COVERS OF INFINITE TRANSLATION SURFACES: RANDOM AND EVIL

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Abstract. We consider how passing to finite covers of an infinite translation surface can interact with ergodic properties of the straight-line flow. We describe a construction of finite covers of an infinite translation surfaces which destroys the ergodicity of the straight-line flow via a mechanism not available in the compact case. We describe a canonical notion of random finite covers of infinite genus translation surfaces. We show that for some infinite translation surfaces, S, that in certain directions, the straight-line flows have ergodicity which persists in passing to random covers. We explain how the notion of random covers gives rise to new Teichmuller flow invariant measures on the moduli space of infinite translation surfaces.

1. Introduction

A *translation surface* is a pair (S, α), where S is a Riemann surface and α is a holomorphic 1-form on S. Let $Z \subset S$ denote the zeros of α. The 1-form α endows $S \setminus Z$ with local coordinates to the plane: for any p we have the locally defined *coordinate chart* to $\mathbb{C}$ given by the local homeomorphism $q \mapsto \int_p^q \alpha$. These coordinate charts differ locally only by translation. A translation surface inherits a metric by pulling back the Euclidean metric on the plane along the coordinate charts. Points in $S \setminus Z$ are locally isometric to the plane, while points in Z are cone singularities with cone angle $2(k+1)\pi$ where $k \geq 1$ is the degree of the zero of α.

We'll say a translation surface is *classical* if S is a closed surface. Here, there is a well known interplay between two types of dynamical systems: *(1)* the dynamics of the translation flow on a translation surface (S, α) given in local coordinates by

$$F^t : S \to S; \quad (x, y) \mapsto (x + t, y),$$

and *(2)* the Teichmüller deformation on the moduli space of translation surfaces, where $g^t(S, \alpha)$ is obtained from (S, α) by postcomposing the coordinate charts with the affine coordinate change $g^t(x, y) = (e^{-t}x, e^t y)$. Namely, the Teichmüller deformation renormalizes the translation flow. A famous consequence of this relationship is given by Masur's Criterion: if the forward orbit of the Teichmüller deformation, $\{g^t(S, \alpha)\}_{t \geq 0}$, has a convergent subsequence $g^{t_n}(S, \alpha)$ with $t_n \to \infty$ (i.e., the orbit is *non-divergent*), then the translation flow is uniquely ergodic [Mas92]. Slow divergence of $g^t(S, \alpha)$ leads to similar conclusions [CE07] [Tre14].

Recently, many results similar in spirit to Masur's criterion have been proven in special cases for translation surfaces of infinite genus. Due to the lack of a well defined moduli space of such surfaces, the standard approach is to define some topological space of surfaces where $SL(2, \mathbb{R})$ acts and prove that non-divergence of a $g^t(S, \alpha)$ orbit within this space

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entails consequences for the translation flow on (S,α) [1]. To date, the primary mechanism for building such a topological space of surfaces uses affine symmetries of the translation surface (S,α), as we will now explain. In §2, we describe the action of $SL(2,\mathbb{R})$ by affine deformations on the collection of all translation surfaces. The stabilizer of (S,α) is called the surface's *Veech group*, $V(S,\alpha) \subset SL(2,\mathbb{R})$. Let $W \subset V(S,\alpha)$ be a discrete subgroup of the Veech group and let $\mathcal{O}(S,\alpha) = SL(2,\mathbb{R})/W$. This quotient has the property that $A(S,\alpha)$ only depends on the equivalence class in $\mathcal{O}(S,\alpha)$. This allows us say that (S,α) is *non-divergent in* $\mathcal{O}(S,\alpha)$ if there is a sequence $t_n \to \infty$ so that g^{t_n} converges in $\mathcal{O}(S,\alpha)$.

We will study the following situation. Suppose (S,α) is a finite area translation surface with infinite topological type, and suppose that it is non-divergent in $\mathcal{O}(S,\alpha)$. Our primary concern is with surfaces of infinite genus. Let $(\tilde{S},\tilde{\alpha})$ be a finite cover of (S,α). Can we conclude that the translation flow on $(\tilde{S},\tilde{\alpha})$ is ergodic or uniquely ergodic? We will provide contexts where the answer is yes, and examples where the answer is no.

In order to say something constructive, we restrict our covers. First we will pick some integer $d \geq 2$, and restrict attention to covers of degree d. Choose an arbitrary non-singular basepoint. Let $(\tilde{S},\tilde{\alpha})$ be a cover of (S,α), where the flat structure on $(\tilde{S},\tilde{\alpha})$ is lifted from the one on (S,α). The fiber of the basepoint can be identified in an arbitrary way with the set $\{1,2,\ldots,d\}$. The *monodromy action* is the natural right action of the fundamental group on fiber of the basepoint. Our identification of the fiber determines a *monodromy representation* $\pi_1(S) \to S_d$, where S_d is the permutation group of $\{1,\ldots,d\}$. We note that a cover can be reconstructed from such a monodromy representation, and two covers are isomorphic if and only if these monodromy representations differ by conjugation by an element of S_d. Now fix a subgroup $G \subset S_d$. We say a cover $(\tilde{S},\tilde{\alpha})$ has *monodromy in* G if it can be realized by a monodromy representation to G. We define $\mathrm{Cov}_G(S,\alpha)$ to be the collection of all covers with monodromy in G (up to translation equivalence). From the discussion above, we have the natural identification

$$\mathrm{Cov}_G(S,\alpha) = S_d \backslash \mathrm{Hom}\big(\pi_1(S \smallsetminus Z), G\big).$$

We note that a cover $(\tilde{S},\tilde{\alpha}) \in \mathrm{Cov}_G(S,\alpha)$ need not be connected, and is connected if and only if the image of the associated monodromy representation acts transitively on $\{1,\ldots,d\}$. In particular, there are connected covers with monodromy in G if and only if G acts transitively on $\{1,\ldots,d\}$.

The main construction of the paper is the definition of the *covers cocycle*. We will give a brief overview here, but the topic is formally treated in §3. We'd like to parameterize covers of surfaces affinely equivalent to (S,α) with monodromy in G. Such a cover can be specified by a pair $\big(A,(\tilde{S},\tilde{\alpha})\big)$, where $A \in SL(2,\mathbb{R})$ and $(\tilde{S},\tilde{\alpha}) \in \mathrm{Cov}_G(S,\alpha)$. This pair would then represent the surface $A(\tilde{S},\tilde{\alpha})$ which covers $A(S,\alpha)$. The subgroup W of the Veech group acts naturally on $\mathrm{Cov}_G(S,\alpha)$. Observe that if $R \in W$, then the pair $\big(AR^{-1}, R(\tilde{S},\tilde{\alpha})\big)$ represents the same surface. Therefore, points in the quotient space

$$\tilde{\mathcal{O}}_G(S,\alpha) = \big(SL(2,\mathbb{R}) \times \mathrm{Cov}_G(S,\alpha)\big)/W \tag{1}$$

[1]Ok, a few issues here. Masur's criterion uses a moduli space, which we don't have for infinite genus. The dificulty is not that we have lack of examples, but that we have a lack of moduli space. Moreover, I think it's misleading to state that it seems like Masur's criterion holds in infinite genus. I think we should be more specific, I think (and I think you may agree) that it holds for infinite genus and finite area and where almost every point is defined for all time. Otherwise, I'm not so sure. Ok, I was trying to be somewhat vague here.

parameterize such affinely equivalent covers. (Here, W acts simultaneously on both factors as just described.) Further, $SL(2,\mathbb{R})$ has a well defined left action, which descends from left multiplication on the $SL(2,\mathbb{R})$ factor of the product $SL(2,\mathbb{R}) \times \mathrm{Cov}_G(S,\alpha)$. This allows us to state our first main result.

I commented out the topological description of $\tilde{\mathcal{O}}_G(S,\alpha)$.

Theorem 1. *Suppose (S,α) is a finite area translation surface with infinite topological type, and suppose that it is non-divergent in $\mathcal{O}(S,\alpha)$. Let $(\tilde{S},\tilde{\alpha})$ be a finite cover of (S,α) with monodromy in G. If there is a sequence $t_n \to \infty$ for which $g^{t_n}\big[I,(\tilde{S},\tilde{\alpha})\big]$ converges to a connected cover in $\tilde{\mathcal{O}}_G(S,\alpha)$, then the translation flow on $(\tilde{S},\tilde{\alpha})$ is uniquely ergodic.*

To produce examples of covers for which the theorem above applies, we establish a notion of a random cover of (S,α). Let $G \subset S_d$ be as above. There is a natural Borel probability measure m_G on $\mathrm{Cov}_G(S,\alpha)$ which is invariant under the action the mapping class group of S. See §4.2. We prove the following.

Theorem 2. *Let (S,α) be a finite area translation surface with infinite topological type, and suppose that it is non-divergent in $\mathcal{O}(S,\alpha)$. Suppose G is a transitive subgroup of the permutation group S_d. Then m_G-almost every cover $(\tilde{S},\tilde{\alpha})$ has uniquely ergodic translation flow.*

Still working on this paragraph. The existence and naturality of the measure m_G on the space of covers $\mathrm{Cov}_G(S,\alpha)$ has a nice consequence in terms of lifting measures. Let μ be a probability measure on $\mathcal{O}(S,\alpha)$ which is invariant under the left action by some subgroup $H \subset SL(2,\mathbb{R})$. Then we can lift the measure to a measure $\tilde{\mu}$ on $SL(2,\mathbb{R})$. Does "lifting" measures seem right? The quotient measure $\tilde{\mu} \times m_G/Q$ is an H-invariant probability measure on $\tilde{\mathcal{O}}_G(S,\alpha)$. So Corollary **??** gives a construction of new Teichmüller deformation-invariant measures.

We also produce some examples of non-ergodic covers. Let (S,α) be a finite area translation surface with infinite topological type which is non-divergent in $\mathcal{O}_G(S,\alpha)$. We call a finite cover $(\tilde{S},\tilde{\alpha})$ (branched over the zeros of α) *evil* if it is connected but its translation flow is not ergodic. We produce examples of evil covers in §6.

We note that evil covers must have interesting properties. Since (S,α) is non-divergent in $\mathcal{O}_G(S,\alpha)$, and because the fibers of the projection $p : \tilde{\mathcal{O}}_G(S,\alpha) \to \mathcal{O}(S,\alpha)$ are compact, there must be accumulation points of the forward orbit $\{g^t(\tilde{S},\tilde{\alpha})\}_{t\geq 0}$ inside of $\tilde{\mathcal{O}}_G(S,\alpha)$. But we can see from Theorem 1 that all such accumulation points must be disconnected. In the examples we build, there is only one disconnected surface in $\tilde{\mathcal{O}}_G(S,\alpha)$, and so our evil covers lie in the stable set of this disconnected surface.

Remark 3 (Slow divergence)**.** *While the condition that all accumulation points be disconnected is a necessary condition for a cover to be evil, we believe it is insufficent. In particular, the main result of* [Tre14] *implies that if $\{g^t(\tilde{S},\tilde{\alpha})\}_{t\geq 0}$ converges sufficiently slowly to a disconnected surface, then the translation flow is still uniquely ergodic. But, we have found no examples of this behavior yet.*

We further remark that evil covers do not exist in the classical setting. If (S,α) is a closed translation surface with non-divergent g^t orbit, then any finite cover $(\tilde{S},\tilde{\alpha})$ also has non-divergent g^t orbit.

1.1. **Context.** There has been an increased interest in the study of the dynamics and geometry of flat surfaces of infinite genus. Unlike compact flat surfaces of finite type, no space has been found which parametrizes flat metrics for all surfaces of a given topological type. This gives the first obstacle to utilizing tools from the finite type world. Different techniques have been developed to overcome this silly but fundamental shortcoming, a shortcoming which prevents us from answering one of the most basic questions: whether the translation flow on a given flat surface of infinite type and finite area is ergodic or not.

Most studies concentrate on the case of the surface having finite or infinite area (an exception being [Hoo10]). Such choice has great implications to the tools used, the results obtained, and the method of construction used to produce examples or to define some "spaces of surfaces". A common tool in both contexts is the use of Veech groups, which are a sort of symmetry groups of the surface. For compact flat surfaces, these are always discrete subgroups of $SL(2,\mathbb{R})$. Flat surfaces of finite type with non-trivial Veech groups (which are lattices in $SL(2,\mathbb{R})$) are part of a very deep theory, so the fact that they can be used in the infinite type setting is encouraging.

To our knowledge, the first papers on dynamics on flat surfaces of infinite type are those which come out of the infinite step polygonal billiards introduced in [Tro99, DEDML98] through the unfolding procedure. All such surfaces considered were of finite area and came from "rational" polygons, i.e., the angles of the billiard from which the surfaces were constructed satisfied some rationaly conditions. Ergodic properties as well as topological results were obtained for a large class of these types of surfaces, but it was done without the use of Veech groups. In fact, it can be proved that the surfaces obtained from an infinite step billiard does not have hyperbolic elements in its Veech group.

The groundbreaking paper of Chamanara [Cha04] introduced a 1-parameter family of flat surfaces of infinite type and of finite area with a non-trivial Veech group. The main results of that paper discussed the types of Veech groups appear in the construction . Most importantly, even though the surfaces constructed possess many symmetries, the Veech group is never a lattice for any surface arising in this construction. We will review Chamanara's construction in section 6 and apply some of our results to spaces of covers thereof.

Another study of flat surfaces of infinite type and finite area has been the construction of Bowman which extends the Arnoux-Yoccoz family of flat surfaces to include a surface of infinite genus and finite area [Bow13]. This surface of infinite genus and finite area admits a pseudo-Anosov diffeomorphism. Moreover, it was found that the Veech group of this surface is isomorphic to $\mathbb{Z} \times \mathbb{Z}_2$ and that the directions preserved by the pseudo-Anosov map correspond to uniquely ergodic translation flows.

The work of the second author [Tre14] addresses the question of ergodicity of translation flow for surfaces of finite area. In particular, it is independent of topological type and therefore can be used to determine when the translation flow of a flat surface of infinite genus and finite area is ergodic or uniquely ergodic. We use this criterion in section **??**.

I'M TEMPTED TO PLUG IN HERE KATHRYN'S AND MY THING WITH BRATTELI DIAGRAMS. THOUGHTS? Go ahead!

The study of translation flows for surfaces of infinite genus and finite area has painted a completely different landscape compared to that of surfaces of infinite type and of finite area. Two themes are uniquitous in this setting: the use of covers and the use of Veech groups.

OK, I'M STOPING HERE FOR NOW BECAUSE I HAVE TO GO.

As it can be seen, Veech groups have played a significant role in the study of translation flows. The properties of translation flows so far obtained for surfaces of infinite genus and finite area are mostly similar to those of compact flat surfaces.

It is known that for practically any subgroup G of $SL(2,\mathbb{R})$ there is a flat surface of infinite type and infinite area whose Veech group is G [PSV11]. Moreover, it is unknown whether there exists a flat surface of infinite type and finite area whose Veech group is a lattice in $SL(2,\mathbb{R})$. It is known that for surfaces of infinite type obtained as an infinite covers of a compact flat surface, there is a relationship between the Veech group of the base surface and that of that cover [GJ00] and that thus some of the properties of the base surface given by the Veech group can be "lifted" to the infinite cover. Infinite covers of finite area have not, as far as we are aware, been considered.

In this paper we address the problem posed by the apparent non-existence of a lattice Veech group for a flat surface of infinite type and finite area by constructing spaces of covers on which we can define a dynamical system, which we call the *cover cocycle*. The spaces constructed are Cantor sets and can be thought of as spaces of random covers, and they cary a probability measure which is invariant under the action of the Veech group (which is defined in §?). As far as we know, this construction is new. Moreover, this construction allows us to determine when the lift of a translation flow to a cover retains ergodic properties which the base surface possesses, such as unique ergodicity. This, in a way, overcomes some of the problems created by the non-existence of lattice Veech groups for surfaces of infinite type and finite area. The main result can be summarized as follows.

Theorem 4 (We probably need to introduce more and perhaps some notation.)**.** *Let S be a flat surface constructed from a graph Γ as above, let V be its Veech group, and $\mathcal{C}$ a space of random coveres. Suppose the Teichmuüller orbit of S is non-divergent in $SL(2,\mathbb{R})/V$. Then almost every cover $p : \hat{S} \to S$ has a translation flow which is uniquely ergodic.*

- Motivation: there seem to be no lattice Veech groups for infinite genus, finite area.
- You get new Teichmuller probability measures by looking at the Bernoulli measures of the cocycle.
- New types of cover constructions.
- In finite genus, covers branched over singular points maintain ergodicity, so our examples are some sort of new phenomena.
- Veech's construction of branching over non-singular points.
- Statement of theorem and corollaries.
- Examples.

2. Background on translation surfaces

Let (S,α) be a translation surface. The *straight line flow* on (S,α) in direction θ is the flow $F^t_\theta : S \to S$ defined for $t \in \mathbb{R}$ given in local coordinates by $F^t_\theta(x,y) = (x + t\cos\theta, y + t\sin\theta)$ away from the zeros of α. We reserve the name *translation flow* for the straight line flow F^t_θ where $\theta = 0$. This is the flow on S determined by the rightward unit vectorfield X. We will use Y denote the upward unit vectorfield.

There is a group of deformations of the flat metric on (S,α) which is parametrized by the group $SL(2,\mathbb{R})$. This is the $SL(2,\mathbb{R})$ action mentioned in the introduction. Fix a matrix $A \in SL(2,\mathbb{R})$. We get new (non-conformal) local coordinates to the plane by postcomposing the charts on (S, α) to $\mathbb{C}$ by the real-linear map

$$A : \mathbb{C} \to \mathbb{C}; \quad x + iy \mapsto a_{1,1}x + a_{1,2}y + i(a_{2,1}x + a_{2,2}y).$$

Then, we get a new Riemann surface structure S' on S by pulling back the complex structure using these deformed charts, and the charts determine a new holomorphic 1-form α' on S'. We define $A(S, \alpha) = (S', \alpha')$.

Let (S, α) and (S', α') be translation surfaces. We say they are *translation equivalent* if there is a homeomorphism $h : S \to S'$ which is a locally a translation in all local coordinate charts. We call the map h a *translation isomorphism*. The *Veech group* of (S, α) is the subgroup $V(S, \alpha) \subset SL(2, \mathbb{R})$ of elements $A \in SL(2, \mathbb{R})$ so that $A(S, \alpha)$ and (S, α) are translation equivalent. An *affine homeomorphism* from a translation surface (S, α) to another translation surface (S', α') is a homeomorphism $\phi : S \to S'$ so that there is a matrix $A \in SL(2, \mathbb{R})$ so that in local coordinates $\phi(z_2) - \phi(z_1) = A(z_2 - z_1)$. The matrix $D(\phi) = A$ is called the *derivative* of the affine homeomorphism. Observe that the statement that $A(S, \alpha)$ and (S', α') are translation equivalent is equivalent to the statement that there is a affine homeomorphism $\phi : (S, \alpha) \to (S', \alpha')$ with derivative A. The *affine automorphism group of* (S, α), $\mathit{Aff}(S, \alpha)$ is the group of all affine homeomorphisms from (S, α) to itself. By the prior observation, we have $D\big(\mathit{Aff}(S, \alpha)\big) = V(S, \alpha)$.

3. The covers cocycle

A *covering* of (S, α) is a pair $(p, \tilde{S})$, where $\tilde{S}$ is a topological surface and $p : \tilde{S} \to S$ is a topological covering. We can then endow $\tilde{S}$ with the Riemann surface structure obtained by pulling back the complex structure from S and endow it with the 1-form obtained by pullback, $\tilde{\alpha} = p^*(\alpha)$. Thus, a covering determines a translation surface $(\tilde{S}, \tilde{\alpha})$ covering (S, α), and the covering respects the translation structure.

Two covers $(p_1, \tilde{S}_1)$ and $(p_2, \tilde{S}_2)$ are *isomorphic* if there is a homeomorphism $h : \tilde{S}_1 \to \tilde{S}_2$ so that $p_1 = p_2 \circ h$. The notion of translation equivalence from the prior section provides a slightly more general notion of equivalence of covers. Namely, we say $(p_1, \tilde{S}_1)$ and $(p_2, \tilde{S}_2)$ are *translation equivalent* if the respective translation surfaces $(\tilde{S}_1, \tilde{\alpha}_1)$ and $(\tilde{S}_2, \tilde{\alpha}_2)$ determined by the coverings are translation equivalent. We observe that with this definition, the coverings $(p_1, \tilde{S}_1)$ and $(p_2, \tilde{S}_2)$ are translation equivalent if and only if there is a homeomorphism $h : \tilde{S}_1 \to \tilde{S}_2$ and a $\psi \in \mathit{ker}\ D$ so that $p_1 = \psi \circ p_2 \circ h$. We call ψ a *translation automorphism*; it is an affine automorphism with derivative I. We use $\mathrm{Cov}(S, \alpha)$ to denote the collection of all translation equivalence classes $[p, \tilde{S}]$ of covers of (S, α). An affine automorphism $\phi \in \mathit{Aff}(S, \alpha)$ acts on the space of covers via

$$\phi(p, \tilde{S}) \mapsto (\phi \circ p, \tilde{S}).$$

The Veech group $V(S, \alpha)$ is canonically homeomorphic to $\mathit{Aff}(S, \alpha)/\mathit{ker}\ D$. So, observe that the action of $\mathit{Aff}(S, \alpha)$ on covers descends to an action of $V(S, \alpha)$ on $\mathrm{Cov}(S, \alpha)$. Namely if $R \in V(S, \alpha)$, then there is a $\phi \in \mathit{Aff}(S, \alpha)$ so that $D(\phi) = R$ and we define

$$R[p, \tilde{S}] = [\phi \circ p, \tilde{S}], \tag{2}$$

which is independent of the choice of ϕ by the normality of $\mathit{ker}\ D \subset \mathit{Aff}(S, \alpha)$.

Definition 5 (Cover cocycle)**.** Let W be an arbitrary discrete subgroup of the Veech group $V(S, \alpha)$, and let $\mathcal{C} \subset \mathrm{Cov}(S, \alpha)$ be W-invariant. We equip the space $SL(2, \mathbb{R}) \times \mathcal{C}$ with a left action by $SL(2,\mathbb{R})$, which acts by left multiplication on the $SL(2,\mathbb{R})$ factor and acts trivially on the right factor. We also equip the space with a right action by W:

$$\big(A,[p,\tilde{S}]\big)R^{-1} = \big(AR^{-1}, R[p,\tilde{S}]\big) \quad \text{for } R \in W.$$

The *cover cocycle* (associated to W and $\mathcal{C}$) is the quotient by this right action,

$$\tilde{\mathcal{O}}_{W,\mathcal{C}}(S,\alpha) := \big(SL(2,\mathbb{R}) \times \mathcal{C}\big)/W, \tag{3}$$

equipped with the left action by $SL(2,\mathbb{R})$.

We will denote by $G^t : \tilde{\mathcal{O}}_{W,\mathcal{C}}(S,\alpha) \to \tilde{\mathcal{O}}_{W,\mathcal{C}}(S,\alpha)$ the action of the diagonal group and refer to it as the *diagonal* cover cocycle. Can't we get away with g^t still? (It is part of an $SL(2,\mathbb{R})$ action and g^t is diagonal subgroup...) If not, maybe G^t? Ok, used G^t instead.

The following clarifies the meaning of the quotient space, $\tilde{\mathcal{O}}_{W,\mathcal{C}}(S,\alpha)$; it's notion of equivalence is coarser than the notion of translation equivalence of $SL(2,\mathbb{R})$ deformations of covers.

Proposition 6. *Let $(p,\tilde{S})$ and $(p',\tilde{S}')$ be coverings of (S,α), and let $(\tilde{S},\tilde{\alpha})$ and $(\tilde{S}',\tilde{\alpha}')$ be the determined translation surfaces. If $A, A' \in SL(2,\mathbb{R})$ and $(A,[p,\tilde{S}])$ and $(A',[p',\tilde{S}'])$ are equivalent up to the W-action (i.e., if they represent the same point in $\tilde{\mathcal{O}}_{W,\mathcal{C}}(S,\alpha)$, then the surfaces $A(\tilde{S},\tilde{\alpha})$ and $A'(\tilde{S}',\tilde{\alpha}')$ are translation equivalent.*

Proof. Suppose $(A,[p,\tilde{S}])$ and $(A',[p',\tilde{S}'])$ are equivalent up to the W-action. Then, there is an $R \in W$ so that $(AR^{-1}, R[p,\tilde{S}]) = (A',[p',\tilde{S}'])$. That is, $AR^{-1} = A'$ and $R[p,\tilde{S}]$ equals $[p',\tilde{S}']$. By the discussion above the definition, this means that $R(\tilde{S},\tilde{\alpha})$ is translation equivalent to $(\tilde{S}',\tilde{\alpha}')$. And since $AR^{-1} = A'$, we know $A(\tilde{S},\tilde{\alpha})$ is translation equivalent to $A'(\tilde{S}',\tilde{\alpha}')$. □

Remark 7 (Passing to subgroups)**.** *I need to edit this comment, and maybe refer to Josh Bowman's work. The Veech group of a translation surface of finite area and infinite topological type can be indiscrete (for some very special surfaces). This is one reason to pass to a subgroup $W \subset V(S,\alpha)$. A second reason, is so that we don't need to understand the full Veech group in order to apply the techniques being introduced.*

4. Finite covers of infinite surfaces

In this section, we work out the theory of spaces of finite degree covers of an translation surface (S,α) of infinite topological type. We describe the topology of the space of covers $\mathrm{Cov}_G(S,\alpha)$, mentioned in the introduction, in subsection 4.1. In subsection 4.2, we place a natural Borel measure on this space. Subsection 4.3 discusses why disconnected covers should be considered rare.

4.1. Spaces of covers.

Let (S,α) be a connected translation surface of infinite topological type. Choose a nonsingular basepoint $s_0 \in S$. The fundamental group $\pi_1(S,s_0)$ is isomorphic to the free group with countably many generators.

We recall idea of the monodromy action from covering space theory. Let $p : \tilde{S} \to S$ be a covering map. The *monodromy action* is the right action on the fiber over the basepoint, $p^{-1}(s_0)$, defined by

$$p^{-1}(s_0) \times \pi_1(S,s_0) \to p^{-1}(s_0); \quad \tilde{s} \cdot \gamma \mapsto \tilde{\alpha}(1),$$

where $\alpha : [0,1] \to S^2$ is a loop in the class of $\gamma \in \pi_1(S, s_0)$, and $\tilde{\alpha} : [0,1] \to \tilde{S}$ is a lift (i.e., $p \circ \tilde{\alpha} = \alpha$) so that $\tilde{\alpha}(0) = \tilde{s}$. It should be noted that the definition of $\tilde{s} \cdot \gamma$ is independent of the choice of α. (Once α is chosen, its lift $\tilde{\alpha}$ is determined based on the condition that $\tilde{\alpha}(0) = \tilde{s}$, and $\tilde{\alpha}(1)$ depends only on γ.)

It is a basic observation from covering space theory that monodromy action determines the cover (up to isomorphism of covers). This includes disconnected covers of the connected surface S. We will briefly how to build a cover from an action. Concretely, given any right action of $\pi_1(S, s_0)$ on a discrete set J, we can build a cover of S with this action as the monodromy action. To see this, fix such an action. For each $j \in J$, let $\mathrm{Stab}(j) \subset \pi_1(S, s_0)$ be the stabilizer of j. We can then build a cover $\tilde{S}_j$ as the quotient of the universal cover of S by $\mathrm{Stab}(j)$. It may be observed that by taking the disjoint union of such surfaces over all orbits $[j]$ under the action, we obtain a cover,

$$\tilde{S} = \bigsqcup_{[j] \subset J} \tilde{S}_j \tag{4}$$

of S with the desired monodromy action. Furthermore, actions on two discrete sets J and J' determine isomorphic covers if and only if the actions are conjugate up to bijection, i.e., if there is a bijection $\beta : J \to J'$ so that $\beta(j \cdot \gamma) = \beta(j) \cdot \gamma$ for all $\gamma \in \pi_1(S, s_0)$ and $j \in J$.

We will now specialize this discussion to finite covers of S. Suppose $p : \tilde{S} \to S$ is a covering map of degree d. Make an arbitrary identification between the fiber $p^{-1}(s_0)$ and the set $\{1, \ldots, d\}$. Let S_d be the symmetric group acting by permutations of $\{1, \ldots, d\}$, and let

$$\ell : \{1, \ldots, d\} \to p^{-1}(s_0)$$

be a labeling (a bijection) to the fiber. The associated *monodromy representation* (which depends on the labeling) is the group homomorphism $M_\ell : \pi_1(S, s_0) \to S_d$ defined so that

$$M_\ell(\gamma)(i) = \ell^{-1}\big(\ell(i) \cdot \gamma^{-1}\big) \quad \text{for all } \gamma \in \pi_1(S, s_0) \text{ and all } i \in \{1, \ldots, d\}.$$

Conversely, such a representation determines an action on $\{1, \ldots, d\}$ and so, from the above discussion, a choice of a monodromy representation $\pi_1(S, s_0) \to S_d$ determines a $d-$fold cover of S. Given two such representations, the covers are isomorphic if and only if they differ by conjugation by an element of S_d, which has the effect of changing the labeling function. Thus, the space of d-fold covers of S up to isomorphism is canonically identified with

$$S_d \backslash \mathrm{Hom}\big(\pi_1(S, s_0), S_d\big), \quad \text{where } S_d \text{ is acting by conjugation.}$$

We endow $\mathrm{Hom}\big(\pi_1(S, s_0), S_d\big)$ with the topology of pointwise convergence on finite sets, and this space of covers gets the quotient topology.

Recall from §3, that $\mathrm{Cov}(S, \alpha)$ denotes the set of covers of (S, α) up to translation equivalence. We need to work with subsets of $\mathrm{Cov}(S, \alpha)$ in order to construct the covers cocycle as in Definition 5. The space of translation equivalence classes of d-fold covers of (S, α) is

$$\mathrm{Cov}_d(S, \alpha) = S_d \backslash \mathrm{Hom}\big(\pi_1(S, s_0), S_d\big) / \mathit{ker}\ D,$$

where $\mathit{ker}\ D \subset \mathit{Aff}(S, \alpha)$ is the group of translation automorphisms of S.

Remark 8 (Finite translation groups)**.** *For translation surfaces of infinite topological type, the translation automorphism group is discrete. (The only simply connected surfaces for which the translation automorphism group is indiscrete are the plane and strips in the plane.*

[2] The letter α is already being hevily used as a holomorphic 1-form. Maybe we can change the standard letter for loops to something else?

So, the only surfaces with indiscrete Veech groups are quotients of these surfaces, which have very simple topological type.) It follows that in the setting of infinite topological type and finite area, the group ker D is finite.

As in the introduction if G is a subgroup of S_d, we say that a cover $\tilde{S}$ has *monodromy in* G if there is a representation $\pi_1(S, s_0) \to G$ which determines a cover isomorphic to $\tilde{S}$. Such covers are thus determined by elements of $\mathrm{Hom}\big(\pi_1(S, s_0), G\big) \subset \mathrm{Hom}\big(\pi_1(S, s_0), S_d\big)$, and we denote the collection of translation equivalence classes of covers with monodromy in G by

$$\mathrm{Cov}_G(S, \alpha) = S_d \backslash \mathrm{Hom}\big(\pi_1(S, s_0), G\big)/\mathit{ker}\ D.$$

It should be noted that $\mathrm{Cov}_G(S, \alpha)$ is a closed subset of $\mathrm{Cov}_d(S, \alpha)$.

4.2. **Measures on spaces of covers.** In this subsection, we will construct some natural measures on our spaces of covers. We begin by describing some abstract constructions. Later in the subsection, we will specialize the discussion to our setting of translation surfaces.

Let Γ be the non-abelian free group with a countable generating set $\{\gamma_i : i \in \mathbb{N}\}$, and let $G \subset S_d$ as above. We endow the space $\mathrm{Hom}(\Gamma, G)$ with its natural product topology, which makes the $\mathrm{Hom}(\Gamma, G)$ homeomorphic to a Cantor set. This is the coarsest topology so that for each $\eta \in \Gamma$ and each $\sigma \in G$, the set of the form

$$\{h : \mathrm{Hom}(\Gamma, G) : h(\eta) = \sigma\}.$$

is open. A pair of collections, $e_1, \ldots, e_k \in \mathbb{N}$ and $\sigma_1, \ldots, \sigma_k \in G$, determine a *cylinder set* in $\mathrm{Hom}(\Gamma, G)$,

$$\mathcal{C}(e_1, \ldots, e_k; \sigma_1, \ldots, \sigma_k) = \{h : \mathrm{Hom}(\Gamma, G) : h(\gamma_{e_i}) = \sigma_i \text{ for } i = 1, \ldots, k\}. \tag{5}$$

Each cylinder set is both closed and open in the product topology, and the collection of cylinder sets generate the topology.

To characterize a Borel measure on $\mathrm{Hom}(\Gamma, G)$, it suffices to describe the measures of the cylinder sets.

Definition 9. The *product measure* μ on $\mathrm{Hom}(\Gamma, G)$ is defined so that for every cylinder set we have

$$\mu\big(\mathcal{C}(e_1, \ldots, e_k; \sigma_1, \ldots, \sigma_k)\big) = \frac{1}{|G|^k}.$$

This is the product measure induced on $\mathrm{Hom}(\Gamma, G)$ by the counting measure on G.

We remark that this measure μ is interesting even in the case when Γ is a finitely generated free group, and related questions remain open [Pud13].

Automorphisms of Γ act on $\mathrm{Hom}(\Gamma, G)$. Concretely, if $\phi : \Gamma \to \Gamma$ is an automorphism, then we can define

$$\phi^* : \mathrm{Hom}(\Gamma, G) \to \mathrm{Hom}(\Gamma, G); \quad h \mapsto h \circ \phi^{-1}. \tag{6}$$

Proposition 10. *The action of any automorphism of* Γ *preserves the product measure* μ *on* $\mathrm{Hom}(\Gamma, G)$. *In particular, the measure* μ *is independent of our choice of generating set.*

Remark 11. *In the case when* G *is abelian,* $\mathrm{Hom}(\Gamma, G)$ *can be identified with the topological group* $G^{\mathbb{N}}$, *and* μ *is Haar measure. In this case, the proposition follows from the naturality of Haar measure.*

Proof. Let $\phi : \Gamma \to \Gamma$ be an automorphism, and let ϕ^* be its action on $\mathrm{Hom}(\Gamma, G)$:

$$\phi^* : \mathrm{Hom}(\Gamma, G) \to \mathrm{Hom}(\Gamma, G); \quad h \mapsto h \circ \phi^{-1}.$$

We will prove that ϕ^* preserves the product measure μ on $\mathrm{Hom}(\Gamma, G)$. It suffices to prove that the measures of cylinder sets are preserved. Let $\mathcal{C} = \mathcal{C}(e_1, \ldots, e_k; \sigma_1, \ldots, \sigma_k)$ be a cylinder set. We will prove that $\mu \circ \phi^*(\mathcal{C}) = 1/|G|^k$.

Let $X = \langle \gamma_{e_1}, \ldots, \gamma_{e_k} \rangle \subset \Gamma$. Since $\Gamma = \langle \gamma_e \ : \ e \in \mathbb{N} \rangle$, there is a finite set $\{e'_1, \ldots, e'_m\} \subset \mathbb{N}$ so that

$$\phi^{-1}(X) \subset \langle \gamma_{e'_1}, \ldots, \gamma_{e'_m} \rangle.$$

We'll call the subgroup on the right hand side of the equation Y. By viewing μ as the product of counting measures, we see

$$\mu \circ \phi^{-1}(\mathcal{C}) = \frac{\#\{h \in \mathrm{Hom}(Y, G) \ : \ h \circ \phi^{-1}(\gamma_{e_i}) = \sigma_i \quad \text{for } 1 \le i \le k\}}{|G|^m}. \tag{7}$$

So it suffices to show that the number of homomorphisms in the numerator is $|G|^{m-k}$.

As above, we can find a finite set $\{e''_1, \ldots, e''_n\} \subset \mathbb{N}$ so that

$$\phi(Y) \subset \langle \gamma_{e''_1}, \ldots, \gamma_{e''_n} \rangle.$$

Call the set on the right hand side Z. Note that $X \subset Z$.

We recall some basic definitions from the theory of free groups. A *basis* of a free group F is a set $x_1, \ldots, x_k$ so that $F = x_1 * \ldots * x_k$. A subgroup H of a free group F is a *free factor* if every (equivalently, some) basis of H can be extended to a basis of F.

Consider X, Y, and Z as above. Observe that X is a free factor in Z. So, X is a free factor in $\phi(Y)$ [Pud13, Claim 2.5]. That is, we can extend $\{\gamma_{e_1}, \ldots, \gamma_{e_k}\}$ to a free generating set of $\phi(Y)$. Using ϕ^{-1}, we can pull this back to a generating set of Y. So, we have $k \le n$, and there is a free generating set of Y given by $\alpha_1, \ldots, \alpha_m$ so that

$$\alpha_i = \phi^{-1}(\gamma_{e_i}) \quad \text{for } 1 \le i \le k.$$

Since this set generates Y, we see that $\mathrm{Hom}(Y, G)$ is in bijective correspondence with the possible images of $\{\alpha_1, \ldots, \alpha_m\}$. The last $m - k$ elements in this basis are irrelevant to the values of $\phi^{-1}(\gamma_{e_i})$, so we see that there are exactly $|G|^{m-k}$ possible values which give homomorphisms in the numerator of equation 7. $\square$

We will use the product measure constructed above to produce a natural measure on $\mathrm{Cov}_G(S, \alpha)$. We will do this in steps.

First, we will produce a measure on the space of isomorphism classes of covers with monodromy in G, $S_d \backslash \mathrm{Hom}\big(\pi_1(S, s_0), G\big)$. Note that $\pi_1(S, s_0)$ is homeomorphic to Γ, the countably generated free group. For $G \subset S_d$, let $\mathcal{G}$ denote the collection of all subgroups of S_d which are conjugate to G. For each $G' \in \mathcal{G}$, we have an associated product measure $\mu_{G'}$ as in Definition 9. The measure $\mu_{G'}$ is defined on $\mathrm{Hom}\big(\pi_1(S, s_0), G'\big) \subset \mathrm{Hom}\big(\pi_1(S, s_0), S_d\big)$. We define the measure

$$\nu_G = \frac{1}{|\mathcal{G}|} \sum_{G' \in \mathcal{G}} \mu_{G'}. \tag{8}$$

We have the following corollary to the proposition above.

Corollary 12. *The measure ν_G on $\mathrm{Hom}\big(\pi_1(S, s_0), S_d\big)$ is invariant under the action of S_d by conjugation and under automorphisms of $\pi_1(S, s_0)$.*

Because ν_G is invariant under the S_d action, it descends to a measure on $S_d \backslash \mathrm{Hom}\big(\pi_1(S,s_0), S_d\big)$ which is supported on $S_d \backslash \mathrm{Hom}\big(\pi_1(S,s_0), G\big)$, the space of isomorphism classes of covers with monodromy in G.

Let Aut denote the automorphism group of $\pi_1(S,s_0)$ and let Inn denote normal subgroup of *inner automorphisms*, which consists of those automorphisms which act by conjugation. The *outer automorphism group* of $\pi_1(S,s_0)$ is $Out = Aut/Inn$. As noted above, Aut acts on $\mathrm{Hom}(\Gamma, G)$ by precomposition. This action descends to an action of the outer automorphism group on $S_d \backslash \mathrm{Hom}\big(\pi_1(S,s_0), G\big)$. By the corollary above, the measure ν_G is invariant under the action of S_d, and so induces a measure ν'_G on the quotient space $S_d \backslash \mathrm{Hom}\big(\pi_1(S,s_0), S_d\big)$ which is supported on $S_d \backslash \mathrm{Hom}\big(\pi_1(S,s_0), G\big)$. We have:

Corollary 13. *The measure ν'_G on $S_d \backslash \mathrm{Hom}\big(\pi_1(S,s_0), G\big)$ is invariant under the action of the outer automorphism group of $\pi_1(S,s_0)$.*

Let $\phi : S \to S$ be a homeomorphism (such as an affine automorphism). We will explain how ϕ acts on the space of isomorphism classes of covers with monodromy in G, $S_d \backslash \mathrm{Hom}\big(\pi_1(S,s_0), G\big)$, and observe how it factors through the outer automorphism group of $\pi_1(S,s_0)$. Most naturally, a homeomorphism ϕ induces an action on paths, which in turn induces an isomorphism between the fundamental group of S with different basepoints. To get an automorphism of $\pi_1(S,s_0)$ to itself, we need to make a choice of a curve β joining s_0 to $\phi(s_0)$. With this choice, we define

$$\phi_\beta : \pi_1(S,s_0) \to \pi_1(S,s_0); \quad [\alpha] \mapsto \big[\beta \bullet (\phi \circ \alpha) \bullet \beta^{-1}\big], \tag{9}$$

where $\alpha : [0,1] \to S$ is a loop with $\alpha(0) = \alpha(1) = s_0$ and $\bullet$ denotes path concatenation. This action depends on β, but the outer automorphism class of ϕ_β is independent of the choice of β. Since outer automorphisms act on $S_d \backslash \mathrm{Hom}\big(\pi_1(S,s_0), G\big)$, we get a well defined action of ϕ on this space. Furthermore by Corollary 13, we see that ν'_G is invariant under the action of ϕ.

Our space $\mathrm{Cov}_G(S,\alpha)$ is the quotient of $S_d \backslash \mathrm{Hom}\big(\pi_1(S,s_0), G\big)$ by the action of $ker\ D \subset Aff(S,\alpha)$. The elements of $ker\ D$ act as homeomorphisms of S, and so leave invariant the measure ν'_G. We let m_G denote the induced measure on the quotient space $\mathrm{Cov}_G(S,\alpha)$. We call m_G the *canonical measure* on $\mathrm{Cov}_G(S,\alpha)$. Because the subgroup $ker\ D$ is normal inside the affine automorphism group, and the Veech group can be identified with $Aff(S,\alpha)/kerD$, we observe:

Theorem 14. *The measure m_G on $Cov_G(S,\alpha)$ is invariant under the action of the Veech group.*

The below is fixed. Let me know if you have comments...

Let W be a discrete subgroup of $V(S,\alpha)$. Recall from the introduction that we defined $\mathcal{O}(S,\alpha) = SL(2,\mathbb{R})/W$. This parameterizes surfaces affinely equivalent to (S,α) (with redundancy if $W \neq V(S,\alpha)$). As in equation (1), covers of these surfaces with monodromy in G are parameterized by the space:

$$\tilde{\mathcal{O}}_G(S,\alpha) = \big(SL(2,\mathbb{R}) \times \mathrm{Cov}_G(S,\alpha)\big)/W. \tag{10}$$

We note that a measure μ on $\mathcal{O}(S,\alpha) = SL(2,\mathbb{R})/W$ descends from a measure $\tilde{\mu}$ to $SL(2,\mathbb{R})$ which is invariant under the right-action of W. By the theorem above, we see:

Corollary 15 (Lifting measures)**.** *Let μ be a Borel probability measure on $\mathcal{O}(S,\alpha)$ which is invariant under the subgroup $H \subset SL(2,\mathbb{R})$. Then, the measure on $\tilde{\mathcal{O}}_G(S,\alpha)$ defined by $\tilde{\mu}_G = (\tilde{\mu} \times m_G)/W$ is an H-invariant Borel probability measure supported on $p^{-1}\big(\mathrm{supp}(\mu)\big)$ and satisfying $p_*(\tilde{\mu}_G) = \mu$, where $p : \tilde{\mathcal{O}}_G(S,\alpha) \to \mathcal{O}(S,\alpha)$ is the natural projection.*

4.3. **Disconnected covers.** Let $G \subset S_d$ and let $h \in \mathrm{Hom}\big(\pi_1(S,s_0), G\big)$. By interpreting h as the monodromy action of the fundamental group of S on the fibers of the basepoint, we obtain a cover $\tilde{S}$ of S as in §4.1. This cover is explicitly described by equation (4), and we can see the following:

Proposition 16. *The cover associated to $h \in \mathrm{Hom}\big(\pi_1(S,s_0), G\big)$ is connected if and only if the image $h\big(\pi_1(S,s_0)\big)$ acts transitively on $\{1,2,\dots,d\}$.*

In particular, in order to have connected covers of (S,α) with monodromy in G, the subgroup $G \subset S_d$ must act transitively on $\{1,2,\dots,d\}$. The goal of this subsection is to formulate the following precise version of the statement that the collection of all disconnected covers is small both topologically and measure theoretically.

Theorem 17. *Let $G \subset S_d$ be a subgroup which acts transitively on $\{1,2,\dots,d\}$. Then, the set of disconnected surfaces in $Cov_G(S,\alpha)$ is closed and nowhere dense in $Cov_G(S,\alpha)$. Moreover, there is a sequence U_n of open subsets of $Cov_G(S,\alpha)$, each of which contains all disconnected surfaces in $Cov_G(S,\alpha)$, so that $\lim_{n\to\infty} m_G(U_n) = 0$.*

Proof. Let $\mathcal{H}$ denote the collection of all subgroups of $H \subset S_d$ so that H does not act transitively on $\{1,\dots,d\}$, but so that H is conjugate to a subgroup of G. Note that $\mathcal{H}$ is a finite set. We will use this to show that the set of $\mathcal{D} \subset \mathrm{Cov}_G(S,\alpha)$ of disconnected surfaces is closed. We think of $\mathcal{D}$ as a subset of $\mathrm{Cov}_d(S,\alpha)$, which is a finite quotient of $\mathrm{Hom}\big(\pi_1(S,s_0), S_d\big)$. Let $\tilde{\mathcal{D}} \subset \mathrm{Hom}\big(\pi_1(S,s_0), S_d\big)$ be the lift of $\mathcal{D}$. Observe that

$$\tilde{\mathcal{D}} = \bigcup_{H\in\mathcal{H}} \mathrm{Hom}\big(\pi_1(S,s_0), H\big).$$

It suffices to show that $\tilde{\mathcal{D}}$ is closed. To see this, let $\langle h_n \in \tilde{\mathcal{D}}\rangle$ be a sequence which converges to $h \in \mathrm{Hom}\big(\pi_1(S,s_0), S_d\big)$. Since $\mathrm{Hom}\big(\pi_1(S,s_0), S_d\big)$ is endowed with the topology of pointwise convergence, the images satisfy

$$h\big(\pi_1(S,s_0)\big) \subset \liminf_{n\to\infty} h_n\big(\pi_1(S,s_0)\big).$$

Indeed, since every image $h_n\big(\pi_1(S,s_0)\big)$ lies in $\mathcal{H}$, one subgroup must appear infinitely often, and $h\big(\pi_1(S,s_0)\big)$ must lie in that subgroup. Thus, $h \in \tilde{\mathcal{D}}$ and $\mathcal{D}$ is closed.

Now we will show that $\mathrm{Cov}_G(S,\alpha)$ is nowhere dense. Let $U \subset \mathrm{Cov}_G(S,\alpha)$ be open. It lifts to an open subset $\tilde{U} \subset \mathrm{Hom}\big(\pi_1(S,s_0), G\big)$. Since the cylinder sets form a basis for the topology, we can find a cylinder set $\mathcal{C} = \mathcal{C}(e_1,\dots,e_k;\sigma_1,\dots,\sigma_k)$ inside of $\tilde{U}$. Choose an arbitrary additional collection $e_{k+1},\dots,e_{k+|G|}$ of natural numbers, and enumerate G by $\sigma_{k+1},\dots,\sigma_{k+|G|}$. Then

$$\mathcal{C}(e_1,\dots,e_{k+|G|};\sigma_1,\dots,\sigma_{k+|G|}) \subset \mathcal{C},$$

and any homomorphism $h \in \mathcal{C}(e_1,\dots,e_{k+|G|};\sigma_1,\dots,\sigma_{k+|G|})$ has image equal to G. So, by the proposition above, the cover associated to each h in this new cylinder set is connected. This h also represents a connected cover in U.

Similar ideas will be used to produce the sets U_n. Note that $\text{Cov}_G(S,\alpha) \subset \text{Cov}_d(S,\alpha)$. We think of the measure m_G as defined on $\text{Cov}_d(S,\alpha)$ and supported on $\text{Cov}_G(S,\alpha)$. We will produce open subsets $V_n \subset \text{Cov}_d(S,\alpha)$ containing all disconnected surfaces in $\text{Cov}_G(S,\alpha)$ so that $m_G(V_n) \to 0$ as $n \to \infty$. The sets $U_n = V_n \cap \text{Cov}_G(S,\alpha)$ will be the sets used in the statement of the theorem.

For each $H \in \mathcal{H}$, we will find open subsets $\tilde{V}_{n,H} \subset \text{Hom}\big(\pi_1(S,s_0), S_d\big)$ containing the subspace $\text{Hom}\big(\pi_1(S,s_0), H\big)$ so that $\mu_{G'}(\tilde{V}_{n,H}) \to 0$ for any $G' \subset S_d$ conjugate to G. Here, $\mu_{G'}$ is the product measure on $\text{Hom}\big(\pi_1(S,s_0), G'\big) \subset \text{Hom}\big(\pi_1(S,s_0), S_d\big)$ defined in Definition 9. Fix $H \in \mathcal{H}$. We define

$$\tilde{V}_{n,H} = \bigcup_{(h_1,\ldots,h_n)\in H^n} \mathcal{C}(1,\ldots,n;h_1,\ldots,h_n) \subset \text{Hom}\big(\pi_1(S,s_0), S_d\big).$$

(The cylinder set $\mathcal{C}(1,\ldots,n;h_1,\ldots,h_n)$ specifies the images of the first n generators.) We observe that $\mu_{G'}(\tilde{V}_{n,H}) \leq \frac{|H|^n}{|G'|^n}$. (This is equality if $H \subset G'$ and is smaller when $H \not\subset G'$.) This sequence tends to zero, because H is a proper subgroup of some conjugate of G: this conjugate of G acts transitively on $\{1,\ldots,d\}$, while H does not. Now recall that the measure ν_G was constructed by averaging over the conjugates of G as in equation (8). Since $\mu_{G'}(\tilde{V}_{n,H}) \to 0$ as $n \to \infty$ for every G', we see that $\nu_G(\tilde{V}_{n,H}) \to 0$ as well. Define $\tilde{V}_n = \bigcup_{H\in\mathcal{H}} \tilde{V}_{n,H}$. By the finiteness of $\mathcal{H}$, $\tilde{V}_n$ is open and $\nu_G(\tilde{V}_n) \to 0$. Also observe that $\tilde{\mathcal{D}} \subset \tilde{V}_n$. Letting $V_n \subset \text{Cov}_d(S,\alpha)$ be the quotient of $\tilde{V}_n$ completes the argument from the prior paragraph. We see that V_n is open, contains $\mathcal{D}$, and $m_G(V_n) \to 0$. □

5. Unique Ergodicity

In this section we will review the ergodicity criterion in [Tre14] which we will use to show that almost every Bernoulli cover has a uniquely ergodic translation flow in the direction associated to a Teichmüller recurrent surface. Let us recall the constructions from sections 3 and 4 in order to use them here.

Let (S,α) be a flat surface, G a subgroup of S_d, and W a discrete subgroup of $V(S,\alpha)$. Recall the diagonal cover cocycle $G^t : \tilde{\mathcal{O}}_{G,W}(S,\alpha) \to \tilde{\mathcal{O}}_{G,W}(S,\alpha)$ where $\tilde{\mathcal{O}}_{G,W}(S,\alpha)$ is the cover bundle defined in (10). Let us explain how we interpret the action of the diagonal cover cocycle, which will be relevant to the proofs in this section.

Choose a fundamental domain for the W-action, $F \subset SL(2,\mathbb{R})$. We make this choice so that F contains the identity element. Then by considering the g^t action on (S,α) for $t > 0$, we get a (finite or infinite) sequence of Veech group elements $\{R_i \in W\}$ as follows. Define $t_0 = 0$ and $R_0 = \text{Id}$. Recursively, let

$$t_{i+1} = \inf\{t > t_i : g_t R_0 R_1 \cdots R_i \notin F\}$$

if it exists and define $R_{i+1} \in W$ to be the element so that $g_{t_{i+1}+\varepsilon} R_1 \cdots R_{i+1} \in F(S,\alpha)$ for all $\varepsilon \geq 0$ small enough. If $t_{i+1} = \infty$, then R_i will be the final element of our sequence, and if each $t_i < \infty$, the sequence $\{R_i\}$ is infinite and is responsible for bringing back the orbit to the fundamental domain of the identity of the action of W. As such, there is a sequence $\{\phi_1, \phi_2, \ldots\}$ of orientation preserving affine homeomorphism of (S,α) with the property that $D\phi_i = R_i$.

For a flat surface (S,α) denote by $dist_t(x,y)$ the distance between $x, y \in S$ with respect to the flat metric on $(S,\alpha_t) = g_t(S,\alpha)$, the one-parameter family of flat surfaces obtained from (S,α) by Teichmüller deformation. Note that the metric extends to the metric completion

$\overline{(S,\alpha)}$ of (S,α). Denote by $\Sigma \subset \overline{(S,\alpha)}$ the set of singularities of (S,α). That is, Σ contains the zero set Z of α as well as points in the metric completion $\overline{(S,\alpha)}$.

Theorem 18 ([Tre14]). *Let (S,α) be a flat surface of finite area. Suppose that for any $\eta > 0$ there exist a function $t \mapsto \varepsilon(t) > 0$, a one-parameter family of subsets*

$$S_{\varepsilon(t),t} = \bigsqcup_{i=1}^{C_t} S_t^i$$

of S made up of $C_t < \infty$ path-connected components, each homeomorphic to a closed orientable surface with boundary, and functions $t \mapsto \mathcal{D}_t^i > 0$, for $1 \leq i \leq C_t$, such that for

$$\Gamma_t^{i,j} = \{\textit{paths connecting } \partial S_t^i \textit{ to } \partial S_t^j\}$$

and

$$\delta_t = \min_{i \neq j} \sup_{\gamma \in \Gamma_t^{i,j}} dist_t(\gamma, \Sigma) \tag{11}$$

the following hold:

(1) $\mathrm{Area}(S \backslash S_{\varepsilon(t),t}) < \eta$ *for all* $t > 0$,
(2) $dist_t(\partial S_{\varepsilon(t),t}, \Sigma) > \varepsilon(t)$ *for all* $t > 0$,
(3) *the diameter of each S_t^i, measured with respect to the flat metric on (S,α_t), is bounded by $\mathcal{D}_t^i$ and*

$$\int_0^\infty \left(\varepsilon(t)^{-2} \sum_{i=1}^{C_t} \mathcal{D}_t^i + \frac{C_t - 1}{\delta_t} \right)^{-2} dt = +\infty. \tag{12}$$

Moreover, suppose the set of points whose translation trajectories leave every compact subset of S has zero measure. Then the translation flow is ergodic.

The theorem above is a geometric criterion for ergodicity. The spirit of the theorem is that if, as ones deforms a flat surface (S,α) using the Teichmüller deformation g_t, the geometry of the surface does not deteriorate too quickly (as measured by the diameter of big components, among other things), the translation flow is ergodic.

Theorem 18 will be used to prove the following.

Theorem 19. *Let (S,α) be a flat surface of finite area and (S_{h_0}, α_{h_0}) a connected cover, for some $h_0 \in \mathrm{Cov}_G(S,\alpha)$. Suppose the orbit of $G^t((\mathrm{Id}, h_0))$ contains an accumulation point with a connected fiber, i.e., there is a subsequence $t_k \to \infty$ such that $G^{t_k}((\mathrm{Id}, h_0)) \to (S, h^*)$ with $S \in Q(S,\alpha)$ and (S_{h^*}, α_{h^*}) a connected surface. Then the translation flow on (S_{h_0}, α_{h_0}) is uniquely ergodic.*

Suppose $g_s \in V(S,\alpha)$ for some $s \neq 0$, i.e., there is a pseudo-Anosov map $\phi : S \to S$ which preserves the vertical and horizontal foliations. Then the cocycle G^t consists of a periodic orbit on the first component of $\mathcal{Q}(S,\alpha,\Gamma,G)$ and the measure-preserving (Proposition 10) induced action of ϕ on $\mathrm{Cov}_G(S,\alpha)$ through (6). Since almost every $h \in \mathrm{Cov}_G(S,\alpha)$ is connected, by the Poincaré recurrence theorem and these observations, we obtain the following result.

Corollary 20. *Suppose $g_s \in V(S,\alpha)$ for some $s \neq 0$. Then for almost every $h \in \mathrm{Hom}(\Gamma, G)$ the horizontal and vertical flows on (S_h, α_h) are uniquely ergodic. Can we get that for every h in a dense G_δ set, the result holds too?*

Let us first review the strategy of the proof of Theorem 19. The Teichmüller orbit of (S,α) in $Q(S,\alpha)$ corresponds to the Teichmüller deformation of the surface $g_t(S,\alpha)$. By assumption, it has a converging subsequence, i.e., there is a sequence of times $t_k \to \infty$ such that $g_{t_k}(S,\alpha) \to (S,\alpha^*)$ in the sense of conformal structures. More precisely, for any $\varepsilon > 0$ there exists a $T > 0$, $\theta \in S^1$, and an orientation-preserving affine homeomorphism ϕ with $D\phi \in V(S,\alpha)$ such that $\phi(g_T(S,\alpha)) = g_s r_\theta(S,\alpha^*)$ with $|s| < \varepsilon$.

Through any subsequence $t_k \to \infty$ as above we can control the deforming geometry of the surface $g_t(S,\alpha)$ along these times. Since these times correspond to applications of renormalizing maps ϕ_i (with $D\phi_i \in V(S,\alpha)$), and these maps act on h by (6), we can control the deforming geometry of the cover (S_h,α_h) by considering its convergence along a subsequence to h^*. Since the convergence of the conformal structure for $g_{t_k}(S,\alpha)$ controls the deforming geometry of $g_t(S,\alpha)$ and there is a convergence on the fibers $h_i \to h^*$, we can control the deforming geometry of the cover along a subsequence. Since the limiting surface (S_{h^*},α_{h^*}) is connected, we can apply Theorem 18 to show that the horizontal flow on (S_h,α_h) is ergodic. The upgrade to unique ergodicity is done through a standard trick. We start with ergodicity.

Proposition 21. *Let (S,α) be a flat surface of finite area and (S_h,α_h) a connected cover, for some $h \in \mathrm{Hom}(\Gamma, G)$. Suppose the orbit of $G^t((\mathrm{Id},h))$ contains an accumulation point with a connected fiber, i.e., there is a subsequence $t_k \to \infty$ such that $G^{t_k}((\mathrm{Id},h)) \to (S,h^*)$ with $S \in Q(S,\alpha)$ and (S_{h^*},α_{h^*}) a connected surface. Then the translation flow on (S_h,α_h) is ergodic.*

Proof. Denote by $(S,\alpha_t) = g_t(S,\alpha)$. By construction, we have that

$$(S_h,\alpha_h^t) := g_t(S_h,\alpha_h) = (S_h, p_h^*\alpha_t). \tag{13}$$

Since the Teichmüller orbit of (S,α) has an accumulation point, for any $\varepsilon > 0$ there exists a $T > 0$, $s \in (-\varepsilon,\varepsilon)$, $\theta \in S^1$, and an orientation-preserving affine homeomorphism ϕ with $D\phi \in V(S,\alpha)$ such that $\phi(g_T(S,\alpha)) = g_s r_\theta(S,\alpha^*)$ with $|s| < \varepsilon$. Let $\varepsilon_k \to 0$ be a decreasing sequence and $t_k \to \infty$ a sequence such that

$$\phi_k(g_{t_k}(S,\alpha)) = g_{s_k} r_{\theta_k}(S,\alpha^*) \tag{14}$$

for some $|s_k| \leq \varepsilon_k$, $\theta_k \in S^1$, and ϕ_k an orientation preserving affine diffeomorphism. Without loss of generality we can assume that this same sequence t_k has the property $(S_k,h_k) \equiv G^{t_k}((\mathrm{Id},h)) \to (S,h^*)$ with (S_{h^*},α_{h^*}) connected and (S_{h_k},α_{h_k}) connected for all $k > 0$.

Define the sets

$$K_{\varepsilon,t} = \{z \in (S,\alpha_t) : \mathrm{dist}(z,\Sigma) \geq \varepsilon\} \qquad \text{and} \qquad K^{\hat h}_{\varepsilon,t} = \{z \in (S_{\hat h},\alpha^t_{\hat h}) : \mathrm{dist}(z,\Sigma) \geq \varepsilon\} \tag{15}$$

for any element $\hat h \in \mathrm{Cov}_G(S,\alpha)$. By (13) we have that $p^{-1}_{\hat h}(K_{\varepsilon,t}) = K^{\hat h}_{\varepsilon,t}$ for any $\hat h \in \mathrm{Cov}_G(S,\alpha)$. These are two families of exhaustive compact sets: for any $t > 0$, for any $\eta > 0$, there exists an $\varepsilon_0 > 0$ such that

$$\frac{\mathrm{Area}(S \backslash K_{\varepsilon,t})}{\mathrm{Area}(S,\alpha)} = \frac{\mathrm{Area}(S_{\hat h} \backslash K^{\hat h}_{\varepsilon,t})}{\mathrm{Area}(S_{\hat h},\alpha_{\hat h})} < \eta \tag{16}$$

for positive $\varepsilon < \varepsilon_0$ and any $\hat h \in \mathrm{Cov}_G(S,\alpha)$.

We will assume, without loss of generality, that the number of components of $K_{\varepsilon,t}$ is always finite. Indeed, for any $t > 0$ and any $\eta > 0$, suppose that (16) holds for $K_{\varepsilon_0,t}$ for some $\varepsilon_0 > 0$

but the number of components of $K_{\varepsilon,t}$ is infinite for any $\varepsilon < \varepsilon_0$. Since the collection $K_{\varepsilon,t}$ is exhaustive and, more importanty, the area of (S,α) is finite, there exists an $\varepsilon_* \leq \varepsilon_0$ such that there is an N^* with

$$K_{\varepsilon_*,t} = \bigsqcup_{i>0} K^i_{\varepsilon_*,t} \qquad \text{and} \qquad \sum_{i=1}^N \frac{\mathrm{Area}(K^i_{\varepsilon,t})}{\mathrm{Area}(S,\alpha)} > 1-\eta \tag{17}$$

for any $N > N^*$. Since we will use these sets for the application of Theorem 18 and we can satisfy (1) in that theorem by (17) using sets with finitely many components, we will always assume sets of the form (15) have finitely many components.

Let $\eta > 0$ be arbitrary and $t_k \to \infty$ the sequence of times such that $((S,\alpha_{t_k}),h_k) = G^{t_k}(\mathrm{Id},h) \to ((S,\alpha^*),h^*)$ with (S_{h^*},α_{h^*}) connected. Choose ε^* small enough so that

$$\Upsilon = \{z \in (S,\alpha^*) : \mathrm{dist}(z,\Sigma) \geq \varepsilon^*\}$$

satisfies (16) and let

$$\Upsilon_k = \{z \in (S,\alpha_{t_k}) : \mathrm{dist}(\phi_k^{-1}(z),\Sigma) \geq \varepsilon^*\}$$

which is a sequence of deformations of Υ which converge to it, where ϕ_k satisfies (14). Denote by C^* the number of components of Υ.

We now choose the appropriate quantities for Theorem 18 along the sequence of times t_k for the surface (S_h,α_h). Let

- $\varepsilon(t_k) = e^{-\varepsilon_1}\varepsilon^*$ for all k;
- $S_{\varepsilon(t_k),t_k} = p_{h^*}^{-1}(\Upsilon_k)$, for k. Note that the number of components $C_{t_k} = C^*$ is taken to be the same for all k and finite by (17). Denote by $\Upsilon_k^j, 1 \leq j \leq C^*$, the connected components of Υ_k;
- $\mathcal{D}^i_{t_k} = \max_{1\leq j\leq C^*}\mathrm{diam}(p_h^{-1}(\Upsilon_k^j)$ for all i. Note that as such we have that $\mathcal{D}^i_{t_k} \leq ne^{\varepsilon_1}\max_j \mathrm{diam}(\Upsilon^j_k) = ne^{\varepsilon_1}\mathcal{D}^*$ for all i;
- $\delta_{t_k} = n^{-1}e^{-\varepsilon_1}\delta_*$ for all k, where δ_* is the analogous quantity for $p_{h^*}^{-1}(\Upsilon)$.

Let $\mu > 0$ be a real number satisfying $2\mu \ll \inf_k(t_k - t_{k-1})$. Using Teichmüller deformations, we can deform the sets $S_{\varepsilon(t_k),t_k}$ in intervals of width $\mu/2$ around t_k and thus, for $t \in (t_k - \frac{\mu}{2}, t_k - \frac{\mu}{2})$, obtain the following bounds

$$\mathcal{D}^i_t \leq e^{\frac{\mu}{2}}\mathcal{D}^i_{t_k} \text{ for all } i, \qquad \varepsilon(t) \geq e^{-\frac{\mu}{2}}\varepsilon(t_k), \qquad \delta_t \geq e^{-\frac{\mu}{2}}\delta_{t_k}. \tag{18}$$

Using these bounds, for any other choice of sets $S_{\varepsilon(t),t}$ and functions $\varepsilon(t)$, $\mathcal{D}^i_t$ for $t \in (\mathbb{R}\setminus\bigcup_k(t_k - \frac{\mu}{2}, t_k + \frac{\mu}{2}))$ we have that

$$\begin{aligned}\int_0^\infty \left(\varepsilon(t)^{-2}\sum_{i=1}^{C_t}\mathcal{D}^i_t + \frac{C_t - 1}{\delta_t}\right)^{-2} dt &\geq \sum_{k>0}\int_{t_k-\frac{\mu}{2}}^{t_k+\frac{\mu}{2}} \left(\varepsilon(t)^{-2}\sum_{i=1}^{C_t}\mathcal{D}^i_t + \frac{C_t-1}{\delta_t}\right)^{-2} dt \\ &\geq \frac{\mu}{n^2}\sum_{k>0}\left(e^{\frac{3\mu}{2}+3\varepsilon_1}\varepsilon^*\mathcal{D}^* + e^{\frac{\mu}{2}+\varepsilon_1}\frac{C^*-1}{\delta_*}\right)^{-2} = \infty.\end{aligned} \tag{19}$$

Since the surface (S_h,α_h) is made up of countably many polygons, the set of trajectories which is not defined for all time is of negligible measure, so Theorem 18 applies and the translation flow on (S_h,α_h) is ergodic with respect to the Lebesgue measure ω_h. □

Proof of Theorem 19. By Proposition 21, the translation flow is ergodic with respect to Lebesgue measure $\omega_h = \Re(\alpha_h)\wedge\Im(\alpha_h)$. Supposing there is another finite ergodic invariant measure ν, we can consider the convex combination $\mu_s = s\omega_h + (1-s)\nu$ for some

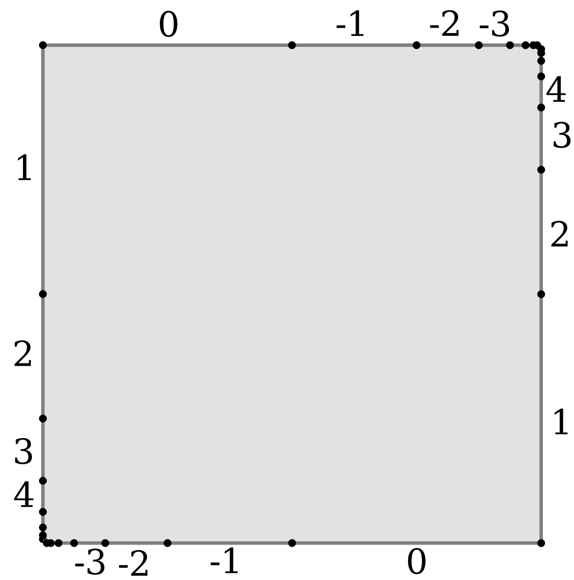

FIGURE 1. Chamanara's surface.

$s \in (0,1)$. There is a translation structure on S_h for which μ_s is in the Lebesgue class. It can be shown[3] that the choice of sets $S_{\varepsilon(t),t}$ which made (12) diverge for ω_h also work to show that the translation flow is ergodic with respect to μ_s. But this contradicts that μ_s is a convex combination of finite, ergodic invariant measures. Therefore, up to scalar multiples, ω_h is the only finite invariant measure. □

6. Examples of evil covers

6.1. **Chamanara's surface.** We introduce a surface first studied by Chamanara in [Cha04]. (See also the related work [CGL06].) The surface is built from a closed 1×1 square with each of the edges subdivided into intervals of length $\frac{1}{2^k}$ for $k \in \mathbb{N}$ as indicated in Figure 1. The vertical intervals of equal length are then glued together by translation, and we do the same to the horizontal intervals. The intervals being identified have been labeled by the integers in the figure. The endpoints of these intervals being glued and the corners of the square are discarded to give the space a translation structure.

For this section, let (S,α) denote Chamanara's surface as shown in the figure. We note that the surface has an affine automorphism ϕ whose derivative is

$$D(\phi) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix} \in V(S,\alpha).$$

Thus, in particular $g^t(S,\alpha)$ is a closed geodesic in the $SL(2,\mathbb{R})$-orbit $\mathcal{O}(S,\alpha)$, which is naturally identified with $SL(2,\mathbb{R})/V(S,\alpha)$ as in the introduction. Corollary 20 then implies that if $d \geq 2$ is an integer and $G \subset S_d$ acts transitively on $\{1,\ldots,d\}$, then almost every G cover of (S,α) has ergodic translation flow (horizontal straight-line flow).

Nonetheless, there are connected covers which are connected and whose translation flow is non-ergodic. Our goal with this subsection is to give a result which can conclude that many connected covers are non-ergodic. Let G be a subgroup of the symmetric group S_d

[3]See for example the proof of [Tre14, Theorem 3] or the proof of [LT14, Theorem 2].

with $d \geq 2$ which acts transitively on $\{1, \ldots, d\}$. Recall that the space of covers of (S, α) with monodromy in G can be thought of as

$$\mathrm{Cov}_G(S, \alpha) = S_d \backslash Hom\big(\pi_1(S, s_0), G\big).$$

(Note that *ker D* is trivial; the surface has no translation automorphisms.) If h is a homomorphism from $\pi_1(S, s_0)$ to G, we use $[h]$ to denote its equivalence class in $\mathrm{Cov}_G(S, \alpha)$. The disconnected covers in surfaces in $\mathrm{Cov}_G(S, \alpha)$ are given by

$$\bigcup_{H \in \mathcal{H}} S_d \backslash Hom\big(\pi_1(S, s_0), H\big),$$

where $\mathcal{H}$ denotes the collection of all subgroups of G which fail to act transitively on $\{1, \ldots, d\}$.

Recall that formally, ϕ^{-1} does not act on $\pi_1(S, s_0)$. To get an action on π_1, we need to select a curve joining s_0 to $\phi^{-1}(s_0)$ as described by equation 9. Because we chose s_0 in the interior of the square near the southwest corner, its image $\phi^{-1}(s_0)$ will also lie near the southwest corner and in the interior of the square. We specify β to be a curve joining s_0 to $\phi^{-1}(s_0)$ while not leaving the interior of the square. Then (as in equation 9), we define

$$\phi_\beta^{-1} : \pi_1(S, s_0) \to \pi_1\big(S, \phi^{-1}(s_0)\big); [\alpha] \mapsto [\beta \bullet (\phi^{-1} \circ \alpha) \bullet \beta^{-1}].$$

Theorem 22 (Non-ergodic covers)**.** *Let $h \in Hom\big(\pi_1(S, s_0), G\big)$. The translation flow on the cover $(\tilde{S}_h, \tilde{\alpha}_h)$ associated to $[h]$ is non-ergodic whenever there is an $H \in \mathcal{H}$ so that every accumulation point of $h \circ \phi_\beta^{-n}$ (as $n \to \infty$) lies in $Hom\big(\pi_1(S, s_0), H\big)$. For any $H \in \mathcal{H}$, there exists a dense collection of connected covers with this property.*

Theorem 23 (Ergodicity through slow divergence)**.** *Suppose G is a subgroup of S_d, and $H_1, H_2 \subset G$ are subgroups which do not act transitively on $\{1, \ldots, d\}$, but the group generated by the elements of $H_1 \cup H_2$ does act transitively. Then, there is a finite covers of (S, α) with monodromy in G whose orbit under ϕ consists only of disconnected surfaces, but so that the translation flow on the cover is uniquely ergodic.*

The key to proving these results is an understanding of the action of ϕ_β^{-1} on the fundamental group. In order to describe this action, we select a generating set. We choose a basepoint s_0 in the interior of the square near the southwest corner of the square in Figure 1. For each integer n, we will let $\gamma_n \in \pi_1(S, s_0)$ be a homotopy class of curves which start and end at the basepoint. If $n \geq 0$, we define γ_n to contain the curves which move downward from the basepoint passing through the horizontal edge labeled n and returning to the basepoint without passing through any other labeled edges. We similarly define γ_n for $n < 0$ to contain the curves which move rightward over the vertical edge labeled n. Observe that $\pi_1(S, s_0)$ is freely generated by $\{\gamma_n \ : \ n \in \mathbb{Z}\}$.

We also define ϕ_β as the inverse of the map ϕ_β^{-1}. The following describes the actions of ϕ_β^{-1} and ϕ_β on our basis for $\pi_1(S, s_0)$.

Proposition 24. *For each $n \in \mathbb{N}$, we have*

$$\phi_\beta^{-1}(\gamma_n) = \begin{cases} \gamma_{n+1}\gamma_1^{-1} & \text{if } n < 0 \\ \gamma_1 & \text{if } n = 0 \\ \gamma_1 \gamma_{n+1} & \text{if } n > 0, \end{cases} \quad \text{and} \quad \phi_\beta(\gamma_n) = \begin{cases} \gamma_{n-1}\gamma_0 & \text{if } n < 1 \\ \gamma_0 & \text{if } n = 1 \\ \gamma_0^{-1}\gamma_{n-1} & \text{if } n > 1. \end{cases}$$

This proposition may be proved by inspecting the action of ϕ_β^{-1}, and we leave the details to the reader.

In order to work with homomorphisms $h : \pi_1(S, s_0) \to G$, we define the *G-sequence* of a homomorphism h to be the bi-infinite sequence of elements of G defined by

$$g_m = h \circ \phi_\beta^{-m}(\gamma_1) \quad \text{for } m \in \mathbb{Z}.$$

This sequence encodes h in a way which is natural with respect to the action of ϕ. Observe that provided the map $h \mapsto \langle g_m \rangle$ is a homeomorphism, it conjugates the action of ϕ_β^{-1} on $Hom\big(\pi_1(S, s_0), G\big)$ to the shift map on $G^{\mathbb{Z}}$. That this map is a homeomorphism follows from the following, which can be used to define the inverse map and provides a more generally useful formula.

Lemma 25. *Let $\langle g_m \rangle$ be the G-sequence of a homomorphisms $h : \pi_1(S, s_0) \to G$. Then, for each $k, n \in \mathbb{Z}$,*

$$h \circ \phi_\beta^{-k}(\gamma_n) = \begin{cases} g_{k+n-1} g_{k+n}^{-1} g_{k+n+1}^{-1} \cdots g_{k-1}^{-1} & \text{if } n < 0 \\ g_{k-1} & \text{if } n = 0 \\ g_k & \text{if } n = 1 \\ g_k^{-1} g_{k+1}^{-1} \cdots g_{k+n-2}^{-1} g_{k+n-1} & \text{if } n > 1, \end{cases}$$

Proof. Observe that by definition $\phi^{-k}(\gamma_1) = g_k$ for all $k \in \mathbb{Z}$. This case of $n = 1$ will serve as a base case for proving the statement holds when $n \geq 1$. Note that the formula given in the case of $n > 1$ can be extended to hold for $n = 1$ if one allows the (empty) product of negations $g_k^{-1} g_{k+1}^{-1} \cdots g_{k+n-2}^{-1}$ to be the identity when $n = 1$. So, suppose our formula holds for some $n \geq 1$ and all k, we will show it holds for $n + 1$ and all k. Using the proposition, we observe that for $n \geq 1$,

$$h \circ \phi_\beta^{-k-1}(\gamma_n) = h \circ \phi_\beta^{-k}(\gamma_1 \gamma_{n+1}) = g_k \cdot h \circ \phi_\beta^{-k}(\gamma_{n+1}).$$

By our inductive hypothesis applied to the left side, we see that

$$g_{k+1}^{-1} g_{k+2}^{-1} \cdots g_{k+n-1}^{-1} g_{k+n} = g_k \cdot h \circ \phi_\beta^{-k}(\gamma_{n+1})$$

which gives us that $\phi_\beta^{-k}(\gamma_{n+1}) = g_k^{-1} \cdots g_{k+n-1}^{-1} g_{k+n}$. This proves our formula for $n + 1$ and and all k. So, by induction, the statement holds for $n \geq 1$.

Now we consider the base case of $n = 0$. Observe that

$$h \circ \phi_\beta^{-k}(\gamma_0) = h \circ \phi_\beta^{-k+1} \circ \phi_\beta^{-1}(\gamma_0) = h \circ \phi_\beta^{-k+1}(\gamma_1) = g_{k-1}.$$

Again observe that if we treat empty products of negations as the identity that the formula for $n < 0$ holds for $n = 0$. Now suppose the formula holds for some $n \leq 0$ and all k. Then, by the proposition and the base case

$$h \circ \phi_\beta^{-k+1}(\gamma_n) = h \circ \phi_\beta^{-k}(\gamma_{n-1}\gamma_0) = h \circ \phi_\beta^{-k}(\gamma_{n-1}) \cdot g_{k-1}.$$

By inductive hypothesis, we see

$$h \circ \phi_\beta^{-k}(\gamma_{n-1}) = h \circ \phi_\beta^{-k+1}(\gamma_n) \cdot g_{k-1}^{-1} = (g_{k+n-2} g_{k+n-1}^{-1} \cdots g_k^{-1}) g_{k-1}^{-1}.$$

This completes the inductive step, proving the statement for all $n \leq 0$. ∎

Now we prove our theorem on the non-ergodicity of covers.

Proof of Theorem 22. Let $h : \pi_1(S, s_0) \to G$ be a homomorphism, and suppose that all accumulation points of $h \circ \phi^{-k}$ as $k \to \infty$ lie in $Hom\big(\pi_1(S, s_0), H\big)$ for some subgroup $H \subset G$, where H does not act transitively on $\{1, \ldots, d\}$. Then, there is a K so that for each $k \geq K$, $h \circ \phi^{-k}(\gamma_1) \in H$. In terms of the G-sequence $\langle g_k \rangle$ of h, we see that $g_k \in H$ for $k \geq K$. But, then by the lemma above,

$$h \circ \phi^{-K}(\gamma_n) = g_K^{-1} g_{K+1}^{-1} \ldots g_{K+n-2}^{-1} g_{K+n-1} \in H$$

for all $n \geq 1$ (where when $n = 1$, the product $g_K^{-1} g_{K+1}^{-1} \ldots g_{K+n-2}^{-1}$ is taken to be the identity). Observe that the horizontal straight-line flow on the surface (S, α) only crosses the intervals with positive label in Figure 1. Now consider the cover associated to $h \circ \phi_\beta^{-K}$. The cover can be built from copies of the square indexed by $\{1, \ldots, d\}$ with edges identified according to $h \circ \phi_\beta^{-K}$. In particular, the intervals with positive label are glued only according to elements of H. Thus, points in copy $i \in \{1, \ldots, d\}$ only can reach the copies of the square indexed by elements of the orbit $H(i)$, and by assumption $H(i) \neq \{1, \ldots, d\}$. In particular the union of the squares indexed by $H(i)$ gives an invariant set with measure strictly between zero and full measure. Note that ϕ^K induces an affine homeomorphism with diagonal derivative from the cover associated to h to the cover associated to $h \circ \phi_\beta^{-K}$, so pulling back this invariant set gives an invariant subset of the straight line flow for the cover associated to h with intermediate measure.

Finally, we note that the collection of sequence $\langle g_k \rangle$ so that there is a K with $g_k \in H$ for $k \geq K$ is dense inside $G^{\mathbb{Z}}$ but for which the collection of all g_ks generates G is dense. Any $h \in Hom\big(\pi_1(S, s_0), G\big)$ whose G-sequence has this property is associated to a connected cover with non-ergodic translation flow. The map which recovers h from its G-sequence is a homeomorphism, so our set of connected but non-ergodic covers is dense. □

Now we will move toward proving our statement about the existence of covers with ergodic translation flow whose ϕ-orbits accumulate only on disconnected covers. Recall that our theorem dealt with the situation when we have two subgroups H_1 and H_2 of G which do not act transitively on $\{1, \ldots, d\}$ but which together act transitively.

Let $a \in \mathbb{Z}$ and $A = \{n \in \mathbb{Z} \ : \ n \geq a\}$. Let $p : A \to \{0, 1, 2\}$ be an arbitrary function with the property that $p(n) + p(n+1) \neq 3$ for each $n \in A$. For such a map p and an $n \in A$, we define the *preimage interval* containing n, $I(p, n) \subset \mathbb{Z}$, to be the maximal collection of consecutive elements of A containing n so that $n \in I(p, n)$ and p is constant on $I(p, n)$. We'll say that a sequence $\langle g_n \rangle \in G^{\mathbb{Z}}$ is *p-ready* if the following two statements hold for each $n \in A$:

- If $p(n) = 0$, then $g_n \in H_1 \cap H_2$.
- If $p(n) \in \{1, 2\}$, then $g_n \in H_{p(n)}$ and $\{g_k \ : \ k \in I(p, n)\}$ generates $H_{p(n)}$.

Let $|I(p, n)|$ denote the number of integers in $I(p, n)$.

Proposition 26 (Criterion for disconnected accumulation points)**.** *Suppose that h has G-sequence $\langle g_n \rangle$, and that this sequence is p-ready. If* $\liminf_{n \to \infty} |I(p, n)| = +\infty$*, then every accumulation point of $h \circ \phi_\beta^{-n}$ as $n \to \infty$ corresponds to a disconnected cover.*

Proof. Suppose to the contrary that there is a subsequence $h \circ \phi^{-k_j}$ which converges to h_∞ representing a connected cover. Let $\langle g_n^\infty \rangle$ be the G-sequence of h_∞. Observe that the G-sequence of $h \circ \phi_\beta^{-k}$ is given by $\langle g_n^k = g_{n+k} \rangle$. So, for each n, there is a J so that $g_{n+k_j} = g_n^\infty$ for $j > J$. Furthermore, the collection $\{g_n^\infty\}$ generates a subgroup of G which acts transitively on $\{1, \ldots, d\}$. Thus, we can find integers $b < c$ so that $\{g_n^\infty \ : \ b \leq n \leq c\}$

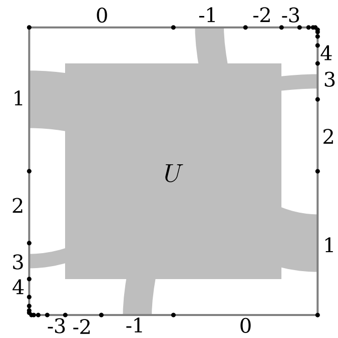

FIGURE 2. A subsurface U when $J = \{-1, 1, 3\}$.

generates a subgroup of G which acts transitively. But then for sufficiently large j, we see that $\{g_{n+k_j} \; : \; b \leq n \leq c\}$ generates a transitive subgroup. Let $I_j = \{b + k_j, \ldots, c + k_j\}$. Observe that since $\langle g_n^\infty \rangle$ is p-ready, it must be true that for such large j with $I_j \subset A$, we have $\{1, 2\} \subset p(I_j)$. But since p takes both the values of 1 and 2 on I_j, it must also take the value (otherwise $p(n) + p(n+1) = 3$ for some n). Moreover, this $m \in I_j$ so that $p(m) = 0$ must separate occurrences of 1 from the occurrences of 2. Thus, we see $I(p, m) \subset I_j$. But then we have found m arbitrarily large with $|I(p, m)|$ less than the constant $c - b + 1 = |I_j|$. This contradicts the assumption in the proposition that $\liminf_{n\to\infty} |I(p, n)| = +\infty$. □

Our proof will combine the above with the following Lemma, which will allow us to apply the ergodicity criterion of given in Theorem 18.

Lemma 27. *Let $G \subset S_d$ with $d \geq 2$ be a subgroup which acts transitively on $\{1, \ldots, d\}$. For each integer finite subset $J \subset \mathbb{Z}$ and for each $\eta > 0$, there is a constant $c > 0$ so that for each $h \in Hom\big(\pi_1(S, s_0), G\big)$ that satisfies the condition that the subgroup generated by $\{h(\gamma_j) \; : \; j \in J\}$ acts transitively on $\{1, \ldots, d\}$, there is a subsurface $\tilde{U}$ of the cover $(\tilde{S}_h, \tilde{\alpha}_h)$ associated to h, so that $Area(\tilde{S}_h \smallsetminus \tilde{U}) < \eta$ and*

$$\epsilon(t)^4 \mathcal{D}_t^{-2} > c \quad \textit{for each } t \in \mathbb{R} \textit{ with } |t| < \frac{\ln(2)}{2},$$

where $\epsilon(t)$ represents a lower bound for the distance from points in $g^t(\tilde{U})$ to points in the completion of $g^t(\tilde{S}_h, \tilde{\alpha}_h)$, and $\mathcal{D}_t$ is an upper bound for the diameter of $g^t(\tilde{U})$.

We note that the expression $\epsilon(t)^4 \mathcal{D}_t^{-2}$ is the function inside the integral in Theorem 18 in the special case of considering a connected subsurface.

Proof. We will explicitly describe how to build $\tilde{U}_h \subset (\tilde{S}_h, \tilde{\alpha}_h)$. We will define U to be a subsurface of (S, α), which only depends on J and η. Recall that (S, α) was built from a single square with edge identifications. To build U, start with a smaller concentric square with area equal to $1 - \frac{\eta}{2d}$. Then for each $j \in J$, consider the edge of the surface labeled j. Attach to the concentric square a "handle" passing through the edge, which stays a bounded distance away from Σ (the points in the metric completion). See Figure 2 for an example. Then, we define $\tilde{U}$ to be the preimage of U under the covering map $\tilde{S}_h \to S$. We note that $\tilde{U}$ is connected because of our condition that $\{h(\gamma_j) \; : \; j \in J\}$ generates G.

Note that the minimal distance from the boundary of $g^t(\tilde{U})$ to the metric completion in $g^t(\tilde{S}_h, \tilde{\alpha}_h)$ is the same as the minimal distance from $g^t(U)$ to the metric completion in $g^t(S, \alpha)$.

This distance is always positive and varies continuously in t, so we can get a uniform lower bound which holds when $|t| \leq \frac{1}{2} \ln 2$. Similarly, the diameter of $g^t(\tilde{U}_h)$ varies continuously in t, so we can get an upper bound on the diameter which is uniform in t with $|t| \leq \frac{1}{2} \ln 2$. Finally observe that up to translation equivalence there are only finitely many $\tilde{U}_h$ as we vary h. (The geometry of $\tilde{U}_h$ only depends on the restriction of h to J.) Thus, we can get an upper bound on the diameter of which is uniform in both h satisfying our condition and t satisfying $|t| \leq \frac{1}{2} \ln 2$. □

We need a mechanism to ensure that h satisfies the condition of the lemma (i.e., for some fixed $J \subset \mathbb{Z}$ the subgroup generated by $\{h(\gamma_j) \ : \ j \in J\}$ acts transitively on $\{1, \ldots, d\}$), given conditions on the G-sequence $\langle g_n \rangle$ of h. This mechanism is a corollary of Lemma 25.

Corollary 28. *Suppose h has G-sequence $\langle g_n \rangle$. Let $m, k \in \mathbb{Z}$ with $m > 0$. Then, if the subgroup generated by $\{g_j \ : \ k - m \leq j \leq k + m\}$ acts transitively on $\{1, \ldots, d\}$, then so does the subgroup generated by $\{h \circ \phi_\beta^{-k}(\gamma_j) \ : \ -m + 1 \leq j \leq m + 1\}$.*

Proof. Using Lemma 25, we can find an expression for each g_j with $k - m \leq j \leq k + m$ in terms of a $\{h \circ \phi_\beta^{-k}(\gamma_j) \ : \ -m + 1 \leq j \leq m + 1\}$. □

Proof of Theorem 23. Let $a \in \mathbb{Z}$ and $A = \{n \in \mathbb{Z} \ : \ a \leq n\}$, and let $p : A \to \{0, 1, 2\}$ be an function so that $p(n) + p(n+1) \neq 3$ for all $n \in A$ as above. We call p *alternating* if whenever $n \in A$ and $p(n) = 0$, we have $p(b - 1) \neq p(c + 1)$, where b and c are the endpoints of the preimage interval $I(p, n) = \{b, b + 1, \ldots, c\}$.

For each positive integer m, let $J_m = \{n \in \mathbb{Z} \ : \ |n| \leq m\}$. Let $\mathrm{Hom}_m \subset \mathrm{Hom}\big(\pi_1(S, s_0), G\big)$ be the collection of all h so that the subgroup generated by $\{h(\gamma_n) \ : \ n \in J_m\}$ acts transitively on $\{1, \ldots, d\}$. Let $\langle \eta_i \rangle_{i>0}$ be a sequence of positive reals tending to zero monotonically as $i \to \infty$. Then applying the prior lemma yields a constant $c_{m,i} > 0$ so that $h \in \mathrm{Hom}_m$ implies that for $\eta = \eta_i$, the function being integrated in Theorem 18 is larger than $c_{m,i}$ on the interval $(-\frac{1}{2} \ln 2, \frac{1}{2} \ln 2)$.

For each $i > 0$ and $m > 0$, choose an integer $N_{m,i} \geq 0$ so that for each i,

$$\sum_{m=1}^{\infty} N_{m,i} c_{m,i} = +\infty.$$

This choice has the consequence that if $h \in \mathrm{Hom}\big(\pi_1(S, s_0), G\big)$, and there is a pairwise disjoint collection of subsets of positive integers K_m for integers $m > 0$, so that for each $k \in K_m$, $h \circ \phi^{-k} \in \mathrm{Hom}_m$, then the integral in Theorem 18 in the case of $\eta = \eta_i$ is infinite. Now define a new sequence of integers by

$$M_m = \max\{N_{m,1}, N_{m,2}, \ldots, N_{m,m}\}.$$

Observe that if h is given and there is a pairwise disjoint collection of subsets of positive integers K_m so that for each $k \in K_m$, $h \circ \phi^k \in \mathrm{Hom}_m$, then the integral in Theorem 18 in the case of $\eta = \eta_i$ is infinite for every i. It follows that the integral is infinite for every $\eta > 0$, because function being integrated for $\eta_i < \eta$ is pointwise larger than the one for η.

CONTINUE FILLING IN DETAILS... □

6.2. **OLD stuff.** Indeed, each such cover is determined by a homomorphism $\pi_1(S, s_0) \to G$, and two such covers are isomorphic (or equivalently, translation equivalent) if they differ by conjugation by an element of S_d. We can naturally identify each $h \in Hom\big(\pi_1(S, s_0), G\big)$

with a bi-infinite sequence $\omega \in G^{\mathbb{Z}}$ defined by $\omega_n = h(\gamma_n)$. We let $[\omega]$ be the equivalence class of ω in $S_d \backslash G^{\mathbb{Z}}$, where S_d is acting by simultaneous conjugation on all elements of the sequence ω.

Knowledge of a homomorphism $h : \pi_1(S, s_0) \to G$ is equivalent to knowing the images of the generators. Indeed,

Proposition 29. *Let $[\omega] \in S_d \backslash G^{\mathbb{Z}}$. Then, $\phi_*([\omega]) = [\eta]$ with $\eta \in G^{\mathbb{Z}}$ given by*

$$\eta_n = \begin{cases} \omega_{n-1}\omega_0 & \text{if } n \leq 0 \\ \omega_0 & \text{if } n = 1 \\ \omega_{n-1}\omega_0^{-1} & \text{if } n \geq 2. \end{cases}$$

This surface, S is built from a unit square with boundary segments of length $\frac{1}{2^n}$ identified in pairs by translation. The identified edges are labeled in the same way in the above figure. Each of these edges represents a saddle connection on our surface, which will call e_i for $i \in \mathbb{Z}$.

For each $i \in \mathbb{Z}$, let γ_i be the homotopy class of loops on S beginning at s_0 and passing only through saddle connection e_i. We take γ_i to be oriented downward if $i \geq 0$ and oriented rightward if $i < 0$. These homotopy classes generate the fundamental group $\Gamma = \pi_1(S, s_0)$. We choose the basepoint $s_0 \in S$ to be the center of the square.

This surface has an affine homeomorphism ϕ with derivative

$$D(\phi) = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

We will describe the action of ϕ on the space of finite covers. In order to describe this action, we need to work out the action of ϕ^{-1} on Γ. To describe this action, we make a choice of a curve α joining s_0 to $\phi^{-1}(s_0)$. This choice was depicted in the figure above. The action of ϕ_*^{-1} on homology (following equation **??**) can then determined by the following action on generators:

- $\phi_*^{-1}(\gamma_n) = \gamma_{n+1}\gamma_0^{-1}$ if $n < -1$.
- $\phi_*^{-1}(\gamma_{-1}) = \gamma_0$.
- $\phi_*^{-1}(\gamma_n) = \gamma_{n+1}\gamma_0$ if $n \geq 0$.

We will work out the action of ϕ on the space of double covers of S. These covers are parameterized by elements of $\mathrm{Hom}\big(\pi_1(S), \mathbb{Z}_2\big)$. This space is naturally homeomorphic to the two-sided shift space $\Omega = (\mathbb{Z}_2)^{\mathbb{Z}}$ on the alphabet $\mathbb{Z}_2 = \{0, 1\}$ under the homeomorphism which sends $h \in \mathrm{Hom}\big(\pi_1(S), \mathbb{Z}_2\big)$ to the sequence

$$\ldots h(\gamma_{-2})\ h(\gamma_{-1})\ \widehat{h(\gamma_0)}\ h(\gamma_1)\ h(\gamma_2) \ldots \in \Omega.$$

Here we have used the hat to indicate the element with index zero. An element $\omega \in \Omega$ then determines a double cover according to the following rule. We take two unit squares with boundary edges labeled as in the construction of Chamanara's surface. We identify edges labeled $i \in \mathbb{Z}$ of the same square when $\omega_i = 0$, and of different squares if $\omega_i = 1$. The map ϕ acts on $\mathrm{Hom}\big(\pi_1(S), \mathbb{Z}_2\big)$ via $\phi^* : h \mapsto h \circ \phi_*^{-1}$. We will abuse notation by also using ϕ^* to denote the corresponding action on the homeomorphic space Ω. Let $\omega = \ldots \omega_{-1}\omega_0\omega_1 \ldots \in \Omega$. We have

$$\phi^*(\omega) = \begin{cases} \ldots \omega_{-1}\omega_0\widehat{\omega_1}\omega_2\omega_3 \ldots & \text{if } \omega_0 = 0. \\ \ldots (\omega_{-2}+1)(\omega_{-1}+1)\omega_0(\widehat{\omega_1+1})(\omega_2+1) \ldots & \text{if } \omega_0 = 1. \end{cases}$$

Observe that the cover corresponding to ω is disconnected if and only if $\omega_i = 0$ for all $i \in \mathbb{Z}$. We'll denote this element of Ω by $\mathbf{0}$. The stable set of the action of ϕ^* is the eventually constant sequences:

$$W_A^s(\mathbf{0}) = \{\omega \in \Omega \ : \ \text{there is an } x \in \mathbb{Z}_2 \text{ and } N \in \mathbb{Z} \text{ so that } \omega_n = x \text{ for all } n > N\}.$$

Theorem 30. *Let $\tilde{S}$ denote the cover of Chamanara's surface, S, associated to $\omega \in \Omega$. Then $\tilde{S}$ has uniquely ergodic vertical straight-line flow if and only if $\omega \notin W_A^s(\mathbf{0})$. Further, if $\omega \in W_A^s(\mathbf{0})$, then no leaf of the vertical foliation of $\tilde{S}$ is dense.*

Proof. First suppose that $\omega \in W_A^s(\mathbf{0})$. Then there is an $x \in \mathbb{Z}_2$ and an integer $N \geq 0$ so that $\omega_n = x$ for $n \geq N$. Observe that

$$\big((\phi^*)^{N+1}(\omega)\big)_n = 0 \quad \text{for all } n \geq 0.$$

The cover associated to $\omega' = (\phi^*)^{N+1}(\omega)$ is formed by gluing edges as in the construction of the Chamanara's surface. It follows that the cover associated to ω' has the property that no horizontal edge of one square is identified to a horizontal edge of the other square. In particular, every vertical trajectory of the straight line flow stays in only one of the squares for all forward and backward time. So, no trajectory is dense.

Conversely, suppose $\omega \notin W_A^s(\mathbf{0})$. Then, consider the following subset of the forward orbit of ω under ϕ^*:

$$O = \big\{(\phi^*)^n(\omega) \ : \ n \geq 0 \text{ and } (\phi^*)^n(\omega)_0 = 1\big\}.$$

It can be observed that $\omega \in W_A^s(\mathbf{0})$ if and only if O is finite. So, we are supposing that O is infinite. Since it is an infinite subset of the compact space Ω, there is an accumulation point $\omega^\infty \in \Omega$. This accumulation point has the property that $\omega_0^\infty = 1$. So, the cover assocated to ω^∞ is connected. So, by Theorem 19, the vertical straight line flow on $\tilde{S}$ is uniquely ergodic. □

6.3. **The ladder surface.** For this subsection, we let (S, α) denote the infinite genus translation surface of Figure 3, which we call the *ladder surface*. (The surface is built using Thurston's construction from a graph resembling a ladder.) It can be constructed from a region in the plane bounded by countably many horizontal and vertical edges. We have labeled the edges using the set $\mathbb{Z}^* = \mathbb{Z} \smallsetminus \{0\}$. Edges with the same label are glued by translation to form the surface. Let φ denote the golden mean, $\frac{\sqrt{5}+1}{2}$. An edge with label $j \in \mathbb{Z}^*$ has length given by $\varphi^{-|j|}$. This information specifies the geometry of the surface. We have also selected a basepoint s_0 for our surface in the figure.

The surface (S, α) admits two affine multi-twists ϕ and ψ with derivatives given by

$$D(\phi) = \begin{bmatrix} 1 & 2\varphi \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad D(\psi) = \begin{bmatrix} 1 & 0 \\ 2\varphi & 1 \end{bmatrix}.$$

The affine automorphism ϕ performs a single right Dehn twist in each of countably many horizontal cylinders on (S, α). These horizontal cylinders are separated by dotted horizontal lines in Figure 3. The affine automorphism ψ performs a single left Dehn twist in each of the vertical cylinders shown in the surface. The surface also admits a Euclidean reflection ρ whose derivative is given by

$$D(\rho) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \text{ and which satisfies } \phi \circ \rho = \rho \circ \psi.$$

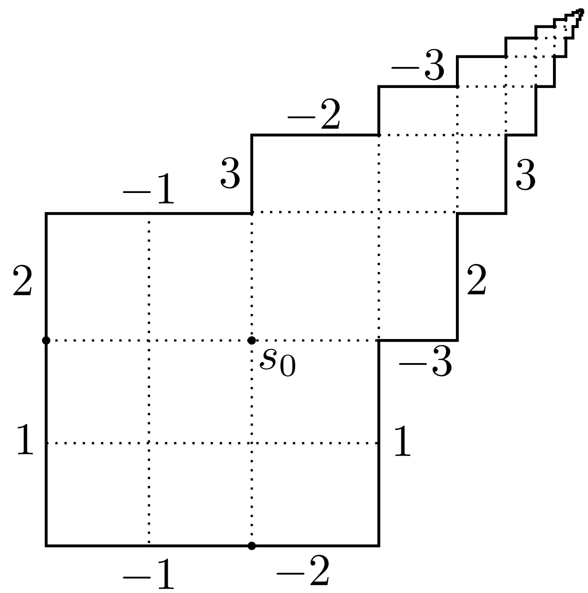

FIGURE 3. The ladder surface.

This is reflection in the line of symmetry of slope one in our figure. All these affine automorphisms fix our basepoint s_0.

For $i \in \mathbb{Z}^*$, we let γ_i be one of the two homotopy class of loops which start at the basepoint, cross only the edge labeled i, and return to the basepoint. These two homotopy classes are inverses in $\Gamma = \pi_1(S, s_0)$. We make the choice of γ_i so γ_i moves rightward across the vertical edge labeled i if $i > 0$, and moves upward over the horizontal edge labeled i if $i < 0$. These curves freely generate the fundamental group, which we denote by $\Gamma = \pi_1(S, s_0)$.

Let $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. The space of double covers $\mathrm{Cov}_2(S, \alpha)$ is precisely $\mathrm{Hom}(\Gamma, \mathbb{Z}_2)$, since the surface admits no translation automorphisms and $\mathbb{Z}_2$ acts trivially on $\mathrm{Hom}(\Gamma, \mathbb{Z}_2)$ by conjugation. Let Aff' be the known affine automorphism group, $\langle \phi, \psi, \rho \rangle$. The orientation preserving part is $\mathit{Aff}'_+ = \langle \phi, \psi \rangle$. Since our basepoint is fixed by the affine automorphisms in Aff', we have a canonical action of each element in Aff' on the space of covers.

Notation 31. Let $h \in \mathrm{Hom}(\Gamma, \mathbb{Z}_2)$. By an abuse of notation, we will write $h(i)$ to abbreviate $h(\gamma_i)$ for $i \in \mathbb{Z}^*$. (The space of covers $\mathrm{Hom}(\Gamma, \mathbb{Z}_2)$ can be identified with the set of all functions $\mathbb{Z}^* \to \mathbb{Z}_2$.)

It can be observed that the generators of Aff' act on $\mathrm{Hom}(\Gamma, \mathbb{Z}_2)$ as follows:

$$\big(\phi_*(h)\big)(\gamma_i) = h \circ \phi^{-1}(\gamma_i) = \begin{cases} h(i) & \text{if } i > 0, \\ h(i) + h(2) & \text{if } i = -1, \\ h(i) + h(2) + h(3) & \text{if } i = -2, \\ h(i) + h(-i-1) + h(-i) + h(-i+1) & \text{if } i < -2. \end{cases} \tag{20}$$

$$\big(\psi_*(h)\big)(i) = \begin{cases} h(i) & \text{if } i < 0, \\ h(i) + h(-2) & \text{if } i = 1, \\ h(i) + h(-2) + h(-3) & \text{if } i = 2, \\ h(i) + h(-i-1) + h(-i) + h(-i+1) & \text{if } i > 2. \end{cases} \tag{21}$$

$$\big(\rho_*(h)\big)(i) = h(-i).$$

Observe that Aff'_+ is a free group, because its derivative is a free subgroup of $SL(2,\mathbb{R})$. However, the action of ϕ_* and ψ_* on the space of covers, $\mathrm{Hom}(\Gamma, \mathbb{Z}_2)$, are each order two. This is because, for example, the action of ϕ_* preserves the value of $h(i)$ for $i > 0$, and adds a sum of values of h evaluated at positive integers to the values of $h(i)$ for $i < 0$. Since adding the same number twice is the same as adding zero in $\mathbb{Z}_2$, ϕ_*^2 acts trivially. Thus, the action of $\langle \phi_*, \psi_* \rangle$ factors through the infinite dihedral group. Including the action ρ, we obtain a homomorphism of Aff' to $\mathit{Isom}(\mathbb{Z})$:

$$\delta : \mathit{Aff}' \to \mathit{Isom}(\mathbb{Z}); \begin{cases} \delta(\rho)(n) = -n \\ \delta(\phi)(n) = 1 - n \\ \delta(\psi)(n) = -1 - n. \end{cases}$$

Proposition 32. *The surjective homomorphism $\delta : \mathit{Aff}' \to \mathit{Isom}(\mathbb{Z})$ has the property that $\xi, \xi' \in \mathit{Aff}'$, we have $\xi_* = \xi'_*$ if and only if $\delta(\xi) = \delta(\xi')$.*

Proof. Let e_* denote the trivial action on $\mathrm{Hom}(\Gamma, \mathbb{Z}_2)$. For the action of Aff', we have the relations $\phi_*^2 = e_*$, $\rho_*^2 = e_*$ and $\psi_* = \rho_*^{-1}\phi_* \circ \rho_*$. These relations generate a group isomorphic to $\mathit{Isom}(\mathbb{Z})$, and δ gives an explicit isomorphism. This shows that $\langle \phi_*, \rho_*, \psi_* \rangle$ is isomorphic to a quotient of $\mathit{Isom}(\mathbb{Z})$. To see it is precisely this group, one needs to observe that $\rho_* \neq e_*$ and that for every integer $k > 0$, $(\phi_* \circ \rho_*)^k \neq e_*$. To see this last statement, it is useful to define $h \in \mathrm{Hom}(\Gamma, \mathbb{Z}_2)$ so that $h(-2) = 1$ and $h(n) = 0$ for $n \in \mathbb{Z}^* \smallsetminus \{-2\}$. Then observe that $(\phi_* \circ \rho_*)^k(h)(-2-k) = 1$ for all $k > 0$. $\square$

We note that the proposition gives an action of $\mathit{Isom}(\mathbb{Z})$ on $\mathrm{Hom}(\Gamma, \mathbb{Z}_2)$.

We can canonically identify $SL(2,\mathbb{R})$ with the unit tangent bundle of the upper half plane. The group $V'_+ = D(\mathit{Aff}'_+)$ acts freely on the upper half plane, and a fundamental domain is shown in Figure 4. This gives a fundamental domain F for the action of V'_+ on $SL(2,\mathbb{R})$ consisting of those $A \in SL(2,\mathbb{R})$ whose associated unit tangent vectors are based at a point in the gray region of the Figure. Note that F is only connected to $D(\phi^{\pm 1})(F)$ and $D(\psi^{\pm 1})(F)$. Let $A \in F$. The orbit of a point $AV'_+ \in SL(2,\mathbb{R})/V'_+$ under the Teichmüller deformation g^t leads to a sequence of elements of V'_+ which successively brings the trajectory back into the fundamental domain as discussed in §3 Update when this discussion is moved.. We recall that this sequence may be finite if the trajectory is divergent, but we will be concerned only with the case when this sequence is infinite. Since our surface has no translation automorphisms, the sequence of Veech group elements determines a unique sequence of affine automorphisms $\{\phi_1, \phi_2, \ldots\}$ in Aff'_+ so that $\{D(\phi_1), D(\phi_2), \ldots\}$ is our sequence in V'_+. Because of our choice of fundamental domain, each $\phi_i \in \{\phi, \phi^{-1}, \psi, \psi^{-1}\}$. Do we have a name for this sequence? We will be mostly interested in the products that show up in §3 Update when this discussion is moved.. We define ξ_0 to be the identity and $\xi_i = \phi_1\phi_2 \ldots \phi_i$ for $i \geq 1$. This is the sequence of orientation preserving affine automorphisms so that $g^{t_i}D(\xi_i) \in F$. Note that $\{\xi_1, \xi_2, \ldots\}$ satisfies the condition that

$$\xi_{j+1} \circ \xi_j^{-1} = \phi_{j+1} \in \{\phi, \phi^{-1}, \psi, \psi^{-1}\} \quad \text{for all } j \geq 1. \tag{22}$$

There is only a single disconnected double cover of (S, α); it is associated to the trivial group homomorphism $\mathbf{0} \in \mathrm{Hom}(\Gamma, \mathbb{Z}_2)$ which sends each element in the fundamental group to the identity, $0 \in \mathbb{Z}_2$. Given a sequence $\{\xi_i\}$ determined by $A \in F$, we define the *stable set*

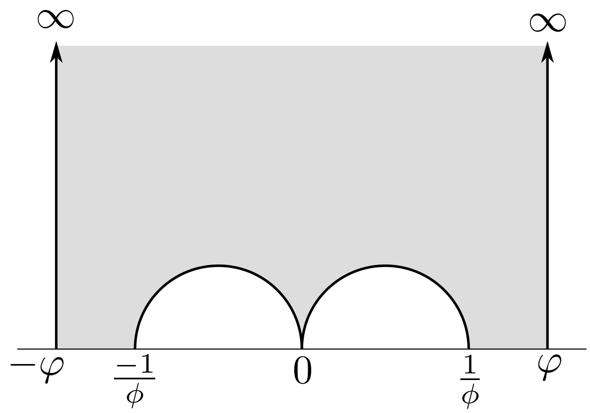

FIGURE 4. A fundamental domain for the action of V'_+ on the upper half plane.

of $\mathbf{0}$ to be

$$W^s_A(\mathbf{0}) = \{h \in \mathrm{Hom}(\Gamma, \mathbb{Z}_2) \ : \ (\xi_i)_*(h) \to \mathbf{0} \text{ as } i \to \infty\}.$$

Theorem 33. *Let $(S', \alpha') = A(S, \alpha)$ for some A in our fundamental domain F. Suppose that the g^t-action applied to AV'_+ in $SL(2,\mathbb{R})/V'_+$ leads to an infinite sequence of affine automorphisms of (S, α), $\{\xi_1, \xi_2, \ldots\}$ taken from Aff'_+ as above.*

1. *If there is an element x of the dihedral group D_∞ so that $\delta(\xi_i) = x$ for infinitely many $i \geq 1$, then the translation flow on any connected double cover of (S', α') is uniquely ergodic.*
2. *If there is a uniform bound on the size of the set $\{i \ : \ \delta(\xi_i)(0) = x\}$, then for any $h \in W^s_A(\mathbf{0}) \subset \mathrm{Hom}(\Gamma, \mathbb{Z}_2)$, the cover $(\tilde{S}'_h, \tilde{\alpha}'_s) = A(\tilde{S}_h, \tilde{\alpha}_h)$ of (S', α') has non-ergodic translation flow. Requires a summability of symmetric difference argument following Masur and Smillie (maybe others).*

Proof of Theorem 33, Statement (1). Choose a double cover $(\tilde{S}_h, \tilde{\alpha}_h)$ of (S, α) where $h \in \mathrm{Hom}(\Gamma, \mathbb{Z}_2)$. Assume $(S', \alpha') = A(S, \alpha)$ with $A \in SL(2, \mathbb{R})$. Then $(\tilde{S}'_h, \tilde{\alpha}'_h) = A(\tilde{S}_h, \tilde{\alpha}_h)$ is an arbitrary double cover of (S', α'). Let $\{\xi_1, \xi_2, \xi_3, \ldots\}$ be the sequence of affine automorphisms obtained from the trajectory $g^t AV'_+$ in $SL(2, \mathbb{R})/V'_+$.

Let $\{i_n\}$ be the set of all i so that $\delta(\xi_i) = x \in D_\infty$. By assumption, this set is infinite, and we arrange it into an increasing sequence. Since the action of $\langle \phi_*, \psi_* \rangle$ on $\mathrm{Hom}(\Gamma, \mathbb{Z}_2)$ factors through the dihedral group, x also has a well defined action on this space of marked covers. Then, $g^{t_{i_n}} A \cdot D(\xi_{i_n})$ lies in the fundamental domain F for all i_n. See §3 UPDATE. As far as the cover cocycle is concerned, the Teichmüller deformation gives rise to the sequence of points in $\mathcal{O}_{V'_+, \mathcal{C}}(S, \alpha)$ with $\mathcal{C} = \mathrm{Hom}(\Gamma, \mathbb{Z}_2)$, the space of double covers:

$$\big(g^{t_{i_n}} A, x(h)\big)/V'_+, \quad \text{which have representatives} \quad \big(g^{t_{i_n}} A \cdot D(\xi_{i_n}), x(h)\big).$$

Note that the second coordinate of these representatives in $SL(2, \mathbb{R}) \times \mathcal{C}$ is always the same, while the first coordinate lies in F. We will explain that by passing to a further subsequence and perturbing the values of t_{i_n} so that $g^{t_{i_n}} A \cdot D(\xi_{i_n})$ remains in F, we can take $g^{t_{i_n}} A \cdot D(\xi_{i_n})$ to lie in a compact subset of F. Since our trajectory crosses the fundamental domain infinitely often, it must be forward asymptotic to the convex core of $SL(2, \mathbb{R})/V'_+$. (That is the geodesic $g^t A$ has an endpoint in the limit set of V'_+.) This convex core is only non-compact in that it contains two cusps corresponding to the elements of the Veech group $D(\phi)$ and $D(\psi)$. When the trajectory enters a cusp, the value of $\delta(\xi_i)$ oscillates between adjacent values. When it exits it returns to a compact set. In particular, when $g^t AV'_+$ is in a cusp with t near t_{i_n}, with $\delta(\xi_{i_n}) = x$, there is a maximal integer $k \geq 0$ so that $\delta(\xi_{i_n+2j}) = x$ for $i = 0, \ldots, k$ and there is a value of t'_{i_n+2k} so that $g^t AD(\xi_{i_n+2k})$ stays within the fundamental domain F for t between t_{i_n+2k} and t'_{i_n+2k}, and so that the later endpoint is in a compact set. So, because we can pass to a subsequence returning to a compact set, we can pass to a further convergent subsequence. So we get unique ergodicity by Theorem 19. □

I am doing revisions below!

In order to prove the second statement in the theorem, we will need to give a characterization of the stable set $W^s_A(\mathbf{0})$. Recall that there is a well defined action of the infinite dihedral group $Isom(\mathbb{Z})$ on $\mathrm{Hom}\big(\pi_1(S, s_0), \mathbb{Z}_2\big)$ based on the definition that

$$\delta(\xi)(h) = \xi_*(h) = h \circ \xi^{-1}.$$

In order to denote elements of $Isom(\mathbb{Z})$, we use τ to denote the translation by 1, so that $\tau^n(j) = n + j$ denotes translation by $n \in \mathbb{Z}$. For each $N \geq 1$, we let define the following subset of $\mathrm{Hom}\big(\pi_1(S, s_0), \mathbb{Z}_2\big)$:

$$Z_N = \{h \ : \ h(-N-1) = h(N) = 1 \text{ and } h(i) = 0 \text{ for } i \in \mathbb{Z}^* \text{ with } -N-1 < i < N\}.$$

Lemma 34. *Let $\{\xi_i\}$ be any sequence for which the set $\{i : \delta(\xi_i) = x\}$ is finite for all $x \in Isom(\mathbb{Z})$. Then, $\{\delta(\xi_i)(0)\}$ either contains all the positive integers (as a subset) or all the negative integers and not both. Suppose $\{\delta(\xi_i)(0)\}$ contains all the positive integers. Then,*

(a) $W^s_A(\mathbf{0}) = \displaystyle\bigcup_{M=0}^{\infty} \bigcup \bigcap_{k=0}^{\infty} \tau\tau^{-2k}(Z_{K+2k}) Z_{2k-M}$. *THINKING ABOUT THIS*

(b)

Remark 35 (Entropy)**.** *The information entropy contained in the space of $W^s_A(\mathbf{0})$ is half that of all sequences in* $\mathrm{Hom}(\Gamma, \mathbb{Z}_2)$*. This is because of statement (3) which determines $\xi_J(h)(m)$ for $m < 0$ based on knowing $\xi_J(h)(n)$ for $n > 0$. Conversely, except for finitely many positive values ($0 < n \leq 4$), we can use whatever we want for $h_J(n)$ for $n > 0$.*

Lemma 36. *For each $N \geq 1$, there is an $h \in Z_N$ so that $\tau^{2k}(h) \in Z_{N+2k}$ for all integers $k \geq 0$. In particular, $\lim_{k\to\infty} \tau^{2k}(h) = \mathbf{0}$.*

The lemma is a basic consequence of the following proposition.

Proposition 37. *If $N \geq 3$, then $\tau^{-2}(Z_N) \subset Z_{N-2}$.*

Proof. Let $N \geq 4$ and $h \in Z_N$. Then $h(-N) = h(N+1) = 1$ and $h(i) = 0$ when $-N < i < N+1$. Observe that $\tau^{-2}(h) = \psi_* \circ \phi_*(h)$. We need to complete the values of this function evaluated at $i \in \mathbb{Z}^*$ with $-N+1 \leq i \leq N-2$. To do this, we use the formulas for ϕ_* and ψ_* in equations 20 and 21. We start with the negative entries in the interval we care about. Let $0 < k \leq N-1$. Observe that $\tau^{-2}(h)(-k) = \phi_*(h)(-k)$, so

$$\tau^{-2}(h)(-k) = \begin{cases} h(-k) + h(2) & \text{if } k = 1 \\ h(-k) + h(2) + h(3) & \text{if } k = 2 \\ h(-k) + h(k-1) + h(k) + h(k+1) & \text{if } k \geq 3. \end{cases}$$

Because $h \in Z_N$, we see that $\tau^{-2}(h)(-k) = 0$ if $k < N-1$ and $\tau^{-2}(h)(-N+1) = 1$. Now consider the positive entries. Suppose $0 < k \leq N-2$. Observe that

$$\tau^{-2}(h)(k) = \begin{cases} \phi_*(h)(1) + \phi_*(h)(-2) & \text{if } k = 1 \\ \phi_*(h)(2) + \phi_*(h)(-2) + \phi_*(h)(-3) & \text{if } k = 2 \\ \phi_*(h)(k) + \phi_*(h)(1-k) + \phi_*(h)(-k) + \phi_*(h)(-k-1) & \text{if } k \geq 3. \end{cases}$$

Then by evaluating ϕ_*, we see:

$$\tau^{-2}(h)(k) = \begin{cases} h(1) + h(2) + h(3) + h(-2) & \text{if } k = 1 \\ h(2) + h(4) + h(-2) + h(-3) & \text{if } k = 2 \\ h(5) + h(-2) + h(-3) + h(-4) & \text{if } k = 3 \\ h(k-2) + h(k+2) + h(-k+1) + h(-k) + h(-k-1) & \text{if } k \geq 4. \end{cases}$$

Then if $k \leq N-3$, we see that $\tau^{-2}(h)(k) = 0$. We also have $\tau^{-2}(h)(N-2) = 1$ since $h(N) = 1$. $\square$

Proof of Lemma 36. Fix an integer $N \geq 1$. Consider the intersection

$$\bigcap_{k=0}^{\infty} \tau^{-2k}(Z_{N+2k}).$$

By the prior proposition, this is a nested sequence. The sets are each compact, so there is an h in the common intersection. This means that $\tau^{2k}(h) \in Z_{N+2k}$ for all k. By definition of Z_{N+2k}, we see that $\tau^{2k}(h) \to \mathbf{0}$ as $k \to \infty$. $\square$

Now we prove a converse to the lemma.

Lemma 38. *Suppose that $h \in \mathrm{Hom}\big(\pi_1(S, s_0), \mathbb{Z}_2\big)$ and $\lim_{k\to\infty} \tau^{2k}(h) = \mathbf{0}$. Then, there are integers M and K so that $\tau^{2k}(h) \in Z_{2k-M}$ for $k > K$.*

In order to prove the theorem, we define a measure of how close an $h \in \mathrm{Hom}\big(\pi_1(S, s_0), \mathbb{Z}_2\big)$ is to zero. For $h \neq \mathbf{0}$, we define

$$\chi(h) = \max\{N \in \mathbb{Z} \ : \ N \geq 0 \text{ and } h(n) = 0 \text{ whenever } |n| < N\}.$$

Observe that if $N = \chi(h)$, then either $h(N) = 1$ or $h(-N) = 1$. Note also that $h \in Z_N$ implies $\chi(h) = N$.

Proposition 39. *Let $N = \chi(h)$ and $\chi\big(\tau^{2k}(h)\big) = N'$. If $N \geq 3$ and $h \notin \mathbb{Z}_N$, then $N' < N$ and $\tau^{2k}(h) \notin Z_{N'}$.*

Proof. We break into cases. Assume $\chi(h) = N$ and $h \notin Z_N$. Since $\chi(h) = N$, we know $h(N) = 1$ or $h(-N) = 1$. Thus, h has one of the following forms:

(1) $h(-N) = 1$.
(2) $h(-N) = 0$, $h(N) = 1$ and $h(-N-1) = 0$.

Consider the case (1) where $h(-N) = 1$. Recall that $\tau^2(h) = \phi_* \circ \psi_*(h)$. Based on the knowledge that $h(k) = 0$ when $|k| < N$, we see by applying our understanding of ψ_* (equation 21) that for $k \in \mathbb{Z}^*$ with $-N \leq k \leq N-1$,

$$\psi_*(h)(k) = \begin{cases} 1 & \text{if } k \in \{-N, N-1\} \\ 0 & \text{if } -N < k < N-1. \end{cases}$$

Now we apply equation 20, which gives a formula for ϕ_*. We see that

$$\tau^2(h)(k) = \begin{cases} 1 & \text{if } k \in \{-N+2, N-1\} \\ 0 & \text{if } -N+2 < k < N-1. \end{cases}$$

We observe that in this case $\chi\big(\tau^2(h)\big) = N-2$ and $\tau^2(h) \notin Z_{N-2}$.

Now consider case (2). Suppose that $h(-N) = 0$, $h(N) = 1$ and $h(-N-1) = 0$. Then,

$$\psi_*(h)(k) = \begin{cases} 1 & \text{if } k = N \\ 0 & \text{if } -N-1 \leq k < N. \end{cases}$$

$$\tau^2(h)(k) = \begin{cases} 1 & \text{if } k \in \{-N+1, N\} \\ 0 & \text{if } -N+1 < k < N. \end{cases}$$

In this case, $\chi\big(\tau^2(h)\big) = N-1$ and $\tau^2(h) \notin Z_{N-1}$. □

Proposition 40. *Let $N = \chi(h)$ and $\chi\big(\tau^2(h)\big) = N'$. If $h \in Z_N$ and $\tau^2(h) \in Z_{N'}$, then $N' = N+2$.*

Proof. Let $h \in Z_N$. Then, $h(N) = h(-N-1) = 1$ and $h(k) = 0$ for $k \in \mathbb{Z}^*$ with $-N-1 < k < N$. We do a calculation similar to the last proof. We have

$$\psi_*(h)(k) = \begin{cases} 1 & \text{if } k = N+1 \\ 0 & \text{if } -N \leq k \leq N. \end{cases}$$

Let $a = \psi_*(h)(N+1)$ and $b = \psi_*(h)(N+2)$. Then,

$$\tau^2(h)(k) = \phi_* \circ \psi_*(h)(k) = \begin{cases} 0 & \text{if } -N < k < N+1 \\ a & \text{if } k\{-N, N+1\} \\ 1+a+b & \text{if } k = -N-1 \\ b & \text{if } k = N+1. \end{cases}$$

Observe that if $a = 1$, then $N' = \chi\big(\tau^2(h)\big) = N$ and $\tau^2(h) \notin Z_{N'}$. If $a = 0$ and $b = 0$, then $N' = N+1$ and $\tau^2(h)(-N-1) = 1$ so $\tau^2(h) \notin Z_{N'}$. The proposition doesn't say anything about either of these cases because $\tau^2(h) \notin Z_{N'}$. Finally, we consider the possibility that $a = 0$ and $b = 1$. Here we see that $N' = N+2$ as required. □

Proof of Lemma 38. Fix an h so that $\lim_{k\to\infty} \tau^{2k}(h) = \mathbf{0}$. Then for any sufficiently large integer N, there is a minimal K so that $\tau^{2k}(h)(n) = 0$ for any $k \geq K$ and $n \in \mathbb{Z}^*$ with $|n| < N$. Choose such an $N \geq 4$ and let it determine K. Set $h' = \tau^{2K}(h)$. For $j \geq 0$, let $\chi_j = \chi\big(\tau^{2j}(h')\big)$. We claim that for $j > 0$, $\tau^{2j}(h') \in Z_{\chi_j}$. Suppose not. Then there is a $j > 0$ so that $\tau^{2j}(h') \notin Z_{\chi_j}$. Then by inductively applying Proposition 39, we see that there is an integer $i \geq 0$ so that

$$\chi_j > \chi_{j+1} > \chi_{j+2} > \ldots > \chi_{j+i} \quad \text{with } \chi_{j+i} \in \{1, 2\}.$$

But this violates our original statement that $\tau^{2k}(h)(n) = 0$ for any n with $|n| < N$.

We have shown that $\tau^{2j}(h') \in Z_{\chi_j}$ for each $j \geq 0$. Then, Proposition 40 implies that $\chi_{j+1} = \chi_j + 2$ for all $j \geq 0$.

□

OLD STUFF Now we will describe the stable set, $W^s_A(\mathbf{0})$. Our characterization for the stable set uses the following observation. Given $\{\xi_i\}$ we get a walk starting at zero on the integers given by $\{\delta(\xi_i)(0)\}$. Successive integers in this sequence always differ by ± 1, by equation 22 and by definition of δ.

Remark 41 (Negative case)**.** *The case where $\{\delta(\xi_i)(0)\}$ contains all negative integers is not much different. Indeed, h lies in $W^s_A(\mathbf{0})$ constructed according to the sequence $\{\xi_i\}$ if and only if $\rho_*(h)$ lies in $W^s_A(\mathbf{0})$ constructed according to the sequence $\{\rho \circ \xi_i \circ \rho\}$. Further, $\{\delta(\xi_i)(0)\}$ contains all negative integers if and only if $\{\delta(\rho \circ \xi_i \circ \rho)(0)\}$ contains all positive integers.*

Proof of Lemma 42. Throughout the proof, we follow the convention that $h_j = (\xi_j)_*(h) = h \circ \xi_j^{-1}$ for all integers $j \geq 0$.

First we will prove that if the four statements are satisfied for some $J \geq 0$, then $h \in W^s_A(\mathbf{0})$. Lets assume the statements are satisfied for $J \geq 0$. In order to prove that $h \in W^s_A(\mathbf{0})$, we will show that the statements are true for infinitely many $K > J$ by induction. Since the map $i \mapsto \delta(\xi_i)(0)$ contains all positive integers and is finite to one, we can define

$$K = \max\{j > J \ : \ \delta(\xi_j)(n) = \delta(\xi_J)(n) + 2 \text{ for all } n \in \mathbb{Z}\}.$$

So, $\delta(\xi_K)$ is a translation which translates by two more than $\delta(\xi_J)$. Because we chose the last time this happens, and the sequence $\{\delta(\xi_i)(0)\}$ is a walk on the integers which in each step moves between integers which differ by ± 1, we see that:

(1') $\delta(\xi_K)$ is a translation by a positive and even integer.
(2') $\delta(\xi_j)(0) > \delta(\xi_K)(0)$ for all $j > K$.

We will now prove a refined version of statement (3). Suppose that $N \geq 4$ and that $h_J(n) = 0$ for each $n \in \mathbb{Z}^*$ with $|n| \leq N$. We will show

(3') $h_K(n) = 0$ for each $n \in \mathbb{Z}^*$ with $|n| \leq N + 1$.

In particular, we claim we get additional zeros. By factoring through the dihedral group, we see

$$h_K = (\xi_K)_*(h) = \phi_* \circ \psi_* \circ h_J. \tag{23}$$

Consider the case when $n > 0$. In this case, $h_K(n) = \psi_* \circ h_J(n)$ because of how ϕ_* acts. Using our formula for ϕ_*, we see the following cases of $n > 0$:

$$h_K(h)(1) = h_J(1) + h_J(-2) = 0.$$
$$h_K(h)(2) = h_J(1) + h_J(-2) + h_J(-3) = 0.$$
$$h_K(n) = h_J(n) + h_J(-1-n) + h_J(-n) + h_J(1-n) \quad \text{if } n > 2.$$

We can show that when $0 < n \leq N+1$, the expression evaluates to zero using our assumption (4). Here,

$$h_K(n) = h_J(n) + h_J(-1-n) + h_J(-n) + h_J(1-n) = h_J(1-n) - h_J(2-n) - h_J(3-n) = 0.$$

Now consider the case when $n < 0$. Here, using equation 24, we get:

$$h_K(h)(-1) = \psi_* \circ h_J(-1) + \psi_* \circ h_J(2) = h_J(-1) + h_J(2) + h_J(-2) + h_J(-3) = 0.$$

$$\begin{aligned} h_K(h)(-2) &= \psi_* \circ h_J(-2) + \psi_* \circ h_J(2) + \psi_* \circ h_J(3) \\ &= h_J(-2) + h_J(-1) + h_J(-4) + h_J(2) + h_J(3) = 0 \end{aligned}$$

$$\begin{aligned} h_K(h)(-3) &= \psi_* \circ h_J(-3) + \psi_* \circ h_J(2) + \psi_* \circ h_J(3) + \psi_* \circ h_J(4) \\ &= h_j(-5) + h_j(2) + h_j(3) + h_j(4) = h_j(-5). \end{aligned}$$

Observe that line

(In the last line we applied statement (4) with $n = 4$.) Now suppose $-N-1 \leq n < -3$. We have,

$$\begin{array}{rcl} h_K(n) & = & \psi_* \circ h_J(n) + \psi_* \circ h_J(-1-n) + \psi_* \circ h_J(-n) + \psi_* \circ h_J(1-n) \\ & = & h_J(-1-n) + h_J(-n) + h_J(1-n) + h_J(n-2) + h_J(n+2) \\ & = & h_J(-n) + h_J(1-n) + h_J(n-2) \\ & = & h_J(-n) + h_J(1-n) + \big(- h_J(n-1) - h_J(n+1) - h_J(n+2) - h(1-n) \big) \\ & = & h_J(-n) + h_J(n-1) + h_J(n+1) + h_J(n+2) = h_J(-n) + h_J(n-1) \\ & = & h_J(-n) + \big(- h_J(n) - h_J(n+2) - h_J(n+3) - h_J(-n) \big) = h_J(n) \\ & = & \big(- h_J(n+1) - h_J(n+3) - h_J(n+4) - h_J(-1-n) \big) = 0. \end{array}$$

Here, we repeatedly make use of the fact that $h_J(m) = 0$ if $|m| < -n$ to eliminate terms, and we have used parenthesis surrounding a sum of negative terms to denote an application of statement (4) (despite the fact that we are working in $\mathbb{Z}_2$). It still remains to verify statement (4) for h_K. Consider the sum

$$S_n = h_K(-1-n) + h_K(-n) + h_K(2-n) + h_K(3-n) + h_K(n) \quad \text{for } n \geq 4.$$

By equation 24, we see:

$$\begin{array}{rcl} S_n & = & \psi_* \circ h_J(-1-n) + \psi_* \circ h_J(-n) + \psi_* \circ h_J(2-n) + \psi_* \circ h_J(3-n) + \\ & & \psi_* \circ h_J(n+2) + \psi_* \circ h_J(n) + \psi_* \circ h_J(n-4) \\ & = & h_J(-3-n) + h_J(-2-n) + h_J(-1-n) + h_J(1-n) + h_J(2-n) + \\ & & h_J(4-n) + h_J(5-n) + h_J(n+2) + h_J(n) + h_J(n-4). \end{array}$$

Then by repeatedly applying rule (4) three times (for values of n given by $n+2$, n and then $n-4$):

$$\begin{array}{rcl} S_n & = & h_J(-1-n) + h_J(-n) + h_J(2-n) + h_J(4-n) + h_J(5-n) + h_J(n) + h_J(n-4) \\ & = & h_J(3-n) + h_J(4-n) + h_J(5-n) + h_J(n-4) \ldots \end{array}$$

CONTINUE □

OLD STATEMENT

Lemma 42. *Let $\{\xi_i\}$ be any sequence for which the set $\{i : \delta(\xi_i) = x\}$ is finite for all $x \in Isom(\mathbb{Z})$. Then, $\{\delta(\xi_i)(0)\}$ either contains all the positive integers (as a subset) or all the negative integers and not both. Suppose $\{\delta(\xi_i)(0)\}$ contains all the positive integers. Then, $h \in \mathrm{Hom}(\Gamma, \mathbb{Z}_2)$ lies in $W^s_A(\mathbf{0})$ if and only if there is an even integer $J \geq 0$:*

1. *$\delta(\xi_J)$ is a translation by an integer that is positive and even.*
2. *$\delta(\xi_j)(0) > \delta(\xi_J)(0)$ for all $j > J$.*
3. *$h_J(n) = 0$ for each $n \in \mathbb{Z}^*$ with $|n| \leq 4$, where $h_J = (\xi_J)_*(h)$.*
4. *$h_J(-1-n) + h_J(-n) + h_J(2-n) + h_J(3-n) + h_J(n) = 0$ for all $n \geq 4$.*

OLD PROOF

Proof of Lemma 42. Throughout the proof, we follow the convention that $h_j = (\xi_j)_*(h) = h \circ \xi_j^{-1}$ for all integers $j \geq 0$.

First we will prove that if the four statements are satisfied for some $J \geq 0$, then $h \in W^s_A(\mathbf{0})$. Lets assume the statements are satisfied for $J \geq 0$. In order to prove that $h \in W^s_A(\mathbf{0})$, we will show that the statements are true for infinitely many $K > J$ by induction. Since the map $i \mapsto \delta(\xi_i)(0)$ contains all positive integers and is finite to one, we can define

$$K = \max\{j > J \ : \ \delta(\xi_j)(n) = \delta(\xi_J)(n) + 2 \text{ for all } n \in \mathbb{Z}\}.$$

So, $\delta(\xi_K)$ is a translation which translates by two more than $\delta(\xi_J)$. Because we chose the last time this happens, and the sequence $\{\delta(\xi_i)(0)\}$ is a walk on the integers which in each step moves between integers which differ by ± 1, we see that:

(1') $\delta(\xi_K)$ is a translation by a positive and even integer.
(2') $\delta(\xi_j)(0) > \delta(\xi_K)(0)$ for all $j > K$.

We will now prove a refined version of statement (3). Suppose that $N \geq 4$ and that $h_J(n) = 0$ for each $n \in \mathbb{Z}^*$ with $|n| \leq N$. We will show

(3') $h_K(n) = 0$ for each $n \in \mathbb{Z}^*$ with $|n| \leq N + 1$.

In particular, we claim we get additional zeros. By factoring through the dihedral group, we see

$$h_K = (\xi_K)_*(h) = \phi_* \circ \psi_* \circ h_J. \tag{24}$$

Consider the case when $n > 0$. In this case, $h_K(n) = \psi_* \circ h_J(n)$ because of how ϕ_* acts. Using our formula for ϕ_*, we see the following cases of $n > 0$:

$$h_K(h)(1) = h_J(1) + h_J(-2) = 0.$$

$$h_K(h)(2) = h_J(1) + h_J(-2) + h_J(-3) = 0.$$

$$h_K(n) = h_J(n) + h_J(-1-n) + h_J(-n) + h_J(1-n) \quad \text{if } n > 2.$$

We can show that when $0 < n \leq N+1$, the expression evaluates to zero using our assumption (4). Here,

$$h_K(n) = h_J(n) + h_J(-1-n) + h_J(-n) + h_J(1-n) = h_J(1-n) - h_J(2-n) - h_J(3-n) = 0.$$

Now consider the case when $n < 0$. Here, using equation 24, we get:

$$h_K(h)(-1) = \psi_* \circ h_J(-1) + \psi_* \circ h_J(2) = h_J(-1) + h_J(2) + h_J(-2) + h_J(-3) = 0.$$

$$\begin{array}{rcl} h_K(h)(-2) & = & \psi_* \circ h_J(-2) + \psi_* \circ h_J(2) + \psi_* \circ h_J(3) \\ & = & h_J(-2) + h_J(-1) + h_J(-4) + h_J(2) + h_J(3) = 0 \end{array}$$

$$\begin{array}{rcl} h_K(h)(-3) & = & \psi_* \circ h_J(-3) + \psi_* \circ h_J(2) + \psi_* \circ h_J(3) + \psi_* \circ h_J(4) \\ & = & h_j(-5) + h_j(2) + h_j(3) + h_j(4) \\ & = & \big(-h_j(-4) - h_j(-2) - h_j(-1) - h_j(4)\big) + h_j(2) + h_j(3) + h_j(4) = 0. \end{array}$$

(In the last line we applied statement (4) with $n = 4$.) Now suppose $-N - 1 \leq n < -3$. We have,

$$\begin{array}{rcl} h_K(n) & = & \psi_* \circ h_J(n) + \psi_* \circ h_J(-1-n) + \psi_* \circ h_J(-n) + \psi_* \circ h_J(1-n) \\ & = & h_J(-1-n) + h_J(-n) + h_J(1-n) + h_J(n-2) + h_J(n+2) \\ & = & h_J(-n) + h_J(1-n) + h_J(n-2) \\ & = & h_J(-n) + h_J(1-n) + \big(-h_J(n-1) - h_J(n+1) - h_J(n+2) - h(1-n)\big) \\ & = & h_J(-n) + h_J(n-1) + h_J(n+1) + h_J(n+2) = h_J(-n) + h_J(n-1) \\ & = & h_J(-n) + \big(-h_J(n) - h_J(n+2) - h_J(n+3) - h_J(-n)\big) = h_J(n) \\ & = & \big(-h_J(n+1) - h_J(n+3) - h_J(n+4) - h_J(-1-n)\big) = 0. \end{array}$$

Here, we repeatedly make use of the fact that $h_J(m) = 0$ if $|m| < -n$ to eliminate terms, and we have used parenthesis surrounding a sum of negative terms to denote an application of statement (4) (despite the fact that we are working in $\mathbb{Z}_2$). It still remains to verify statement (4) for h_K. Consider the sum

$$S_n = h_K(-1-n) + h_K(-n) + h_K(2-n) + h_K(3-n) + h_K(n) \quad \text{for } n \geq 4.$$

By equation 24, we see:

$$\begin{aligned} S_n &= \psi_* \circ h_J(-1-n) + \psi_* \circ h_J(-n) + \psi_* \circ h_J(2-n) + \psi_* \circ h_J(3-n) + \\ &\quad \psi_* \circ h_J(n+2) + \psi_* \circ h_J(n) + \psi_* \circ h_J(n-4) \\ &= h_J(-3-n) + h_J(-2-n) + h_J(-1-n) + h_J(1-n) + h_J(2-n) + \\ &\quad h_J(4-n) + h_J(5-n) + h_J(n+2) + h_J(n) + h_J(n-4). \end{aligned}$$

Then by repeatedly applying rule (4) three times (for values of n given by $n+2$, n and then $n-4$):

$$\begin{aligned} S_n &= h_J(-1-n) + h_J(-n) + h_J(2-n) + h_J(4-n) + h_J(5-n) + h_J(n) + h_J(n-4) \\ &= h_J(3-n) + h_J(4-n) + h_J(5-n) + h_J(n-4) \ldots \end{aligned}$$

CONTINUE □

References

[Bow13] Joshua P. Bowman, *The complete family of Arnoux-Yoccoz surfaces*, Geometriae Dedicata **164** (2013), no. 1, 113–130 (English).

[CE07] Yitwah Cheung and Alex Eskin, *Unique ergodicity of translation flows.*, Forni, Giovanni (ed.) et al., Partially hyperbolic dynamics, laminations, and Teichmüller flow. Selected papers of the workshop, Toronto, Ontario, Canada, January 2006. Providence, RI: American Mathematical Society (AMS); Toronto: The Fields Institute for Research in Mathematical Sciences. Fields Institute Communications 51, 213-221 (2007)., 2007.

[CGL06] R. Chamanara, F. P. Gardiner, and N. Lakic, *A hyperelliptic realization of the horseshoe and baker maps*, Ergodic Theory Dynam. Systems **26** (2006), no. 6, 1749–1768. MR 2279264 (2008j:37088)

[Cha04] R. Chamanara, *Affine automorphism groups of surfaces of infinite type*, In the tradition of Ahlfors and Bers, III, Contemp. Math., vol. 355, Amer. Math. Soc., Providence, RI, 2004, pp. 123–145. MR 2145060 (2006b:30077)

[DEDML98] Mirko Degli Esposti, Gianluigi Del Magno, and Marco Lenci, *An infinite step billiard*, Nonlinearity **11** (1998), no. 4, 991–1013. MR 1632594 (99i:58092)

[GJ00] Eugene Gutkin and Chris Judge, *Affine mappings of translation surfaces: geometry and arithmetic*, Duke Math. J. **103** (2000), no. 2, 191–213. MR 1760625 (2001h:37071)

[LT14] Kathryn Lindsey and Rodrigo Treviño, *Flat surface models of ergodic systems*, Preprint.

[Mas92] Howard Masur, *Hausdorff dimension of the set of nonergodic foliations of a quadratic differential.*, Duke Math. J. **66** (1992), no. 3, 387–442 (English), Contains Masur's Criterion for ergodicity.

[PSV11] Piotr Przytycki, Gabriela Schmithüsen, and Ferrán Valdez, *Veech groups of Loch Ness monsters.*, Ann. Inst. Fourier **61** (2011), no. 2, 673–687 (English).

[Pud13] Doron Puder, *Primitive words, free factors and measure preservation*, Israel Journal of Mathematics (2013), 1–49 (English).

[Tre14] Rodrigo Treviño, *On the ergodicity of flat surfaces of finite area*, Geometric and Functional Analysis (2014), 1–27 (English).

[Tro99] Serge Troubetzkoy, *Billiards in infinite polygons*, Nonlinearity **12** (1999), no. 3, 513–524. MR 1690190 (2000b:37040)

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