Bronx Community College Colloquium
April 17, 2018

Refraction in the trihexagonal tiling

arXiv:1609.00772

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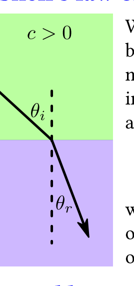
on joint work with **Diana Davis** (Swarthmore)

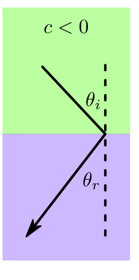
Snell's law on refraction:



When light passes between transparent media, the angles of incidence and refraction are related by

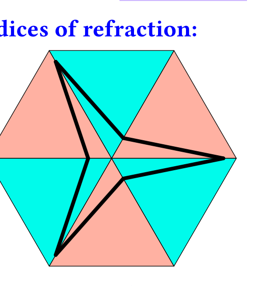
$$\frac{\sin\theta_i}{\sin\theta_r} = c,$$

where $c = \frac{n_r}{n_i}$ is the ratio of the indices of refraction of the media.



Equal but opposite indices of refraction:

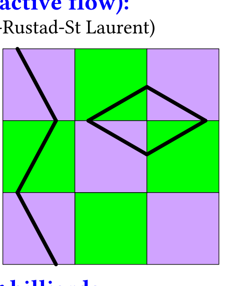
When the refraction coefficient $c = -1$, a trajectory in a line ℓ moving between media continues by following $R(\ell)$ where R is the reflection in the the tangent line at the point of intersection.



Tiling Billiards (or refractive flow):

(introduced by Davis-DiPietro-Rustad-St Laurent)

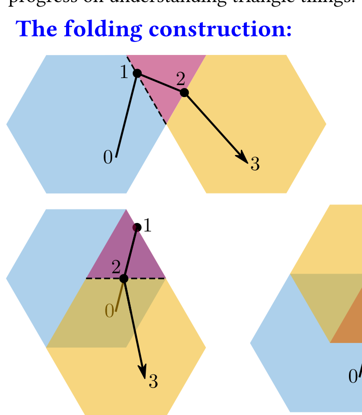
Given a tiling of the plane by regions with C^1 boundary, consider unit speed geodesics which bend according to the refraction coefficient -1 along the smooth boundary points of regions.



A brief history of tiling billiards:

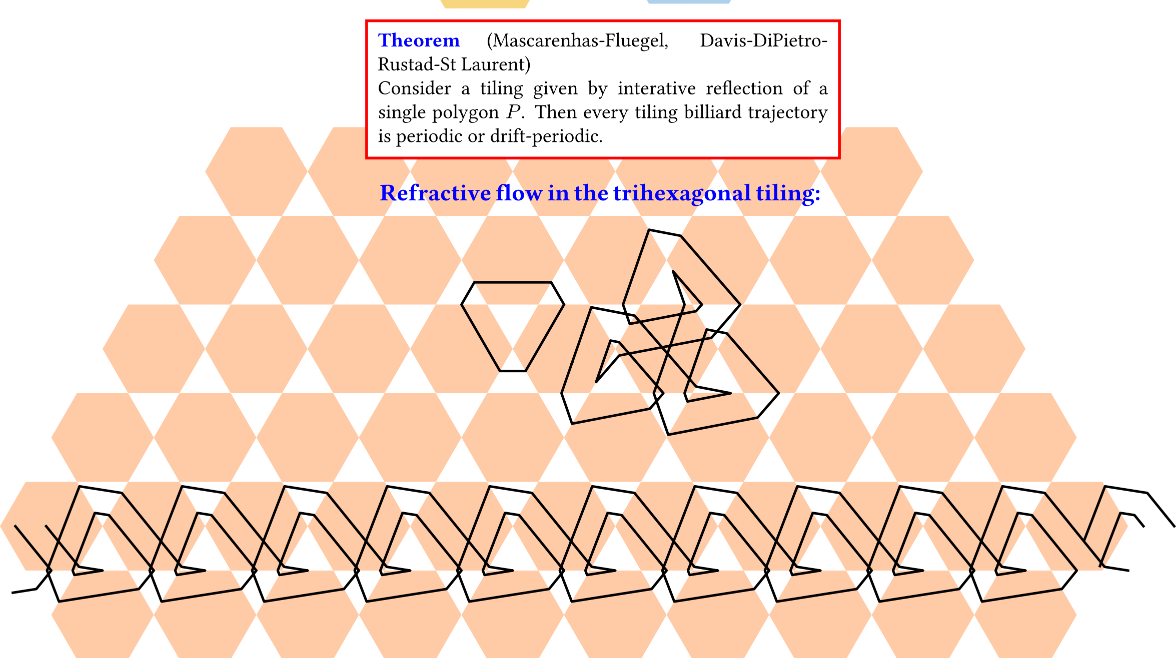
- 1967: Negative indices of refraction theorized by Russian physicist Victor Veselago.
- 2001: Shelby, Smith and Shultz discover such metamaterials.
- 2010: Physicists Mascarenhas and Fluegel study tiling billiards in the regular triangle and square tilings and the 30-60-90-triangle tiling.
- 2012: Sergei Tabachnikov poses some questions about tiling billiards to undergraduates at Summer@ICERM.
- 2012: Summer@ICERM Undergraduates Engelman and Kimball prove results on the local behaviors of trajectories around a vertex in a tiling.
- 2013: Summer@ICERM Undegraduates DiPietro, Rustad and St Laurent working with Davis study various tilings (triangle tilings, trihexagonal tiling) find mechanisms for generating a lot of periodic and unbounded trajectories (to appear).
- 2016: Williams College Undergraduates Baird-Smith, Fromm and Iyer working with Davis make further progress on understanding triangle tilings.

The folding construction:



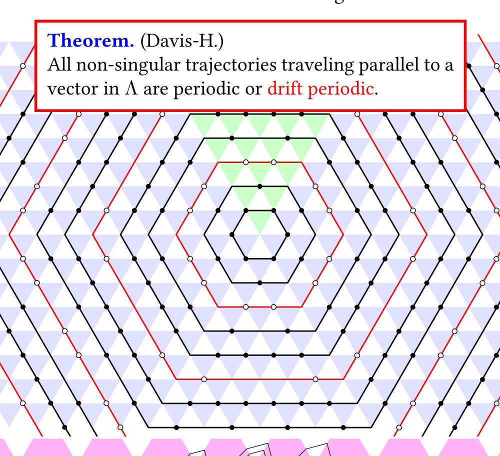
Theorem (Mascarenhas-Fluegel, Davis-DiPietro-Rustad-St Laurent)
Consider a tiling given by interative reflection of a single polygon P. Then every tiling billiard trajectory is periodic or drift-periodic.

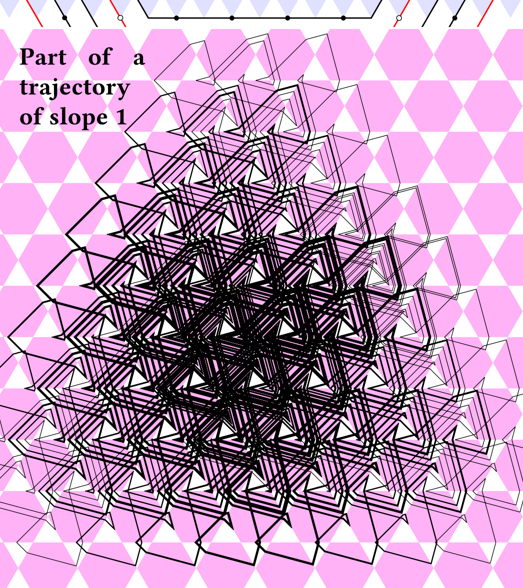
Refractive flow in the trihexagonal tiling:



Let Λ be the lattice of Eisenstein integers.

Theorem. (Davis-H.)
All non-singular trajectories traveling parallel to a vector in Λ are periodic or drift periodic.

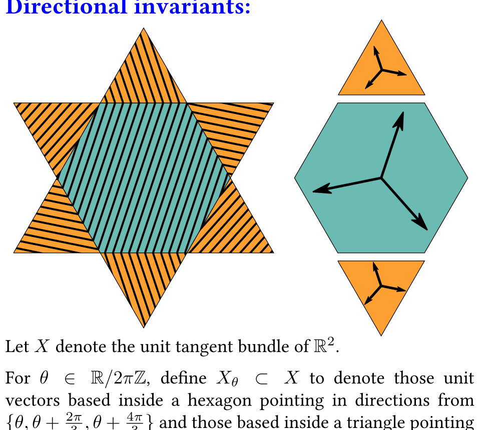




In fact, almost every trajectory of slope 1 is dense in an open periodic region of infinite area.

Invariant sets:

Directional invariants:



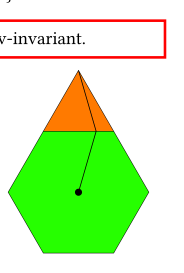
Let X denote the unit tangent bundle of $\mathbb{R}^2$.

For $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, define $X_\theta \subset X$ to denote those unit vectors based inside a hexagon pointing in directions from $\{\theta, \theta + \frac{2\pi}{3}, \theta + \frac{4\pi}{3}\}$ and those based inside a triangle pointing in directions from $\{\pi - \theta, \frac{\pi}{3} - \theta, \frac{5\pi}{3} - \theta\}$.

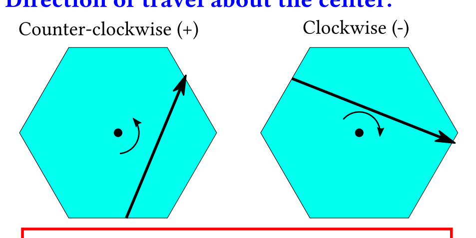
For each θ, the set X_θ is refractive flow-invariant.

Centers of hexagons are singular:

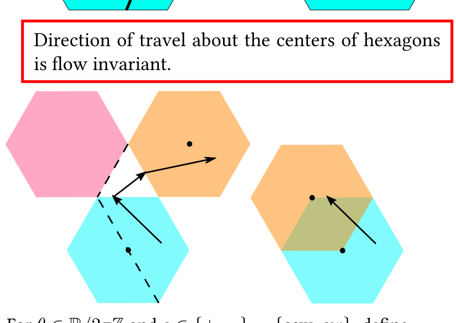
Any trajectory passing through the center of a hexagon hits singularities in forward and backward time.



Direction of travel about the center:

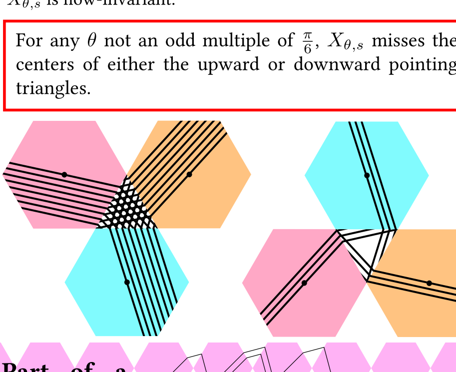


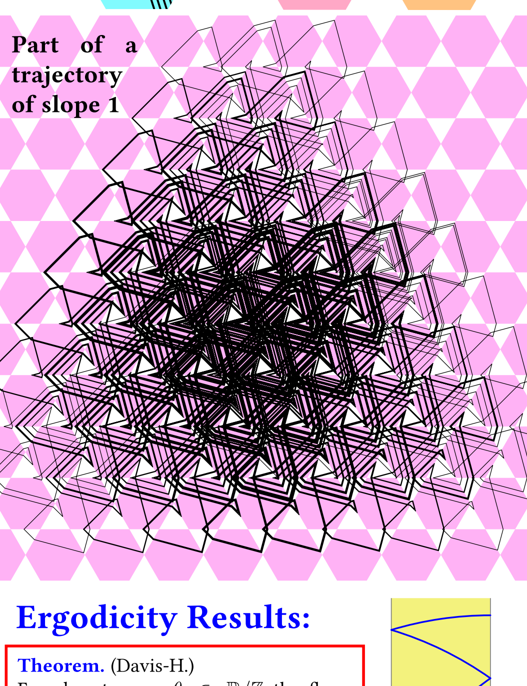
Direction of travel about the centers of hexagons is flow invariant.



For $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ and $s \in \{+, -\} = \{ccw, cw\}$, define $X_{\theta,s}$ to be those vectors in X_θ which when inside hexagons travel with sign s around the centers. Then $X_{\theta,s}$ is flow-invariant.

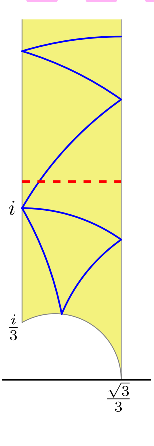
For any θ not an odd multiple of $\frac{\pi}{6}$, $X_{\theta,s}$ misses the centers of either the upward or downward pointing triangles.





Ergodicity Results:

Theorem. (Davis-H.)
For almost every $\theta \in \mathbb{R}/\mathbb{Z}$ the flows obtained by restricting to $X_{\theta,+}$ and $X_{\theta,-}$ (endowed with pull-backs of Lebesgue measure) are ergodic.

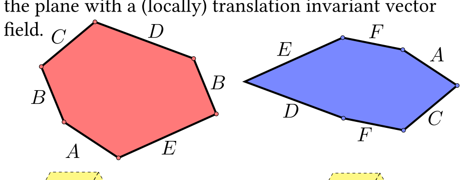


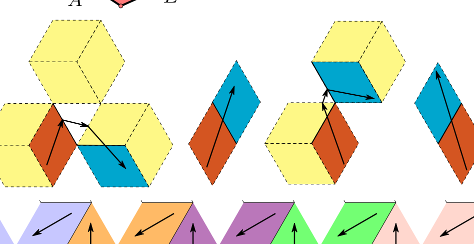
A related billiard table:

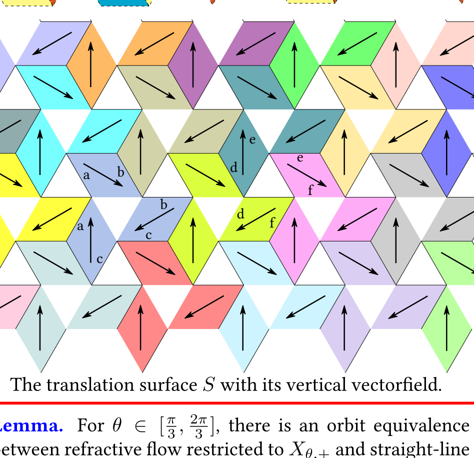
Theorem. (Davis-H.)
Ergodicity occurs for $\theta \in [\frac{\pi}{3}, \frac{\pi}{2}]$ if the billiard trajectory leaving i at angle 2θ from the vertical travels infinitely often above the line $y = \frac{2}{\sqrt{3}}$.

Translation surface:

A *translation surface* is a surface locally isometric to the plane with a (locally) translation invariant vector field.

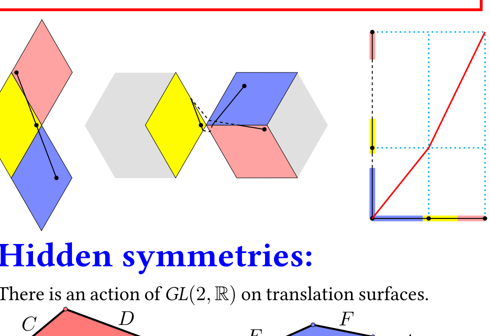






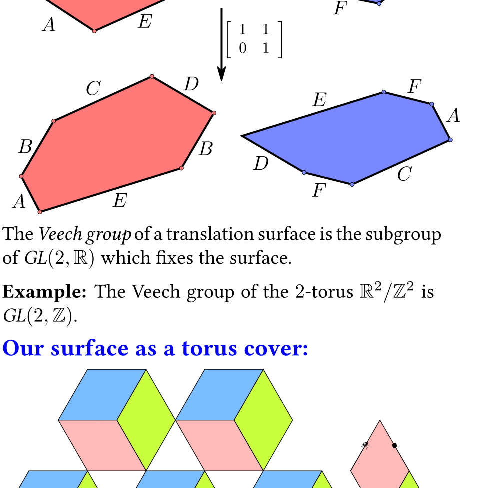
The translation surface S with its vertical vectorfield.

Lemma. For $\theta \in [\frac{\pi}{3}, \frac{2\pi}{3}]$, there is an orbit equivalence between refractive flow restricted to $X_{\theta,+}$ and straight-line flow on S in direction θ.



Hidden symmetries:

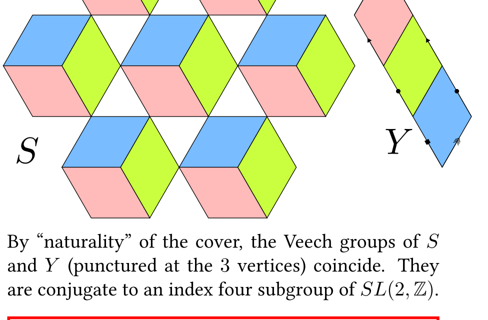
There is an action of $GL(2, \mathbb{R})$ on translation surfaces.



The *Veech group* of a translation surface is the subgroup of $GL(2, \mathbb{R})$ which fixes the surface.

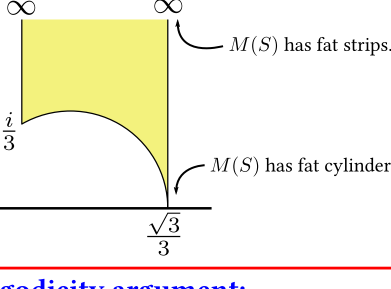
Example: The Veech group of the 2-torus $\mathbb{R}^2/\mathbb{Z}^2$ is $GL(2, \mathbb{Z})$.

Our surface as a torus cover:



By "naturality" of the cover, the Veech groups of S and Y (punctured at the 3 vertices) coincide. They are conjugate to an index four subgroup of $SL(2, \mathbb{Z})$.

Lemma. The Veech groups of S and Y are the reflection groups in the hyperbolic triangle below.



Ergodicity argument:

(Following Pascal Hubert & Barak Weiss)

Skew products:

Let Λ be the Eisenstein lattice, which is the deck group of the cover $S \to Y$.

Fix θ not parallel to a vector in Λ. Let $F : Y \to Y$ be straight-line flow on the torus, which is ergodic for lebesgue measure m. There is a cocycle

$$\alpha : Y \times \mathbb{R} \to \Lambda \quad \text{satisfying} \quad \alpha(y, t+s) = \alpha(y,t) + \alpha\big(F^t(y), s\big)$$

so that straight-line flow on S in direction θ is measurably conjugate to

$$\tilde{F}_t : Y \times \Lambda \to Y \times \Lambda; \quad (y, \lambda) \mapsto \big(F^t(y), \lambda + \alpha(y,t)\big).$$

A value $\lambda \in \Lambda$ is an *essential value* for $\tilde{F}$ if for every $A \subset Y$ with $m(A) > 0$, there is a subset $T \subset \mathbb{R}$ of positive Lebesgue measure so that for all $t \in T$,

$$m\{y \in A : F^t(y) \in A \quad \text{and} \quad \alpha(y,t) = \lambda\} > 0.$$

Theorem. (K. Schmidt)
$\tilde{F}^t$ is ergodic if and only if there are essential values generating Λ.

Theorem. (Hubert-Weiss)
If (θ, λ) is "well-approximated by strips" then λ is an essential value for $\tilde{F}$.