Billiards in right triangles are unstable

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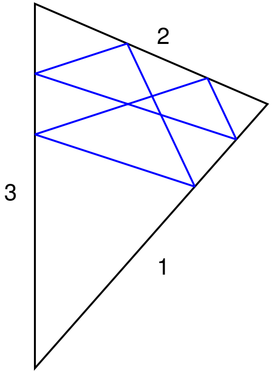
Comparing billiard paths in different triangles

- Mark the edges of our triangles by $\{1, 2, 3\}$.
- The **orbit-type** $\mathcal{O}(\widehat{\gamma})$ of a periodic billiard path $\widehat{\gamma}$ is the bi-infinite periodic sequence of markings corresponding to the edges hit.



A periodic billiard path $\widehat{\gamma}$ with orbit type $\mathcal{O}(\widehat{\gamma}) = \overline{123123}$.

Comparing billiard paths in different triangles

Open Question

Does every triangle have a periodic billiard path?

- Let $\mathcal{T}$ be the space of marked triangles up to similarities preserving the markings.
- We coordinatize $\mathcal{T}$ by the angles of the triangles:

$$\mathcal{T} = \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \mid \alpha_1 + \alpha_2 + \alpha_3 = \pi \text{ and each } \alpha_i > 0\}.$$

- The **tile** of a periodic billiard path $\widehat{\gamma}$ is the subset *tile*$(\widehat{\gamma}) \subset \mathcal{T}$ consisting of all triangles $\Delta \in \mathcal{T}$ with periodic billiard paths $\hat{\eta}$ with the same orbit type as $\widehat{\gamma}$.
- The question above becomes equivalent to "Can $\mathcal{T}$ be covered by tiles?"

Theorem (Classification of tiles)

Let $\widehat{\gamma}$ be a periodic billiard path in a triangle Δ. Then either

1. *tile$(\widehat{\gamma})$ is an open subset of $\mathcal{T}$, or*
2. *tile$(\widehat{\gamma})$ is an open subset of a* **rational line** *of the form*

$$\{(\alpha_1, \alpha_2, \alpha_3) \in \mathcal{T} \mid n_1\alpha_1 + n_2\alpha_2 + n_3\alpha_3 = 0\}$$

   for some integers n_1, n_2, n_3 (not all zero).

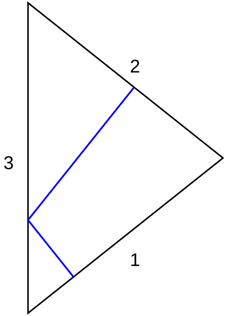
- In the first case, $\widehat{\gamma}$ is called **stable**. In any sufficiently small perturbation of Δ, we can find a periodic billiard path with the same orbit-type.
- Almost every triangle only has stable periodic billiard paths!
- But for example, right triangles may have unstable periodic billiard paths. Since if $\alpha_1 = \frac{\pi}{2}$, then

$$\alpha_1 - \alpha_2 - \alpha_3 = 0.$$

Example of an unstable periodic billiard path

- A periodic billiard path $\widehat{\gamma}$ with orbit type $\mathcal{O}(\widehat{\gamma}) = \overline{1323}$ is unstable.
- By similar triangles, *tile*$(\widehat{\gamma})$ is the collection of isosceles triangles with base marked '3'.

$$\textit{tile}(\widehat{\gamma}) = \{(\alpha_1, \alpha_2, \alpha_3) \in \mathcal{T} \mid \alpha_1 - \alpha_2 = 0\}.$$



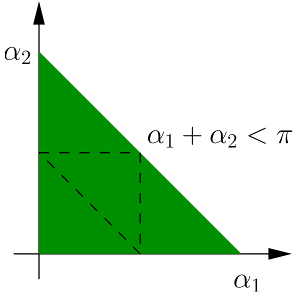
Theorems on stability in right triangles

The following settles a conjecture of Vorobets, Galperin, and Stepin.

Theorem (H)

Right triangles do not have stable periodic billiard paths.

Let $\mathcal{R} \subset \mathcal{T}$ denote the collection of right triangles.

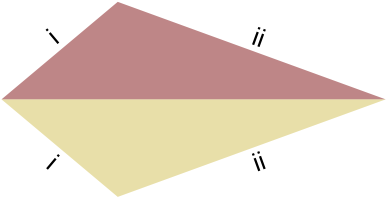


Theorem (H)

If $\widehat{\gamma}$ is a stable periodic billiard path in a triangle, then tile$(\widehat{\gamma})$ is contained in one of the connected components of $\mathcal{T} \smallsetminus \mathcal{R}$.

When are billiard paths stable? (1 of 4)

- Given a triangle Δ we can build a Euclidean cone surface $\mathcal{D}(\Delta)$ by doubling the triangle across its boundary. (A triangular pillowcase.)
- Let Σ denote the collection of cone singularities on $\mathcal{D}(\Delta)$.



- There is a natural folding (or ironing) map $\mathcal{D}(\Delta) \to \Delta$.
- The billiard flow on Δ can be pulled back to the geodesic flow on $\mathcal{D}(\Delta) \setminus \Sigma$.
- Each periodic billiard path $\widehat{\gamma}$ on Δ pulls back to a closed geodesic γ on $\mathcal{D}(\Delta)$.

When are billiard paths stable? (2 of 4)

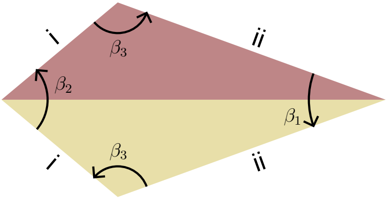
- Let $\theta : T_1\mathbb{R}^2 \to \mathbb{R}/2\pi\mathbb{Z}$ be the function which measures angle.
- The closed 1-form $d\theta$ on $T_1\mathbb{R}^2$ is invariant under the action of $\mathrm{Isom}_+(\mathbb{R}^2)$.
- It pulls back to closed 1-form on the unit tangent bundle of any locally Euclidean surface.
- If γ is a closed geodesic on a locally Euclidean surface then

$$\int_{\gamma'} d\theta = 0.$$

- Because $d\theta$ is closed, this is a homological invariant of the curve γ' in the unit tangent bundle.

When are billiard paths stable? (3 of 4)

- In our case, the surface is the double of a triangle $\mathcal{D}(\Delta)$.



- The homology group of the unit tangent bundle, $H_1\big(T_1\mathcal{D}(\Delta) \setminus \Sigma, \mathbb{Z}\big) \cong \mathbb{Z}^3$, is generated by derivatives of curves which travel around the punctures.

- If $[\![x]\!] = n_1\beta_1' + n_2\beta_2' + n_3\beta_3' \in H_1(T_1\mathcal{D}(\Delta) \setminus \Sigma, \mathbb{Z})$ then

$$\int_{[\![x]\!]} d\theta = 2n_1\alpha_1 + 2n_2\alpha_2 + 2n_3\alpha_3.$$

When are billiard paths stable? (4 of 4)

- Consequently, in order for a homology class $[\![x]\!] = n_1\beta_1' + n_2\beta_2' + n_3\beta_3' \in H_1(T_1\mathcal{D}(\Delta) \smallsetminus \Sigma, \mathbb{Z})$ to contain the derivative of a closed geodesic, it must be

$$n_1\alpha_1 + n_2\alpha_2 + n_3\alpha_3 = 0.$$

- Suppose, $\widehat{\gamma}$ is a stable periodic billiard path. Let γ be the pull back to $\mathcal{D}(\Delta)$. Then γ' must be homologous to zero in $T_1\mathcal{D}(\Delta) \smallsetminus \Sigma$.

- **Remark:** In fact, this is a sufficient condition for stability. This can be seen by checking that all remaining conditions for a homotopy class in $\mathcal{D}(\Delta) \smallsetminus \Sigma$ to contain a geodesic are open conditions.

Theorem (Classification of tiles)

Let $\widehat{\gamma}$ be a periodic billiard path in a triangle Δ. Then tile($\widehat{\gamma}$) is stable iff γ' is null homologous. Otherwise, tile($\widehat{\gamma}$) is an open subset of a rational line.

Translation surfaces (1 of 2)

- A **translation surface** is a Euclidean cone surface whose cone angles are all in $2\pi\mathbb{N} \cup \{\infty\}$.
- These surfaces appear naturally from the point of view of the previous discussion.
- Consider the group homomorphism

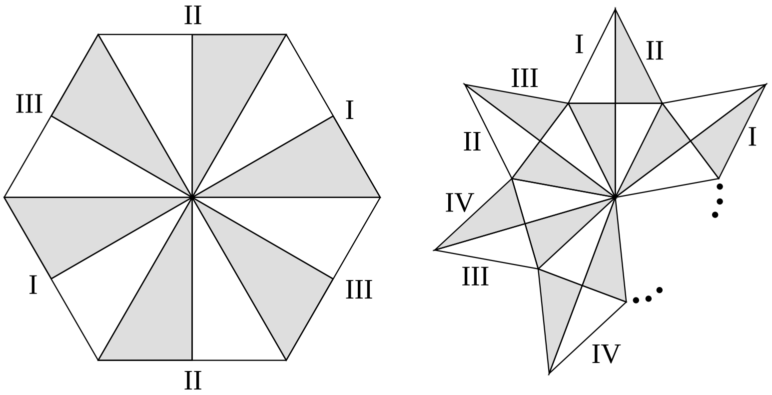
$$\Theta : \pi_1(T_1\mathcal{D}(\Delta) \smallsetminus \Sigma) \to \mathbb{R} : [x] \mapsto \int_x d\theta.$$

- Let $\phi : T_1\mathcal{D}(\Delta) \smallsetminus \Sigma \to \mathcal{D}(\Delta) \smallsetminus \Sigma$.
- Let $G = \phi_*(ker\ \Theta) \subset \pi_1(\mathcal{D}(\Delta) \smallsetminus \Sigma)$.
- A homotopy class $[\gamma]$ in $\mathcal{D}(\Delta) \smallsetminus \Sigma$ must lie in G in order to contain a geodesic.
- The cover of $\mathcal{D}(\Delta)$ branched over Σ associated to G is a translation surface $MT(\Delta)$.

Translation surfaces (2 of 2)

- We call $MT(\Delta)$ the minimal translation surface cover of $\mathcal{D}(\Delta)$.
- We will now describe a less technical definition of $MT(\Delta)$.
- Let H be the group $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ generated by r_1, r_2, r_3.
- Let $\rho : H \to \mathrm{Isom}\, \mathbb{R}^2$ which sends each generator r_i to reflection in the i-th side of Δ.
- $MT(\Delta)$ is $\{h(\Delta) \mid h \in H\}$ with some identifications:
  1. Identify $h_1(\Delta)$ and $h_2(\Delta)$ along the edge i if $h_1 \circ h_2^{-1} = r_i$.
  2. Identify triangles $h_1(\Delta)$ and $h_2(\Delta)$ if $\rho(h_1 \circ h_2^{-1})$ is a translation.

Example minimal translation surfaces



- The minimal translation surface for the 30-60-90 triangle is a hexagonal torus.
- The minimal translation surface of a right triangle whose other angles are not rational multiples of π is an infinite union of rombi.

Closed geodesics on translation surfaces

- Every closed geodesic on a Euclidean cone surface $\mathcal{D}(\Delta)$ lifts to the minimal translation surface cover of $MT(\Delta)$.
- The direction map $\theta : T_1\mathbb{R}^2 \to \mathbb{R}/2\pi\mathbb{Z}$ lifts to a map $\theta : T_1 MT(\Delta) \to \mathbb{R}/2\pi\mathbb{Z}$.
- The direction map θ is invariant under the geodesic flow. Thus, closed geodesics on $MT(\Delta)$ never intersect.
- Moreover, two geodesics which travel in the same direction can not intersect.
- This is the main idea behind the proof of

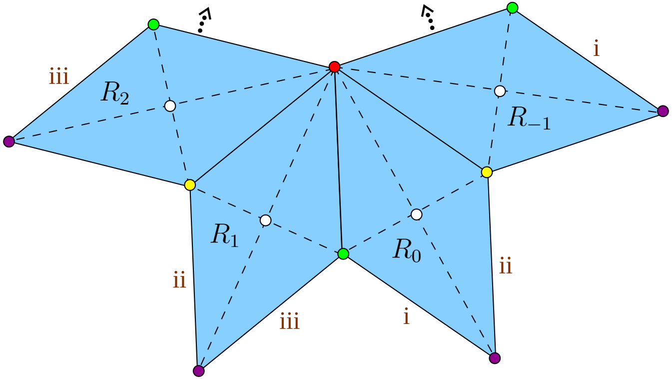
Theorem

Right triangles don't have stable periodic billiard paths.

We will now discuss the proof.

No stable trajectories (1 of 4)

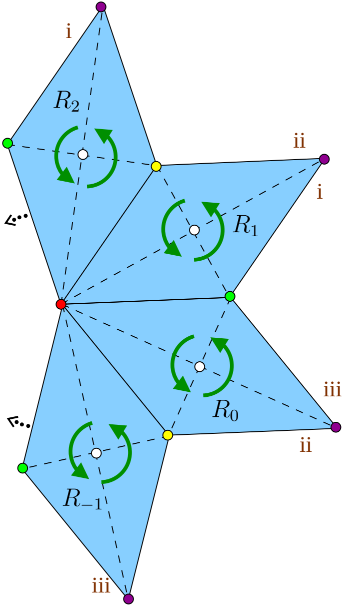
- It is sufficient to prove that a right triangle Δ whose other angles are not rational multiples of π has no stable periodic billiard paths.
- A periodic billiard path $\widehat{\gamma}$ in Δ lifts to a closed geodesic γ in $\mathcal{D}(\Delta)$.
- γ lifts to a closed geodesic $\tilde{\gamma}$ in $MT(\Delta)$.



No stable trajectories (2 of 4)

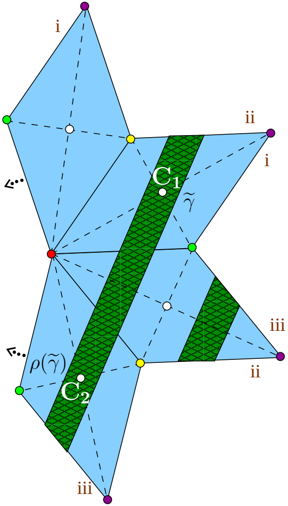
- $MT(\Delta)$ supports a rotation by π, ρ, which preserves each rhombus.
- This rotation by π, ρ, is an automorphism of the cover

$$MT(\Delta) \to \mathcal{D}(\Delta)$$

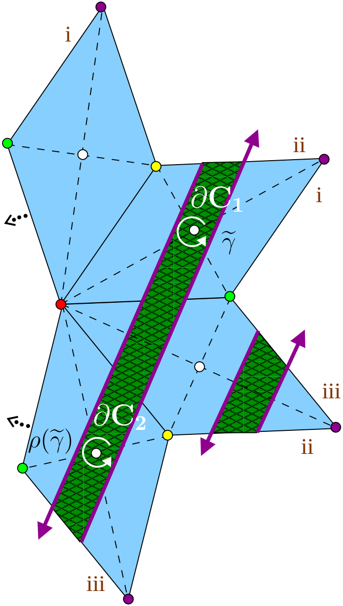


No stable trajectories (3 of 4)

- **Sketch of proof:** Right triangles only have unstable periodic billiard paths.
- We will show that the pair of curves $\tilde{\gamma}$ and $\rho(\tilde{\gamma})$ bound a cylinder in $MT(\Delta)$ containing two centers of rhombi, C_1 and C_2.



No stable trajectories (4 of 4)



- Then in homology on MT_Δ,

$$[\![\tilde{\gamma}]\!] + [\![\rho(\tilde{\gamma})]\!] = [\![\partial C_1]\!] + [\![\partial C_2]\!]$$

- Under the covering map

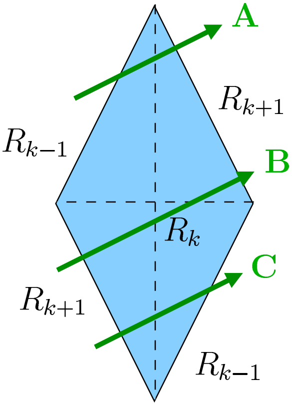
$$\phi : MT_\Delta \to \mathcal{D}_\Delta,$$

  - $\phi([\![\tilde{\gamma}]\!]) = \phi([\![\rho(\tilde{\gamma})]\!]) = [\![\gamma]\!]$
  - $\phi([\![\partial C_1]\!]) = \phi([\![\partial C_2]\!])$
  - Thus,

$$[\![\gamma]\!] = \phi([\![\partial C_1]\!]) \neq 0$$

- The geodesic γ is **unstable**!

Finding the cylinder (1 of 3)

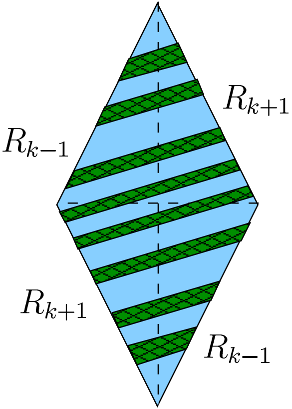


Claim 1: $\widetilde{\gamma} \cup \rho(\widetilde{\gamma})$ intersects each edge of each rhombus an even number of times.

Proof:

- Fixing the direction $\widetilde{\gamma}$ travels, there are only 3 possible ways $\widetilde{\gamma}$ can cross each rhombus R_k.
- The claim is equivalent to showing that the number of type **A** crossings of $\widetilde{\gamma}$ equals the number of type **C** crossings of $\widetilde{\gamma}$.
- But, $\widetilde{\gamma}$ must close up. So, each time it passes from R_{k+1} to R_{k-1} it must later pass from R_{k-1} to R_{k+1}.

Finding the cylinder (2 of 3)



Claim 2: $\widetilde{\gamma} \cup \rho(\widetilde{\gamma})$ disconnects $MT(\Delta)$. At least one component contains no singularities with infinite cone angle.

Proof:

- By claim 1, each rhombus $R_k \smallsetminus (\widetilde{\gamma} \cup \rho(\widetilde{\gamma}))$ may be colored so that each infinite cone point is blue, and colors alternate blue/green by adjacency.
- The colorings of each rhombus agree along the boundaries of the rhombi. So, the green and blue components are distinct.

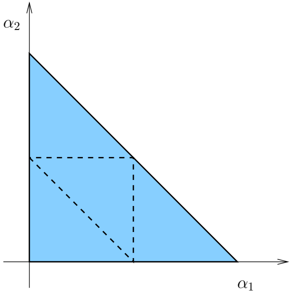
  $\square$

Finding the cylinder (3 of 3)

- The only oriented Euclidean surface with one or two geodesic boundary components is a cylinder.
- We still must show that the cylinder contains two centers of rhombi.
- By construction, the rotation by π must preserve the cylinder. A rotation by π of a cylinder has 2 fixed points. The only fixed points of the rotation by π of $MT(\Delta)$ are the centers. $\square$

The generalization

- The right triangles consist of three lines ℓ_1, ℓ_2, and ℓ_3 in the space of triangles $\mathcal{T}$.

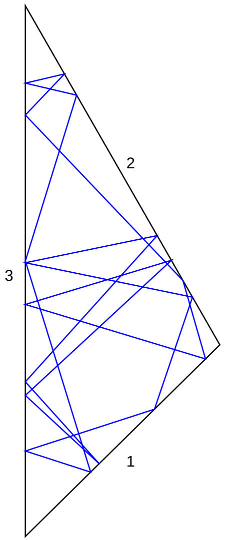


Theorem (H)

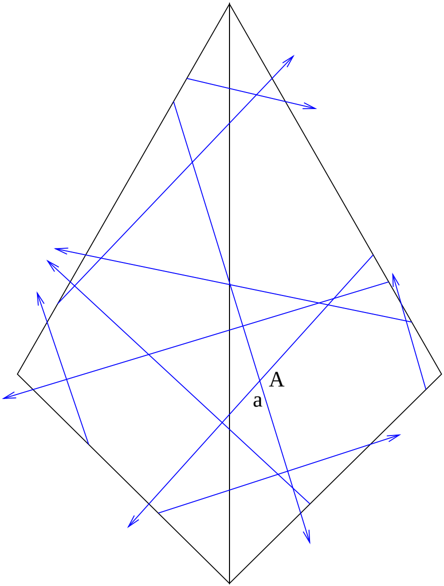
If $\widehat{\gamma}$ is a **stable** *periodic billiard path in a triangle, then tile$(\widehat{\gamma})$ is contained in one of the four components of $\mathcal{T} \setminus (\ell_1 \cup \ell_2 \cup \ell_3)$.*

The argument in action (1 of 2)

- Here is a **stable** periodic billiard path in a slightly obtuse triangle.
- Let's prove that a periodic billiard path with the same orbit type can not appear in a triangle where this obtuse angle becomes right or acute.

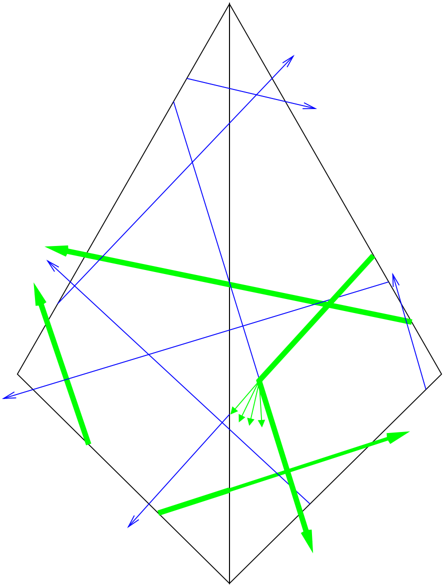


The argument in action (2 of 2)



- The proof follows from the "general principle" that intersections between geodesics on locally Euclidean surfaces are "essential."
- For every triangle Δ with a geodesic in this homotopy class on $\mathcal{D}(\Delta)$, we can find an intersection A with similar topological properties.

The argument in action (3 of 2)



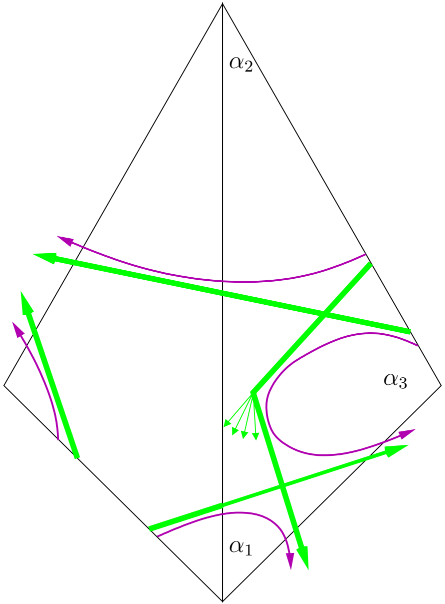
- This angle a must satisfy $0 < a < \pi$.
- We compute this angle using a "detecting curve" η' on the unit tangent bundle.

$$a = \int_{\eta'} d\theta$$

- Thus, for all $\Delta \in tile(\widehat{\gamma})$,

$$0 < \int_{\eta'} d\theta_{\Delta} < \pi$$

The argument in action (4 of 4)



- We compute

$$\begin{aligned} a &= \int_\eta d\theta \\ &= 2\alpha_3 - 2\alpha_1 - 2\alpha_2 \\ &= 4\alpha_3 - 2\pi \end{aligned}$$

- For all $\Delta \in tile(\widehat{\gamma})$,

$$0 < a < \pi$$

So,

$$\frac{\pi}{2} < \alpha_3 < \frac{3\pi}{4}$$

The argument in action (5 of 5)

Iterating over all intersections gives a convex bounding box for the tile.