its geometric structure. Given some symbolics class of curves, $[\gamma]_s$, on Σ, we define *tile*$([\gamma]_s)$ to be the set of all marked triangles Δ for which there is a geodesic in the symbolics class $[\gamma]_s$ on Σ_Δ. We will say that a geodesic γ on Σ_Δ is *stable* if *tile*$([\gamma]_s)$ contains an open neighborhood of Δ. By the above discussion, if the periodic billiard path $\widehat{\gamma}$ on Δ is the image of the geodesic γ on Σ_Δ, then *tile*$(s_{\widehat{\gamma}})$ = *tile*$([\gamma])$. Therefore, stable periodic billiard paths correspond to stable geodesics and vice versa.

2 Translation Surfaces

A locally Euclidean surface S has a holonomy homomorphism defined by developing a loop into the plane using analytic continuation:

$$hol : \pi_1(S) \to Isom_+(\mathbb{R}^2) \tag{1}$$

Here, the holonomy homeomorphism is defined after noticing that if X is an open simply connected space which is locally isometric to the plane, then it immerses isometrically into the plane. First choose a base point P and an open neighborhood $U_P \ni P$ together with an isometry $\phi : U_p \to \mathbb{R}^2$. Suppose we are given a loop $\alpha : [0,1] \to S$ so that $\alpha(0) = \alpha(1) = P$. We extend α by taking an open 2-disk $U_\alpha \supset [0,1]$ together with an immersion $\widehat{\alpha} : U_\alpha \to S$. Choose components $U_0, U_1 \subset \widehat{\alpha}^{-1}(U_P)$ so that $0 \in U_0$ and $1 \in U_1$. When endowed with the pull back metric, U_α becomes locally isometric to the plane. Then because U_α is simply connected, there is an isometric immersion $\psi : U_\alpha \to R^2$. We make the unique choice for ψ so that $\psi|_{U_0} = \phi \circ \widehat{\alpha}|_{U_0}$. Choose $hol(\alpha)$ so that $hol(\alpha) \circ \phi \circ \widehat{\alpha}|_{U_1} = \psi|_{U_1}$. This construction is invariant under homotopies of α preserving the condition that $\alpha(0) = \alpha(1) = P$. Further, hol defines a group homomorphism.

A *translation surface* is a locally Euclidean surface where the image $hol(\pi_1(S))$ consists only of translations. This implies that the surface can be constructed by gluing subsets of the plane together by translations. In particular, directions make sense on a translation surface. In other words,

Proposition 3 *There is a fibration from the unit tangent bundle of a translation surface S to the unit circle in the plane,*

$$\theta : T_1 S \to S^1$$

which is an isometry when restricted to the unit tangent circle at any point and is invariant under the geodesic flow.

In particular, the image under θ of unit tangent vectors to a geodesic is constant. That is, geodesics travel in a unique direction on translation surfaces. Furthermore, the angle of intersection between two geodesics is determined by directions the two geodesics are traveling.

Proposition 4 *Geodesics on translation surfaces satisfy the following:*

1. *Geodesics have no self intersections, and two distinct geodesics traveling in the same direction never intersect.*

2. *Algebraic signs of intersection are determined by their directions. For example, each intersection between two geodesics has the same algebraic sign.*

Given our locally Euclidean surface Σ_Δ, we can construct coverings which are translation surfaces. A trivial example is the universal cover, $\widetilde{\Sigma}_\Delta$. Another translation surface covering is the universal abelian cover,

$$AC_\Delta = \widetilde{\Sigma}_\Delta / [\pi_1(\Sigma_\Delta), \pi_1(\Sigma_\Delta)] \tag{2}$$

Here, $[\pi_1(\Sigma_\Delta), \pi_1(\Sigma_\Delta)]$ is the commutator subgroup of $\pi_1(\Sigma_\Delta)$. The universal abelian cover is a translation surface because the commutator subgroup of $Isom_+(\mathbb{R}^2)$ is the group of translations. There is always a *minimal translation surface covering*, which we define as

$$MT_\Delta = \widetilde{\Sigma}_\Delta / hol_\Delta^{-1}(\mathbb{R}^2)$$

where $hol_\Delta^{-1}(\mathbb{R}^2)$ is the preimage of the group of translations of the plane in $\pi_1(\Sigma_\Delta)$. We call MT_Δ minimal because every translation surface which covers Σ_Δ also covers MT_Δ.

Remark 5 *MT_Δ can also be built directly out of copies of Δ. Consider the group action of $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ generated by reflections in the sides of Δ. Take all images of Δ under this group and identify any two triangles which differ by a translation. Identify two triangles along an edge if they differ by reflection in that edge, up to a translation. This tautologically builds MT_Δ.*

In summary, we have discussed the following sequence of covers:

$$\widetilde{\Sigma}_\Delta \to AC_\Delta \to MT_\Delta \to \Sigma_\Delta \tag{3}$$

It should be noted that each cover is regular. The cover $AC_\Delta \to \Sigma_\Delta$ is regular because the commutator subgroup of the fundamental group is normal. Thus the automorphism group of this cover is

$$H_1(\Sigma_\Delta, \mathbb{Z}) = \pi_1(\Sigma_\Delta, \mathbb{Z}) / [\pi_1(\Sigma_\Delta), \pi_1(\Sigma_\Delta)] \cong \mathbb{Z}^2 \tag{4}$$

Since this group is abelian, the intermediate covers, $AC_\Delta \to MT_\Delta \to \Sigma_\Delta$, will be regular as well.

3 Geodesics on translation surfaces

If we develop a geodesic on Σ_Δ into the plane, we see a line. The holonomy around this geodesic must preserve this line and is therefore a translation. A loop γ on Σ_Δ whose holonomy is a translation must lift to a loop $\widetilde{\gamma}$ on the minimal translation surface MT_Δ. The main idea of this paper is to use this to build obstructions to the existence of geodesics on Σ_Δ in a fixed symbolics class.

Proposition 4 implies that a collection of geodesics which all are traveling the same direction on a translation surface are simple and disjoint. Using objects called train-tracks, Thurston discovered a technique to list collections of disjoint simple closed curves on surfaces. See [PH92] for more on train tracks. We follow this idea in spirit here.

A second look at proposition 4 reveals that a collection of geodesics which all are traveling the same direction have at most one algebraic sign of intersection with each edge. In particular, the magnitude of the algebraic intersection number between this curve family and any edge of the triangulation of MT_Δ equals the geometric intersection number. This algebraic intersection number is a homological invariant.

For a slightly more general setup, let S be a translation surface. Assume S admits a triangulation with edges consisting of saddle connections. That is, the geodesics have no end points (or alternately have singular end points). Let $X = \{\gamma_1, \ldots, \gamma_n\}$ be a collection of geodesics on S which are all traveling in the same direction. Let $[\![X]\!]$ be the collection's homology class

$$[\![X]\!] = [\![\gamma_1]\!] + \ldots + [\![\gamma_n]\!]$$

We will show that we can recover the symbolics classes of the curves in X from its homology class, $[\![X]\!]$.

The homology class $[\![X]\!]$ is determined by its intersection numbers with the edges of the triangulation of S. Because the triangulation of S is by saddle connections, S is homotopic to the dual graph to this triangulation. Let Γ be the dual graph to the triangulation. A homology class of a graph is an assignment of orientations and non-zero integral weights to finitely many edges of the graph subject to a vertex condition. The vertex condition is that the sum of the weights of edges oriented inward at a vertex is equal to the sum of weights of edges oriented outward. Given such an assignment of orientations and weights, for each triangle there is a unique topological way to draw disjoint oriented arcs connecting the edges of the triangle so that

1. orientations of arcs crossing an edge of a triangle match the orientation of the edge of Γ crossing the same edge of the triangle, and

2. the number of arcs crossing an edge of a triangle equals the weight of the edge of Γ crossing that same edge.

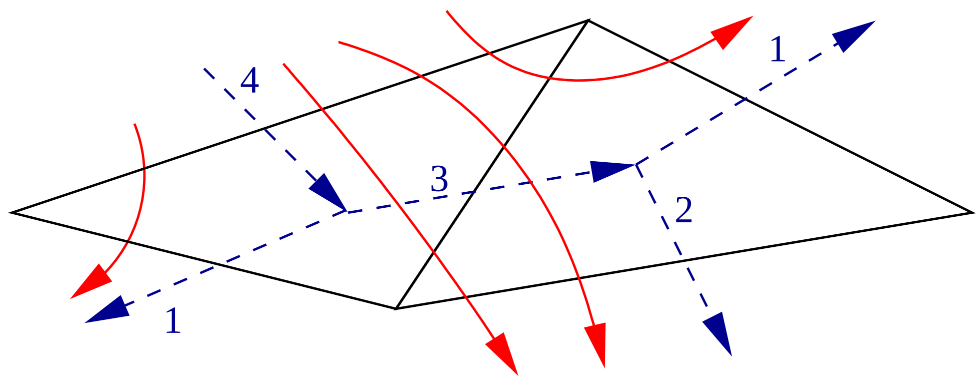


Figure 3: Turning a homology class into simple curves

There is a unique way to glue together the arcs of two adjacent triangles. See figure 3 for a local picture. The result of gluing the arcs together is a finite union of disjoint oriented simple closed curves in the right homology class. We call this construction "combing out the homology class."

We say that a *symbolics class* of curves on the surface S triangulated by saddle connections is a collection of loops which hit the saddle connections in the same sequence.

Now let us summarize this situation. Let $H_1(S)$ denote the first homology group with coefficients in $\mathbb{Z}$. Let $\mathcal{H}(S)$ denote the collection of all finite sets of symbolics classes of (non-homotopically trivial) oriented curves. This construction gives us a map

$$\Psi : H_1(S) \to \mathcal{H}(S)$$

which sends a homology class $[\![X]\!]$ to a set of symbolics classes of curves $[X]_s$ in that homology class. Furthermore $[X]_s$ can always be realized by disjoint simple curves which intersect each edge of the triangulation of S with the same algebraic sign. By uniqueness of the construction,

Proposition 6 *Let $[X]_s$ be finite set of (nontrivial) symbolics classes of oriented curves on S. Then $[X]_s$ can be realized by disjoint simple curves which intersect each edge of the triangulation of S with the same algebraic sign if and only if $[X]_s = \Psi(\llbracket X \rrbracket)$.*

The purpose for this is

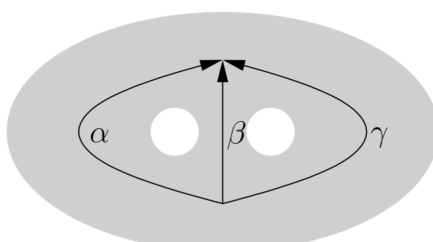
Corollary 7 *If X is a collection of closed geodesics that are all traveling in the same direction on S then $[X]_s = \Psi(\llbracket X \rrbracket)$ or equivalently $[X]_s \in img(\Psi)$.*

Proof: Combine propositions 4 and 6. $\diamondsuit$

To apply this corollary to our situation, we will need to better understand the translation surfaces AC_Δ and MT_Δ.

4 The universal abelian cover

Note that Σ_Δ is a three punctured sphere, and thus is homotopic to the Θ-graph, which we will think of as embedded in Σ_Δ. See the left side of figure 4. Because Σ_Δ is a thickening of the Θ-graph, AC_Δ will be a thickening of the universal abelian cover of the Θ-graph, AC_Θ.



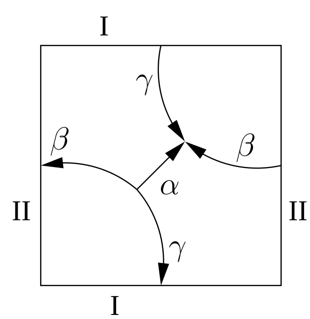


Figure 4: *Left:* Edge markings on the theta graph. *Right:* The theta graph embedded inside the torus with a hexagon in the complement. Roman numerals indicate edge identifications on the torus.

The universal abelian cover of the Θ-graph, AC_Θ, is the graph consisting of the edges of the usual tiling of the plane by hexagons. To see this, mark and orient the edges of the theta graph as in figure 4. Then we can glue a hexagonal 2-cell to the theta graph by using the concatenation of paths, $\alpha\beta^{-1}\gamma\alpha^{-1}\beta\gamma^{-1}$, as a boundary curve. This builds an embedding of the theta graph into the torus, $i : \Theta \hookrightarrow T^2$. The loop bounding this hexagon is inside the commutator because

$$[\alpha\beta^{-1}, \gamma\beta^{-1}] = (\alpha\beta^{-1})(\gamma\beta^{-1})(\beta\alpha^{-1})(\beta\gamma^{-1}) = \alpha\beta^{-1}\gamma\alpha^{-1}\beta\gamma^{-1} \tag{5}$$

This together with the fact that the commutator $[\pi_1(T^2), \pi_1(T^2)] = 0$ is trivial implies that the commutator $[\pi_1(\Theta), \pi_1(\Theta)] = 0$ is precisely the kernel of the induced map $i_* : \pi_1(\Theta) \to \pi_1(T^2)$. Hence the universal abelian cover of the theta graph is the cover of the theta graph induced by the universal cover of T^2. As claimed, the universal abelian cover of the theta graph is the graph consisting of edges of the usual tiling of the plane by hexagons. The automorphisms of this cover act by translations of the plane.

We mentioned that Σ_Δ is a thickening of the Θ-graph. We now understand that the Θ-graph is the graph coming from the tiling of the plane by hexagons modulo the set of translations preserving the tiling. We redraw the embedding of the Θ-graph into Σ_Δ in figure 5 to make our two pictures more clearly compatible.

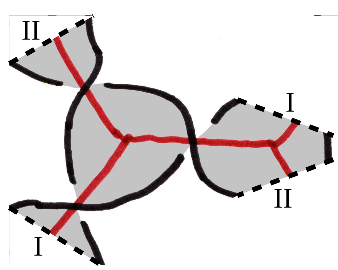


Figure 5: The embedding $\Theta \hookrightarrow \Sigma_\Delta$. This surface is just the thrice punctured sphere. Each edge of the Θ-graph is thickened to a half twisted band.

Translations of figure 5 glue up the surface to a thrice punctured sphere. We lift this thickening of the Θ-graph to the universal abelian cover. We see AC_Δ is homeomorphic to the surface shown in figure 6. Each edge of AC_Θ is thickened to a half twisted band.

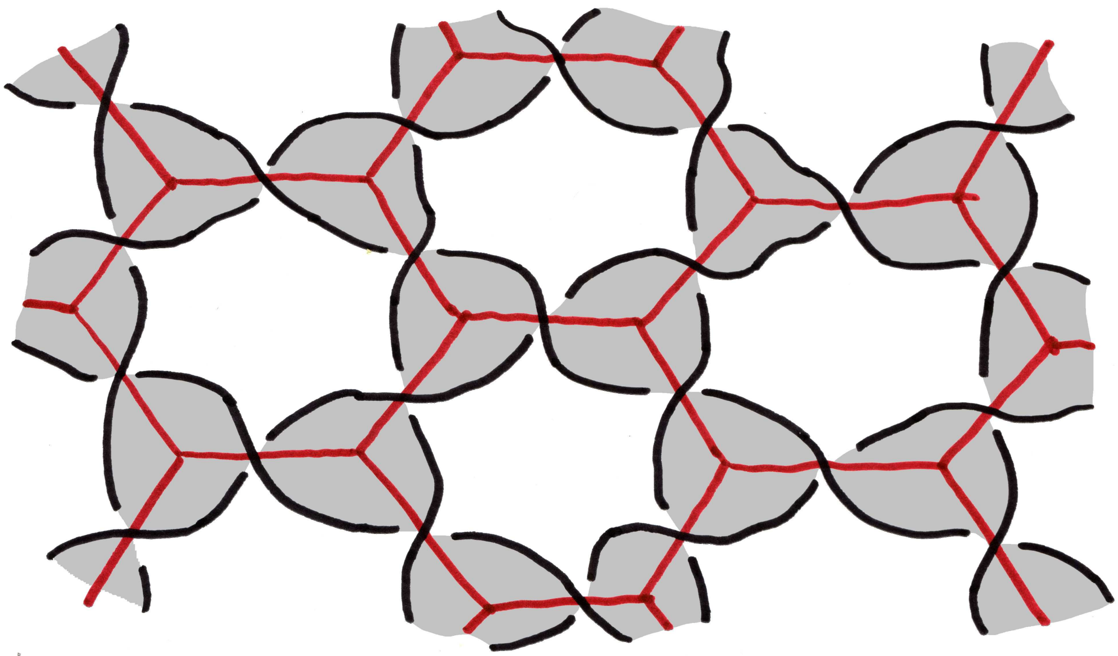


Figure 6: The embedding $AC_\Theta \hookrightarrow AC_\Delta$

5 The minimal translation surface cover

MT_Δ is always an intermediate cover between AC_Δ and Σ_Δ. Recall, the cover $AC_\Delta \to \Sigma_\Delta$ is regular and has an automorphism group $H_1(\Sigma_\Delta)$ isomorphic to $\mathbb{Z}^2$. Since MT_Δ is intermediate, it is regular and arises as

$$MT_\Delta = AC_\Delta / K_\Delta \tag{6}$$

for some subgroup $K_\Delta \subset H_1(\Sigma_\Delta)$.

By definition, $MT_\Delta = \widetilde{\Sigma}_\Delta / hol_\Delta^{-1}(\mathbb{R}^2)$. Thus K_Δ is the image of the subgroup $hol_\Delta^{-1}(\mathbb{R}^2) \subset \pi_1(\Sigma_\Delta)$ inside $H_1(\Sigma_\Delta)$. K_Δ can be understood as the kernel of the abelianization of the holonomy map, hol_Δ. Consider the commutative diagram, where the down arrows correspond to abelianizations,

$$\begin{array}{ccc} \pi_1(\Sigma_\Delta) & \xrightarrow{hol_\Delta} & Isom_+(\mathbb{R}^2) \\ \downarrow & & \downarrow \\ H_1(\Sigma_\Delta) & \xrightarrow{hol_\Delta^{ab}} & Isom_+(S^1) \end{array} \tag{7}$$

Here $Isom_+(S^1)$ is the abelianization of $Isom_+(\mathbb{R}^2)$, because the commutator subgroup of $Isom_+(\mathbb{R}^2)$ is precisely the subgroup of translations. We can think of the abelianization of an isometry of the plane as its rotational part. Now,

$$K_\Delta = ker(hol_\Delta^{ab}) \quad \text{and} \quad MT_\Delta = AC_\Delta / ker(hol_\Delta^{ab}) \tag{8}$$

Using the direction map θ of proposition 3, we can give a clean geometric interpretation of the abelianized holonomy map hol_Δ^{ab}. Consider an automorphism ζ of the cover $AC_\Delta \to \Sigma_\Delta$. Because ζ preserves the local Euclidean structure, it must act as a rotation on the direction map. $hol_\Delta^{ab}(\zeta)$ describes this rotation. In other words, the following diagram commutes

$$\begin{array}{ccc} T_1 AC_\Delta & \xrightarrow{\theta} & S^1 \\ \zeta \downarrow & & \downarrow hol_\Delta^{ab}(\zeta) \\ T_1 AC_\Delta & \xrightarrow{\theta} & S^1 \end{array} \tag{9}$$

So, MT_Δ is AC_Δ modulo those automorphisms which act trivially on the directions in AC_Δ. Therefore, to understand MT_Δ, we will first need to understand $hol^{ab}_{\Delta} : H_1(\Sigma_\Delta) \to Isom_+(S^1)$. The group $H_1(\Sigma_\Delta)$ is the free abelian group generated by the homology classes of the curves β_1 and β_2 which travel around the punctures as in figure 2. If α_i is the angle in radians of the triangle Δ at the vertex surrounded by β_i, then it is clear that the holonomy around β_i acts as a rotation by $2\alpha_i$ around the puncture surrounded by β_i. Consequently,

$$hol^{ab}_{\Delta} : r[\![\beta_1]\!] + s[\![\beta_2]\!] \mapsto 2r\alpha_1 + 2s\alpha_2 \in \mathbb{R}/2\pi\mathbb{Z} = Isom_+(S^1) \tag{10}$$

There is another way we could have arrived at the map hol^{ab}_{Δ}. If $[\![\gamma]\!] \in H_1(\Sigma_\Delta)$ and $\gamma \in [\![\gamma]\!]$ is a smooth curve, then hol^{ab}_{Δ} measures the total turning angle of γ modulo 2π. In other words,

$$hol^{ab}_{\Delta}([\![\gamma]\!]) = \Big(\int_{\gamma'} d\theta\Big) \ (mod\ 2\pi) = \Big(\int_{\gamma} \kappa_g\, ds\Big) \ (mod\ 2\pi) \tag{11}$$

Here γ' represents the curve on the unit tangent bundle, $T_1\Sigma_\Delta$, obtained by taking the unit tangents at each point of γ. The form $d\theta$ is the derivative of the direction map $\theta : T_1\Sigma_\Delta \to \mathbb{R}/2\pi\mathbb{Z}$ described by proposition 3. $\kappa_g\, ds$ denotes the geodesic curvature with respect to arc length. Since our surface is locally Euclidean, this just measures the rate at which the unit tangents change directions.

The *(mod 2π)* term is needed for these integrals to yield homological invariants over $H_1(\Sigma_\Delta)$. Consider for example, a homotopically trivial simple closed loop with total curvature 2π. Nonetheless, we would like to remove *(mod 2π)* from the equation. Note that $d\theta$ is a closed form on $T_1\Sigma_\Delta$. Therefore integrating $d\theta$ over a curve in $T_1\Sigma_\Delta$ is a homological invariant. We define the "lift" of hol^{ab}_{Δ} to be the map

$$\widetilde{hol}^{ab}_{\Delta} : H_1(T_1\Sigma_\Delta) \to \mathbb{R} : [\![\gamma']\!] \mapsto \int_{\gamma'} d\theta \tag{12}$$

Let $\pi : T_1\Sigma_\Delta \to \Sigma_\Delta$ be projection to the base of the unit tangent bundle and let $\pi_* : H_1(T_1\Sigma_\Delta) \to H_1(\Sigma_\Delta)$ be the map induced on homology. Then $\widetilde{hol}^{ab}_{\Delta}$ is a lift of hol^{ab}_{Δ} in the sense of the following commuting diagram:

$$\begin{array}{ccc} H_1(T_1\Sigma_\Delta) & \xrightarrow{\widetilde{hol}^{ab}_{\Delta}} & \mathbb{R} \\ {\scriptstyle\pi_*}\big\downarrow & & \big\downarrow{\scriptstyle (mod\ 2\pi)} \\ H_1(\Sigma_\Delta) & \xrightarrow{hol^{ab}_{\Delta}} & \mathbb{R}/2\pi\mathbb{Z} \end{array} \tag{13}$$

We now go about understanding the map $\widetilde{hol}^{ab}_\Delta$. Let β_1, β_2, and β_3 be loops which travel around each puncture of Σ_Δ as depicted in figure 2. Let β_i' be their respective derivatives. It is easy to see that

$$\widetilde{hol}^{ab}_\Delta([\![\beta_i']\!]) = \int_{\beta_i'} d\theta_\Delta = 2\alpha_i \tag{14}$$

Where α_i is angle of the triangle at the vertex that β_i surrounds. This in fact determines the map $\widetilde{hol}^{ab}_\Delta$, because

Proposition 8 *The homology classes $[\![\beta_1']\!]$, $[\![\beta_2']\!]$, and $[\![\beta_3']\!]$ generate the homology group $H_1(T_1\Sigma_\Delta)$. The kernel of the map $H_1(T_1\Sigma_\Delta) \to H_1(\Sigma_\Delta)$ induced by the projection $T_1\Sigma_\Delta \to \Sigma_\Delta$ is the subgroup generated by $[\![\beta_1']\!]+[\![\beta_2']\!]+[\![\beta_3']\!]$.*

Proof: Since $T_1\Sigma_\Delta$ is a circle bundle over Σ_Δ, we know

$$0 \to H_1(S^1) \to H_1(T_1\Sigma_\Delta) \to H_1(\Sigma_\Delta) \to 0$$

is an exact sequence. These maps are induced by the inclusion $S^1 \to T_1\Sigma_\Delta$ as a unit tangent circle to a single point P and the projection to the base $p : T_1\Sigma_\Delta \to \Sigma_\Delta$. Therefore,

$$H_1(T_1\Sigma_\Delta) \cong H_1(S^1) \oplus H_1(\Sigma_\Delta)$$

The homology group from the three punctured sphere, $H_1(\Sigma_\Delta)$, is isomorphic to $\mathbb{Z}^2$ and generated by $[\![\beta_1]\!]$ and $[\![\beta_2]\!]$ of figure 2. Since, $\pi : \beta_i' \mapsto \beta_i$, we have the $H_1(\Sigma_\Delta)$ factor in the group $\langle [\![\beta_1']\!], [\![\beta_2']\!], [\![\beta_3']\!] \rangle$. Next, we must show that the $H_1(S^1)$ factor is included.

Let $\sigma : S^1 \to T_1\Sigma_\Delta$ be a loop which travels once around the tangent circle to a single point P. See figure 7. We will show that

$$[\![\beta_1']\!] + [\![\beta_2']\!] + [\![\beta_3']\!] = [\![\sigma]\!] \tag{15}$$

Essentially, this is the Poincaré-Hopf index theorem. We will prove this by explicitly showing that the set of loops $\{\beta_1', \beta_2', \beta_3', -\sigma\}$ bound a surface in $T_1\Sigma_\Delta$. This surface is a vector field on the sphere with four singularities coming from the boundaries of this surface. See figure 8. $\diamondsuit$

We can use this proposition to show derive a necessary condition for stability of a geodesic in Σ_Δ.

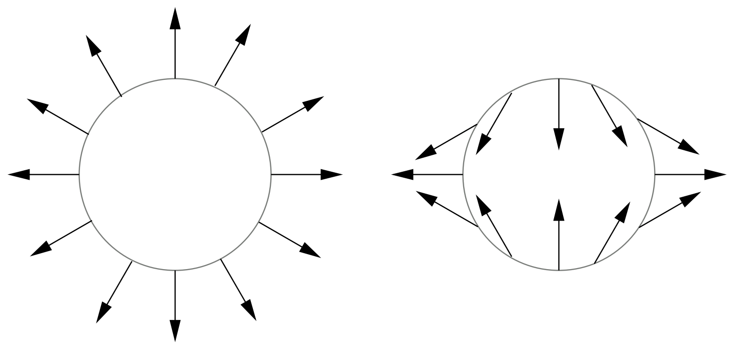


Figure 7: $\sigma : S^1 \to T_1\Sigma_\Delta$ is an assignment of unit vectors to points in the circle. This map is depicted on the left. The loop $-\sigma$ with opposite orientation is depicted to the right.

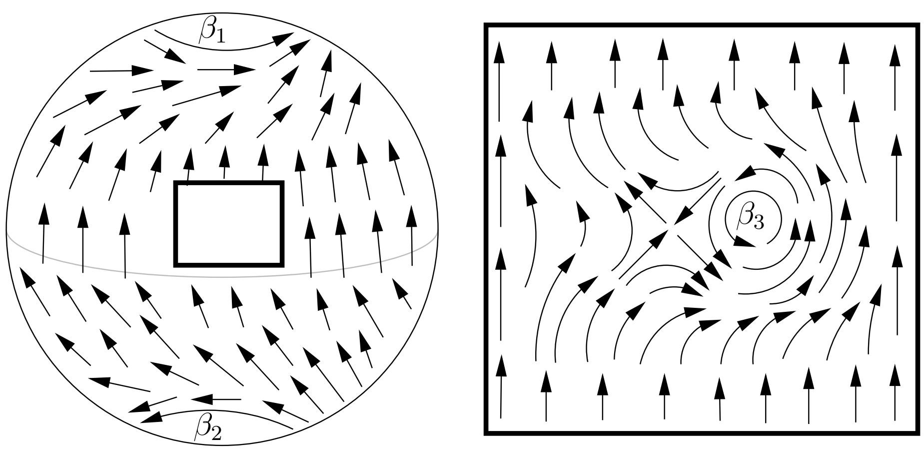


Figure 8: There is a vector field on the sphere with two singularities each with index 1. Singularities of indices 1 and -1 "cancel" and can be added to the vector field.

Corollary 9 *A stable geodesic γ on Σ_Δ is null-homologous. Its derivative γ' is null-homologous on $T_1\Sigma_\Delta$.*

Proof: Let $\gamma' \subset T_1\Sigma_\Delta$ be the derivative of a stable geodesic γ on Σ_Δ. By definition, there is an open set U of triangles for which the homology class $[\![\gamma']\!]$ contains a loop which is the derivative of a geodesic. Using proposition 8, we know that there are $n_1, n_2, n_3 \in \mathbb{Z}$ so that

$$[\![\gamma']\!] = n_1[\![\beta_1']\!] + n_2[\![\beta_2']\!] + n_3[\![\beta_3']\!]$$

By equation 14, the integral of $[\![\gamma']\!]$ with respect to $d\theta$ depends on the angles of the triangle.

$$\int_{[\![\gamma']\!]} d\theta_{(\alpha_1,\alpha_2,\alpha_3)} = 2n_1\alpha_1 + 2n_2\alpha_2 + 2n_3\alpha_3 \tag{16}$$

This equation must be zero for there to be a geodesic loop on the surface $\Sigma_{(\alpha_1,\alpha_2,\alpha_3)}$ whose derivative is in the homology class $[\![\gamma']\!]$. But α_1, α_2, and α_3 vary independently with the choice of the triangle subject to the single constraint that $\alpha_1 + \alpha_2 + \alpha_3 = \pi$. If equation 16 is identically zero on an open set, then $n_1 = n_2 = n_3 = 0$. Therefore, γ' is null homologous on $T_1\Sigma_\Delta$.

Furthermore, the homology class $[\![\gamma]\!]$ is the image of $[\![\gamma']\!]$ under the map $H_1(T_1\Sigma_\Delta) \to H_1(\Sigma_\Delta)$ induced by the projection to the base of the circle bundle. Therefore, $[\![\gamma]\!]$ is null homologous on Σ_Δ. $\diamondsuit$

Equation 10 tells us that the map hol^{ab} is bilinear map from $\mathcal{T} \times H_1(\Sigma_\Delta)$ to $\mathbb{R}/2\pi$. Let us change notation, to make this bilinear map look more standard. We will use Σ to refer to the triangulated three punctured sphere when we only care about topology. Let $\Delta = (\alpha_1, \alpha_2, \alpha_3)$, then we can denote the map $hol^{ab} : \mathcal{T} \times H_1(\Sigma) \to \mathbb{R}/2\pi\mathbb{Z}$ as

$$hol^{ab}((\alpha_1,\alpha_2,\alpha_3), r[\![\beta_1]\!] + s[\![\beta_2]\!]) = 2r\alpha_1 + 2s\alpha_2 \in \mathbb{R}/2\pi\mathbb{Z} \tag{17}$$

Likewise, $\widetilde{hol}^{ab}$ is a bilinear map from $\mathcal{T} \times H_1(T_1\Sigma)$ to $\mathbb{R}$. By equation 14 and proposition 8. We write

$$\widetilde{hol}^{ab} : \big((\alpha_1,\alpha_2,\alpha_3), r[\![\beta_1']\!] + s[\![\beta_2']\!] + t[\![\beta_3']\!]\big) \mapsto 2r\alpha_1 + 2s\alpha_2 + 2t\alpha_3 \tag{18}$$

Our non-degenerate bilinear map, $\widetilde{hol}^{ab}$, induces a duality between $H_1(T_1\Sigma)$ and even integer linear functions $\mathcal{T} \to \mathbb{R}$. Let $\widetilde{hol}^{ab}(\star, \zeta)$ denote the even integer linear map $\Delta \mapsto \widetilde{hol}^{ab}(\Delta, \zeta)$, where where $\zeta \in H_1(T_1\Sigma)$. Note that every even integer linear map from $\mathcal{T}$ to $\mathbb{R}$ can be written in a unique way as $\widetilde{hol}^{ab}(\star, \zeta)$.

Theorem 10 *Consider the map ϕ from the space of even integer linear functions on $\mathcal{T}$ to $H_1(\Sigma)$ given by*

$$\phi : \widetilde{hol}^{ab}(\star, \zeta) \mapsto \pi_*(\zeta) \tag{19}$$

Fixing any $\Delta \in \mathcal{T}$, the restricted map

$$\phi : \{\widetilde{hol}^{ab}(\star, \zeta) \big| \widetilde{hol}^{ab}(\Delta, \zeta) = 0\} \to H_1(S) \tag{20}$$

is a bijection to $ker(hol^{ab}_\Delta)$.

Proof: First we will check that this map does indeed send even integer functions which kill Δ to elements in $ker(hol^{ab}_\Delta)$. Assume $\widetilde{hol}^{ab}(\Delta, \zeta) = 0$. Then because $\widetilde{hol}^{ab}$ is a lift of hol^{ab},

$$hol^{ab}(\Delta, \pi_*(\zeta)) = \widetilde{hol}^{ab}(\Delta, \zeta) \ (mod\ 2\pi) = 0 \in \mathbb{R}/2\pi\mathbb{Z} \tag{21}$$

Second, we will check that the restricted ϕ is 1-1. Consider any ζ_1 and ζ_2 with $\pi_*(\zeta_1) = \pi_*(\zeta_2)$. We will show that the $\widetilde{hol}^{ab}(\Delta, \zeta_i)$ can not both be zero unless $\zeta_1 = \zeta_2$. We know that $\zeta_1 - \zeta_2 \in ker(\pi_*)$, so by proposition 8, $\zeta_1 - \zeta_2$ lies in the subgroup generated by $[\![\beta_1']\!] + [\![\beta_2']\!] + [\![\beta_3']\!]$. Therefore, there exists an integer n so that

$$\zeta_1 - \zeta_2 = n\big([\![\beta_1']\!] + [\![\beta_2']\!] + [\![\beta_3']\!]\big) \tag{22}$$

Using linearity and the definition of equation 18, we see

$$\widetilde{hol}^{ab}(\Delta, \zeta_1) - \widetilde{hol}^{ab}(\Delta, \zeta_2) = 2n\alpha_1 + 2n\alpha_2 + 2n\alpha_3 = 2n\pi \tag{23}$$

So, $\widetilde{hol}^{ab}(\Delta, \zeta_1) = \widetilde{hol}^{ab}(\Delta, \zeta_2)$ implies $\zeta_1 = \zeta_2$.

Finally, we must check that the restricted ϕ is onto. Assume $\pi_*(\zeta) \in ker(hol^{ab}_\Delta)$. Then because $\widetilde{hol}^{ab}$ is a lift of hol^{ab}, there is an integer n so that $\widetilde{hol}^{ab}(\Delta, \zeta) = 2n\pi$. Then

$$\widetilde{hol}^{ab}(\Delta, \zeta - n\big([\![\beta_1']\!] + [\![\beta_2']\!] + [\![\beta_3']\!]\big)) = 0 \tag{24}$$

But of course, $[\![\beta_1']\!] + [\![\beta_2']\!] + [\![\beta_3']\!] \in ker(\pi_*)$, so

$$\pi_*(\zeta - n\big([\![\beta_1']\!] + [\![\beta_2']\!] + [\![\beta_3']\!]\big)) = \pi_*(\zeta) \tag{25}$$

Therefore $\pi_*(\zeta)$ is the image of $\widetilde{hol}^{ab}(\star, \zeta - n([\![\beta_1']\!] + [\![\beta_2']\!] + [\![\beta_3']\!]))$ which kills Δ as desired. $\diamondsuit$

Essentially, theorem 10 tells us that even integer linear functions which vanish on Δ and elements in $ker(hol^{ab})$ are dual under the bilinear map $\widetilde{hol}^{ab}$. Recall that $ker(hol^{ab}_\Delta) \subset H_1(\Sigma_\Delta) \cong \mathbb{Z}^2$. Therefore, $ker(hol^{ab}_\Delta)$ is isomorphic to either 0, $\mathbb{Z}$, or $\mathbb{Z}^2$. We can use theorem 10 to determine the isomorphism class of $ker(hol^{ab}_\Delta)$ by looking at the angles of Δ. One way to differentiate between these groups is by looking at the number of maximal non-trivial one-generator subgroups (0, 1, or ∞ respectively). Maximal non-trivial one-generator subgroups of $\mathbb{Z}^2$ correspond to sets of elements of $\mathbb{Z}^2$ which lie on a line with rational slope. In other words, maximal non-trivial one-generator subgroups are in one to one correspondence with the projectivization of $\mathbb{Z}^2$. We now define a dual notion:

Definition 11 *A* rational line ℓ *in* $\mathcal{T}$ *is a subset determined by three integers* n_1, n_2, n_3*, not all zero:* $\ell = \{(\alpha_1, \alpha_2, \alpha_3) | n_1\alpha_1 + n_2\alpha_2 + n_3\alpha_3 = 0\} \subset \mathcal{T}$*.*

Note that a rational line is a projective notion, in the sense that the line determined by the integers n_1, n_2, n_3 is the same as the line determined by the integers kn_1, kn_2, kn_3. Also, we can assume the n_i are all even and we get the same set of lines. The duality of theorem 10 gives us a bijection between maximal non-trivial one-generator subgroups of $ker(hol^{ab}_\Delta)$ and rational lines containing Δ. We introduce

Definition 12 (A classification of triangles) We say a triangle is *irrational* if it lies on no rational lines, and *generic in a rational line* ℓ if ℓ is the unique rational line it lies on. Otherwise, the triangle lies on infinitely many rational lines. In this case, each angle of the triangle is a rational multiple of π and we call it a *rational* triangle.

According to the duality just mentioned this classification of triangles also classifies the isomorphism class of $ker(hol^{ab}_\Delta)$. In particular, $ker(hol^{ab}_\Delta) = \{0\}$ if Δ is irrational, $ker(hol^{ab}_\Delta) \cong \mathbb{Z}$ if Δ is generic in a rational line, and $ker(hol^{ab}_\Delta) \cong \mathbb{Z}^2$ if Δ is rational.

AC_Δ was described as a twisted thickening of the graph coming from the tiling of the plane by regular hexagons. The cover automorphisms of $AC_\Delta \to \Sigma_\Delta$ act by translations of the plane preserving the hexagonal tiling.

We deduce that MT_Δ can be described as a twisted thickening of the hexagonal tiling of the Euclidean surface, $\mathbb{R}^2$ modulo the action of $ker(hol^{ab}_\Delta)$. To summarize, we give a vague description of MT_Δ corresponding to the classification of Δ.

Corollary 13 (Topology of MT_Δ) *MT_Δ is homeomorphic to a twisted thickening of the graph coming from a regular hexagonal tiling of an Euclidean surface $E_\Delta = \mathbb{R}^2/ker(hol^{ab}_\Delta)$. The topology of E_Δ and the isomorphism class of $ker(hol^{ab}_\Delta)$ depend on the classification of Δ:*

1. *If Δ is* irrational*, then $ker(hol^{ab}_\Delta) = \{0\}$ and E_Δ is the plane. Thus, we know $MT_\Delta = AC_\Delta$.*
2. *If Δ is* rational*, then $ker(hol^{ab}_\Delta) \cong \mathbb{Z}^2$ and E_Δ is a torus. The covering $MT_\Delta \to \Sigma_\Delta$ is finite.*
3. *Otherwise, Δ is* generic in some rational line ℓ*, $ker(hol^{ab}_\Delta) \cong \mathbb{Z}$, and E_Δ is an Euclidean cylinder.*

Remark 14 *If ℓ is a rational line, then each MT_Δ is homeomorphic for a generic $\Delta \in \ell$. We will denote this homeomorphism class by MT_ℓ, and use $MT_{\ell,\Delta}$ as an alternate notation for MT_Δ when Δ is generic in ℓ. We think of MT_ℓ only as a topological surface with a triangulation and $MT_{\ell,\Delta}$ as that topological surface rigidified by setting the geometry of each triangle to match Δ. Then if Δ is rational triangle contained in ℓ, $MT_{\ell,\Delta}$ is a translation surface which covers MT_Δ.*

Figure 9 shows some examples of MT_Δ.

6 A bounding box

The first goal of this section is to understand the ramifications of corollary 7. Sets of symbolics classes of curves on MT_Δ which can be realized by geodesics all traveling in the same direction lie in the image of the comb-out map $\Psi : H_1(MT_\Delta, \mathbb{Z}) \to \mathcal{H}(MT_\Delta)$. To make discussing this notion easier we introduce a definition:

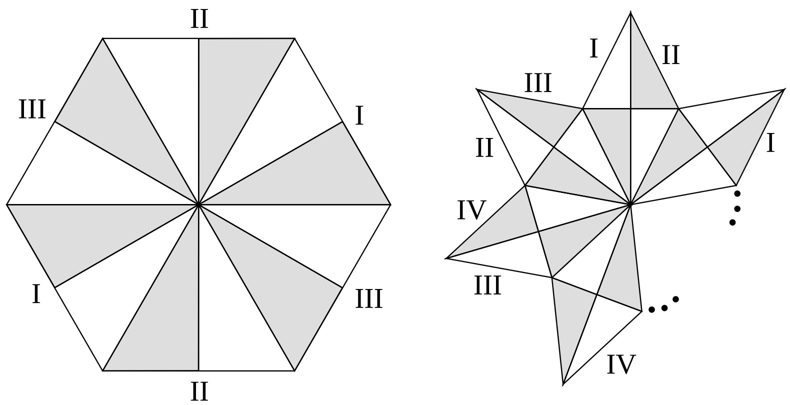


Figure 9: This figure describes some examples of minimal translation surfaces. If Δ is the 30-60-90 triangle, then the associated translation surface is a torus obtained by gluing opposite sides of a hexagon. If Δ is generic in the line of right triangles, then the translation surface can be decomposed into rhombi which spiral around a single vertex. The edges on the boundary of this spiral of rhombi can be identified in parallel pairs.

Definition 15 *Let S be a surface which is triangulated by saddle connections (such as MT_Δ or AC_Δ). Let $\{[\eta_1]_s, \ldots, [\eta_n]_s\}$ be a set of symbolics classes of oriented curves on S. We say that $\{[\eta_1]_s, \ldots, [\eta_n]_s\}$ is* friendly *if*

$$\{[\eta_1]_s, \ldots, [\eta_n]_s\} = \Psi\Big(\llbracket \eta_1 \rrbracket + \ldots + \llbracket \eta_n \rrbracket\Big)$$

By proposition 6, this is equivalent to the statement that $\{[\eta_1]_s, \ldots, [\eta_n]_s\}$ can be realized by a set of curves $\{\eta_1, \ldots, \eta_n\}$ with $\eta_i \in [\eta_i]_s$ so that

1. *no η_i has a self-intersection,*
2. *the $\{\eta_1, \ldots, \eta_n\}$ are all disjoint, and*
3. *the set of curves $\{\eta_1, \ldots, \eta_n\}$ intersect each edge of the triangulation of S with at most one algebraic sign.*

If $\{[\eta_i]_s\}$ is not friendly, then we call it unfriendly*.*

Remark 16 *The second definition of friendliness uses only topological notions of intersections. Therefore, if $\phi_1 : S_1 \to S_2$ is a homeomorphism* *between surfaces which respects the triangulation, then $\{[\eta_i]_s\}$ is friendly on S_1 if and only if $\{\phi_1([\eta_i]_s)\}$ is friendly on S_2. Further, if $\phi_2 : S_3 \to S_4$ is a covering map respecting the triangulations, then if $\{\phi_2([\eta_i]_s)\}$ is friendly, then so is $\{[\eta_i]_s\}$. To see this, realize the $\{\phi_2([\eta_i]_s)\}$ by curves on S_4 which satisfy the definition and lift them back to S_3. Since, there were no faulty intersections on S_4, there can be none on S_3.*

We are interested in applying this concept to the study of stable geodesics on Σ_Δ. By corollary 9, stable geodesics must lift to the universal abelian cover AC_Δ. Since each AC_Δ is homeomorphic, we use AC to denote this homeomorphism class. Let $[\widetilde{\gamma}]_s$ be a symbolics class of curves on AC. Let $\phi_\Delta : AC \to MT_\Delta$ be the covering map. There might (depending on the choice of Δ) be an automorphism $\rho : MT_\Delta \to MT_\Delta$ of the cover $MT_\Delta \to \Sigma_\Delta$ which acts on the direction map by a rotation by π. That is, ρ also acts on $T_1 MT_\Delta$, and the map $\theta \circ \rho \circ \theta^{-1}$ is well defined and rotates elements in S^1 by π. We call ρ a *rotation by* π. If x is a geodesic on MT_Δ, and ρ is a rotation by π, then x and $-\rho(x)$ (the image of x under ρ with the opposite orientation) both travel in the same direction.

Remark 17 *A minimal translation surface can admit at most one rotation by π. Suppose ρ_1 and ρ_2 are both rotations by π. Then $\rho_1 \circ \rho_2^{-1}$ acts trivially on directions in the minimal translation surface, and hence must be the trivial automorphism of the cover.*

Definition 18 *Let ϕ_Δ be the covering $AC \to MT_\Delta$. Given a symbolics class of curves $[\widetilde{\gamma}]_s$ on AC, the* unfriendly set of $[\widetilde{\gamma}]_s$ *is*

$$UF([\widetilde{\gamma}]_s) = \left\{ \Delta \in \mathcal{T} \;\middle|\; \begin{array}{l} \{\phi_\Delta([\widetilde{\gamma}]_s)\} \textit{ is unfriendly on } MT_\Delta \textit{ or} \\ \{\phi_\Delta([\widetilde{\gamma}]_s), -\rho \circ \phi_\Delta([\widetilde{\gamma}]_s)\} \textit{ is unfriendly on } MT_\Delta \\ \textit{which admits } \rho \textit{, a rotation by } \pi. \end{array} \right\}$$

It is clear by corollary 7, that if $[\gamma]_s$ is a symbolics class on Σ and $[\widetilde{\gamma}]_s$ is a lift to AC then $UF([\widetilde{\gamma}]_s)$ and $tile([\gamma]_s)$ are disjoint.

Determining whether $\Delta \in UF([\widetilde{\gamma}]_s)$ requires first determining if MT_Δ admits a rotation by π. Recall the definition of the topological surface MT_ℓ given in remark 14.

Proposition 19 (Existence of ρ) *There is a rotation by π of MT_Δ if and only if Δ lies on a rational line ℓ which can be determined by three odd integers n_1, n_2, and n_3.*

$$\ell = \{(\alpha_1, \alpha_2, \alpha_3) \in \mathcal{T} | n_1\alpha_1 + n_2\alpha_2 + n_3\alpha_3 = 0\}$$

The rotation by π of MT_Δ lifts to an automorphism ρ of the cover $MT_\ell \to \Sigma$, which acts as a rotation by π on $MT_{\ell,\Delta'}$ for all $\Delta' \in \ell$.

Proof: First, if ρ is a rotation by π of MT_Δ, then it lifts to a rotation by π of AC_Δ. Call this lift $\widetilde{\rho} \in H_1(\Sigma_\Delta)$, and it satisfies $hol^{ab}(\Delta, \widetilde{\rho}) = \pi$. Assume $\widetilde{\rho} = a[\![\beta_1]\!] + b[\![\beta_2]\!]$. Then, there is an integer n so that

$$2a\alpha_1 + 2b\beta_1 = (2n+1)\pi$$

Therefore, $(2a - 2n - 1)\alpha_1 + (2b - 2n - 1)\alpha_2 - (2n+1)\alpha_3 = 0$. The odd integers $n_1 = 2a - 2n - 1$, $n_2 = 2b - 2n - 1$ and $n_3 = -2n - 1$ suffice to determine the rational line ℓ containing Δ.

Now, we will prove that if odd n_1, n_2, and n_3 determining a rational line ℓ containing Δ, then MT_Δ admits a rotation by π. Of course, saying that there exist odd integers with $n_1\alpha_1 + n_2\alpha_2 + n_3\alpha_3 = 0$, is equivalent to say that there exist even integers $e_1 = n_1 + 1$, $e_2 = n_2 + 1$, and $e_3 = n_3 + 1$ with

$$e_1\alpha_1 + e_2\alpha_2 + e_3\alpha_3 = \pi$$

First assume such e_i exist. Let $\zeta = \frac{e_1}{2}[\![\beta_1']\!] + \frac{e_2}{2}[\![\beta_2']\!] + \frac{e_3}{2}[\![\beta_3']\!] \in H_1(T_1\Sigma_\Delta, \mathbb{Z})$, so that $\widetilde{hol}^{ab}(\Delta, \zeta) = \pi$ by equation 18. Therefore, $hol^{ab}(\Delta, \pi_*(\zeta)) = \pi$ and $\pi_*(\zeta)$ is a rotation by π of AC_Δ. Because the covers $AC_\Delta \to MT_\Delta \to \Sigma_\Delta$ are abelian, $\pi_*(\zeta)$ descends to an automorphisms of the cover $MT_\Delta \to \Sigma_\Delta$ which also acts by a rotation by π.

Continuing the argument, $\pi_*(\zeta)$ also descends to a rotation by π of $MT_{\ell,\Delta}$. Since we used nothing about Δ besides that $\Delta \in \ell$, $\pi_*(\zeta)$ acts by a rotation by π on $MT_{\ell,\Delta'}$ for all $\Delta' \in \ell$. $\diamondsuit$

In order to deal with the unfriendly set, we also need a technique to show when a self intersection or an intersection between two curves must happen regardless of the choice of loops in a symbolics class of curves. We introduce the following two propositions which follow the philosophy that intersections between geodesics on non-positively curved surfaces are "essential."

Proposition 20 *Suppose γ is a geodesic on a locally Euclidean surface S covering Σ_Δ. If γ is not simple, then no curve in the symbolics class $[\gamma]_s$ is simple.*

Proposition 21 *Suppose γ_1 and γ_2 are geodesics on a locally Euclidean surface S covering Σ_Δ. If γ_1 and γ_2 are not disjoint, then no curve taken from the symbolics class $[\gamma_1]_s$ is disjoint from a curve from the symbolics class $[\gamma_2]_s$.*

Proof of Propositions 20 and 21: We provide the proof of proposition 21. The other follows the same logic.

Let P be an intersection point between geodesics γ_1 and γ_2 on S. We lift the geodesics passing through this point to two geodesics, $\widetilde{\gamma_1}$ and $\widetilde{\gamma_2}$ on the universal cover $\widetilde{S}$ so that each passes though a lift of P, which we call $\widetilde{P}$.

We will develop a neighborhood N of this intersection at $\widetilde{P}$ into the plane. First let N be the triangle containing $\widetilde{P}$. Whenever a point in the pair $\partial N \cap \widetilde{\gamma_1}$ lies on the same edge as a point in the pair $\partial N \cap \widetilde{\gamma_2}$, add the triangle on the other side of this edge to N. Eventually, this process must terminate, because the developed images of $\widetilde{\gamma_1}$ and $\widetilde{\gamma_2}$ are lines intersecting in the plane. Therefore, the lines are growing further apart, but the triangles are all isometric and hence have a common bounded diameter. The result is a simply connected chain of triangles similar to the one shown in figure 10. We think of this intersection as forced because the end points $\partial N \cap \widetilde{\gamma_1}$ separate $\partial N \cap \widetilde{\gamma_2}$ on ∂N.

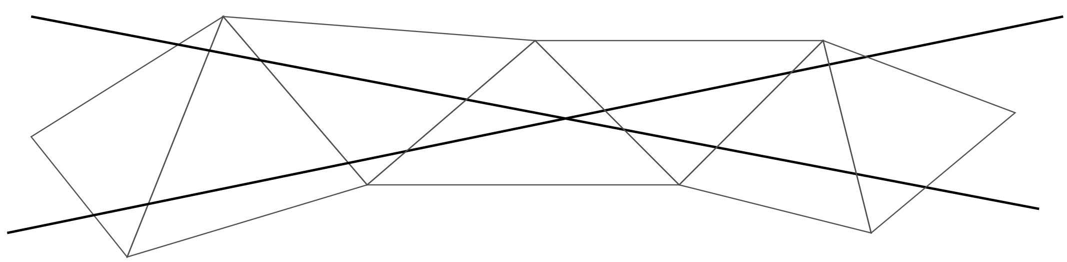


Figure 10: An intersection developed into the plane. The intersection is forced because one arc comes in the bottom-left and leaves out the top-right, but the other comes in the top-left and leaves out the bottom-right.

Now suppose $\eta_1 \in [\gamma_1]_s$ and $\eta_2 \in [\gamma_2]_s$. We take the same lifts $\widetilde{\eta_1}$ and $\widetilde{\eta_2}$ to $\widetilde{S}$. By definition of the symbolics class, they must intersect the edges in the same order. Therefore, the end points $\partial N \cap \widetilde{\eta_1}$ separate $\partial N \cap \widetilde{\eta_2}$ on ∂N. Since N is just a disk, $\widetilde{\eta_1}$ and $\widetilde{\eta_2}$ must intersect. We push this intersection back down to S to see an intersection between η_1 and η_2 on S. $\diamondsuit$

We now describe our general approach for working with the unfriendly set, $UF([\widetilde{\gamma}]_s)$. We assume that γ is a geodesic on Σ_Δ for some Δ, and that it lifts to a loop $\widetilde{\gamma}$ on AC_Δ. Now if Δ' is another triangle, then we know there is a covering map $\phi_{\Delta'} : AC_{\Delta'} \to MT_{\Delta'}$. We build a new locally Euclidean surface, $MT_{\Delta'}(\Delta)$, out of copies of Δ by replacing every copy of Δ' in $MT_{\Delta'}$ with Δ. We abuse notation by also using $\phi_{\Delta'}$ to denote the covering $AC_\Delta \to MT_{\Delta'}(\Delta)$. Similarly, if the automorphism $\rho : MT_{\Delta'} \to MT_{\Delta'}$ of the cover $MT_{\Delta'} \to \Sigma_{\Delta'}$ is a rotation by π, we also obtain an automorphism $\rho : MT_{\Delta'}(\Delta) \to MT_{\Delta'}(\Delta)$ of the cover $MT_{\Delta'}(\Delta) \to \Sigma_\Delta$.

Propositions 20 and 21 tell us that

Lemma 22 (Witnessing Unfriendliness) *If γ is a geodesic on Σ_Δ which lifts to a loop $\widetilde{\gamma}$ on AC_Δ, then $\Delta' \in UF([\widetilde{\gamma}]_s)$ if and only if one of the following holds:*

1. *$\phi_{\Delta'}(\widetilde{\gamma})$ is not simple on $MT_{\Delta'}(\Delta)$.*

2. *$\phi_{\Delta'}(\widetilde{\gamma})$ intersects the same edge of $MT_{\Delta'}(\Delta)$ twice but with opposite orientations.*

3. *There is a rotation by π, $\rho : MT_{\Delta'} \to MT_{\Delta'}$, and $\phi_{\Delta'}(\widetilde{\gamma})$ is not disjoint from $\rho \circ \phi_{\Delta'}(\widetilde{\gamma})$ on $MT_{\Delta'}(\Delta)$.*

4. *There is a rotation by π, $\rho : MT_{\Delta'} \to MT_{\Delta'}$, and $\phi_{\Delta'}(\widetilde{\gamma})$ and $-\rho \circ \phi_{\Delta'}(\widetilde{\gamma})$ intersect the same edge of the triangulation of $MT_{\Delta'}(\Delta)$ with opposite orientations.*

Proof: It is easier to prove the contrapositive. That is, we will show that all four statements above are false if and only if Δ' is in $\mathcal{T} \setminus UF([\widetilde{\gamma}]_s)$.

First suppose that there is no rotation by π of $MT_{\Delta'}$. Then if statements 1 and 2 are false, then by definition, $\{[\phi_{\Delta'}(\widetilde{\gamma})]_s\}$ is friendly on $MT_{\Delta'}(\Delta)$ which is homeomorphic to $MT_{\Delta'}$, so that $\Delta' \in \mathcal{T} \setminus UF([\widetilde{\gamma}]_s)$.

Now, suppose $\rho : MT_{\Delta'} \to MT_{\Delta'}$ is a rotation by π of $MT_{\Delta'}$. By the negation of statements 1-4, the pair of symbolics classes $\{[\phi_{\Delta'}(\widetilde{\gamma})]_s, [\rho \circ \phi_{\Delta'}(\widetilde{\gamma})]_s\}$ is friendly on $MT_{\Delta'}(\Delta)$ and thus on $MT_{\Delta'}$ as well. Again, $\Delta' \in \mathcal{T} \setminus UF([\widetilde{\gamma}]_s)$.

Now suppose there is no rotation by π of $MT_{\Delta'}$ and $\Delta' \in \mathcal{T} \setminus UF([\widetilde{\gamma}]_s)$. Then by definition of $UF([\widetilde{\gamma}]_s)$, $\{\phi_{\Delta'}[\widetilde{\gamma}]_s\}$ is friendly. This implies that there is a loop $\eta \in \phi_{\Delta'}[\widetilde{\gamma}]_s$ which is simple and intersects each edge of the triangulation with only one orientation. But $\phi_{\Delta'}(\widetilde{\gamma}) \in \phi_{\Delta'}[\widetilde{\gamma}]_s$ as well. Since they lie in the same symbolics class, they must intersect the same sequence of edges, so statement 2 is false. Also, by proposition 20, a self intersection of $\phi_{\Delta'}(\widetilde{\gamma})$ forces a self intersection of η. Therefore, statement 1 is false.

Finally, assume there is a rotation by π of $MT_{\Delta'}$ called ρ and that $\Delta' \in \mathcal{T} \setminus UF([\widetilde{\gamma}]_s)$. By definition, the pair $\{\phi_{\Delta'}[\widetilde{\gamma}]_s, \rho \circ \phi_{\Delta'}[\widetilde{\gamma}]_s\}$ is friendly. Therefore, there exists $\eta_1 \in \phi_{\Delta'}[\widetilde{\gamma}]_s$ and $\eta_2 \in -\rho \circ \phi_{\Delta'}[\widetilde{\gamma}]_s$ which are simple, disjoint, and intersect each edge of the triangulation of $MT_{\Delta'}$ with only one orientation. We conclude that statements 2 and 4 are true because $\phi_{\Delta'}(\widetilde{\gamma})$ and $-\phi_{\Delta'}(\widetilde{\gamma})$ lie in the same symbolics classes as η_1 and η_2 respectively. Again, both must be simple by 20, so that statement 1 is false. By 21, they must be disjoint. Therefore, statement 3 is false. $\diamondsuit$

In the next section we will use this lemma to prove the following theorem:

Theorem 23 *Let γ be a closed geodesic on Σ_Δ which lifts to a closed geodesic $\widetilde{\gamma}$ on AC_Δ. The unfriendly set, $UF([\widetilde{\gamma}]_s)$, is a finite union of rational lines in $\mathcal{T}$.*

We will also connect this set to the *tile*$([\gamma]_s)$ introduced in section 1. By corollary 7, we know *tile*$([\widetilde{\gamma}]_s)$ is disjoint from $UF([\widetilde{\gamma}]_s)$. More is true.

Theorem 24 (Bounding Box Theorem) *Let γ be a closed geodesic on Σ_Δ which lifts to a closed geodesic $\widetilde{\gamma}$ on AC_Δ. The set tile$([\gamma]_s])$ is contained in one component of $\mathcal{T} \setminus UF([\widetilde{\gamma}]_s)$.*

This theorem will also be proved in the next section.

7 Angles at intersections

Given two oriented lines intersecting at a point P in the plane we associate a canonical angle θ_P with measure between 0 and π. This choice of angle is shown in figure 11.

Suppose $\widehat{\gamma}$ is a stable periodic billiard on Δ, and γ is its geodesic lift to Σ_Δ. Now suppose there is an intersection X of γ on Σ_Δ. We can measure the

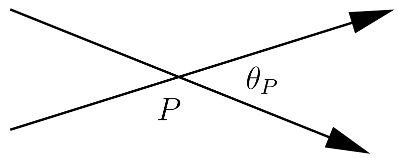


Figure 11: An intersection, P, between two oriented lines and the canonical choice of angle θ_P.

angle of intersection θ_X at this point using a protractor, or alternately using the Gauss-Bonnet formula. Consider the bent geodesic L_X which follows γ until it hits this intersection X, then it makes a left turn and closes up. See figure 12. Let L'_X be the loop on $T_1\Sigma_\Delta$ which follows the derivative of L_X except at the bend, where it interpolates between the tangents by rotating around the unit tangent circle to the left by angle θ_X. Because all the turning of L'_X occurs at this intersection point we see

$$\int_{L'_X} d\theta = \widetilde{hol}^{ab}(\Delta, [\![L'_X]\!]) = \theta_X \tag{26}$$

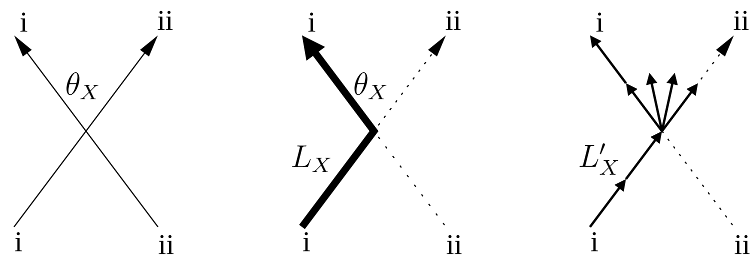


Figure 12: *Left:* A local picture of a geodesic self-intersection X on a locally Euclidean surface. θ_X is the angle associated to this intersection. *Center:* The loop L_X on the surface. *Right:* The loop L'_X on the unit tangent bundle of the surface.

Clearly, $0 < \theta_X < \pi$. The value of $\widetilde{hol}^{ab}(\Delta', [\![L'_X]\!])$ varies linearly in the angles of the triangle Δ' as Δ' varies (Recall equation 18). More generally,

Lemma 25 *Suppose X is a self-intersection of the geodesic γ on Σ_Δ. Then if $\Sigma_{\Delta'}$ has a geodesic γ' with the same symbolic dynamics, then*

$$0 < \widetilde{hol}^{ab}(\Delta', [\![L'_X]\!]) < \pi$$

Proof: We prove this by exhibiting an intersection Y of γ' so that $[\![L'_X]\!] = [\![L'_Y]\!]$. This is an application of the general principle that intersections between geodesics on non-positively curved surfaces are essential. It follows much of the logic of the proof of proposition 20.

Consider the intersection X of γ. We can lift the two arcs of γ passing through X to the universal cover $\widetilde{\Sigma_\Delta}$ obtaining two geodesic arcs γ_1 and γ_2. We choose the lifts so that they each pass through the lift $\widetilde{X}$ of X. We produce a simply connected union of triangles $U \subset \widetilde{\Sigma_\Delta}$ as in figure 10. The union U has the property that the four geodesic rays out of $\widetilde{X}$ leave U out different edges. Their end points $\partial U \cap \gamma_1$ and $\partial U \cap \gamma_2$ separate each other on ∂U. This is forced by the symbolic dynamics.

Since γ' has the same symbolic dynamics, we can find a simply connected union of triangles, $U' \subset \widetilde{\Sigma_{\Delta'}}$, homeomorphic to U and lifts γ'_1 and γ'_2 of γ' which pass through U' in the same manner. Again, their end points separate each other on $\partial U'$ (they are ordered the same on the boundary, because this order is determined by the symbolic dynamics). Therefore γ'_1 and γ'_2 must have an intersection, $\widetilde{Y}$. Name the image of the intersection $\widetilde{Y}$ on $\Sigma_{\Delta'}$ Y.

There is a homeomorphism $U \to U'$ which sends $\gamma_1 \mapsto \gamma'_1$ and $\gamma_2 \mapsto \gamma'_2$. We can build L_X out of the derivative of γ and a local picture about the intersection point X. This local picture lifts to $\widetilde{X} \subset U$. The mentioned homeomorphism tells us that the local picture at Y matches that of X. Moreover, the fact that the symbolic dynamics agree tells us that the constructed curves L'_X and L'_Y are homotopic. Therefore $[\![L'_X]\!] = [\![L'_Y]\!]$, and

$$\theta_Y = \widetilde{hol}^{ab}(\Delta', [\![L'_X]\!]) \tag{27}$$

Of course, $0 < \theta_Y < \pi$. $\diamondsuit$

For the next case, suppose that e is an (oriented) edge of the triangulation of Σ_Δ. Let γ be a geodesic which intersects the edge e twice but with opposite signs. Call these two intersections A and B and their associated angles θ_A and θ_B respectively. We wish to find a curve in $T_1\Sigma_\Delta$ which when integrated with respect to $d\theta$ yields $\theta_A + \theta_B$.

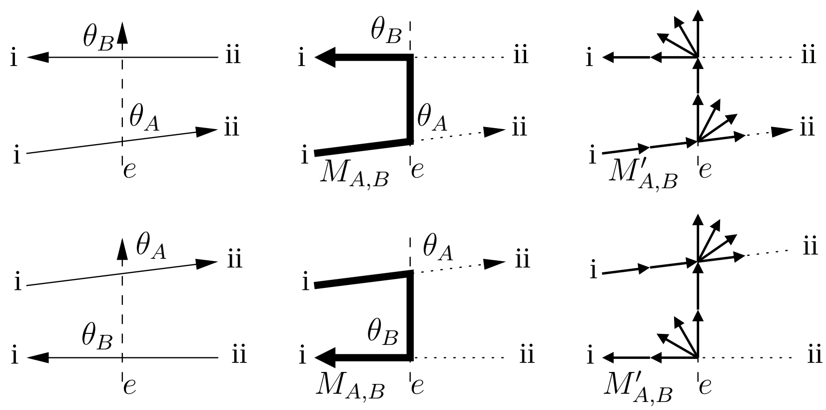


Figure 13: *Left:* A local picture of a pair of intersections with opposite orientations, A and B, between a closed geodesic and an edge of the triangulation of Σ_Δ. Two possibilities are shown. *Center:* The loop $M_{A,B}$ on the surface. *Right:* The loop $M'_{A,B}$ on the unit tangent bundle of the surface.

In this case their are two distinct pictures which appear, shown as rows in figure 13. The differences will not matter for our argument. Rotate the edge e so it is oriented vertically. Assume the arc of γ containing A comes in from the left and the arc of γ containing B comes from the right. We define the loop $M_{A,B}$ which travels along γ until it hits A then travels along e to B and then follows γ again and closes up. We define the loop $M'_{A,B}$ on $T_1\Sigma_\Delta$ to be the loop which agrees with the derivative of γ while $M_{A,B}$ travels along γ and agrees with the derivative of e while $M_{A,B}$ travels along e. It rotates the vectors by θ_A and θ_B at the intersection points. This loop is the derivative of a geodesic except when it rotates by θ_A and θ_B. We see that

$$\int_{M'_{A,B}} d\theta = \widetilde{hol}^{ab}(\Delta, [\![M'_{A,B}]\!]) = \theta_A + \theta_B \tag{28}$$

Of course $0 < \theta_A + \theta_B < 2\pi$. More generally,

Lemma 26 *Suppose A and B are intersections with different orientations of the geodesic γ with an edge e on Σ_Δ. Then if $\Sigma_{\Delta'}$ has a geodesic γ' with the same symbolic dynamics, then*

$$0 < \widetilde{hol}^{ab}(\Delta', [\![M'_{A,B}]\!]) < 2\pi$$

Proof: The symbolic dynamics of γ determine the sequence of edges it hits and the orientations of these intersections. Therefore, there must be corresponding intersections A' and B' of γ'. The loop $M_{A,B}$ and the corresponding loop $M_{A',B'}$ can be easily verified to be homotopic. Similarly, $M'_{A,B}$ and $M'_{A',B'}$ are homotopic on $T_1\Sigma$. $\diamondsuit$

We will consider one final case of intersection angles. Suppose C and D are intersections on Σ_Δ between a closed geodesic γ and an edge e with the same orientation. We will define a loop $N'_{C,D}$ on $T_1\Sigma_\Delta$, so that when we integrate $d\theta$ over the loop we obtain $\theta_C - \theta_D$.

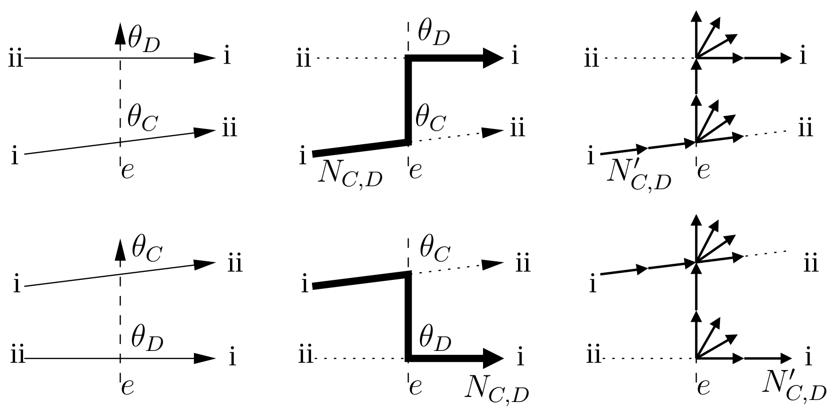


Figure 14: *Left:* A local picture of a pair of intersections with the same orientations, C and D, between a closed geodesic and an edge of the triangulation of Σ_Δ. *Center:* The loop $N_{C,D}$ on the surface. *Right:* The loop $N'_{C,D}$ on the unit tangent bundle of the surface.

Define $N_{C,D}$ to be the loop which runs along γ to the intersection C, then travels along e to the intersection D, and finally runs along γ again to close up. We define the loop $N'_{C,D}$ to agree with the derivative of γ while $N_{C,D}$ travels along γ and agree with the derivative of e while it travels along e. $N'_{C,D}$ rotates by θ_C at the intersection C and by $-\theta_D$ at the intersection D. We see

$$\int_{N'_{C,D}} d\theta = \widetilde{hol}^{ab}(\Delta, [\![N'_{C,D}]\!]) = \theta_C - \theta_D \tag{29}$$

And $-\pi < \theta_C - \theta_D < \pi$.

Lemma 27 *Suppose C and D are intersections with the same orientations of the geodesic γ with an edge e on Σ_Δ. Then if $\Sigma_{\Delta'}$ has a geodesic γ' with the same symbolic dynamics, then*

$$-\pi < \widetilde{hol}^{ab}(\Delta', [\![N'_{C,D}]\!]) < \pi$$

Proof: As in the proof of lemma 26, there must be corresponding intersections C' and D' of γ'. The loop $N_{C,D}$ on Σ_Δ is homotopic to $N_{C',D'}$ on $\Sigma_{\Delta'}$ and the loop $N'_{C,D}$ on $T_1\Sigma_\Delta$ is homotopic to $N'_{C',D'}$ on $T_1\Sigma_{\Delta'}$. $\diamondsuit$

Definition 28 *Given a geodesic loop γ on Σ_Δ. We determine a convex subset of $\mathcal{T}$, the space of marked triangles up to similarity parameterized by angles, by considering all such restrictions on the angles of the triangles. Define the* billiard-like tile of γ, *bl-tile(γ), to be the open convex polygon consisting of triangles Δ' satisfying the conditions:*

1. *For each self intersection X of γ,*

$$0 < \widetilde{hol}^{ab}(\Delta', [\![L'_X]\!]) < \pi$$

2. *For each pair of intersections, A and B, between γ and an edge e of opposite orientations,*

$$0 < \widetilde{hol}^{ab}(\Delta', [\![M'_{A,B}]\!]) < 2\pi$$

3. *For each pair of intersections, C and D, between γ and an edge e with the same orientation,*

$$-\pi < \widetilde{hol}^{ab}(\Delta', [\![N'_{C,D}]\!]) < \pi$$

The lemmas 25, 26, and 27 combine to yield:

Theorem 29 *Given a geodesic loop γ on Σ_Δ, the billiard-like tile bl-tile(γ) contains tile($[\gamma]_s$), the set of all triangles with periodic billiard paths with symbolic dynamics $[\gamma]_s$.*

In order to prove theorems 23 and 24 we need to recall the connection between $\widetilde{hol}^{ab}$ and hol^{ab}. The projection $p : T_1\Sigma_\Delta \to \Sigma_\Delta$ sends a curve η' on the unit tangent bundle to a curve η on Σ_Δ. By diagram 13, regardless of the choice of a triangle Δ' we know

$$hol^{ab}(\Delta', [\![\eta]\!]) = \widetilde{hol}^{ab}(\Delta', [\![\eta']\!]) \pmod{2\pi}$$

Essentially, the unfriendly set detects phenomena that occur on the level of hol^{ab} while our definition of the billiard-like tile uses the stronger notion of $\widetilde{hol}^{ab}$.

The following lemma will yield both theorems 23 and 24.

Lemma 30 (The Unfriendly Set) *Suppose γ is a closed geodesic on Σ_Δ which lifts to a closed geodesic $\widetilde{\gamma}$ on AC_Δ. The unfriendly set, $UF([\widetilde{\gamma}]_s)$, is the set of all triangles Δ' so that at least one of the following holds:*

1. *There is a self intersection X of γ so that*

$$hol^{ab}(\Delta', [\![L_X]\!]) = 0 \in \mathbb{R}/2\pi\mathbb{Z}$$

2. *There is a pair of intersections, A and B, between γ and an edge e of opposite orientations so that*

$$hol^{ab}(\Delta', [\![M_{A,B}]\!]) = 0 \in \mathbb{R}/2\pi\mathbb{Z}$$

3. *There is a self intersection X of γ so that*

$$hol^{ab}(\Delta', [\![L_X]\!]) = \pi \in \mathbb{R}/2\pi\mathbb{Z}$$

4. *There is a pair of intersections, C and D, between γ and an edge e with the same orientation so that*

$$hol^{ab}(\Delta', [\![N_{C,D}]\!]) = \pi \in \mathbb{R}/2\pi\mathbb{Z}$$

Proof: There is a correspondence between the conditions of this lemma and those of the Witnessing Unfriendliness Lemma (lemma 22). We will show that each statement i from this lemma is equivalent to statement i from the Witnessing Unfriendliness Lemma (we call this statement WUL-i).

Recall that, $MT_{\Delta'}(\Delta)$ is an intermediate cover of $AC_\Delta \to \Sigma_\Delta$ and can be defined as:

$$MT_{\Delta'}(\Delta) = AC_\Delta / \{x \in H_1(\Sigma) \mid hol^{ab}(\Delta', x) = 0 \in S^1\} \tag{30}$$

We have the intermediate covers $\phi_{\Delta'} : AC_\Delta \to MT_{\Delta'}(\Delta)$ and $\psi : MT_{\Delta'}(\Delta) \to \Sigma_\Delta$.

(WUL-1 $\implies$ 1): Suppose $\phi_{\Delta'}(\widetilde{\gamma})$ is not simple on $MT_{\Delta'}(\Delta)$. Then it has a self intersection $\widetilde{X}$ on $MT_{\Delta'}(\Delta)$. Call X the self intersection of γ, $\psi(\widetilde{X})$. L_X is built on Σ_Δ by a surgery on γ localized at X. Since this surgery is local, we can also perform it at $\widetilde{X}$ on $\phi_{\Delta'}(\widetilde{\gamma})$, building a new loop $\widetilde{L}_X$. Since L_X lifts to $\widetilde{L}_X$ on $MT_{\Delta'}(\Delta)$, by definition (equation 30) it must be that $hol^{ab}(\Delta', [\![L_X]\!]) = 0 \in \mathbb{R}/2\pi\mathbb{Z}$.

(1 $\implies$ WUL-1): If $hol^{ab}(\Delta', [\![L_X]\!]) = 0 \in \mathbb{R}/2\pi\mathbb{Z}$, then L_X lifts to a loop $\widetilde{L}_X$. Since $\psi \circ \phi_{\Delta'}(\widetilde{\gamma}) = \gamma$, this lift $\widetilde{L}_X$ can be chosen to be a subset of $\phi_{\Delta'}(\widetilde{\gamma})$. Therefore, the point at which $\widetilde{L}_X$ bends is an self intersection of $\phi_{\Delta'}(\widetilde{\gamma})$. So, $\phi_{\Delta'}(\widetilde{\gamma})$ is not simple.

(WUL-2 $\implies$ 2): Suppose that $\phi_{\Delta'}(\widetilde{\gamma})$ intersects an edge $\widetilde{e}$ of the triangulation of $MT_{\Delta'}(\Delta)$ with opposite orientations. Call these intersections $\widetilde{A}$ and $\widetilde{B}$. Call A the intersection $\psi(\widetilde{A})$ and $B = \psi(\widetilde{B})$ which occur between γ and an edge $e = \psi(\widetilde{e})$. We build $M_{A,B}$ and it must lift to a loop $\widetilde{M_{A,B}}$ on $MT_{\Delta'}(\Delta)$. Since this loop lifts, by definition of $MT_{\Delta'}(\Delta)$ we know that $hol^{ab}(\Delta', [\![M_{A,B}]\!]) = 0 \in \mathbb{R}/2\pi\mathbb{Z}$.

(2 $\implies$ WUL-2): Suppose that $hol^{ab}(\Delta', [\![M_{A,B}]\!]) = 0 \in \mathbb{R}/2\pi\mathbb{Z}$. Then, $M_{A,B}$ lifts to a loop $\widetilde{M_{A,B}}$ on $MT_{\Delta'}(\Delta)$. $M_{A,B}$ consists of two pieces, part of the edge e and part of the geodesic γ. We choose the lift $\widetilde{M_{A,B}}$ so that the lifted portion of γ is contained in $\phi_{\Delta'}(\widetilde{\gamma})$. The lifts of the end points of this arc A and B to $\widetilde{A}$ and $\widetilde{B}$ must be contained in the same edge. Therefore, $\phi_{\Delta'}(\widetilde{\gamma})$ intersects this lifted edge with opposite orientations.

Statements 3 and 4 involve loops η with $hol^{ab}(\Delta', [\![\eta]\!]) = \pi$. Now, η no longer lifts to $MT_{\Delta'}(\Delta)$. However, the loop which travels around η twice which we denote by 2η satisfies:

$$hol^{ab}(\Delta', [\![2\eta]\!]) = 0 \in \mathbb{R}/2\pi\mathbb{Z}$$

So 2η lifts to a curve $\widetilde{2\eta}$ on $MT_{\Delta'}(\Delta)$. Furthermore, cover automorphisms of $AC \to \Sigma$ are in bijection with elements of $H_1(\Sigma)$. Given a homology class $[\![\eta]\!]$ with $hol^{ab}(\Delta', [\![\eta]\!]) = \pi$, we know that the corresponding cover automorphism $\widetilde{\rho} : AC_{\Delta'} \to AC_{\Delta'}$ is a rotation by π. This cover automorphism descends to the unique rotation by π of the minimal translation surface, $\rho : MT_{\Delta'} \to MT_{\Delta'}$. Of course ρ also acts as a cover automorphism of $MT_{\Delta'}(\Delta) \to \Sigma_\Delta$. Since $hol^{ab}(\Delta', [\![\eta]\!]) = \pi$, ρ preserves the loop $\widetilde{2\eta}$, and acts as the non-trivial automorphism of the double cover $\psi : \widetilde{2\eta} \to \eta$.

(WUL-3 $\implies$ 3): Suppose that $\phi_{\Delta'}(\widetilde{\gamma})$ and $\rho \circ \phi_{\Delta'}(\widetilde{\gamma})$ intersect at $\widetilde{X}_1$. Here ρ is a cover automorphism and acts fix point freely. Furthermore, ρ interchanges the two loops. Let, $\widetilde{X}_2 = \rho(\widetilde{X}_1)$ which must be a second distinct intersection point. Now, we build a new loop, called $\widetilde{2L_X}$ which travels along $\phi_{\Delta'}(\widetilde{\gamma})$ from $\widetilde{X}_1$ to $\widetilde{X}_2$ and then travels from $\widetilde{X}_2$ to $\widetilde{X}_1$ along $\rho \circ \phi_{\Delta'}(\widetilde{\gamma})$. We travel along each curve $\phi_{\Delta'}(\widetilde{\gamma})$ and $\rho \circ \phi_{\Delta'}(\widetilde{\gamma})$ along the direction of the orientation. Then $\rho(\widetilde{2L_X}) = \widetilde{2L_X}$ swapping the two arcs of the bi-gon. Let $X = \psi(\widetilde{X}_1) = \psi(\widetilde{X}_2)$ be the self intersection point of γ. The image $\psi(\widetilde{2L_X})$ wraps twice around the curve L_X obtained from this intersection point. The holonomy around L_X can be described by ρ, in the sense that if we try to lift L_X to $MT_{\Delta'}(\Delta)$, we obtain an arc connecting $\widetilde{X}_1$ to $\widetilde{X}_2$. Therefore $hol^{ab}(\Delta', [\![L_X]\!]) = \pi$.

(3 $\implies$ WUL-3): Suppose that $hol^{ab}(\Delta', [\![L_X]\!]) = \pi$. This homology class determines a rotation by π of $AC_{\Delta'}$, which descends to a rotation by π, $\rho : MT_{\Delta'} \to MT_{\Delta'}$. The doubled curve $2L_X$ lifts to a curve $\widetilde{2L_X}$ on $MT_{\Delta'}(\Delta)$ which is preserved by ρ. We can choose this lift to ensure that one arc of the bi-gon is contained in $\phi_{\Delta'}(\widetilde{\gamma})$. Then, because $hol^{ab}(\Delta', [\![L_X]\!]) = \pi$, the other arc is contained in $\rho \circ \phi_{\Delta'}(\widetilde{\gamma})$. Therefore, the two curves $\phi_{\Delta'}(\widetilde{\gamma})$ and $\rho \circ \phi_{\Delta'}(\widetilde{\gamma})$ intersect.

(WUL-4 $\implies$ 4): Suppose that $\phi_{\Delta'}(\widetilde{\gamma})$ and $\rho \circ \phi_{\Delta'}(\widetilde{\gamma})$ intersect the same edge of $MT_{\Delta'}(\Delta)$ with the same orientation. (This is equivalent to $\phi_{\Delta'}(\widetilde{\gamma})$ and $-\rho \circ \phi_{\Delta'}(\widetilde{\gamma})$ intersecting the edge with opposite orientations.) Call this oriented edge $\widetilde{e}_1$. Use $\widetilde{C}_1$ to denote the intersection between $\widetilde{e}_1$ and $\phi_{\Delta'}(\widetilde{\gamma})$, and use $\widetilde{D}_1$ to denote the intersection of $\widetilde{e}_1$ with $\rho \circ \phi_{\Delta'}(\widetilde{\gamma})$. Let $\widetilde{e}_2 = \rho(\widetilde{e}_1$. The intersection $\widetilde{C}_2 = \rho(\widetilde{C}_1$ occurs between $\rho \circ \phi_{\Delta'}(\widetilde{\gamma})$ and $\widetilde{e}_2$, and $\widetilde{D}_2 = \rho(\widetilde{D}_1$ occurs between $\phi_{\Delta'}(\widetilde{\gamma})$ and $\widetilde{e}_2$. Define the loop $\widetilde{2N_{C,D}}$ which travels along $\phi_{\Delta'}(\widetilde{\gamma})$ from $\widetilde{D}_2$ to $\widetilde{C}_1$, then along $\widetilde{e}_1$ to $\widetilde{D}_1$, then along $\rho \circ \phi_{\Delta'}(\widetilde{\gamma})$ to $\widetilde{C}_2$, and finally closes up by traveling along $\widetilde{e}_2$ to $\widetilde{D}_2$. The automorphism ρ preserves the loop $\widetilde{2N}_{C,D}$, swapping the edge contained in $\phi_{\Delta'}(\widetilde{\gamma})$ with that contained in $\rho \circ \phi_{\Delta'}(\widetilde{\gamma})$. The image $\psi(\widetilde{2N}_{C,D})$ wraps twice around the loop $N_{C,D}$ obtained from the intersections of γ $C = \psi(\widetilde{C}_1) = \psi(\widetilde{C}_2)$ and $D = \psi(\widetilde{D}_1) = \psi(\widetilde{D}_2)$ with the edge $e = \psi(\widetilde{e}_1) = \psi(\widetilde{e}_2)$. The holonomy around $N_{C,D}$ is described by ρ so that $hol^{ab}(\Delta', [\![N_{C,D}]\!]) = \pi$.

(4 $\implies$ WUL-4): Suppose that $hol^{ab}(\Delta', [\![N_{C,D}]\!]) = \pi$. We obtain a rotation by π, $\rho : MT_{\Delta'} \to MT_{\Delta'}$, which describes the holonomy of the loop $N_{C,D}$. Its double $2N_{C,D}$ lifts to a loop $\widetilde{2N}_{C,D}$ on $MT_{\Delta'}(\Delta)$, which is preserved by ρ. We choose the lift so that one of the arcs of $2N_{C,D}$ contained in γ lifts to a subset of $\phi_{\Delta'}(\widetilde{\gamma})$. The lift of the other arc must lie in $\rho \circ \phi_{\Delta'}(\widetilde{\gamma})$. The arcs of $2N_{C,D}$ contained in the edge lift to two edges up in $MT_{\Delta'}(\Delta)$. We see that $\phi_{\Delta'}(\widetilde{\gamma})$ and $\rho \circ \phi_{\Delta'}(\widetilde{\gamma})$ intersect each of these edges with the same orientation. Or equivalently, $\phi_{\Delta'}(\widetilde{\gamma})$ and $-\rho \circ \phi_{\Delta'}(\widetilde{\gamma})$ intersect with opposite orientations. $\diamondsuit$

The remaining results from the previous section follow easily.

Proof of Theorem 23: Lemma 30 places finitely conditions on the unfriendly set each of the form

$$hol^{ab}(\Delta', [\![\eta]\!]) = 0 \ \mathit{or}\ \pi \in \mathbb{R}/2\pi\mathbb{Z} \tag{31}$$

for some loop η. Equivalently, we could choose a loop η' on $T_1\Sigma$ so that its image under the projection $T_1\Sigma \to \Sigma$ is η and require that

$$\widetilde{hol}^{ab}(\Delta', [\![\eta']\!]) \equiv 0 \ \mathit{or}\ \pi \pmod{2\pi}$$

But, for any choice of Δ', $0 < \widetilde{hol}^{ab}(\Delta', [\![\beta_i']\!]) < \pi$. Therefore, the potential values for $\widetilde{hol}^{ab}(\Delta', [\![\eta']\!])$ as Δ' varies are bounded by the size of $[\![\eta]\!]$. Thus, there are only finitely many values of k so that

$$\widetilde{hol}^{ab}(\Delta', [\![\eta']\!]) = (0 \ \mathit{or}\ \pi) + 2\pi k \tag{32}$$

could be true for some Δ'. Thus each condition in the form of equation 31 contributes finitely many lines to the unfriendly set. $\diamondsuit$

Proof of the Bounding Box Theorem: We will show that the billiard-like tile is a single component of $\mathcal{T} \smallsetminus UF([\widetilde{\gamma}]_s)$. Then theorem 29 states that $tile([\gamma]_s)$ is contained in this one component.

It is easy to verify that the boundaries of the billiard-like tile (see definition 28) are contained in the unfriendly set as described by lemma 30. Conversely, each rational line in the unfriendly set is one of a family of parallel rational lines in the form of equation 32 and determined by the lemma. Theorem 29 forces the billiard-like tile to be contained in one complement of these parallel lines. $\diamondsuit$

8 Right triangles are unfriendly

Now that we have proved the bounding box theorem, we can discuss the idea of the proof of Theorem 1. We assume that γ is a closed geodesic on the surface Σ_Δ (for arbitrary Δ) which lifts to a closed geodesic $\widetilde{\gamma}$ on AC_Δ. We need to show that $tile([\gamma]_s)$ is contained in at most one component of $\mathcal{T}$ with the right triangles removed (see figure 1).

Let ℓ be any one of the right triangle lines, The set of right triangles where $\alpha_3 = \pi/2$ is

$$\{(\alpha_1, \alpha_2, \alpha_3) \in \mathcal{T} | \alpha_1 + \alpha_2 - \alpha_3 = 0\} \tag{33}$$

Therefore, by proposition 19, MT_ℓ admits a rotation by π which we call ρ. This map simultaneously rotates by π around each lift of the right angled vertex of Δ to $MT_\ell(\Delta)$. We let ϕ_ℓ be the covering map $AC \to MT_\ell$.

We will have to prove the following lemma:

Lemma 31 (Right triangles are unfriendly) *Suppose γ is a closed geodesic on Σ_Δ (for arbitrary Δ) which lifts to a closed geodesic $\widetilde{\gamma}$ on AC_Δ. The pair of oriented symbolics class of curves $\{[\phi_\ell(\widetilde{\gamma})]_s, [-\rho \circ \phi_\ell(\widetilde{\gamma})]_s\}$ is unfriendly on MT_ℓ for each right triangle line ℓ.*

In other words, the lemma claims that each right triangle line ℓ is contained in $UF([\widetilde{\gamma}])$. Given this lemma, the proof of Theorem 1 follows directly from the bounding box theorem.

Let $\gamma_1 = \phi_\ell(\widetilde{\gamma})$ and $\gamma_2 = -\rho \circ \phi_\ell(\widetilde{\gamma})$. To prove this lemma, we suppose by contradiction, that the pair of loops $x = \{\gamma_1, \gamma_2\}$ is friendly on MT_ℓ. Then by proposition 6, the symbolics class of the pair of curves is determined uniquely by its homology class $[\![x]\!] = [\![\gamma_1]\!] + [\![\gamma_2]\!]$. We immediately notice two trivial things about this homology class:

1. The homology class $[\![x]\!]$ lifts to a homology class on AC.
2. $\rho([\![x]\!]) = -[\![x]\!]$

First we will use triviality 1. We know that AC is homotopic to the graph corresponding to the hexagonal tiling of the plane, AC_Θ. With hexagonal 2-cells, we can fill in these hexagons to get something simply connected (the plane). Consequently, we can think of a homology class on AC_Θ as the boundary of a finite union of weighted hexagonal 2-cells. MT_ℓ is homotopic to a graph coming from the hexagonal tiling of a cylinder. The hexagonal 2-cells from AC_Θ push down to fill in the hexagons in this graph building a cylinder. In particular, we see that the image of the map $H_1(AC, \mathbb{Z}) \to H_1(MT_\ell, \mathbb{Z})$ is not everything. Instead, $H_1(MT_\ell, \mathbb{Z})$ modulo the image hexagons is isomorphic to $\mathbb{Z}$, generated by a loop which goes once around the cylinder.

We can understand the graph homotopic to MT_ℓ and action of ρ on this graph by looking at the translation surface itself. The surface decomposes into rhombi consisting of four copies of the triangle as in figure 15. In the figure, edges of the dual graph are dashed according to which edge of the triangle they cross. The Θ-graph is the hexagonal tiling graph of the plane modulo translations. So, parallel edges all cross the same type of edge of the triangulation. Indeed, because the dual graph to the triangles in a rhombus is a quadrilateral only crossing two edge types, the dual graph must be the hexagonal graph in the plane modulo a translation by two units upward. Similarly, it can be checked that the rotation by π preserving the rhombus, acts as a translation by one unit upward.

Pictorially, we can describe the homology class of these curves by assigning integral weights (only finitely many nonzero) to the hexagons of the cylinder. These integers represent the weights assigned to the hexagonal 2-cells. Refer to figure 16. The hexagonal tiling of the cylinder we mention above is obtained as the quotient of the hexagonal tiling of the plane by the translation by two units upward. Our rotation by π, ρ, acts on a hexagonal 2-cell by translating it one unit upward.

The homology class described by figure 16 is a candidate for $[\![x]\!]$. This class lifts to a homology class on AC and $\rho([\![x]\!]) = -[\![x]\!]$. As in the figure, weights assigned to the hexagonal 2-cells in a column are opposites. This forces weights assigned to the horizontal edges in a column to be even and opposite. This is an essential ingredient in the proof.

Figure 16 showed the homology class on the hexagonal graph on the cylinder. We would also like to understand what this homology class looks

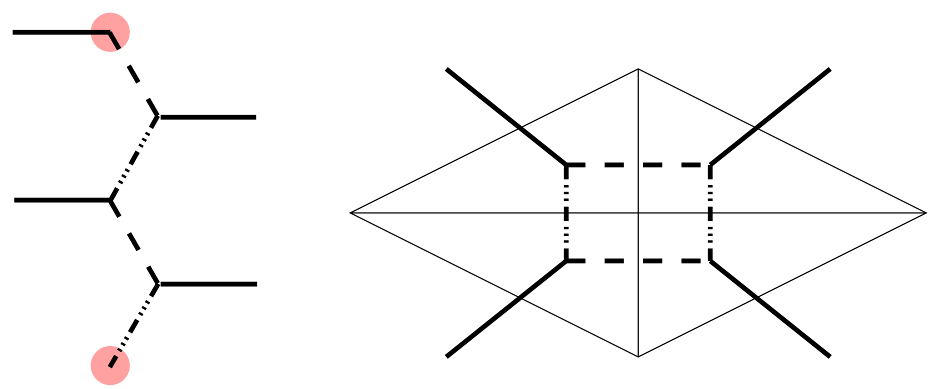


Figure 15: On the left, we show a subset of the graph coming from the hexagonal tiling of the cylinder. The two dotted vertices are identified by the translation which we quotient the plane by to get the cylinder. On the right, we show the same graph, the dual graph to the triangulated MT_ℓ.

like on the translation surface MT_ℓ. The homology class of figure 16 displayed as the dual graph to the triangulation is shown in figure 17.

It is important to note that a loop which travels once around the puncture at the center of a rhombus travels once around the cylinder in the other picture. These curves do not lift to AC.

Let $Y = \Psi([\![x]\!])$ be the set of symbolics classes curves obtained by combing out the homology class $[\![x]\!]$ as in the proof of proposition 6.

It is useful to fill in the punctures at the center of each rhombus, constructing a larger surface which we will call $\overline{MT}_\ell$. We have a natural inclusion $i : MT_\ell \hookrightarrow \overline{MT}_\ell$. From what we have already said, we can deduce the following:

Proposition 32 *There are an even number of curves in Y. Pushed into $\overline{MT}_\ell$, the curves $i(Y)$ come in homotopic pairs.*

Proof: The push forward $i_*([\![x]\!]) \in H_1(\overline{MT}_\ell, \mathbb{Z})$ can be combed out in a similar manner as in the proof of proposition 6. We can see a homology class in $H_1(\overline{MT}_\ell, \mathbb{Z})$ as an assignment of integer weights and orientations to the dual graph to the decomposition of $\overline{MT}_\ell$ into rhombi. The weight assigned by $i_*([\![x]\!])$ to an edge of the dual graph which crosses a boundary of a rhombus is the same as the weight assigned by $[\![x]\!]$ to the edge of the dual graph to the

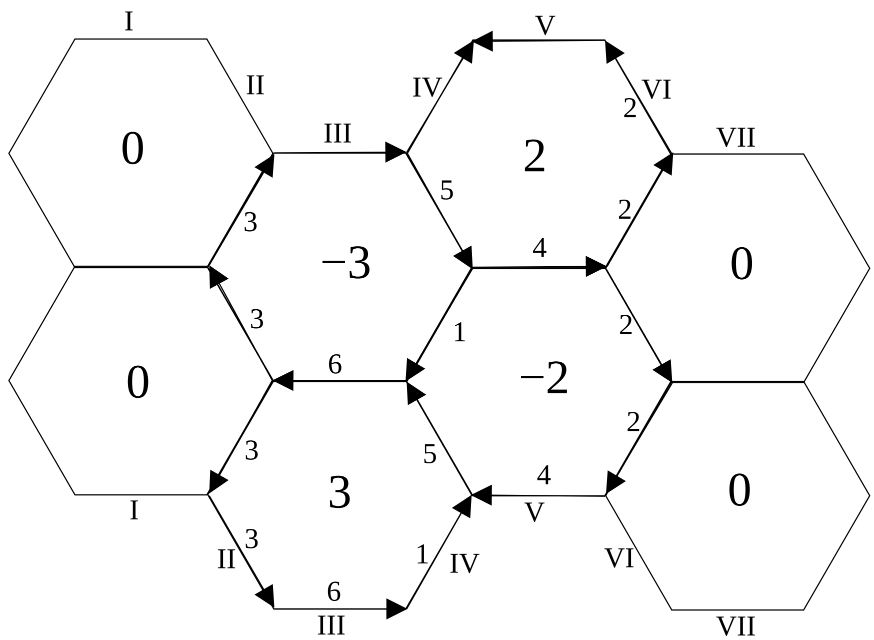


Figure 16: A homology class on the graph homotopic to MT_ℓ that lifts to AC_Θ is determined by assigning integer weights to the hexagonal 2-cells in the complement of the graph. This graph is the quotient of the hexagonal graph in the plane by a translation that translates two units upward. Roman numerals indicate identifications used to reconstruct this graph on the cylinder.

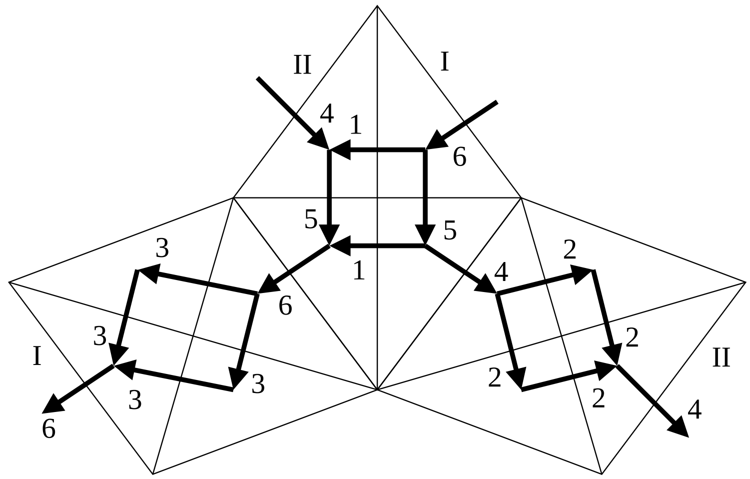


Figure 17: This is the same homology class as figure 16, only this time it is displayed with the triangulation of MT_ℓ. The roman numerals I and II indicate edge identifications.

triangulation which crosses that same boundary edge of the rhombus. The edges of the dual graph which cross the hypotenuse of the right triangle are horizontal when the graph is depicted as the hexagonal graph on the cylinder (see figure 15). So, all edge weights of $i_*(\llbracket x \rrbracket)$ are even. Weights assigned to edges which cross opposite sides of a rhombus are equal in magnitude but orientations are opposite (one points in, the other out).

Note that if $\llbracket x \rrbracket$ assigned orientations to edges as in figure 18, then we could not comb out the homology class to arcs in a unique way. Fortunately, because $\rho(\llbracket x \rrbracket) = -\llbracket x \rrbracket$ this assignment of orientations never occurs.

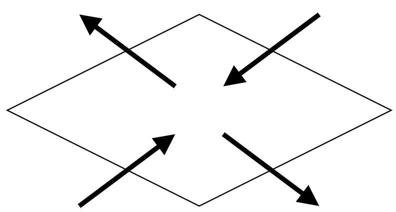


Figure 18: $i_*(\llbracket x \rrbracket)$ never assigns orientations to edges crossing the boundary of a rhombus as pictured.

Instead, opposite edges are given opposite orientations. Therefore, we can comb out in a unique way as in figure 19.

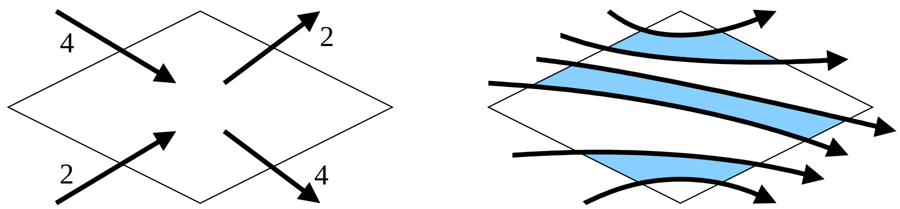


Figure 19: Possible edge weights assigned by $i_*(\llbracket x \rrbracket)$ to edges crossing the boundary of a rhombus. On the right, we show the unique comb out.

By uniqueness, the comb out construction constructs a set of curves in the same homotopy class as $i(Y)$. Because an even weight is assigned to each edge, we can partition the arcs which cross a boundary edge of a rhombus into adjacent pairs. (This partition is unique.) The curves travel together across the rhombus and leave as an adjacent pair, since all edge weights are even. Induction tells us that adjacent pairs stay adjacent for all time. Thus, we have partitioned the curves in $i(Y)$ into pairs which are homotopic inside $\overline{MT_\ell}$. We comb out the example homology class of figures 16 and 17 in figure 20. $\diamondsuit$

Proof of Lemma 31 We continue to use the notation of this section. Suppose the pair of curves $x = \{\gamma_1, \gamma_2\}$ is friendly on MT_ℓ. Then by proposition 6, combing out the homology class $\Psi(\llbracket x \rrbracket)$ yields a pair of curves in the same symbolics class as x. Proposition 32 tells us that γ_1 and γ_2 are homotopic inside $\overline{MT_\ell}$. Since they are disjoint, they bound an annulus $\mathcal{A} \subset \overline{MT_\ell}$.

The rotation by π, ρ, preserves the annulus $\mathcal{A}$ and switches its boundary components since $\gamma_2 = -\rho(\gamma_1)$. We claim that ρ fixes two points in $\mathcal{A}$. This follows from Lefshetz's fixed point theorem. Recall that the Lefshetz number associated to the map $\rho : \mathcal{A} \to \mathcal{A}$ is

$$L(\rho) = \sum_i (-1)^i tr(\rho_{H_i}) \tag{34}$$

where $tr(\rho_{H_i})$ is the trace of the map $\rho_{H_i} : H_i(\mathcal{A}, \mathbb{Z}) \to H_i(\mathcal{A}, \mathbb{Z})$ induced by ρ. In our case, the homology of an annulus is

$$H_0(\mathcal{A}, \mathbb{Z}) = H_1(\mathcal{A}, \mathbb{Z}) = \mathbb{Z} \text{ and } H_i(\mathcal{A}, \mathbb{Z}) = 0 \text{ for all } i > 1 \tag{35}$$

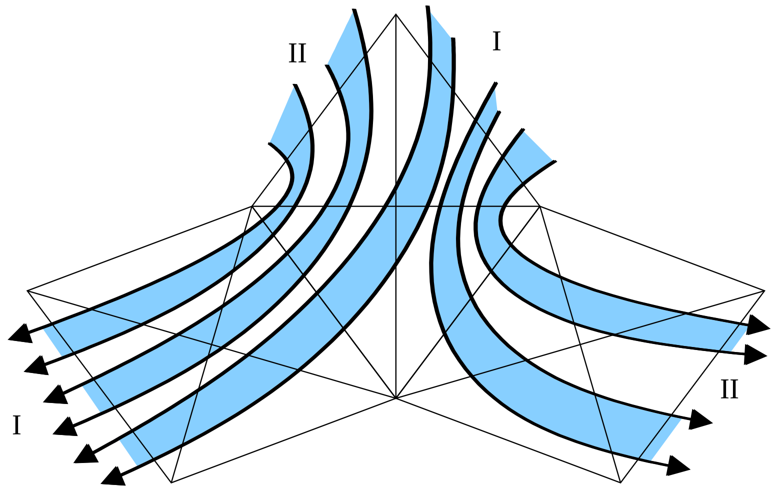


Figure 20: The comb out operation applied to the homology class of figures 16 and 17.

We see $tr(\rho_{H_0}) = 1$ and $tr(\rho_{H_1}) = -1$, because ρ is homotopic to a homeomorphism from the circle to itself which switches orientation. Thus, $L(\rho) = 2$. Lefshetz's theorem tells us that the Lefshetz number can also be computed as

$$L(\rho) = \sum_{\rho(P)=P} sign(det(I - D_P(\rho))) \tag{36}$$

where I and $D_P(\rho)$ are the actions of the identity map and the derivative of ρ on the tangent plane to $\mathcal{A}$ at P respectively. In our case, the only fixed points of the action of ρ on $\overline{MT_\ell}$ are the centers of the rhombi. At the center of each rhombus, ρ acts by a rotation by π. Thus $L(\rho) = 2$ is also the number of centers of rhombi contained in $\mathcal{A}$.

Now we will show that the curve η_1 can not lift to AC, giving us our contradiction. Note that if γ_1 lifts, then $\rho(\gamma_1) = -\gamma_2$ must lift as well since ρ is a deck transformation of the cover $MT_\ell \to S(\Delta)$. Consequently, the homology class $[\![\eta_1]\!] + [\![\rho(\gamma_1)]\!]$ must lift to a homology class in $H_1(AC, \mathbb{Z})$. From the previous paragraph, we know $\gamma_1 \cup \rho(\gamma_1)$ is the (oriented) boundary of the annulus $\mathcal{A}$ on $\overline{MT_\ell}$. On MT_ℓ, we know that γ_1 and $\rho(\gamma_1)$ together with two curves α and β which travel around punctures with the same orientations bound a subsurface. See figure 21. Then, we know that

$$[\![\gamma_1]\!] + [\![\rho(\gamma_1)]\!] = -([\![\alpha]\!] + [\![\beta]\!]) \in H_1(MT_\ell, \mathbb{Z}) \tag{37}$$

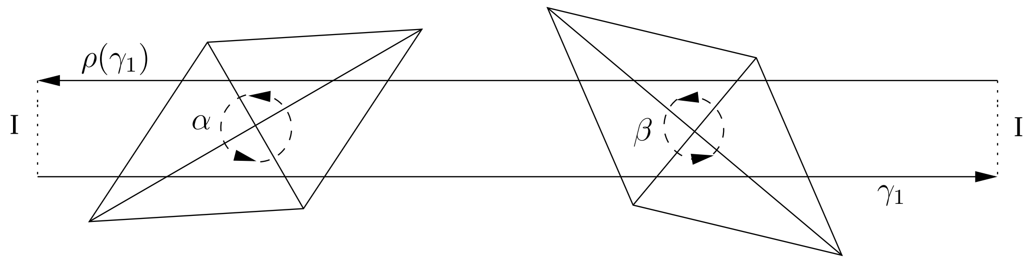


Figure 21: In the proof of lemma 31, γ_1 and $\rho(\gamma_1)$ bound a punctured annulus. The annulus here is the horizontal strip with vertical dotted lines identified. The punctures come from the centers of rhombi.

Recall the discussion at the beginning of this section. Curves which travel around the punctures correspond to curves which travel around the core of the cylinder. These curves do not lift to AC. The automorphism group of the cover $MT_\ell \to \Sigma$ acts transitively on the rhombuses. It acts on the dual graph as translations on the cylinder that preserve the hexagonal tiling. Thus because α and β travel around punctures with the same orientation, they travel around the core of the cylinder in the same direction. The image of the homology class $[\![\alpha]\!]+[\![\beta]\!]$ under the covering $MT_\ell \to \Sigma$ sends $[\![\alpha]\!]+[\![\beta]\!]$ to the homology class of a loop which travels around the puncture of Σ corresponding to the right angled vertex four times. This homology class on $H_1(\Sigma)$ is not null-homologous, so $[\![\alpha]\!] + [\![\beta]\!]$ can not lift to a homology class on AC. This is our contradiction. $\diamondsuit$

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