NOTES ON DEFORMING THE STAIRCASE SURFACE

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The intuitive idea is that if two (infinite) translation surfaces have "large" affine automorphism groups which act in the same way, then they should have their geodesics should have the same combinatorics.

Here's a way to make this rigorous. Let X and Y be infinite translation surfaces. Let $\Sigma(X)$ and $\Sigma(Y)$ denote the singularities on these surfaces. Let X_Σ and Y_Σ be the surfaces X and Y with all the singularities identified. (This is so that saddle connections become closed curves which can be detected by their homotopy classes.) Now suppose we have We say a homeomorphism $\phi : X_\Sigma \to Y_\Sigma$ *preserves saddle connections (up to homotopy)* if the following two statements are satisfied.

- Given any saddle connection σ in X_Σ there is a saddle connection σ' in the homotopy class $[\phi(\sigma)]$.
- Given any saddle connection σ' in Y_Σ there is a saddle connection σ in the homotopy class $[\phi^{-1}(\sigma')]$.

(Existence of such a ϕ is an equivalence relation among infinite translation surfaces. We say X and Y *have the same saddle connections.*)

Now suppose the translation surface X has set of disjoint saddle connections S_X which triangulate X and that $\phi : X_\Sigma \to Y_\Sigma$ is a homeomorphism which preserves saddle connections. It follows that the set of saddle connections of Y given by

$$S_Y = \{\sigma' \ : \ \sigma' \in [\phi(\sigma)] \text{ for } \sigma \in S_X\}$$

is a disjoint set of saddle connections which triangulates Y. Moreover we have the following.

Theorem 1 (Same coding theorem)**.** *Given any straight-line trajectory (finite or infinite or bi-infinite) T in X consider the sequence $\langle \sigma_i \in S_X \rangle$ of saddle connections in S_X crossed by T. Then there is a straight-line trajectory T' in Y which crosses the corresponding sequence of saddle connections $\langle \sigma_i' \in S_Y \rangle$ with $\sigma_i' \in [\phi(\sigma_i)]$ for all i.*

A variant of this theorem was stated as lemmas 8 and 11 in my paper "Dynamics on an infinite surface with the lattice property." The proof is essentially the same.

An essential idea in the paper was to find a deformation of the translation surface into surfaces that have the same saddle connections. Intuitively, in these cases you have the lattice property, so you can try to build these surfaces by preserving the topological action of the affine automorphism group.

Let X_0 denote your staircase translation surface depicted in figure 1 below. We will find a deformation of X_0 to a family of surfaces X_t with $t \in \mathbb{R}$ which have affine automorphism groups which act in the same way as the affine automorphism group of X_0. (Actually, the deformed surfaces will only preserve an index two subgroup of the affine automorphism group. This is the subgroup which preserves the two "ends" of the staircase.)

Consider the affine automorphisms $\phi^h : X_0 \to X_0$ and $\phi^v : X_0 \to X_0$ which act by simultaneous right Dehn twist in the horizontal and vertical cylinders respectively. Veech's

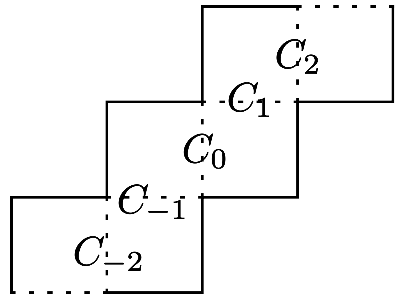

FIGURE 1. The staircase surface X_0 has a horizontal and vertical cylinder decompositions. These cylinders are named C_j with $i \in \mathbb{Z}$. When j is even C_j is horizontal, and when j is odd C_j is vertical.

necessary and sufficient conditions to have such actions is that the horizontal cylinders must all be the same modulus and the vertical cylinders must all be the same modulus. We will build homeomorphic surfaces X_t which also have this property. By conjugating by an element of $SL(2,\mathbb{R})$, we may assume that these surfaces also have a horizontal and vertical cylinder decomposition. Also, we assume that all these cylinders (both horizontal and vertical) have the same modulus. Thus, we can specify the geometry of the surface by determining the widths of the cylinders.

The cylinders are named so that C_j represents a horizontal cylinder when i is even and a vertical cylinder when j is odd. And, the cylinder C_j intersects only the cylinders C_{j-1} and C_{j+1}. Let w_j denote the width of the cylinder C_j. Then the modulus of cylinder C_j is given by

$$M_j = \frac{w_j}{w_{j-1}+w_{j+1}}$$

So, we will assume that there is an M such that for all j,

$$Mw_j = w_{j-1}+w_{j+1} \quad \text{and} \quad w_j > 0$$

I believe the only solutions to this set of equations are given by

$$w_j = ae^{jt}+be^{-jt} \quad \text{and} \quad M = e^t + e^{-t} \tag{1}$$

with $t \in \mathbb{R}$ and $a \geq b \geq 0$, but not both $a = b = 0$.

Now we consider the remainder of the affine automorphism groups. There are translations that preserve the staircase X_0. There are also rotations by ± 90 degrees that send horizontal cylinders to vertical cylinders and vertical to horizontal. The key here is to allow affine automorphisms whose derivatives lie in $GL(2,\mathbb{R})$. Consider the affine automorphism ψ which sends each cylinder C_i to C_{i+1}. The affine automorphism must scale the area of each cylinder by a constant. The area of cylinder C_i is given by $A_j = w_j(w_{j-1}+w_{j+1})$. If $\frac{A_{j+1}}{A_j}$ is a constant independent of j then the only reasonable solutions from equation 1 are

$$w_j = e^{jt}$$

for some $t \in \mathbb{R}$ up to a uniform scalar constant.

Summarizing, we have the following.

Proposition 2. *The surfaces X_t with $t \in \mathbb{R}$ and cylinder widths given by $w_j = e^{jt}$ have affine automorphism groups which act in the topologically the same way as the portion of* *the affine automorphism group of X_0 which preserves the two ends of the staircase. (Here* topologically the same *means up to conjugacy by a fixed homeomorphism and isotopy.)*

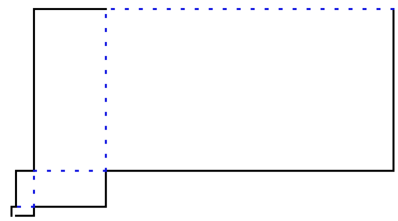

FIGURE 2. The surface X_t with $t = \log 2$.

The affine automorphism groups can be used to understand the saddle connections of the surfaces X_t. The action of this group can be shown that these surfaces all have the same saddle connections. (This is also how I did it for my surface in my article.) It follows that they have the surfaces have the same geodesics (in the coding sense of theorem 1).

The affine automorphism ψ^2 sends each cylinder C_i to C_{i+2}. The surface X_t/ψ^2 is interesting. See figure 3.

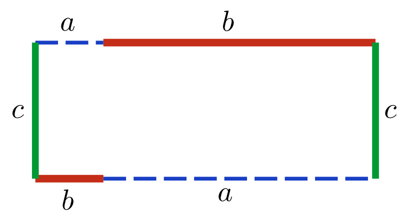

FIGURE 3. The quotient X_t/ψ^2 with $t = \log 2$. The edges marked a are glued together affinely, by scaling the small interval up to the size of the big interval to identify points. The same with the edges marked b. The vertical edges marked c are glued by translation.

The surface is naturally a torus with a singular affine structure. Note that ψ does not commute with the geodesic flow Φ_s (with s the time parameter). Rather, it satisfies $\psi \circ \Phi_s = \Phi_{se^t} \circ \psi$. So, the geodesic flow is not well defined on the quotient X_t/ψ^2 but the notion of a trajectory is well defined.

Since we have the notion of a trajectory, we do have the notion of a first return to the horizontal edges $S^1 = a \cup b$ of figure 3. Consider the flow in directions of slope m. Let the map $r_t^m : S^1 \to S^1$ denote the first return map of the trajectories of slope m to the horizontal edges $S^1 = a \cup b$ of the surface X_t/ψ^2.

Now, you can now see these 1-parameter invariant measures for the staircase surface X_0. Fix an irrational slope m on X_0. The return map $r_0^m : S^1 \to S^1$ has rotation number $\frac{2}{m}$.

Given t find the slope $m(t)$ such that the return map $r_t^{m(t)} : S^1 \to S^1$ has the same rotation number as r_0^m, $\frac{2}{m}$. (The rotation number of r_t^m varies continuously and monotonically in the slope m. This implies existence of $m(t)$. The affine automorphism groups can be used to demonstrate uniqueness of the slope $m(t)$.) Since the rotation numbers are the same, there is a topological conjugacy between the return maps r_0^m and $r_t^{m(t)}$.

Recall that the surfaces X_0 and X_t have the same geodesics. The rotation number of a geodesic on X_t (with rotation number measured on X_t/ψ^2) is determined by the sequence of saddle connections it crosses. It is also determined by the slope of the geodesic. It follows that there is a homeomorphism $h : X_0 \to X_t$ which sends trajectories of slope m in X_0 to trajectories of slope $m(t)$ in X_t which preserves the codings of geodesics in the sense of theorem 1. If we push this identification of trajectories down to X_0/ψ^2 and X_t/ψ^2 this gives a topological conjugacy of the type guaranteed by the previous paragraph.

Note that the new invariant measure the geodesic flow on X_0 in the direction of slope m is not Lebesgue measure on X_t with the flow in direction $m(t)$, because the return times to the lifts of the horizontal segments in X_t are different than those in X_0. We correct this minor problem below.

Now look at the first return map $\tilde{r}_t^m$ of the flow in the direction of slope m in the surface X_t to the boundaries of the horizontal cylinders. (This is a skew product over r_t^m.) The homeomorphism $h : X_0 \to X_t$ may be taken to preserve the boundaries of horizontal cylinders. The restriction of h to the boundaries of the horizontal cylinders induces a conjugacy between the return maps $\tilde{r}_0^m$ and $\tilde{r}_t^{m(t)}$. Then Lebesgue measure in X_t on the boundaries of the horizontal cylinders pulls back to an alternate invariant measure for $\tilde{r}_0^m$. This measure can be suspended to give an alternate invariant measure for the geodesic flow on X_0 in the direction of slope m.