IMMERSIONS AND TRANSLATION STRUCTURES ON THE DISK

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Abstract. A translation structure on a surface is an atlas of charts to the plane so that the transition functions are translations. We will investigate the moduli space of pointed translation structures on the disk by considering immersions, maps between the structures which respect the basepoints and locally preserve the translation structure. We use immersions to define a topology this moduli space, and prove that the topology behaves nicely with respect to many natural operations on translation structures. Such immersions give rise to a partial order on this moduli space, and we show that the one-point compactification of this moduli space has the structure of a complete lattice in the sense of order theory. We show that for any $\epsilon > 0$, the set of surfaces in the moduli space for which the open ϵ-ball about the basepoint is isometric to the Euclidean ϵ-ball is compact in moduli space.

1. Introduction

A *translation structure* on a surface Σ is an atlas of charts to the plane so that the transition functions are translations. This article is part of a sequence of at least two articles which will introduce a topology on the space of all translation structures on pointed surfaces modulo isomorphism. In particular, this space will incorporate translation structures on all types of surfaces: finite genus or infinite genus.

In this article, we place a topology on the space of translation structures on the open disk Δ with basepoint x_0. We call this topology the *immersive topology* because it involves studying the possibility of immersing one subset of a translation structure into another translation structure (in a manner which respects the local structure).

For the author's point of view, an understanding of translation structures on the disk is fundamentally important, because the disk is homeomorphic to the universal cover of any surface admitting a translation structure. By lifting to the universal cover, a translation structure on any surface induces a translation structure on the disk. In the subsequent article [Hoo13a], we explain how to pass from a topology on translation structures on the disk to a topology on the space of translation structures (on any surface).

2. Connections and motivation

We have taken some pains to distinguish translation structures (as defined above), from the notion of translation surface. A *translation surface* can be thought of as a pair (X, ω) consisting of a Riemann surface X with a (non-zero) holomorphic 1-form ω on X. The 1-form ω can be integrated to obtain charts to the plane, which are canonical up to translation. These charts are local homeomorphisms away from the set $Z \subset X$ of zeros of ω, where cone singularities with cone angles in $2\pi\mathbb{Z}$ appear. A translation surface (X, ω) gives rise to a translation structure on $X \setminus Z$.

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Our understanding of translation surfaces has developed extensively since the pioneering work of Masur, Rauzy and Veech in the late 1970s and early 1980s. The field has attracted researchers from diverse fields in mathematics including Teichmüller theory, algebraic geometry, and dynamical systems. Indeed, the interplay between these subjects has driven great progress in the field since its inception, and the field continues to be vibrant today.

We briefly discuss some well-established ideas relating to the study of the space of translation surfaces of genus $g \geq 1$, and to the convergence of a sequence of translation surfaces. For more detail see the survey articles [MT02, §2] and [Zor06, §3.3]. Consider the moduli space of all translation surfaces of fixed genus. The collection of surfaces in this space with a fixed number of cone singularities with fixed cone angles is called a *stratum* of translation surfaces. There is a well-studied way to place locally coordinates on a stratum given via period coordinates. These coordinates give the stratum the structure of an orbifold with a locally affine structure. Of course, the cone singularities of a sequence of translation surfaces in a stratum can collide in the limit. In this case, there can still be a limiting translation surface but it lies outside the stratum. We can take such a limit by viewing the space of translation surfaces of genus g as identified with the collection of pairs (X, ω), where X is a Riemann surface and ω is a holomorphic 1-form. As such the space of translation surface of genus g has the structure of a vector bundle over the moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g. Sequences of translation surfaces of genus g can still leave this space. For instance, a separating subsurface such as a cylinder could collapse to a point under a sequence. In order to take limits of such sequences, one can consider the Deligne-Mumford compactification of $\mathcal{M}_g$, and make appropriate considerations for corresponding degenerations of holomorphic 1-forms. See [MT02, Definition 4.7].

The approaches described in the above paragraph give a beautiful complex analytic structure to the space of translation surfaces (of bounded genus). This paper and the sequel [Hoo13a] sacrifice much of this structure. But, we hope to describe a natural notion in which a sequence of translation structures can converge. We want to be able to take limits of sequences of infinite genus surfaces, and we want to allow a sequences of surfaces of growing genus to limit on a surface of infinite genus. This has been done (informally) in [Bow13] and [Hoo13b], for instance.

This work is motivated by growing interest in the geometry and dynamics of translation surfaces of infinite topological type, which lead to an international conference on the subject. (See the acknowledgments at the end of the paper.) Many works share a common interest in topological aspects of the space of all translation surfaces, e.g. to take limits of a sequence or to consider a continuously varying family. Examples of papers in which this idea appears include [Cha04] and [HHW13]. This paper and [Hoo13a] provide a firm basis for making such topological statements.

From the author's point of view, it is becoming increasingly clear that many of the techniques in the subject of translation surfaces are applicable to the study of infinite translation surfaces. For instance, there is widespread interest in infinite abelian branched covers of translation surfaces. Example articles include [DHL11], [FU11], [HLT11], [HS10], [HW13], [RT12], [RT13] and [Sch11]. More relevantly, there is some work to suggest that many methods in use are applicable to infinite translation surfaces which do not arise from covering constructions. In [Tre12], a criterion was described for unique ergodicity of the straight-line flow in translation surfaces of infinite topological type but finite area. In the finite genus case, Masur's criterion [Mas82] says that if the orbit of a translation surface under the Teichmüller flow recurs than the vertical straight line flow is uniquely ergodic. To make such a statement in the infinite case, we need a topology. The article [Hoo10] concludes ergodic theoretic results about infinite surfaces from a notion of recurrence in the spirit of Masur's criterion, but the author believes that Trevino's approach [Tre12] is more appropriate for making such a topological statement.

3. The set of translation structures on the disk

In this section, we will give a set-theoretic description of the moduli space of translation structures on the disk. We also introduce the canonical (set-theoretic) disk bundle over this moduli space.

3.1. **Translation structures on the disk.** Let X and Y be topological spaces. A map $f : X \to Y$ is a *local homeomorphism* if for every $x \in X$, there is an open neighborhood U of x so that the image $f(U)$ is open and $f|_U$ is a homeomorphism onto its image.

Let Δ denote the open oriented topological 2-dimensional disk with basepoint $x_0 \in \Delta$. A *translation structure* on Δ is an atlas of charts $\{(U_j, \phi_j) \ : \ j \in \mathcal{J}\}$ so that:

1. Each $U_j \subset \Delta$ is an open set.
2. The collection of sets $\{U_j \ : \ j \in \mathcal{J}\}$ covers Δ.
3. Each $\phi_j : U_j \to \mathbb{R}^2$ is an orientation preserving local homeomorphism.
4. For each $i, j \in \mathcal{J}$ and each $x \in U_i \cap U_j$, there is a neighborhood U of x so that $\phi_i|_U$ differs from $\phi_j|_U$ by postcomposition by a translation.

Two translation structures are the *same* if the union of the two atlases still determines a translation structure. We say the translation structures determined by atlases $\{(U_j, \phi_j) : j \in \mathcal{J}\}$ and $\{(V_k, \psi_k) : k \in \mathcal{K}\}$ are *translation isomorphic* if there is an orientation preserving homeomorphism $h : \Delta \to \Delta$ so that the structure determined by $\{\big(h(U_j), \phi_j \circ h^{-1}\big) : j \in \mathcal{J}\}$ is the same as the structure determined by $\{(V_k, \psi_k) : k \in \mathcal{K}\}$. The homeomorphism h is called a *translation isomorphism* from the first structure to the second. If h fixes the basepoint (i.e., $h(x_0) = x_0$), then we call h an *isomorphism* and say the structures are *isomorphic*.

3.2. **The set-theoretic moduli space.** Let $\{(U_j, \phi_j) : j \in \mathcal{J}\}$ be an atlas of charts determining a translation structure on the disk Δ. Because Δ is simply connected, by analytic continuation there is a unique map $\phi : \Delta \to \mathbb{R}^2$ so that

1. $\phi(x_0) = \mathbf{0}$, where $\mathbf{0} = (0, 0) \in \mathbb{R}^2$.
2. For each $j \in \mathcal{J}$, the map $\phi|_{U_j}$ agrees with ϕ_j up to postcomposition with a translation.

This map ϕ is a local homeomorphism called the *developing map* in the language of (G, X) structures. See [Thu97]. Observe that the single chart (Δ, ϕ) determines a translation surface structure on Δ which is the same as the original structure. Conversely, each orientation preserving local homeomorphism $\phi : \Delta \to \mathbb{R}^2$ determines a translation structure: the one determined by the atlas $\{(\Delta, \phi)\}$.

We will say a *pointed local homeomorphism* (from (Δ, x_0) to $(\mathbb{R}^2, \mathbf{0})$) is an orientation preserving local homeomorphisms $\phi : \Delta \to \mathbb{R}^2$ so that $\phi(x_0) = \mathbf{0}$. We use PLH to denote the collection of all such maps. Observe that our developing maps lie in PLH, and PLH is naturally identified with the collection of all translation structures on (Δ, x_0) modulo sameness.

Let $\mathrm{Homeo}_+(\Delta, x_0)$ denote the group of orientation preserving homeomorphisms $\Delta \to \Delta$ which fix the basepoint x_0. Note that the translation structures on Δ determined by ϕ and ψ in PLH are isomorphic if and only if there is an $h \in \mathrm{Homeo}_+(\Delta, x_0)$ so that $\psi = \phi \circ h^{-1}$. Thus, the (set-theoretic) moduli space of all translation structures on (Δ, x_0) modulo isomorphism is given by

$$\tilde{\mathcal{M}} = \mathrm{PLH}/\mathrm{Homeo}_+(\Delta, x_0).$$

We use the notation $[\phi]$ to indicate the $\mathrm{Homeo}_+(\Delta, x_0)$-equivalence class of $\phi \in \mathrm{PLH}$.

Remark 1 (Quotient topology)**.** *One can endow PLH with the compact-open topology and $\tilde{\mathcal{M}}$ with the resulting quotient topology. This is not what we do in this paper, because the resulting topology is not Hausdorff. The open disk and the plane can be considered to be points in $\tilde{\mathcal{M}}$, and every open set containing the disk in the quotient topology also contains the plane.*

3.3. **The disk bundle over moduli space.** The group $\mathrm{Homeo}_+(\Delta, x_0)$ naturally acts on $\mathrm{PLH} \times \Delta$ by

$$(\phi, y) \mapsto \big(\phi \circ h^{-1}, h(y)\big).$$

The *canonical disk bundle over $\tilde{\mathcal{M}}$* is given by

$$\tilde{\mathcal{E}} = \mathrm{PLH} \times \Delta/\mathrm{Homeo}_+(\Delta, x_0).$$

We denote the $\mathrm{Homeo}_+(\Delta, x_0)$-equivalence class of (ϕ, y) by $[\phi, y] \in \tilde{\mathcal{E}}$.

Because of the description of the $\mathrm{Homeo}_+(\Delta, x_0)$-action, there is a canonical map

$$Dev : \tilde{\mathcal{E}} \to \mathbb{R}^2; \quad [\phi, y] \mapsto \phi(y).$$

We call this map the *(bundle-wide) developing map.* There is also a natural projection from $\tilde{\mathcal{E}}$ on to the moduli space $\tilde{\mathcal{M}}$ given by

$$\tilde{\pi} : \tilde{\mathcal{E}} \to \tilde{\mathcal{M}}; \quad [\phi, y] \mapsto [\phi].$$

3.4. **Structures on the fibers.** We will call each fiber $\tilde{\pi}^{-1}([\phi])$ *a planar surface.* Observe that the choice of a representative $\phi \in [\phi]$ yields an identification of the planar surface $\tilde{\pi}^{-1}([\phi])$ with Δ:

$$i_\phi : \Delta \to \tilde{\pi}^{-1}([\phi]); \quad y \mapsto [\phi, y]. \tag{1}$$

We use i_ϕ to push the topology from Δ onto the fiber, and note that the resulting topology is independent of the choice of $\phi \in [\phi]$. This map can also be used to push the basepoint $x_0 \in \Delta$ onto the planar surface. We treat $[\phi, x_0]$ as the basepoint of the planar surface, and note that this point is also independent of the choice of ϕ. Finally, we note that the restriction of the developing map to the fiber is a local homeomorphism.

Convention 2. We follow some conventions to simplify notation by effectively removing the need to discuss equivalence classes. Formally, the planar surfaces are parametrized by equivalence classes $[\phi] \in \tilde{\mathcal{M}}$ via $P = \tilde{\pi}^{-1}([\phi])$. We will identify the objects P and $[\phi]$, and thus we can more simply write $P \in \tilde{\mathcal{M}}$. We will typically denote the basepoint of P by $o_P = [\phi, x_0]$. We will denote points of P by letters such as $p, q \in P$. Note that points of P are also points of $\tilde{\mathcal{E}}$. But, we will rarely refer to a point $p \in \tilde{\mathcal{E}}$ without referring to the planar surface $P = \tilde{\pi}(p) \in \tilde{\mathcal{M}}$ which contains p. Therefore, we will redundantly refer to points of $\tilde{\mathcal{E}}$ as pairs (P, p) where P is a planar surface and $p \in P$ is a point in this surface (i.e., $P = \tilde{\pi}(p)$).

3.5. **Action of homeomorphisms of the plane.** Let $\text{Homeo}(\mathbb{R}^2, \mathbf{0})$ denote the homeomorphisms of $\mathbb{R}^2$ which fix the origin, $\mathbf{0}$. This group naturally decomposes into the orientation preserving elements, $\text{Homeo}_+(\mathbb{R}^2, \mathbf{0})$, and the orientation reversing elements $\text{Homeo}_-(\mathbb{R}^2, \mathbf{0})$.

The collection $\text{Homeo}_+(\mathbb{R}^2, \mathbf{0})$ forms a group and acts naturally on PLH by postcomposition. As such, it commutes actions of $\text{Homeo}_+(\Delta, x_0)$ by precomposition. Thus, this gives rise to an action on both $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{E}}$. If $H \in \text{Homeo}_+(\mathbb{R}^2, \mathbf{0})$, these actions are given by:

$$H : \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}; \quad [\phi] \mapsto [H \circ \phi], \quad \text{and} \quad H : \tilde{\mathcal{E}} \to \tilde{\mathcal{E}}; \quad [\phi, y] \mapsto [H \circ \phi, y].$$

The action of elements $H \in \text{Homeo}_-(\mathbb{R}^2, \mathbf{0})$ is slightly more subtle. We need to choose an arbitrary orientation reversing homeomorphism $j : \Delta \to \Delta$ which fixes x_0. Then, the action is given by

$$H : \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}; \quad [\phi] \mapsto [H \circ \phi \circ j^{-1}],$$
$$H : \tilde{\mathcal{E}} \to \tilde{\mathcal{E}}; \quad [\phi, y] \mapsto [H \circ \phi \circ j^{-1}, j(y)].$$

Here, the choice of j irrelevant because the choice is canceled when quotienting by precomposition with $\text{Homeo}_+(\Delta, x_0)$.

Remark 3 (Affine maps)**.** *The action of Homeo*$(\mathbb{R}^2, \mathbf{0})$ *generalizes the affine action of* $GL(2; \mathbb{R}) \subset$ *Homeo*$(\mathbb{R}^2, \mathbf{0})$ *on translation structures on the disk. Of course, the* $GL(2; \mathbb{R})$ *action is distinguished because it always descends to an action on moduli spaces of quotient translation structures.*

We conclude by making two observations about how the action of $\text{Homeo}(\mathbb{R}^2, \mathbf{0})$ interacts with the natural functions we have defined. First, we remark that the action of any $H \in \text{Homeo}(\mathbb{R}^2, \mathbf{0})$ commutes with the bundle-wide developing map. This is because regardless of orientation, we have

$$Dev \circ H([\phi, y]) = H \circ \phi(y) = H \circ Dev([\phi, y]).$$

We can write this using the convention mentioned above as

$$Dev \circ H(P, p) = H \circ Dev(P, p). \tag{2}$$

Second, the actions of $H \in \text{Homeo}(\mathbb{R}^2, \mathbf{0})$ commutes with the projection map $\tilde{\pi} : \tilde{\mathcal{E}} \to \tilde{\mathcal{M}}$,

$$\tilde{\pi} \circ H(P, p) = H \circ \tilde{\pi}(P, p) = H(P). \tag{3}$$

4. Immersions and a topology on the moduli space

In this section, we introduce the idea of immersing a subset of one planar surface into another planar surface. We use this idea establish topologies on $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{E}}$.

4.1. **Definition of immersion.** Let P be a planar surface, following the discussion in §3 including Convention 2. We define $\text{PC}(P)$ to be the collection of all path-connected subsets of P which contain the basepoint $o_P \in P$.

Let P and Q be planar surfaces, and choose $A \in \text{PC}(P)$ and $B \in \text{PC}(Q)$. We say "A *immerses* into B" and write "$A \rightsquigarrow B$" if there is a continuous map $\iota : A \to B$ so that $\iota(o_P) = o_Q$ and

$$Dev|_P(p) = Dev|_Q \circ \iota(p) \quad \text{for all } p \in A. \tag{4}$$

We call the map ι an *immersion (respecting the translation structures)*. We will write "$A \not\rightsquigarrow B$" to indicate that A does not immerse in B, and will write "$\exists \iota : A \rightsquigarrow B$" as shorthand for the phase "there exists an immersion ι from A to B."

Proposition 4 (Uniqueness of immersions)**.** *There is at most one immersion from A into B.*

Proof. Note that we must have $\iota(o_P) = o_Q$. Because $Dev|_P$ and $Dev|_Q$ are local homeomorphisms, analytic continuation can be used to determine the map ι if it exists. $\square$

Corollary 5. *Let P and Q be planar surfaces. If $P \rightsquigarrow Q$ and $Q \rightsquigarrow P$, then $P = Q$.*

Proof. Let $\iota_1 : P \rightsquigarrow Q$ and $\iota_2 : Q \rightsquigarrow P$. Then $\iota_2 \circ \iota_1$ is an immersion of P into itself. Since the identity map on P is also an immersion of P into itself, by the uniqueness of immersions, $\iota_2 \circ \iota_1 = id_P$. Similarly, $\iota_1 \circ \iota_2 = id_Q$. Thus these maps are inverses of one another, and hence are homeomorphisms. Furthermore, the fact that the maps respect the basepoints and satisfy equation 4 imply that they are isomorphisms. $\square$

These two results as well as the fact that compositions of immersions are immersions imply that the relation $\rightsquigarrow$ is a partial order on the moduli space $\tilde{\mathcal{M}}$.

Let A and B be path connected subsets of planar surfaces which contain basepoints. An *embedding* of A in B is an injective immersion $e : A \to B$. If such a map exists, we say "A embeds in B" and write "$A \hookrightarrow B$." We follow notational conventions as for immersions.

4.2. **The topology on moduli space.** We will specify the topology on the moduli space of all planar surfaces, $\tilde{\mathcal{M}}$, by specifying a subbasis for the topology. That is, we will be concerned with the coarsest topology which makes a collection of sets open.

Let $P \in \tilde{\mathcal{M}}$ be a planar surface and let $K \in \mathrm{PC}(P)$ be compact. We define the following subsets of the moduli space $\tilde{\mathcal{M}}$:

$$\tilde{\mathcal{M}}_{\rightsquigarrow}(K) = \{Q \in \tilde{\mathcal{M}} \ : \ K \rightsquigarrow Q\} \quad \text{and} \quad \tilde{\mathcal{M}}_{\hookrightarrow}(K) = \{Q \in \tilde{\mathcal{M}} \ : \ K \hookrightarrow Q\}. \tag{5}$$

Sets of this form will be open in our topology. However, they are insufficient to form a subbasis for a Hausdorff topology, because they fail to isolate points. For instance, the plane (interpreted as a planar surface) lies in each set $\tilde{\mathcal{M}}_{\rightsquigarrow}(K)$. Also, any set of the form $\tilde{\mathcal{M}}_{\hookrightarrow}(K)$ which contains the unit disk also contains the plane.

Let P be a planar surface, and let $U \in \mathrm{PC}(P)$ be open. We define:

$$\tilde{\mathcal{M}}_{\not\rightsquigarrow}(U) = \{Q \in \tilde{\mathcal{M}} \ : \ U \not\rightsquigarrow Q\} \quad \text{and} \quad \tilde{\mathcal{M}}_{\not\hookrightarrow}(U) = \{Q \in \tilde{\mathcal{M}} \ : \ U \not\hookrightarrow Q\}. \tag{6}$$

These sets will also be open in our topology.

We would like to describe a subbasis for our topology which consists of sets which are fairly easy to work with. So, for any planar surface P, we will distinguish two natural subsets of $\mathrm{PC}(P)$. We define $\overline{\mathrm{Disk}}(P)$ to be those sets in $\mathrm{PC}(P)$ which are homeomorphic to a closed 2-dimensional disk and contain the basepoint o_P in their interior. We define $\mathrm{Disk}(P)$ to be the set of sets in $\mathrm{PC}(P)$ which are homeomorphic to an open 2-dimensional disk and contain the basepoint.

Definition 6. The *immersive topology* on $\tilde{\mathcal{M}}$ is the coarsest topology so that sets of either of the two forms below are open:

1. Sets of the form $\tilde{\mathcal{M}}_{\rightsquigarrow}(K)$ where $K \in \overline{\mathrm{Disk}}(P)$ for some $P \in \tilde{\mathcal{M}}$.
2. Sets of the form $\tilde{\mathcal{M}}_{\not\hookrightarrow}(U)$ where $U \in \mathrm{Disk}(P)$ for some $P \in \tilde{\mathcal{M}}$.

Below, we attempt to provide an easily provable criterion for convergence in the immersive topology:

Proposition 7 (Criterion for convergence in $\tilde{\mathcal{M}}$.)**.** *Let $P \in \tilde{\mathcal{M}}$ be a planar surface and let $\langle P_n \rangle_{n \in \mathbb{N}}$ be a sequence of planar surfaces. Suppose the following two statements hold:*

(A) *If $D \in \overline{Disk}(P)$, then $D \rightsquigarrow P_n$ for all but finitely many n.*
(B) *If Q is a planar surface, and $Q \hookrightarrow P_n$ for infinitely many n, then $Q \rightsquigarrow P$.*

Then, $\langle P_n \rangle$ converges to P in the immersive topology on $\tilde{\mathcal{M}}$.

Our choice of subbasis for the immersive topology gives rise to other natural open sets:

Theorem 8 (Open sets in $\tilde{\mathcal{M}}$)**.** *Let P be a planar surface. If $K \in PC(P)$ is compact, then both $\tilde{\mathcal{M}}_{\rightsquigarrow}(K)$ and $\tilde{\mathcal{M}}_{\hookrightarrow}(K)$ are open in the immersive topology on $\tilde{\mathcal{M}}$. If $U \in PC(P)$ is open, then both $\tilde{\mathcal{M}}_{\not\rightsquigarrow}(U)$ and $\tilde{\mathcal{M}}_{\not\hookrightarrow}(U)$ are open.*

We record the following, which will be used in [Hoo13a]. This result also justifies the name "immersive topology."

Corollary 9 (Immersion subbasis)**.** *The following collection of sets is a subbasis for the immersive topology on $\tilde{\mathcal{M}}$:*

(1) *Sets of the form $\tilde{\mathcal{M}}_{\rightsquigarrow}(K)$ where $K \in \overline{Disk}(P)$ for some $P \in \tilde{\mathcal{M}}$.*
(2) *Sets of the form $\tilde{\mathcal{M}}_{\not\rightsquigarrow}(U)$ where $U \in Disk(P)$ for some $P \in \tilde{\mathcal{M}}$.*

4.3. **The topology on the canonical disk bundle.** We recall that in Convention 2 we have identified the canonical disk bundle $\tilde{\mathcal{E}}$ with the collection of pairs (P, p), where P is a planar surface and $p \in P$.

Let P be a planar surface, let $K \in \mathrm{PC}(P)$ be compact, and let $U \subset K^\circ$ be an open set (not necessarily connected or containing the basepoint). Define the following subset of $\tilde{\mathcal{E}}$:

$$\tilde{\mathcal{E}}_+(K, U) = \{(Q, q) \in \tilde{\mathcal{E}} \ : \ \exists \iota : K \rightsquigarrow Q \text{ and } q \in \iota(U)\}.$$

The *immersive topology on $\tilde{\mathcal{E}}$* is the coarsest topology so that the projection $\tilde{\pi} : \tilde{\mathcal{E}} \to \tilde{\mathcal{M}}$ is continuous and each set of the form $\tilde{\mathcal{E}}_+(K, U)$ is open.

In the remainder of this section, we describe some basic results about this topology. The following provides a criterion for convergence in $\tilde{\mathcal{E}}$:

Proposition 10 (Criterion for convergence in $\tilde{\mathcal{E}}$)**.** *Let $\langle (P_n, p_n) \in \tilde{\mathcal{E}} \rangle$ be a sequence and let $(P, p) \in \tilde{\mathcal{E}}$. Suppose the following statements hold:*

- *The sequence $\langle P_n \rangle$ converges to P in the immersive topology on $\tilde{\mathcal{M}}$.*
- *For every $K \in \overline{Disk}(P)$ and every open $U \subset K^\circ$ so that $p \in U$, we have $(P_n, p_n) \in \tilde{\mathcal{E}}_+(K, U)$ for all but finitely many n.*

Then, $(P_n, p_n) \to (P, p)$ in the immersive topology on $\tilde{\mathcal{E}}$.

The above seems somewhat technical. The following seems more practical, but is it really equivalent.

Proposition 11 (Criterion for convergence in $\tilde{\mathcal{E}}$)**.** *Let $\langle (P_n, p_n) \in \tilde{\mathcal{E}} \rangle$ be a sequence and let $(P, p) \in \tilde{\mathcal{E}}$. Suppose the following statements hold:*

- *The sequence $\langle P_n \rangle$ converges to P in the immersive topology on $\tilde{\mathcal{M}}$.*
- *There is a $K \in \overline{Disk}(P)$ containing p in its interior so that for every $U \subset K^\circ$ with $p \in U$ so that $(P_n, p_n) \in \tilde{\mathcal{E}}_+(K, U)$ for all but finitely many n.*

Then, $(P_n, p_n) \to (P, p)$ in the immersive topology on $\tilde{\mathcal{E}}$.

Add proof.

There is a canonical section of the projection $\tilde{\pi} : \tilde{\mathcal{E}} \to \tilde{\mathcal{M}}$ given by

$$\tilde{\sigma} : \tilde{\mathcal{M}} \to \tilde{\mathcal{E}}; P \mapsto (P, o_P), \tag{7}$$

where o_P denotes the basepoint of P.

Proposition 12. *The section $\tilde{\sigma}$ is continuous.*

Now let P be a planar surface, let $K_2 \in \mathrm{PC}(P)$ be compact and let $K_1 \subset K_2$ be closed. We define:

$$\tilde{\mathcal{E}}_-(K_2, K_1) = \{(Q, q) \in \tilde{\mathcal{E}} \ : \ \exists \iota : K_2 \rightsquigarrow Q \text{ and } q \notin \iota(K_1)\}. \tag{8}$$

Proposition 13. *Sets of the form $\tilde{\mathcal{E}}_-(K_2, K_1)$ are open in the immersive topology on $\tilde{\mathcal{E}}$.*

Finally, we will show that the maximal collection of immersions of an open set is continuous.

Proposition 14 (Continuity of immersions)**.** *Let Q be a planar surface, and let $U \in PC(Q)$ be open. Let $\mathcal{I}(U) \subset \tilde{\mathcal{M}}$ denote the closed set of planar surfaces P so that $U \rightsquigarrow P$. For $P \in \mathcal{I}(U)$, let $\iota_P : U \rightsquigarrow P$ be the associated immersion. Then the function*

$$I_U : \mathcal{I}(U) \times U \to \tilde{\mathcal{E}}; \quad (P, q) \mapsto \big(P, \iota_P(q)\big)$$

is continuous.

5. Main results

Now that we have defined the topologies on $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{E}}$, we will describe the main results of this article. The following gives us unique limits for sequences:

Theorem 15. *The immersive topologies on $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{E}}$ are second countable and Hausdorff.*

We remark that one way a sequence $\langle P_n \rangle$ of planar surfaces can leave every compact set of $\tilde{\mathcal{M}}$ is if the radius of largest open Euclidean metric ball we can immerse in P_n centered at the basepoint tends to zero as $n \to \infty$. In fact, this is the only way a sequence can leave every compact set. Consider the following in the case where P is a planar surface which is isomorphic to a Euclidean ball:

Theorem 16 (Compactness)**.** *Let P be a planar surface. The set of surfaces*

$$\tilde{\mathcal{M}} \smallsetminus \tilde{\mathcal{M}}_{\not\leftarrow}(P) = \{Q \in \tilde{\mathcal{M}} \ : \ P \rightsquigarrow Q\}$$

is compact.

Since we are working with pointed surfaces is natural to consider what happens when we move the basepoint. Let P be a planar surface and p be a point in P. We define P^p to be the translation isomorphic planar surface, so that the translation isomorphism $\phi : P \to P^p$ caries p to the basepoint of P^p. Formally, the two surfaces P and P^p are (typically different) fibers of the projection $\pi : \tilde{\mathcal{E}} \to \tilde{\mathcal{M}}$. The *basepoint changing map* is given by

$$\widetilde{\mathrm{BC}} : \tilde{\mathcal{E}} \to \tilde{\mathcal{M}}; \quad (P, p) \mapsto P^p.$$

Given $(P, p) \in \tilde{\mathcal{E}}$, we have also described a *basepoint changing (translation) isomorphism* $\tilde{\beta}_p : P \mapsto P^p$ which carries $p \in P$ to the basepoint $o_{P^p} \in P^p$.

Theorem 17 (Basepoint Change)**.** *The basepoint changing map, $\widetilde{BC}$, is continuous. The basepoint changing isomorphism, $q \mapsto \tilde{\beta}_p(q)$, is jointly continuous in the choice of $(P,p) \in \tilde{\mathcal{E}}$ and the choice of $q \in P$. The map $q \in P^p \mapsto \beta_p^{-1}(q)$ is also jointly continuous in p and q.*

We also investigate the continuity of the $\mathrm{Homeo}(\mathbb{R}^2, \mathbf{0})$ actions on $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{E}}$, which we first described in §3.5. We prove:

Theorem 18 (Continuity of $\mathrm{Homeo}(\mathbb{R}^2, \mathbf{0})$ actions)**.** *The following two functions are continuous:*

$$\mathit{Homeo}(\mathbb{R}^2, \mathbf{0}) \times \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}; \quad (H, P) \mapsto H(P),$$
$$\mathit{Homeo}(\mathbb{R}^2, \mathbf{0}) \times \tilde{\mathcal{E}} \to \tilde{\mathcal{E}}; \quad \big(H, (P,p)\big) \mapsto H(P,p).$$

Finally, observe that the statement $P \rightsquigarrow Q$ places a partial ordering on the space $\tilde{\mathcal{M}}$ of planar surfaces. We will now describe some of the order structure. If $P \rightsquigarrow Q$ we think of Q as larger than P. The content of the following is that every collection of planar surfaces has a least upper bound:

Theorem 19 (Fusion Theorem)**.** *Let $\mathcal{P}$ denote any non-empty collection of planar surfaces. Then there is a unique planar surface R which satisfies the following statements:*

(I) *For each $P \in \mathcal{P}$, $P \rightsquigarrow R$.*
(II) *For all planar surfaces Q, if $P \rightsquigarrow Q$ for all $P \in \mathcal{P}$, then $R \rightsquigarrow Q$.*

We say that the planar surface R from the above theorem is the *fusion* of $\mathcal{P}$ and write $R = \bigvee \mathcal{P}$. If $\mathcal{P}$ is a finite set, such as $\mathcal{P} = \{P_1, P_2, P_3\}$, we write $P_1 \curlyvee P_2 \curlyvee P_3$ for the fusion.

Consider the one-point compactification of $\bar{\mathcal{M}} = \tilde{\mathcal{M}} \cup \{O\}$. By the Compactness Theorem, a sequence of surfaces $\langle P_n \in \tilde{\mathcal{M}} \rangle$ converges to O if and only if there is no surface Q so that $Q \rightsquigarrow P_n$ for all n. We extend the definition of the immersion relation $\rightsquigarrow$ to $\bar{\mathcal{M}}$ by saying $O \rightsquigarrow P$ for all $P \in \bar{\mathcal{M}}$ and $P \rightsquigarrow O$ if and only if $P = O$. We see that in the space $\bar{\mathcal{M}}$, all sets have greatest lower bounds:

Corollary 20. *Let $\mathcal{P} \subset \bar{\mathcal{M}}$. Then, there is a surface $R \in \bar{\mathcal{M}}$ which satisfies the following statements:*

(I') *For each $P \in \mathcal{P}$, $R \rightsquigarrow P$.*
(II') *If $Q \in \bar{\mathcal{M}}$ and $Q \rightsquigarrow P$ for all $P \in \mathcal{P}$, then $Q \rightsquigarrow R$.*

The proof is a standard observation in order theory, but we give it for completeness. This and the result above indicate that $\bar{\mathcal{M}}$ is a complete lattice in the sense of order theory. See [Bir64] or [Grä11] for background on complete lattices.

Proof. Consider the set $\mathcal{S} = \{S \in \tilde{\mathcal{M}} ~:~ S \rightsquigarrow P \text{ for all } P \in \mathcal{P}\}$. If this set is empty, then we can take $R = O$. The statements (I') and (II') are clearly satisfied.

If $\mathcal{S} \neq \emptyset$, then let $R = \bigvee \mathcal{S}$. We will now prove that (I') is satisfied. Fix $P \in \mathcal{P}$. Observe that $S \rightsquigarrow P$ for all $S \in \mathcal{S}$. Therefore $R \rightsquigarrow P$ by statement (II) of the Fusion theorem. We now prove that (II') is satisfied. Suppose $Q \in \bar{\mathcal{M}}$ and suppose $Q \rightsquigarrow P$ for every $P \in \mathcal{P}$. Then $Q \in \mathcal{S}$. So $Q \rightsquigarrow R$ by statement (I) of the Fusion Theorem. □

We call the $R \in \bar{\mathcal{M}}$ produced in the above corollary the *core* of the set $\mathcal{P}$, and denote this by $R = \bigwedge \mathcal{P}$. Finally, we make two observations about how these concepts relate to our topology.

Proposition 21 (Direct limit). *Suppose $\langle P_n \in \tilde{\mathcal{M}} \rangle_{n \geq 1}$ is a sequence satisfying*

$$P_1 \rightsquigarrow P_2 \rightsquigarrow P_3 \rightsquigarrow \ldots.$$

Then, the sequence converges to $\curlyvee\{P_n\}$.

Proposition 22 (Inverse limit). *Suppose $\langle P_n \in \bar{\mathcal{M}} \rangle_{n \geq 1}$ is a sequence satisfying*

$$\ldots \rightsquigarrow P_3 \rightsquigarrow P_2 \rightsquigarrow P_1.$$

Then, the sequence converges to $\curlywedge\{P_n\}$.

6. Outline of remainder of paper

In the remainder of the paper, we state the results in an order of dependence of the proofs, which is different than the order of results above.

In §7, we state some basic results which are fundamental for our further work on with planar surfaces throughout the remainder of the paper. We use these basic results to prove the convergence criteria given by Propositions 7 and 10.

In §8, we investigate the fusion operation on collections of translation surfaces. We describe a generalization of the Fusion Theorem, and a constructive description of the fusion. We also investigate a number of relevant properties of the operation.

In §9, we introduce a tool for working with the topology. Namely, we study subsets of planar surfaces which are finite unions of rectangles. We use this idea to prove that our list of open sets in $\tilde{\mathcal{M}}$ (given by Theorem 8) are indeed open. We also prove that the topologies are Hausdorff and second countable, and offer proofs of Corollary 9 and Propositions 12, 13, and 14.

In §10, we investigate the continuity of basepoint change. We carefully restate and prove Theorem 17 in pieces.

In §11, we investigate the action of $\text{Homeo}(\mathbb{R}^2, \mathbf{0})$ on $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{E}}$. We prove Theorem 18.

In §12, we prove the propositions relating to direct and inverse limits (Propositions 21 and 22). We also prove the Compactness Theorem.

7. Basics of immersions and planar surfaces

7.1. **Basic properties of immersions.** We collect some basic properties of immersions and embeddings:

Proposition 23 (Basic properties).

1. (Generalization of subset) *If $A, B \in PC(P)$ and $A \subset B$, then $A \hookrightarrow B$.*
2. (Transitivity) *For $i = 1, 2, 3$, let P_i be planar surfaces and choose $A_i \in PC(P_i)$. Then:*
   - *If $A_1 \rightsquigarrow A_2$ and $A_2 \rightsquigarrow A_3$, then $A_1 \rightsquigarrow A_3$.*
   - *If $A_1 \hookrightarrow A_2$ and $A_2 \hookrightarrow A_3$, then $A_1 \hookrightarrow A_3$.*
3. (Suprema) *Let P and Q be a planar surfaces and $B \in PC(Q)$. Suppose that $\langle A_j \in PC(P) \rangle_{j \in \mathbb{N}}$ is an increasing sequence of open subsets, i.e. $A_j \subset A_{j+1}$ for all $j \in \mathbb{N}$. Let $U = \bigcup_{j \in \mathbb{N}} A_j \in PC(P)$. Then:*
   - *If $A_j \rightsquigarrow B$ for all $j \in \mathbb{N}$, then $U \rightsquigarrow B$.*
   - *If $A_j \hookrightarrow B$ for all $j \in \mathbb{N}$, then $U \hookrightarrow B$.*

Proof. Statements (1) and (2) are observations left to the reader. We prove statement (3). Suppose $\exists \iota_j : A_j \rightsquigarrow B$ for all $j \in \mathbb{N}$. The fact that these immersions are unique implies that for all $j < k$ and all $p \in A_j$, we have $\iota_j(p) = \iota_k(p)$, since $e_j = e_k|_{A_j}$. Therefore, we may define a limiting map $\iota : U \to B$ by $\iota(p) = \iota_j(p)$ whenever $p \in A_j$. The preceding argument indicates that this map is well defined. We must check that it is an immersion. Because the A_j are open, continuity of each ι_j implies continuity of ι. Similarly, ι satisfies the developing map condition of equation 4, because each ι_j satisfies this condition. The embedding case also requires checking injectivity. To prove this from the fact that each ι_j is injective, choose any distinct $p, q \in U$. Then $p, q \in \iota_j(A_j)$ for some j. Then by definition of ι and injectivity of ι_j,

$$\iota(p) = \iota_j(p) \neq \iota_j(q) = \iota(q).$$

□

7.2. **Subsets homeomorphic to disks.** In order to work with disks in planar surfaces, we will utilize some structure coming from Schoenflies' theorem:

Theorem 24 (Schoenflies)**.** *Let C be a simple closed curve in the open topological disk Δ. Then, there is a homeomorphism $h : \Delta \to \mathbb{R}^2$ so that $h(C)$ is the unit circle in $\mathbb{R}^2$.*

We translate this theorem into our setting as follows.

Corollary 25. *Let P be a planar surface and $K \in \overline{Disk}(P)$. Then, there is a homeomorphism $h : P \to \mathbb{R}^2$ so that $h(K)$ is the closed unit ball and $h(o_P) = \mathbf{0}$.*

We use this to impart the following structure.

Proposition 26. *Let P be a planar surface. For each set $K \in \overline{Disk}(P)$, there is a family of sets $\{K_t \in \overline{Disk}(P) \ : \ t > 0\}$ so that the following statements hold.*

(1) $K_1 = K$.
(2) $\bigcap_t K_t = \{o_P\}$.
(3) $P = \bigcup_t K_t^\circ$.
(4) *For each* $t > 0$, $K_t^\circ = \bigcup_{t' < t} K_{t'}^\circ$.
(5) *For each* $t > 0$, $K_t = \bigcap_{t' > t} K_{t'}$.
(6) *There is a continuous surjective function $\alpha : \mathbb{R}/2\pi\mathbb{Z} \times [0, \infty) \to P$, which is injective except that $\alpha(\mathbb{R}/2\pi\mathbb{Z} \times \{0\}) = \{o_P\}$ and satisfies $\alpha(\mathbb{R}/2\pi\mathbb{Z} \times \{t\}) = \partial K_t$.*

Proof. Let $h : \Delta \to \mathbb{R}^2$ be the homeomorphism guaranteed to exist by Corollary 25. The family given by $K_t = h^{-1}\big(\{\mathbf{v} \in \mathbb{R}^2 \ : \ \|\mathbf{v}\| \leq t\}\big)$ satisfies the proposition. The function α can be taken to be the pull back of polar coordinates on $\mathbb{R}^2$. □

We will call any family of sets $\{K_t \in \overline{\mathrm{Disk}}(P) \ : \ t > 0\}$ formed as above a *closed disk family* in the planar surface P. Closed disk families give a natural way to understand immersions and the failure to immerse, and we use them throughout the paper. The next subsection uses them to prove our convergence criteria.

7.3. **Proofs of convergence criteria.** The following is the proof of our convergence criterion for $\tilde{\mathcal{M}}$:

Proof of Proposition 7. We will suppose statements (A) and (B) of the Proposition are satisfied and prove that for any closed disk K, $P \in \tilde{\mathcal{M}}_{\rightsquigarrow}(K)$ implies that $P_n \in \tilde{\mathcal{M}}_{\rightsquigarrow}(K)$ for all but finitely many n, and that for any open disk U, $P \in \tilde{\mathcal{M}}_{\not\rightsquigarrow}(U)$ implies $P_n \in \tilde{\mathcal{M}}_{\not\rightsquigarrow}(U)$ for all but finitely many n.

Let K be a closed disk in a planar surface Q and suppose that $P \in \tilde{\mathcal{M}}_{\rightsquigarrow}(K)$. Then there is an immersion $\iota : K \rightsquigarrow P$. By taking a closed disk family in P, we can find a closed disk

D in that family so that $\iota(K) \subset D$. From (A), we get immersions $\iota_n : D \rightsquigarrow P_n$ for all but finitely many n. Whenever ι_n is defined, the composition $\iota_n \circ \iota$ is an immersion $K \rightsquigarrow P_n$. This proves $P_n \in \tilde{\mathcal{M}}_{\rightsquigarrow}(K)$ for all but finitely many n.

Now suppose that U is an open disk in a planar surface Q and that $P \in \tilde{\mathcal{M}}_{\not\hookrightarrow}(U)$. Then, $U \not\hookrightarrow P$. Suppose it is not true that $P_n \in \tilde{\mathcal{M}}_{\not\hookrightarrow}(U)$ for all but finitely many n. Then there is an increasing sequence of integers $\langle n_k \rangle$ and embeddings $e_k : U \hookrightarrow P_{n_k}$. Since U is an open disk, it is isomorphic to a planar surface which we abuse notation by also denoting U. So, by (B) applied to U, we know that there is an immersion $\iota : U \rightsquigarrow P$. Suppose it is not an embedding. Then there are points $u, v \in U$ so that $\iota(u) = \iota(v)$. Let $K \subset U$ be a closed disk in Q containing both u and v. Then $\iota(K)$ is compact. As in the prior paragraph, we can find a closed disk D so that $\iota(K) \subset D$. Then, by (A) we get immersions $\iota_n : D \rightsquigarrow P_n$ for all but finitely many n. In particular, for sufficiently large k, we have $\iota_{n_k} \circ \iota|_K : K \rightsquigarrow P_{n_k}$. But we also get such an immersion as a restriction of an embedding, $e_k|_K : K \hookrightarrow P_{n_k}$. It follows that for sufficiently large k, that $\iota|_K$ is an embedding. But this contradicts the statement that $\iota(u) = \iota(v)$. We conclude that ι is an embedding of U into P contradicting the second sentence in the paragraph. □

We now prove our criterion for convergence in $\tilde{\mathcal{E}}$:

Proof of Proposition 10. By definition, we have $(P_n, p_n) \to (P, p)$ if we have convergence $\tilde{\pi}(P_n, p_n) = P_n \to \tilde{\pi}(P, p) = P$, and if whenever Q be a planar surface, $K \in \overline{\text{Disk}}(Q)$, $U \subset K^\circ$ is open, and $(P, p) \in \tilde{\mathcal{E}}_+(K, U)$, then there is an N so that $n > N$ implies $(P_n, p_n) \in \tilde{\mathcal{E}}_+(K, U)$. Because the first necessary condition already appears in our statements, we just need to consider the second condition. So fix a planar surface Q, a $K \in \overline{\text{Disk}}(Q)$, an open $U \subset K^\circ$ is open, and suppose $(P, p) \in \tilde{\mathcal{E}}_+(K, U)$.

Since $(P, p) \in \tilde{\mathcal{E}}_+(K, U)$, there is an immersion $\iota : K \to P$ and $p \in \iota(U)$. By taking a closed disk family, we can find a $D \in \overline{\text{Disk}}(P)$ so that $\iota(K) \subset D^\circ$. Let $V = \iota(U)$, which lies inside of D°. We claim that $\tilde{\mathcal{E}}_+(D, V) \subset \tilde{\mathcal{E}}_+(K, U)$. To see this, let $(R, r) \in \tilde{\mathcal{E}}_+(D, V)$. Then, there is an immersion $j : D \rightsquigarrow R$ and $r \in j(V)$. Observe that $j \circ \iota : K \to R$ is an immersion and that $r \in j \circ \iota(U) = j(V)$. Thus, $(R, r) \in \tilde{\mathcal{E}}_+(K, U)$.

By the second statement in the proposition, there is an N so that $(P_n, p_n) \in \tilde{\mathcal{E}}_+(D, V)$ for $n > N$. By the inclusion from the above paragraph, we also have $(P_n, p_n) \in \tilde{\mathcal{E}}_+(K, U)$ for $n > N$. □

7.4. **Embedding radii.** Let B_r denote the open ball of radius r centered at $\mathbf{0}$ in the plane, with $\mathbf{0}$ taken as the basepoint. This ball is isomorphic to a planar surface, and we abuse notation by identifying B_r with this planar surface. The *embedding radius of* $(P, p) \in \tilde{\mathcal{E}}$ is

$$ER(P, p) = \max \{r > 0 \ : \ B_r \hookrightarrow P^p\}.$$

Proposition 27. *The embedding radius function, $ER : \tilde{\mathcal{E}} \to \mathbb{R}$, is well defined and positive.*

Proof. Since P^p is locally homeomorphic to $\mathbb{R}^2$, there is a $r > 0$ so that $B_r \hookrightarrow P^p$. Furthermore by the uniqueness of embeddings, $B_r \hookrightarrow P^p$ if and only if $B_{r'} \hookrightarrow P^p$ for all $r' < r$. Thus,

$$\{r > 0 \ : \ B_r \hookrightarrow P^p\} = (0, r]$$

for some $r > 0$. In this case $ER(P, p) = r$. □

If $r \leq ER(P,p)$, there is a natural direction preserving isometric embedding of B_r into P so that $\mathbf{0}$ is mapped to $p \in P$. Let $e : B_r \hookrightarrow P^p$ be the embedding from knowledge of the embedding radius, and let $\tilde{\beta}_p : P \to P^p$ be the basepoint changing isomorphism defined in §5. We define the *ball embedding*

$$\widetilde{BE}_p = \tilde{\beta}_p^{-1} \circ e : B_r \to P.$$

This map respects the translation structures and sends the center $\mathbf{0} \in B_r$ to $p \in P$.

Proposition 28. *The restriction of the embedding radius function to a planar surface is* 1*-Lipschitz.*

Proof. Let $p, q \in P$ and suppose that $|ER(P,p) - ER(P,q)| > d_P(p,q)$. Without loss of generality, assume that $ER(P,p) > ER(P,q)$ so that

$$ER(P,p) - ER(P,q) > d_P(p,q). \tag{9}$$

Let $r = ER(P,p)$ and consider the ball embedding $\widetilde{BE}_p : B_r \to P$. Since $d_P(p,q) < ER(P,p)$, there is a $\mathbf{v} \in B_r$ so that $\widetilde{BE}_p(\mathbf{v}) = q$. Let $B' \subset B_r$ be the set of points within distance $r' = ER(P,p) - d_P(p,q)$ of $\mathbf{v}$. Then $\widetilde{BE}_p(B')$ is an isometric embedding of a ball of radius r' into P centered at q. Therefore,

$$ER(P,q) \geq r' = ER(P,p) - d_P(p,q).$$

But this contradicts equation 9. □

In particular, the embedding radius is continuous and positive. It follows that there is a minimal positive embedding radius on any compact set $K \subset P$. We denote this by

$$ER(K) = \min\{ER(P,k) \ : \ k \in K\} > 0. \tag{10}$$

It is useful to note that if $U \in \mathrm{Disk}(P)$, then we can also define the embedding radius of by treating U as a planar surface. As such we get an embedding e from the planar surface associated to U to P. Let $u \in U$ and let $K \subset U$ be compact. We define

$$ER(u \in U) = ER\big(e^{-1}(u)\big) \quad \text{and} \quad ER(K \subset U) = ER\big(e^{-1}(K)\big). \tag{11}$$

Finally, we remark that the embedding radius goes non-strictly upward under immersions.

Proposition 29. *Let P be a planar surface and let U be an open disk in P. Let $u \in U$ and let $K \subset U$ be compact. Let Q be another planar surface and suppose there is an immersion $\iota : U \rightsquigarrow Q$. Then,*

$$ER\big(\iota(u) \in \iota(U)\big) \geq ER(u \in U) \quad \textit{and} \quad ER\big(\iota(K) \subset \iota(U)\big) \geq ER(K \subset U).$$

Proof. It suffices to prove the first inequality, since the inequality involving K results from taking the maximum of each side over $u \in K$. Let $r = ER(u \in U)$. Then $\exists e : B_r \hookrightarrow U^u$. We can embed B^r into $\tilde{\beta}_{\iota(u)} \circ \iota(U)$ under the composition:

$$B_r \xrightarrow{e} U^u \xrightarrow{\tilde{\beta}_u^{-1}} U \xrightarrow{\iota} \iota(U) \xrightarrow{\tilde{\beta}_{\iota(u)}} \tilde{\beta}_{\iota(u)} \circ \iota(U).$$

The individual maps respect the translation structures, and the composition sends $\mathbf{0}$ to $\iota(u)$, so this describes an immersion. Furthermore, the developing map applied to B_r is injective, so the composition must be injective. This proves that the embedding radius at $\iota(U)$ into U is greater than or equal to r as desired. □

8. Fusing planar surfaces

In this section, we prove the Fusion Theorem and investigate properties of the fusion operation. The notion of fusion is most naturally applied to surfaces of arbitrary topological type equipped with what we call a *trivial structure*, which can be succinctly defined as a translation structure with trivial translational monodromy. We define these structures first. In subsection 8.2, we state the Generalized Fusion Theorem, which handles trivial structures. In subsection 8.3, we give a constructive version of this theorem. The remainder of this section is spent proving these results.

8.1. **Trivial structures.** Let Σ be an oriented topological surface with basepoint x_0. We will say that a *trivial structure* on a surface is an atlas of orientation preserving local homeomorphisms (charts) to the plane so that the transition functions are restrictions of the identity map on the plane and so that the image of the basepoint x_0 is always mapped to $\mathbf{0} \in \mathbb{R}^2$.

Because the transition functions are the identity map, the image of a point $y \in \Sigma$ under a chart is independent of the choice of the chart. It follows that a trivial structure on a surface is equivalent to an orientation preserving local homeomorphism $\phi : \Sigma \to \mathbb{R}^2$ so that $\phi(x_0) = \mathbf{0}$. Two such local homeomorphisms ϕ and ψ are said to yield *isomorphic trivial structures* if there is an orientation preserving homeomorphism $h : \Sigma \to \Sigma$ so that $h(x_0) = x_0$ and $\phi \circ h^{-1} = \psi$. The homeomorphism h is called an *isomorphism* between the structures. The *(set-theoretic) moduli space of trivial structures on* (Σ, x_0) is the collection isomorphism-equivalence classes of trivial structures on (Σ, x_0). We can construct a canonical Σ-bundle over the moduli space of trivial structures on (Σ, x_0), as for translation structures on the disk. We say that a *trivial surface (homeomorphic to* (Σ, x_0)*)* is a fiber of the projection from the bundle to the moduli space. We endow the trivial surfaces with a pointed trivial structure in the canonical way.

The collection of all trivial surfaces is the collection of fibers of such projections taken over all homeomorphism-equivalence classes of pointed surfaces (Σ, x_0). The developing map *Dev* is well defined on the union of all trivial surfaces, and the restriction to a single trivial surface is a local homeomorphism to $\mathbb{R}^2$ which carries the basepoint to $\mathbf{0}$.

The notions of immersion and embedding carry over trivially to subsets of trivial surfaces. In particular, we note that if P and Q are trivial surfaces, then $P \rightsquigarrow Q$ and $Q \rightsquigarrow P$ implies that $P = Q$.

8.2. **The Fusion Theorem.** The main goal of this section is to prove the following theorem.

Theorem 30 (Generalized Fusion Theorem)**.** *Let $\mathcal{P}$ denote any non-empty collection of trivial surfaces. Then there is a unique trivial surface R which satisfies the following statements:*

(I) *For each $P \in \mathcal{P}$, $P \rightsquigarrow R$.*
(II) *For all planar surfaces Q, if $P \rightsquigarrow Q$ for all $P \in \mathcal{P}$, then $R \rightsquigarrow Q$.*

We call R the fusion of the surfaces in $\mathcal{P}$, and use notation for R as described under the statement of Theorem 19. We establish some simple properties of the fusion operation.

Corollary 31 (Properties of the Fusion)**.** *Let P and Q be arbitrary trivial surfaces. Then, the following statements hold:*

(1) *$P \rightsquigarrow Q$ if and only if $P \curlyvee Q = Q$.*
(2) *If the there is a planar surface R so that $P \hookrightarrow R$ and $Q \rightsquigarrow R$ then $P \hookrightarrow (P \curlyvee Q)$.*

(3) (generalized union) *If A, B and $A \cap B$ are path connected open subsets of a trivial surface P containing the basepoint, then $A \curlyvee B$ is isomorphic to $A \cup B$.*

Proof. We will prove statement (1). Suppose $P \rightsquigarrow Q$. We claim that Q is the fusion of P and Q. We must show that the statements of the fusion theorem are satisfied. Statement (I) is trivially true since $P \rightsquigarrow Q$ and $Q \rightsquigarrow Q$. Now consider statement (II). We see that if $Q \rightsquigarrow S$, then automatically $P \rightsquigarrow S$ by composition. Thus, the statement reduces to the tautology: if $Q \rightsquigarrow S$, then $Q \rightsquigarrow S$. Conversely, suppose that $P \curlyvee Q = Q$. Then, by statement (I) of the Fusion theorem, $P \rightsquigarrow Q$.

Now consider statement (2). We have two immersions of P into R. We are given an embedding $P \hookrightarrow R$. Also by the Fusion Theorem, we have the composition $P \rightsquigarrow (P \curlyvee Q) \rightsquigarrow R$. Since immersions are unique, this composition must be an embedding and hence injective. Therefore $P \rightsquigarrow (P \curlyvee Q)$ is an injective immersion (i.e., an embedding).

We will now prove statement (3). Suppose $R = A \curlyvee B$. It follows from (2) that $A \hookrightarrow R$ and $B \hookrightarrow R$. The restriction of these embeddings to $A \cap B$ is an embedding of $A \cap B$ into R. Since $A \cap B$ is path connected, such an embedding is unique. In particular, the embeddings restricted to $A \cap B$ are the same. It follows that the union of the embeddings gives a well defined immersion $(A \cup B) \rightsquigarrow R$. Also, $A \hookrightarrow (A \cup B)$ and $B \hookrightarrow (A \cup B)$. So by definition of the fusion, $R \rightsquigarrow (A \cup B)$. But we have immersions in both directions between $A \cup B$ and R, so they are isomorphic. □

We are especially interested in the case when P and Q are planar surfaces (trivial surfaces homeomorphic to disks). The following shows that our original Fusion Theorem (Theorem 19) follows from the Generalized Fusion Theorem.

Proposition 32. *If P and Q are trivial surfaces homeomorphic to disks, then so is $P \curlyvee Q$.*

Proof. Suppose to the contrary that $R = P \curlyvee Q$ is not simply connected. Let $\tilde{R}$ be the universal cover of R and let $\pi : \tilde{R} \to R$ be the covering. We will prove that $\tilde{R}$ also satisfies statements (1) and (II) of the Fusion theorem, contradicting uniqueness.

Since P and Q simply connected, the immersions $P \rightsquigarrow R$ and $Q \rightsquigarrow R$ lift to immersions $P \rightsquigarrow \tilde{R}$ and $Q \rightsquigarrow \tilde{R}$. Thus (I) is satisfied by $\tilde{R}$. Now suppose that $P \rightsquigarrow S$ and $Q \rightsquigarrow S$. By statement (II) for R, we know $R \rightsquigarrow S$. The covering map $\tilde{R} \to R$ is an immersion, so by composition with the covering map, $\tilde{R} \rightsquigarrow S$. Thus, (II) is satisfied. □

We will prove the Generalized Fusion Theorem in several steps. We first prove the uniqueness (assuming existence).

Proof of uniqueness in the Fusion Theorem. Suppose there are two trivial surfaces R_1 and R_2 which satisfy statements (I) and (II) of the Theorem. Then by statement (I), for each $j \in \{1, 2\}$ and each $P \in \mathcal{P}$, we have $P \rightsquigarrow R_j$. Then by statement (II), we have $R_1 \rightsquigarrow R_2$ and $R_2 \rightsquigarrow R_1$. So $R_1 = R_2$ by Corollary 5, which may be seen to hold for trivial surfaces. (The same proof works as for planar surfaces.) □

8.3. **Construction of the fusion.** We construct the fusion of $\mathcal{P}$ as a quotient of the disjoint union $\bigsqcup_{P \in \mathcal{P}} P$. We make this a topological space by making each open set in a $P \in \mathcal{P}$ open in the disjoint union.

The bundle-wide developing map *Dev* is defined on the union of all trivial surfaces, so by restriction, it is also defined on the disjoint union $\bigsqcup_{P \in \mathcal{P}} P$.

Let $\sim$ be an equivalence relation on $\bigsqcup_{P \in \mathcal{P}} P$. Make the following choices:

- Let P_1 and P_2 be a pair of (not necessarily distinct) surfaces in $\mathcal{P}$.
- Let $r_1 \in P_1$ and $r_2 \in P_2$ be so that $r_1 \sim r_2$.
- Let $\gamma_1 : [0,1] \to P_1$ and $\gamma_2 : [0,1] \to P_2$ be paths so that $\gamma_1(0) = r_1$, $\gamma_2(0) = r_2$ and
$$Dev \circ \gamma_1(t) = Dev \circ \gamma_2(t) \quad \text{for all } t \in [0,1].$$

We say that $\sim$ is *path invariant* if for every choice made as above, we have $\gamma_1(1) \sim \gamma_2(1)$.

Theorem 33 (Constructive Fusion Theorem)**.** *Let $\sim$ be the smallest path invariant equivalence relation on $\bigsqcup_{P \in \mathcal{P}} P$ so that for each pair of surfaces P and Q in $\mathcal{P}$, we have $o_P \sim o_Q$. Then, $\bigsqcup_{P \in \mathcal{P}} P/\sim$ is isomorphic to the fusion of $\bigcurlyvee \mathcal{P}$. Here, the local homeomorphism from $\bigsqcup_{P \in \mathcal{P}} P/\sim$ to $\mathbb{R}^2$ is provided by the developing map, which descends to this quotient.*

We establish a corollary, which reveals some structure of the fusion of infinitely many surfaces which is unapparent in the original Fusion Theorem. This corollary will be used later in the proof of the Direct Limit Proposition.

Corollary 34. *Let $\mathcal{P}$ be an infinite collection of trivial surfaces. Let $R = \bigcurlyvee \mathcal{P}$. For $P \in \mathcal{P}$, let $\iota_P : P \rightsquigarrow R$ be the immersion guaranteed by statement (I) of the Fusion Theorem. Let p be a point in $P \in \mathcal{P}$ and q be a point in $Q \in \mathcal{P}$, and suppose that $\iota_P(p) = \iota_Q(q)$. Then there is a finite subset $\mathcal{F} \subset \mathcal{P}$ containing P and Q so that the immersions $j_P : P \rightsquigarrow \bigcurlyvee \mathcal{F}$ and $j_Q : Q \rightsquigarrow \bigcurlyvee \mathcal{F}$ satisfy $j_P(p) = j_Q(q)$.*

Proof. Let $\sim$ be the equivalence relation from the Constructive Fusion Theorem. We think of equivalence relations on $\bigsqcup_{P \in \mathcal{P}} P$ as subsets of $(\bigsqcup_{P \in \mathcal{P}} P)^2$. We will construct a sequence of equivalence relations $\sim_n$ on $\bigsqcup_{P \in \mathcal{P}} P$ so that $\bigcup_n \sim_n = \sim$. Then, the finiteness result follows if the finiteness result is proved for each $\sim_n$.

We define $\sim_n$ inductively beginning with $\sim_0$. Let $p, q \in \bigsqcup_{P \in \mathcal{P}} P$. We define $p \sim_0 q$ if $p = q$ or if p and q are both basepoints of surfaces in $\mathcal{P}$. This can be seen to be an equivalence relation. Now suppose that $\sim_n$ is defined and let $p \in P \in \mathcal{P}$ and $q \in Q \in \mathcal{P}$ be points in $\bigsqcup_{P \in \mathcal{P}} P$. We say p is $n+1$-related to q (denoted $p \equiv_{n+1} q$) if there are curves $\gamma_1 : [0,1] \to P$ and $\gamma_2 : [0,1] \to Q$ so that $\gamma_1(0) \sim_n \gamma_2(0)$, $\gamma_1(1) = p$, $\gamma_2(1) = q$ and $Dev \circ \gamma_1(t) = Dev \circ \gamma_2(t)$ for all $t \in [0,1]$. Observe that $\sim_n \subset \equiv_{n+1}$. We define $\sim_{n+1}$ to be the smallest equivalence relation containing $\equiv_{n+1}$. Since $\equiv_{n+1}$ is reflexive and symmetric, we can concretely say that $p \sim_{n+1} q$ if $p \equiv_{n+1} q$ or if there is a finite collection $p_1, p_2, \ldots, p_k \in \bigsqcup_{P \in \mathcal{P}} P$ so that the following holds:

$$p \equiv_{n+1} p_1 \equiv_{n+1} p_2 \equiv_{n+1} \ldots \equiv_{n+1} p_k \equiv_{n+1} q. \tag{12}$$

Observe that by definition of $\sim$ we have $\bigcup_n \sim_n = \sim$.

We now prove our finiteness statement by induction. Let $p \in P \in \mathcal{P}$ and $q \in Q \in \mathcal{P}$. If p and q are the same point, then $P = Q$, we can take $\mathcal{F} = \{P\}$ so that $P = \bigcurlyvee \mathcal{F}$, and the identity map $P \rightsquigarrow P$ sends p and q to the same point. If p and q are basepoints of P and Q, respectively, then the immersions $P \rightsquigarrow (P \curlyvee Q)$ and $Q \rightsquigarrow (P \curlyvee Q)$ carry these points to the basepoint of $P \curlyvee Q$ by definition of immersion. This proves the finiteness statement for $\sim_0$.

Now suppose $\sim_n$ satisfies the finiteness statement, and suppose that $p \equiv_{n+1} q$. Then, there must be curves $\gamma_1 : [0,1] \to P$ and $\gamma_2 : [0,1] \to Q$ as above. Then $\gamma_1(0) \sim_n \gamma_2(0)$, so there is a finite set $\mathcal{F} \subset \mathcal{P}$ containing P and Q so that the immersions $\iota_P : P \rightsquigarrow \bigcurlyvee \mathcal{F}$ and $\iota_Q : P \rightsquigarrow \bigcurlyvee \mathcal{F}$ satisfy $\iota_P \circ \gamma_1(0) = \iota_Q \circ \gamma_2(0)$. Then by the Constructive Fusion Theorem applied to $\bigcurlyvee \mathcal{F}$, we see that $\iota_P \circ \gamma_1(1) = \iota_Q \circ \gamma_2(1)$ as well. This proves the finiteness statement for $\equiv_{n+1}$.

Now suppose that $\equiv_{n+1}$ satisfies the finiteness statement, and suppose that $p \sim_{n+1} q$. Then there are points $p_1, p_2, \ldots, p_k \in \bigsqcup_{P \in \mathcal{P}} P$ satisfying equation 12.. Let $p_0 = p$ and $p_{k+1} = q$. Let $P_j \in \mathcal{P}$ be the surface containing p_j for each j. Then for all $j \in \{0, \ldots, k\}$, there is a finite set $\mathcal{F}_j$ so that the immersions $\iota_j : P_j \rightsquigarrow \bigcurlyvee \mathcal{F}_j$ and $\iota'_j : P_{j+1} \rightsquigarrow \bigcurlyvee \mathcal{F}_j$ satisfies $\iota_j(p_j) = \iota'_j(p_{j+1})$. Let $\mathcal{F} = \bigcup_{j=0}^{k} \mathcal{F}_j$. Then, there are immersions $\jmath_j : \bigcurlyvee \mathcal{F}_j \rightsquigarrow \bigcurlyvee \mathcal{F}$ for all j. The immersions $P_j \rightsquigarrow \bigcurlyvee \mathcal{F}$ can be given by $\jmath_j \circ \iota_j$ for $j \leq k$ and by $\jmath_{j-1} \circ \iota'_{j-1}$ for $j \geq 1$. It follows that the image of p_j inside $\bigcurlyvee \mathcal{F}$ is independent of $j \in \{0, \ldots, k+1\}$. □

8.4. **The fusion is a trivial surface.** We will begin by proving that the quotient space described in the theorem is really a trivial surface.

Lemma 35. *Let $\sim$ be the equivalence relation from the Constructive Fusion Theorem. Then, the quotient $\bigsqcup_{P \in \mathcal{P}} P/\sim$ has the structure of a trivial surface with the associated immersion to $\mathbb{R}^2$ given by*

$$\phi([r]) = Dev(r) \quad \textit{for all } r \in P \in \mathcal{P}.$$

We will devote the remainder of the section to the proof of this fact. We now describe our plan. We will check that ϕ is a well defined map. Then we will prove that ϕ is a local homeomorphism. This demonstrates that $P \sqcup Q/\sim$ is locally modeled on $\mathbb{R}^2$. Finally, we will show that $P \sqcup Q/\sim$ is Hausdorff. This proves that it is a surface, and ϕ gives this surface a trivial planar structure.

Proof that ϕ is well defined. The basepoint of $\bigsqcup_{P \in \mathcal{P}} P/\sim$ is given by the equivalence class $[o_P]$ for some (any) $P \in \mathcal{P}$. We note that the developing map o_P to zero. By induction, we can see that the points we are forced to identify by the path invariance property also have the same image under the developing map. Therefore, ϕ is well defined. □

In order to prove the remainder of the lemma, it is useful to use the following:

Proposition 36. *Let $r_1 \in P_1$ for some $P_1 \in \mathcal{P}$. Let $\epsilon \leq ER(r_1)$, and let $B \subset P_1$ be the open ball of radius ϵ centered at r_1. Let*

$$U = \{r_2 \in \bigsqcup_{P \in \mathcal{P}} P \ : \ r_2 \sim b \textit{ for some } b \in B\}.$$

Then, U is open in $\bigsqcup_{P \in \mathcal{P}} P$, and so by definition $B' = \{[b] \ : \ b \in B\}$ is open in $\bigsqcup_{P \in \mathcal{P}} P/\sim$.

Proof. We remind the reader of the topology we placed on $\bigsqcup_{P \in \mathcal{P}} P$. We need to show that $U \cap P_2$ is open in P_2 for all $P_2 \in \mathcal{P}$. Let $r_2 \in U \cap P_2$. Then, $r_2 \sim b$ for some $b \in B$. Observe that B is isometric to an open Euclidean ball of radius ϵ. Since the image under the developing map is an $\sim$-invariant, we have

$$|Dev(r_1) - Dev(r_2)| = |Dev(r_1) - Dev(b)| \leq \epsilon,$$

because b lies in the ball B of radius ϵ about r_1. Choose

$$\epsilon' = \min\{ER(r_2), \epsilon - |Dev(r_1) - Dev(r_2)|\}.$$

Let D denote the open ball of radius ϵ' about r_2 in P_2. Let $r_3 \in D$. Then there is a path γ_1 of length less than ϵ' joining r_2 to r_3. Similarly, there is a path γ_2 starting at b and contained in B so that $Dev \circ \gamma_1 = Dev \circ \gamma_2$. This path stays within B because triangle inequality prohibits it from leaving. It follows that $r_3 \in U$. Thus $U \cap P_2$ is open as desired. □

Proof that ϕ is a local homeomorphism. Choose any $[r_1] \in \bigsqcup_{P \in \mathcal{P}} P/\sim$, and choose $r_1 \in [r_1]$. Let U be as in the above proposition. Then the set B' at the end of the proposition is an open set containing $[r_1]$. Furthermore, since each point in B' has a unique representative in B, we know that $\phi|_{B'}$ is one-to-one and onto its image, which is an open ball in $\mathbb{R}^2$.

We must prove that $\phi|_{B'}$ is continuous. This also follows from the proposition. Let $\mathbf{v}$ be a point in $\phi(B')$, and let $[r_2] = \phi^{-1}(\mathbf{v})$. We can choose the representative $r_2 \in [r_2] \cap B$. Then the neighborhood of radius $\epsilon - |Dev(r_1) - Dev(r_2)|$ about r_2 is isometric to a Euclidean ball. Applying the proposition to this choice of center r_2 and radius produces an open set containing r_2 and contained in B'.

The fact that $(\phi|_{B'})^{-1}$ is continuous is a tautology, because of the topology we placed on $\bigsqcup_{P \in \mathcal{P}} P$. Recall that the union of equivalence classes in B' is open. Call this union U as in the lemma above. Now let $C' \subset B'$ be a smaller open set. This by definition means that its union of equivalence classes $V' \subset U'$ is open. That is, $V' \cap P$ is open for each $P \in \mathcal{P}$. Moreover by definition of ϕ, we have

$$\phi|_{B'}(C') = Dev(V') = \bigcup_{P \in \mathcal{P}} Dev|_P(V' \cap P).$$

But, the image of any open set in a planar surface under the developing map is open in $\mathbb{R}^2$, and any union of open sets is open. □

Proof that $\bigsqcup_{P \in \mathcal{P}} P/\sim$ is Hausdorff. Let $[r_1], [r_2] \in \bigsqcup_{P \in \mathcal{P}} P/\sim$ be distinct. We will separate these points by open sets. First suppose that $\phi([r_1]) \neq \phi([r_2])$. Then by constructing neighborhoods around each of $[r_1]$ and $[r_2]$ using Proposition 36 with radius less than or equal to $\frac{1}{2}|\phi([r_1]) - \phi([r_2])|$ produces open sets which can discerned to be disjoint because their images under ϕ are disjoint.

Now suppose that $[r_1]$ and $[r_2]$ are distinct but that $\phi([r_1]) = \phi([r_2])$. Choose representatives $r_1 \in [r_1]$ and $r_2 \in [r_2]$. Choose $\epsilon = \min\big(ER(r_1), ER(r_2)\big)$. Let $B_1, B_2 \subset P \sqcup Q$ be balls about r_1 and r_2, respectively. They determine open sets $B_1', B_2' \subset \bigsqcup_{P \in \mathcal{P}} P/\sim$ by Proposition 36. We claim that they are disjoint. Otherwise, there is a $[r_3] \in B_1' \cap B_2'$. By the proposition, we can then find points $b_1 \in B_1 \cap [r_3]$ and $b_2 \in B_2 \cap [r_3]$. Since $b_1 \sim b_2$, they have the same image under Dev. Parameterize the line segments joining b_1 to r_1 within B_1 and joining b_2 to r_2 within B_2 in the same way. Then, path invariance guarantees that $r_1 \sim r_2$. This contradicts the distinctness of $[r_1]$ and $[r_2]$. □

8.5. **Proof of the fusion theorem.** We now prove the Constructive Fusion Theorem. Note that this immediately implies the Generalized Fusion Theorem (Theorem 30), and the original version of the Fusion Theorem (Theorem 19) follows from Proposition 32.

Proof of Theorem 33. Let $P \in \mathcal{P}$. Let $R = \bigsqcup_{P \in \mathcal{P}} P/\sim$, where $\sim$ is equivalence relation described in the theorem. We will prove that R has the properties described in the Fusion Theorem.

Statement (I) simply requires proving that the natural map $P \to R$ respect the basepoints, respect the developing maps, and are local homeomorphisms. Basepoints are respected by construction construction. By definition of ϕ, the developing map is respected. This proves that the natural map $P \to R$ is an immersion. Finally, the fact that $P \to R$ is a local homeomorphism follows from the fact that the developing maps are respected and are local homeomorphisms.

Statement (II) reduces to a statement about equivalence relations. Suppose $P \rightsquigarrow S$ for all $P \in \mathcal{P}$. Let $j : \bigsqcup_{P \in \mathcal{P}} P \to S$ be the simultaneous immersion of all planar surfaces $P \in \mathcal{P}$ into S. Then, we define an equivalence relation on $\bigsqcup_{P \in \mathcal{P}} P$ by $p \approx q$ for $p \in P \in \mathcal{P}$ and $q \in Q \in \mathcal{P}$ if $j(p) = j(q)$. Then all basepoints are equivalent and $\approx$ is path invariant. Since $\sim$ is the smallest such relation, each $\sim$-equivalence class is contained in an $\approx$-equivalence class. This gives a canonical map $R \to S$. By construction, it is an immersion. □

9. Rectangular unions

9.1. **Definition of rectangular union.** A closed *rectangle in the plane* is a subset of $\mathbb{R}^2$ of the form

$$[a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \ : \ a \leq x \leq b \text{ and } c \leq y \leq d\},$$

where $a < b$ and $c < d$. Similarly, an *open rectangle* is a set of the form $(a, b) \times (c, d)$. We call such a rectangles *rational* if a, b, c and d are rational numbers.

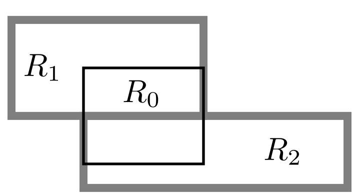
Let P be a planar surface. We call $R \subset P$ a *closed (resp. open) rectangle* if $Dev(R)$ is a closed (resp. open) rectangle and the restriction $Dev|_R : R \to Dev(R)$ is a homeomorphism. We say R is *rational* if $Dev(R)$ is.

A closed (resp. open) *rectangular union* is a finite union of closed (resp. open) rectangles in a planar surface which is connected and whose boundary is a disjoint collection of curves. We call a rectangular union *rational* if it can be constructed as a union of rational rectangles.

Proposition 37. *The closure of an open rectangular union with compact closure is a closed rectangular union. The interior of a closed rectangular union is an open rectangular union.*

Proof. Let P be a planar surface. Suppose $\{R_i \subset P\}$ is a collection of open rectangles whose union is an open rectangular union U with compact closure. Then $\bar{U} = \bigcup_i \bar{R}_i$ is a closed rectangular union.

Now suppose $\mathcal{R} = \{R_i \subset P\}$ is a collection of closed rectangles whose union is a closed rectangular union K. Then $\bigcup_i R_i^\circ$ may not be as large as K°. We will construct a larger finite collection of closed rectangles $\mathcal{R}' \supset \mathcal{R}$ so that $K^\circ = \bigcup_{R \in \mathcal{R}'} R^\circ$. We will describe an algorithm for constructing $\mathcal{R}'$ by adding rectangles beginning with $\mathcal{R}' = \mathcal{R}$. Let $\Lambda = K^\circ \smallsetminus \bigcup_i R_i^\circ$. A point $p \in \Lambda$ is either a vertex of a rectangle in $\mathcal{R}$ or lies in the interior of an edge of an $\mathcal{R}$. If $p \in \Lambda$ is a vertex, then it has a neighborhood which lies in $\bigcup_i R_i^\circ$, so we can add a small rectangle to $\mathcal{R}$ which contains p and is contained entirely in K. We add such a rectangle to $\mathcal{R}'$ for each vertex in Λ. Now suppose $p \in \Lambda$ is not a vertex. Then, it must lie in the common boundary of two rectangles R_1 and R_2 whose edges intersect in an interval. We can construct a closed rectangle, R_0, which is contained in $R_1 \cup R_2$ and contains the overlap $R_1 \cap R_2$. See below:



We add such a rectangle to $\mathcal{R}'$ for all pairs of rectangles in $\mathcal{R}$ which intersect in an interval. The resulting $\mathcal{R}'$ has the desired property. □

Proposition 38. *An open rectangular union is homeomorphic to a finitely punctured disk.*

Proof. By definition, an open rectangular union is a connected surface in a topological disk. So, it is homeomorphic to a punctured disk, but the number of punctures may be infinite. So, it suffices to prove that an open rectangular union has finite Euler characteristic.

We will show that a *union of open rectangles*, i.e., an arbitrary union of open rectangles in a planar surface, has finite Euler characteristic. We define the *complexity* of such a union to be the smallest number of rectangles necessary to write the set as a union. In fact, we will prove that if a union of rectangles U has complexity less than n, then $|\chi(U)| < 2^n$. For $n = 0$, $\chi(U) = 0$, and for $n = 1$, $\chi(U) = 1$. These provide a base case, and we proceed by induction in n. Suppose the statement $|\chi(U)| < 2^n$ for all U of complexity n. Let U' be a union of complexity $n + 1$. Then $U' = U \cup R$, where U has complexity n and R is another open rectangle. By the inclusion-exclusion principle,

$$\chi(U \cup R) = \chi(U) + \chi(R) - \chi(U \cap R) = \chi(U) - \chi(U \cap R) + 1. \tag{13}$$

Note that the intersection of two open rectangles in a planar surface is either open or another open rectangle. In particular, $U \cap R$ is a union of rectangles of complexity no more than n. By inductive hypothesis, $|\chi(U)| < 2^n$ and $|\chi(U \cap R)| < 2^n$. So by equation 13,

$$|\chi(U \cup R)| \leq |\chi(U)| + |\chi(U \cap R)| + 1 \leq (2^n - 1) + (2^n - 1) + 1 < 2^{n+1}.$$

□

9.2. **A finiteness condition.** Let P and Q be planar surfaces. Let $A \in \mathrm{PC}(P)$ and $B \in \mathrm{PC}(Q)$. We say A and B are *isomorphic* if $A \rightsquigarrow B$ and $B \rightsquigarrow A$. This defines an equivalence relation on

$$\mathrm{PC} = \bigcup_{P \in \tilde{\mathcal{M}}} \mathrm{PC}(P).$$

We note that sets of the form $\tilde{\mathcal{M}}_{\rightsquigarrow}(K)$, $\tilde{\mathcal{M}}_{\hookrightarrow}(K)$, $\tilde{\mathcal{M}}_{\not\rightsquigarrow}(U)$ and $\tilde{\mathcal{M}}_{\not\hookrightarrow}(U)$ only depend on the isomorphism classes of K and U.

Proposition 39. *Let $A_1, \ldots, A_n$ be a collection of (open or closed) rectangles in the plane. There are only finitely many isomorphism classes of sets $U \in PC$ so that U is the union of rectangles $R_i \subset U$ (i.e. $U = \bigcup_{i=1}^n \tilde{R}_i$) so that $A_i = Dev(R_i)$ for all i.*

Proof. Fix $A_1, \ldots, A_n$. Let $U = \bigcup_{i=1}^n R_i$ be such a rectangular union. Suppose U lives in the planar surface P with basepoint o_P. We associate U to two pieces of information. First there is a subset $\mathcal{S}(U) \subset \{1, \ldots, n\}$ consisting of those i so that $o_P \in R_i$. Also, we can associate to U a subgraph $\mathcal{G}(U)$ of the complete graph K_n with vertex set $\{1, \ldots, n\}$. We define this subgraph by the condition that there is an edge between distinct $i, j \in \{1, \ldots, n\}$ if $R_i \cap R_j \neq \emptyset$.

Note that we can recover U up to isomorphism from $\mathcal{S}(U)$ and $\mathcal{G}(U)$. Consider the disjoint union $\bigsqcup_i A_i$. Inclusion of each A_i into $\mathbb{R}^2$ is a natural map $\pi : \bigsqcup_i A_i \to \mathbb{R}^2$. Define the equivalence relation $\sim$ on the disjoint union by $p \in A_i$ is equivalent to $q \in A_j$ if $\pi(p) = \pi(q)$ and the edge $\bar{ij}$ lies in $\mathcal{G}(U)$. There is a natural identification between U and $\bigsqcup_i A_i / \sim$ which picks out the isomorphism class of U. The collection of points $p \in A_i$ with $i \in \mathcal{S}(U)$ and $Dev(p) = \mathbf{0}$ is an equivalence class of $\bigsqcup_i A_i / \sim$. This corresponds to the basepoint of U.

Let U_1 and U_2 be rectangular unions coming from the same choices of rectangles $A_1, \ldots, A_n$. We remark that there is an immersion $\iota : U_1 \to U_2$ if and only if $\mathcal{S}(U_1) \subset \mathcal{S}(U_2)$ and $\mathcal{G}(U_1)$

is a subgraph of $\mathcal{S}(U_2)$. Viewing

$$U_1 = \bigsqcup_i A_i / \sim_1 \quad \text{and} \quad U_2 = \bigsqcup_i A_i / \sim_2,$$

we observe that these conditions imply that the identity map $\bigsqcup_i A_i \to \bigsqcup_i A_i$ induces a well defined map from $U_1 \to U_2$. This is the needed immersion.

It follows that the collection of rectangular unions satisfying the proposition is finite: There are no more than the number of choices of $\mathcal{S}(U) \subset \{1, \ldots, n\}$ and subgraphs $\mathcal{G}(U)$ of K_n. □

Corollary 40. *There are only countably many isomorphism classes of (open or closed) rational rectangular unions in PC.*

Proof. This follows from the proposition above, because there are only countably many finite collections of rational rectangles in the plane. □

Corollary 41. *Let U be an open or closed rectangular union in a planar surface. Then there are only finitely many images of U under immersions up to isomorphism.*

Proof. Let $U = \bigcup_{i=1}^n R_i$ be a rectangular union in a planar surface P. Let $A_i = Dev(R_i)$. Given an immersion $\iota : U \rightsquigarrow Q$, we have $\iota(U) = \bigcup_{i=1}^n \iota(R_i)$. This writes $\iota(U)$ as a union of lifts of rectangles $A_i \subset \mathbb{R}^2$ for $i = 1, \ldots, n$. There are only finitely many possibilities for $\iota(U)$ by Proposition 39. □

9.3. **Rectilinear curves.** We will say a closed curve $\gamma : \mathbb{R}/L\mathbb{Z} \to \mathbb{R}^2$ is *rectilinear* if there are $0 = t_0 < t_1 < \ldots < t_{2k} = L$ so that

$$\gamma'(t) = \begin{cases} (\pm 1, 0) & \text{if } t_j < t < t_{j+1} \text{ with } j \text{ even}, \\ (0, \pm 1) & \text{if } t_j < t < t_{j+1} \text{ with } j \text{ odd}. \end{cases}$$

We say the rectilinear curve γ is a *rational* if the points $\gamma(t_j)$ are rational. We will need the following two lemmas.

Lemma 42. *Let γ be a closed immersed rectilinear curve in $\mathbb{R}^2$. Then, lifts of γ to planar surfaces bound at most finitely many (isomorphism classes) of disks.*

Lemma 43. *Suppose $\tilde{\gamma}$ is a simple closed curve in a planar surface P, and that $\gamma = Dev|_P \circ \tilde{\gamma}$ is a rectilinear curve. Then $\tilde{\gamma}$ bounds a rectangular union which is homeomorphic to a disk in P. Furthermore, if $\tilde{\gamma}$ is rational then so is the rectangular union.*

Proof of both lemmas. Consider the rectilinear curve γ in $\mathbb{R}^2$. If it bounds an immersed disk, then γ can be oriented so that the winding number around any point in the plane is non-negative. By rectilinearity, we can divide the bounded components of $\mathbb{R}^2 \setminus \gamma$ into rectangles. Furthermore, if γ is rational, these rectangles can be chosen all to be rational. Let $\mathcal{R}$ be the collection of such closed rectangles with multiplicity corresponding to the winding number.

Each immersed disk bounded by γ can be assembled by identifying boundary edges of rectangles in $\mathcal{R}$. In particular by Proposition 39, there are only finitely many immersed disks with boundary γ. Furthermore, from this construction we see that each such disk is a rectangular union. If γ was rational, then the disk is a rational rectangular union. □

Corollary 44. *Let P be a planar surface and let $K \in PC(P)$ be a closed rectangular union. Then, there is a smallest $D \in \overline{Disk}(P)$ so that $K \subset D$. Furthermore, D is a rectangular union.*

Proof. By definition of rectangular union, ∂K is a union of disjoint simple closed curves, each of which bounds a disk in P. Let A be the unique unbounded component of $P \setminus K$. Then A can only touch one boundary component of P, so A is homeomorphic to an annulus, and $\partial A \subset P$ consists of this one component. The developed image $Dev(\partial A)$ is a rectilinear curve. So, it bounds a rectangular union in $\overline{\mathrm{Disk}}(P)$ by the lemma above. $\square$

If $K \in \mathrm{PC}(P)$ is a closed rectangular union, then we call the disk D provided by the corollary the *smallest closed disk containing* K.

9.4. **Constructing rectangular unions.**

Theorem 45. *Let P be a planar surface, and let $K_1, K_3 \in PC(P)$ be compact. Suppose that $K_1 \subset K_3^\circ$. Then, there is a closed rational rectangular union $K_2 \in PC(P)$ so that $K_1 \subset K_2^\circ$ and $K_2 \subset K_3^\circ$. Furthermore, if K_3 lies in $\overline{Disk}(P)$, then we can arrange that $K_2 \in \overline{Disk}(P)$.*

Proof. We will give a construction of K_2 in the case when $K_3 \in \mathrm{PC}(P)$. For every $p \in K_1$, choose a closed rational rectangle R_p containing p and contained in K_3°. Then $\{R_p^\circ \,:\, p \in K_1\}$ is an open cover of K_1. By compactness, there is a finite subcover. Let $\{R_1, \ldots, R_n\}$ be the corresponding collection of closed rectangles. Let $K_2 = \bigcup_{i=1}^n R_i$. This set is path connected and contains the basepoint because K_1 does. Also by construction, we have $K_1 \subset K_2^\circ$ and $K_2 \subset K_3^\circ$.

It may not be true that K_2 is a rectangular union because ∂K_2 could fail to be bounded by disjoint curves. This can only happen if two rectangles meet only at a vertex as depicted below:

We can fix this problem, by adding a rectangle centered at the common vertex which is small enough to be contained in K_3° and only intersects K_2 in two rectangles containing the common vertex as a vertex.

Now suppose that $K_3 \in \overline{\mathrm{Disk}}(P)$. Replacing the K_2 constructed above by the smallest closed disk containing K_2 gives the last statement. See Corollary 44. $\square$

Let P be a planar surface and $U \in \mathrm{PC}(P)$ be open. Then there is a *smallest open disk containing* U. We can construct this disk by considering the collection $\mathcal{K}$ of compact components of $P \setminus U$. Then this smallest open disk is given by $U \cup \bigcup_{K \in \mathcal{K}} K$.

Proposition 46. *Let U be an open rectangular union in P. Then there are only finitely many smallest open disks of embedded images of U up to isomorphism.*

Proof. Let $D \subset P$ be the smallest open disk containing U. We can think of D as a planar surface. Let $\{K_t \,:\, t > 0\}$ be a closed disk family for D. By Proposition 38, there is an $a > 0$ so that $D \setminus U \subset K_a^\circ$. Then there is a closed rectangular union $K \in \overline{\mathrm{Disk}}(D)$ so that $K_a \subset K^\circ$ and $K \subset K_{a+1}^\circ$ by Theorem 45. Furthermore U is homeomorphic to a finitely punctured disk by Proposition 38. Observe that $K^\circ \cap U$ is also homeomorphic to a finitely punctured disk. Let $\tilde{\gamma} = \partial K$. Then $\gamma = Dev|_P(\tilde{\gamma})$ is a closed rectilinear curve. So, lifts of γ

bounds at most finitely many isomorphism classes of open disks by Lemma 42. Let $\mathcal{V}$ denote the collection of such isomorphism classes.

Now suppose that R is a planar surface and there is an embedding $e : U \hookrightarrow R$. Let V be the open disk bounded by $e(\tilde{\gamma})$. The isomorphism class of V lies in $\mathcal{V}$. Observe that $e(U) \cap V$ is homeomorphic to $U \cap K^\circ$. The set $e(U) \cap V$ is therefore homeomorphic to an finitely punctured disk, and in particular is connected. It then follows from statement (3) of Proposition 31 that the smallest closed disk containing $e(U)$ (namely $e(U) \cup V$) is isomorphic to the fusion $U \curlyvee V$.

It follows then that the collection of isomorphism classes of smallest open disks is a subset of $\{U \curlyvee V \ : V \in \mathcal{V}\}$, which is a finite set. □

9.5. **Open sets in $\tilde{\mathcal{M}}$.** Recall that Theorem 8 claims that sets of the form $\tilde{\mathcal{M}}_{\rightsquigarrow}(K)$, $\tilde{\mathcal{M}}_{\hookrightarrow}(K)$, $\tilde{\mathcal{M}}_{\not\rightsquigarrow}(U)$ and $\tilde{\mathcal{M}}_{\not\hookrightarrow}(U)$ are open in $\tilde{\mathcal{M}}$.

Proof of Theorem 8. We prove Theorem 8 using the definition of the immersive topology. That is, we only assume set of the form $\tilde{\mathcal{M}}_{\rightsquigarrow}(K)$ and $\tilde{\mathcal{M}}_{\not\hookrightarrow}(U)$ are open when K and U are closed and open topological disks, respectively.

Let P be a planar surface and $K \in \mathrm{PC}(P)$ be compact. We will show that $\tilde{\mathcal{M}}_{\rightsquigarrow}(K)$ is open. Choose $Q \in \tilde{\mathcal{M}}_{\rightsquigarrow}(K)$. Then by definition, there is an immersion $\iota : K \rightsquigarrow Q$. Let $K_1 = \iota(K)$. By choosing a closed disk family in Q, we can find a $K_3 \in \overline{\mathrm{Disk}}(Q)$ so that $K_1 \subset K_3^\circ$. Then Theorem 45 guarantees that there is a rectangular union $K_2 \in \overline{\mathrm{Disk}}(Q)$ so that $K_1 \subset K_2^\circ$ and $K_2 \subset K_3^\circ$. Since $\tilde{\mathcal{M}}_{\rightsquigarrow}(K_2)$ is open, it suffices to prove that $\tilde{\mathcal{M}}_{\rightsquigarrow}(K_2) \subset \tilde{\mathcal{M}}_{\rightsquigarrow}(K)$. Let $R \in \tilde{\mathcal{M}}_{\rightsquigarrow}(K_2)$. Then, there is an immersion $j : K_2 \rightsquigarrow R$. The composition $j \circ \iota : K \rightsquigarrow R$ is the immersion needed to prove that $R \in \tilde{\mathcal{M}}_{\rightsquigarrow}(K)$.

Let P be a planar surface and $U \in \mathrm{PC}(P)$ be open. We will show that $\tilde{\mathcal{M}}_{\not\hookrightarrow}(U)$ is open. Choose $Q \in \tilde{\mathcal{M}}_{\not\hookrightarrow}(U)$. Let $\tilde{U}$ be the universal cover of U with covering map π. Then $\tilde{U}$ has the structure of a planar surface. Let $\{\tilde{D}_t \ : \ t > 0\}$ be a closed disk family for $\tilde{U}$ and set $D_t = \pi(\tilde{D}_t)$. By Proposition 23, if each D_t° embedded in Q, then U would embed in Q. So there is an $a > 0$ so that $D_a \not\hookrightarrow Q$. By Theorem 45, there is a rectangular union $K_2 \in \overline{\mathrm{Disk}}(P)$ so that $D_a \subset K_2^\circ$ and $K_2 \subset D_{a+1}^\circ$. Because $D_a^\circ \subset K_2^\circ \subset U$, we have

$$Q \in \tilde{\mathcal{M}}_{\not\hookrightarrow}(D_a^\circ) \subset \tilde{\mathcal{M}}_{\not\hookrightarrow}(K_2^\circ) \subset \tilde{\mathcal{M}}_{\not\hookrightarrow}(U).$$

It suffices to show that $\tilde{\mathcal{M}}_{\not\hookrightarrow}(K_2^\circ)$ is open. Let $\mathcal{D}$ be the collection of all isomorphism classes of smallest open disks containing embedded images of K_2°. This set is finite by Proposition 46. If K_2° embeds in a planar surface R, then there is an element $D \in \mathcal{D}$ which also embeds. It follows that

$$\tilde{\mathcal{M}}_{\not\hookrightarrow}(K_2^\circ) = \bigcap_{D \in \mathcal{D}} \tilde{\mathcal{M}}_{\not\hookrightarrow}(D),$$

which is open by definition of the topology.

Let P be a planar surface and $U \in \mathrm{PC}(P)$ be open. We will now show that $\tilde{\mathcal{M}}_{\not\rightsquigarrow}(U)$ is open. Choose $Q \in \tilde{\mathcal{M}}_{\not\rightsquigarrow}(U)$. By the same reasoning as above, we can find an closed rectangular union $K_2 \in \overline{\mathrm{Disk}}(P)$ so that $K_2^\circ \subset U$ and $K_2^\circ \not\rightsquigarrow Q$. Furthermore, $\tilde{\mathcal{M}}_{\not\rightsquigarrow}(K_2^\circ) \subset \tilde{\mathcal{M}}_{\not\rightsquigarrow}(U)$. Let $\mathcal{V}$ be the collection of all immersed images of K_2°. The set $\mathcal{V}$ is finite by Corollary 41. From the above paragraph, we know that $\tilde{\mathcal{M}}_{\not\hookrightarrow}(V)$ is open for every $V \in \mathcal{V}$. Thus,

$$\tilde{\mathcal{M}}_{\not\rightsquigarrow}(K_2^\circ) = \bigcap_{V \in \mathcal{V}} \tilde{\mathcal{M}}_{\not\hookrightarrow}(V)$$

is open.

Finally, let P be a planar surface and $K \in \mathrm{PC}(P)$ be compact. We will show that $\tilde{\mathcal{M}}_{\hookrightarrow}(K)$ is open. Choose any $Q \in \tilde{\mathcal{M}}_{\hookrightarrow}(K)$. Then there is an embedding $e : K \hookrightarrow Q$. Let $K_1 = e(K)$. Choose $K_3 \in \overline{\mathrm{Disk}}(Q)$ so that $K_1 \subset K_3^\circ$. Then we can find a $K_2 \in \overline{\mathrm{Disk}}(Q)$ which is a closed rectangular union and satisfies $K_1 \subset K_2^\circ$ and $K_2 \subset K_3^\circ$. Let $\mathcal{L}$ be the collection of all immersed images of K_2° up to isomorphism. This collection is finite by Corollary 41. Let $\mathcal{L}_0$ be $\mathcal{L}$ with the equivalence class of K_2° itself removed. Then if $L \in \mathcal{L}_0$, the immersion $K_2^\circ \rightsquigarrow L$ is not an embedding. Let R be a planar surface. Suppose that there is an immersion $\iota : K_2 \rightsquigarrow R$ and that for each $L \in \mathcal{L}_0$, we have $L \not\hookrightarrow R$. Note that by restriction of ι, we have $K_2^\circ \rightsquigarrow R$. Then by definition of $\mathcal{L}_0$, our immersion $K_2^\circ \rightsquigarrow R$ must actually be an embedding. So, by restriction, $\iota|_{K_1} : K_1 \hookrightarrow R$, and by composition $\iota \circ e : K \hookrightarrow R$. It follows that

$$Q \in \tilde{\mathcal{M}}_{\rightsquigarrow}(K_2) \cap \bigcap_{L \in \mathcal{L}_0} \tilde{\mathcal{M}}_{\not\hookrightarrow}(L) \subset \tilde{\mathcal{M}}_{\hookrightarrow}(K).$$

This provides an open neighborhood of Q contained in $\tilde{\mathcal{M}}_{\hookrightarrow}(K)$. □

Recall that Corollary 9 claimed we could use immersions alone to define a subbasis for the immersive topology on $\tilde{\mathcal{M}}$.

Proof of Corollary 9. Let $\mathcal{T}$ denote the immersive topology as defined in this paper. Let $\mathcal{T}'$ denote the topology generated by sets listed in the statement of the corollary. So, $\mathcal{T}'$ is generated by sets of the form $\tilde{\mathcal{M}}_{\rightsquigarrow}(K)$, where K is a closed disk, and $\tilde{\mathcal{M}}_{\not\rightsquigarrow}(U)$, where U is an open disk. As a consequence of Theorem 8, we see the sets in this list are open in $\mathcal{T}$. Thus, $\mathcal{T}' \subset \mathcal{T}$, viewing these topologies as their collection of open subsets of $\tilde{\mathcal{M}}$.

Now we will show that $\mathcal{T} \subset \mathcal{T}'$. We will show that the elements of the subbasis used to define $\mathcal{T}$ are open in $\mathcal{T}'$. Sets of the form $\tilde{\mathcal{M}}_{\rightsquigarrow}(K)$ are open in both topologies. The other sets used to define $\mathcal{T}$ are of the form $\tilde{\mathcal{M}}_{\not\hookrightarrow}(U)$ where U is an open disk. Fix an open disk $U \in \mathrm{Disk}(P)$, and let $Q \in \tilde{\mathcal{M}}_{\not\hookrightarrow}(U)$. It suffices to find an open set in $\mathcal{T}'$ which contains Q and is contained in $\tilde{\mathcal{M}}_{\not\hookrightarrow}(U)$. We consider two possibilities. First, it could be that $U \not\rightsquigarrow Q$. In this case, $Q \in \tilde{\mathcal{M}}_{\not\rightsquigarrow}(U) \subset \tilde{\mathcal{M}}_{\not\hookrightarrow}(U)$. Now suppose that there is an immersion $\iota : U \rightsquigarrow Q$, but the immersion is not an embedding. Then, we can find a compact subset $K \subset U$ so that $\iota|_K$ is not an embedding. Let D be a closed disk in Q which contains $\iota(K)$. Then, $Q \in \tilde{\mathcal{M}}_{\rightsquigarrow}(D)$. We claim that $\tilde{\mathcal{M}}_{\rightsquigarrow}(D) \subset \tilde{\mathcal{M}}_{\not\hookrightarrow}(U)$. This will conclude the proof since $\tilde{\mathcal{M}}_{\rightsquigarrow}(D)$ is open in $\mathcal{T}'$. Suppose not. Then, there is a planar surface R for which $U \hookrightarrow R$ and for which there is an immersion $j : D \rightsquigarrow R$. By restriction of the embedding of U into R, we see that $K \hookrightarrow R$. Observe that $\iota(K) \subset D$. So, the unique immersion of K into R can be obtained as a composition $j \circ \iota|_K : K \rightsquigarrow R$. But, this immersion cannot be an embedding because $\iota|_K$ is not. This contradicts the statement that $K \hookrightarrow R$. □

Proposition 47 (Explicit second countability)**.** *The subsets of the following two forms give a countable subbasis for the topology on $\tilde{\mathcal{M}}$:*

- *Sets of the form $\tilde{\mathcal{M}}_{\rightsquigarrow}(K)$ where $K \in PC$ is an isomorphism class of a closed rational rectangular union.*
- *Sets of the form $\tilde{\mathcal{M}}_{\not\hookrightarrow}(U)$ where $U \in PC$ is an isomorphism class of an open rational rectangular union.*

Proof. Countability follows from Corollary 40.

The sets listed are clearly open by Theorem 8. We must prove that they form a subbasis for the topology. We will show that the sets in the subbasis used to define the immersive topology are open in the topology $\mathcal{T}$ generated by the sets listed in this proposition.

Let P be a planar surface and let $K \in \overline{\text{Disk}}(P)$. We will show $\tilde{\mathcal{M}}_{\rightsquigarrow}(K)$ is open in $\mathcal{T}$. Let $Q \in \tilde{\mathcal{M}}_{\rightsquigarrow}(K)$. Then there is an immersion $\iota : K \rightsquigarrow Q$. Let $K_1 = \iota(K)$. Using a closed disk family, we can find a $K_3 \in \overline{\text{Disk}}(Q)$ so that $K_1 \subset K_3^\circ$. Then, Theorem 45 guarantees that there is a rectangular union $K_2 \in \overline{\text{Disk}}(Q)$ so that $K_1 \subset K_2^\circ$ and $K_2 \subset K_3^\circ$. Observe that $Q \in \tilde{\mathcal{M}}_{\rightsquigarrow}(K_2)$ and $\tilde{\mathcal{M}}_{\rightsquigarrow}(K_2) \subset \tilde{\mathcal{M}}_{\rightsquigarrow}(K)$. It follows that $\tilde{\mathcal{M}}_{\rightsquigarrow}(K)$ is open in $\mathcal{T}$.

Let P be a planar surface and let $U \in \text{Disk}(P)$. We will show that $\tilde{\mathcal{M}}_{\not\rightsquigarrow}(U)$ is open in $\mathcal{T}$. Let $Q \in \tilde{\mathcal{M}}_{\not\rightsquigarrow}(U)$. Then $U \not\rightsquigarrow Q$. Since U is an open disk, we can think of it as a planar surface. Choose a closed disk family $\{K_t \ : \ t > 0\}$ for U. By Proposition 23, there is an $a > 0$ so that $K_a^\circ \not\rightsquigarrow Q$. Let $K \in \overline{\text{Disk}}(P)$ be a rectangular union satisfying $K_a \subset K^\circ$ and $K \subset K_{a+1}^\circ$. We have $Q \in \tilde{\mathcal{M}}_{\not\rightsquigarrow}(K^\circ)$ and $\tilde{\mathcal{M}}_{\not\rightsquigarrow}(K^\circ) \subset \tilde{\mathcal{M}}_{\not\rightsquigarrow}(U)$. So, $\tilde{\mathcal{M}}_{\not\rightsquigarrow}(U)$ is open in $\mathcal{T}$. ☐

Proof that the immersive topology on $\tilde{\mathcal{M}}$ is Hausdorff. Let $P, Q \in \tilde{\mathcal{M}}$ be distinct planar surfaces. Then by Corollary 5, either $P \not\rightsquigarrow Q$ or $Q \not\rightsquigarrow P$. Without loss of generality, assume $P \not\rightsquigarrow Q$. Let $\{K_t\}$ be a closed disk family for P. If each $K_t^\circ \rightsquigarrow Q$, then $P \rightsquigarrow Q$ by statement (3) of Proposition 23. Therefore, there is a t so that $K_t^\circ \not\rightsquigarrow Q$. We conclude that $P \in \tilde{\mathcal{M}}_{\rightsquigarrow}(K_t)$ and $Q \in \tilde{\mathcal{M}}_{\not\rightsquigarrow}(K_t^\circ)$. These open sets are disjoint since if $K_t \rightsquigarrow R$ then restriction gives an immersion $K_t^\circ \rightsquigarrow R$. ☐

9.6. **Open sets in $\tilde{\mathcal{E}}$.** Recall that the map $\tilde{\sigma} : \tilde{\mathcal{M}} \to \tilde{\mathcal{E}}$ sends $P \in \tilde{\mathcal{M}}$ to (P, o_P), where o_P denotes the basepoint of P. This map is continuous:

Proof of Proposition 12. Because $\tilde{\pi} \circ \tilde{\sigma}$ is the identity on $\tilde{\mathcal{M}}$, postcomposition with $\tilde{\pi}$ is continuous. So, it suffices to show that each set of the form $\tilde{\sigma}^{-1}\big(\tilde{\mathcal{E}}_+(K, U)\big)$ is open. Let $Q \in \tilde{\sigma}^{-1}\big(\tilde{\mathcal{E}}_+(K, U)\big)$. Then there is an immersion $\iota : K \rightsquigarrow Q$ and $o_Q \in \iota(U)$. Observe that $Q \in \tilde{\mathcal{M}}_{\hookrightarrow}\big(\iota(K)\big)$ and $\tilde{\mathcal{M}}_{\hookrightarrow}\big(\iota(K)\big) \subset \tilde{\sigma}^{-1}\big(\tilde{\mathcal{E}}_+(K, U)\big)$. The later holds because if $R \in \tilde{\mathcal{M}}_{\hookrightarrow}\big(\iota(K)\big)$ then there is an embedding $e : \iota(K) \hookrightarrow R$ and we get an immersion $K \rightsquigarrow R$ from the composition $e \circ \iota$. Furthermore, by definition of embedding $o_R = e(o_Q) \in e \circ \iota(U)$. ☐

We now prove that sets of the form $\tilde{\mathcal{E}}_-(K_2, K_1)$ are open in $\tilde{\mathcal{E}}$. These sets were defined in equation 8.

Proof of Proposition 13. Let P be a planar surface. Let $K_2 \in \text{PC}(P)$ be compact, and let $K_1 \subset K_2$ be closed. We will prove that $\tilde{\mathcal{E}}_-(K_2, K_1)$ is open. Let $(Q, q) \in \tilde{\mathcal{E}}_-(K_2, K_1)$. Then, there is an immersion $\iota : K_2 \rightsquigarrow Q$ and $q \notin \iota(K_1)$. Using a closed disk family in Q, we can find a $K_4 \in \overline{\text{Disk}}(Q)$ so that $\iota(K_2) \subset K_4^\circ$ and $q \in K_4^\circ$. Let $K_3 = \iota(K_1)$. Then $q \notin K_3$. Let $U = K_4^\circ \smallsetminus K_3$. Then,

$$(Q, q) \in \tilde{\mathcal{E}}_+(K_4, U) \cap \tilde{\pi}^{-1}\big(\tilde{\mathcal{M}}_{\hookrightarrow}(K_4)\big).$$

We will now show that $\tilde{\mathcal{E}}_+(K_4, U) \cap \tilde{\pi}^{-1}\big(\mathcal{M}_{\hookrightarrow}(K_4)\big) \subset \tilde{\mathcal{E}}_-(K_2, K_1)$. It follows that $\tilde{\mathcal{E}}_-(K_2, K_1)$ is open. Suppose that $(R, r) \in \tilde{\mathcal{E}}_+(K_4, U) \cap \tilde{\pi}^{-1}\big(\tilde{\mathcal{M}}_{\hookrightarrow}(K_4)\big)$. We will show that $(R, r) \in \tilde{\mathcal{E}}_-(K_2, K_1)$. Since $R \in \tilde{\mathcal{M}}_{\hookrightarrow}(K_4)$, there is an embedding $e : K_4 \hookrightarrow R$. Since $(R, r) \in \tilde{\mathcal{E}}_+(K_4, U)$, $r \in e(U)$. By composition, we get an immersion $e \circ \iota : K_2 \rightsquigarrow R$. Since $U = K_4^\circ \smallsetminus \iota(K_1)$, so $r \in e\big(K_4^\circ \smallsetminus \iota(K_1)\big)$. Since e is an embedding, $r \notin e \circ \iota(K_1)$. So, $(R, r) \in \tilde{\mathcal{E}}_-(K_2, K_1)$, as desired. ☐

Proof that the immersive topology on $\tilde{\mathcal{E}}$ is Hausdorff. Suppose (P,p) and (Q,q) are distinct points in $\tilde{\mathcal{E}}$. We will find open sets separating these points. If P and Q are distinct planar surfaces, then we can use the Hausdorff property of the embedding topology on $\tilde{\mathcal{M}}$ to separate P and Q open sets. The preimages of these open sets under $\tilde{\pi}$ are open in $\tilde{\mathcal{E}}$ and separate our points.

Otherwise, we have $P=Q$, and $q \in P$. By distinctness, we have $p \neq q$. Choose a compact set $K \in \overline{\text{Disk}}(P)$ so that $p, q \in K^\circ$. Since K is homeomorphic to a closed disk, we can find disjoint open sets U and V in K so that $p \in U$ and $q \in V$. Then we have

$$(P,p) \in \tilde{\pi}^{-1}\big(\tilde{\mathcal{M}}_{\hookrightarrow}(K)\big) \cap \tilde{\mathcal{E}}_+(K,U) \quad \text{and} \quad (Q,q) \in \tilde{\pi}^{-1}\big(\tilde{\mathcal{M}}_{\hookrightarrow}(K)\big) \cap \tilde{\mathcal{E}}_+(K,V).$$

Moreover, these sets can be seen to be disjoint because if $R \in \tilde{\mathcal{M}}_{\hookrightarrow}(K)$, then there is an embedding $e: K \hookrightarrow R$ and $e(U) \cap e(V) = \emptyset$ by injectivity of e. $\square$

The following gives an explicit countable subbasis for the immersive topology on $\tilde{\mathcal{E}}$.

Proposition 48 (Second countability of $\tilde{\mathcal{E}}$)**.** *A countable subbasis for the immersion topology on $\tilde{\mathcal{E}}$ is given by the union of the collection of preimages under $\tilde{\pi}$ of the subbasis provided by Proposition 47 together with the collection sets of the form $\tilde{\mathcal{E}}_+(K,U)$ where $K \in PC$ is an isomorphism class of a closed rational rectangular union and $U \subset K^\circ$ is an open rational rectangle.*

Proof. The potential subbasis described is clearly a countable collection of open sets. We must show that it generates the topology. By Proposition 47, the map $\tilde{\pi}$ is continuous in the generated topology. To conclude the proof, we must show that $\tilde{\mathcal{E}}_+(D,V)$ is open for an arbitrary $D \in \overline{\text{Disk}}(P)$ and arbitrary $U \subset D^\circ$ open. Choose a $(Q,q) \in \tilde{\mathcal{E}}_+(D,V)$. Then there is an immersion $\iota: D \rightsquigarrow Q$ and $q \in \iota(V)$. By taking a closed disk family in Q and applying Theorem 45, we can produce a closed rational rectangular union $K \in \overline{\text{Disk}}(Q)$ so that $\iota(D) \subset K^\circ$. Also since $q \in \iota(V)$ and $\iota(V)$ is open, we can find an open rational rectangle U so that $q \in U \subset \iota(V)$. Then, $(Q,q) \in \tilde{\mathcal{E}}_+(K,U)$. We also claim that $\tilde{\mathcal{E}}_+(K,U) \subset \tilde{\mathcal{E}}_+(D,V)$. Suppose $(R,r) \in \tilde{\mathcal{E}}_+(K,U)$. Then, there is an immersion $j: K \rightsquigarrow R$ and $r \in j(U)$. By composition, we have an immersion $j \circ \iota : D \rightsquigarrow R$. Furthermore, since $U \subset \iota(V)$, we have $r \in j(U) \subset j \circ \iota(V)$. $\square$

Finally, we give a proof of our result on the (joint) continuity of immersions.

Proof of Proposition 14. Take a sequence $P_n \in \mathcal{I}(U)$ converging to $P \in \mathcal{I}(U)$, and a sequence $q_n \in U$ converging to $q \in U$. Let $\iota_n : U \rightsquigarrow P_n$ and $\iota: U \rightsquigarrow P$ be the immersions coming from the fact that $P_n, P \in \mathcal{I}(U)$. We will prove that $\big(P_n, \iota_n(q_n)\big)$ converges to $\big(P, \iota(q)\big)$ by applying Proposition 10. Let $D \in \overline{\text{Disk}}(P)$, let $V \subset D^\circ$ be open, and assume that $\iota(q) \in V$. We will show that $\big(P_n, \iota_n(q_n)\big) \in \tilde{\mathcal{E}}_+(D,V)$ for sufficiently large n.

The set $\{q_n : n \in \mathbb{N}\} \cup \{q\}$ is a compact set contained in U, so we can find a compact set $K_1 \in \text{PC}(Q)$ so that $K_1 \subset U$ and $\{q_n : n \in \mathbb{N}\} \cup \{q\} \subset K_1$. Let $D' = D \cup \iota(K_1)$. This is a union of two compact sets in $\text{PC}(P)$, so $D' \in \text{PC}(P)$ is compact. Since $P_n \to P$, there is an N_1 so that $n > N_1$ implies that there is an embedding $e_n : D' \hookrightarrow P_n$. We claim that for $n > N_1$, $\iota_n(q_n) = e_n \circ \iota(q_n)$. This is because both ι_n and $e_n \circ \iota$ restrict to immersions defined on $K_1 \in \text{PC}(Q)$ and $q_n \in K_1$. So by uniqueness of immersions, $\iota_n(r) = e_n \circ \iota(r)$ for any $r \in K_1$.

The set $\iota^{-1}(V)$ is open and contains q. Therefore, there is an N_2 so that $n > N_2$ implies $q_n \in \iota^{-1}(V)$. We conclude that for $n > \max\{N_1, N_2\}$,

$$\iota_n(q_n) = e_n \circ \iota(q_n) \in e(V).$$

This proves that $\big(P_n, \iota_n(q_n)\big) \in \tilde{\mathcal{E}}_+(D, V)$ for sufficiently large n as desired. $\square$

10. Changing the basepoint

10.1. **Formal basepoint changes.** In order to formally define the maps relating to basepoint change, we need to use the definitions of $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{M}}$ given in §3. Let $(P, p) \in \tilde{\mathcal{E}}$. The pair (P, p) is an equivalence class of pairs consisting of a $\phi \in \mathrm{PLH}$ and a $y \in \Delta$. Given (ϕ, y), we define a new pointed local homeomorphism

$$\phi^y(x) = \phi \circ j^{-1}(x) - \phi(y), \tag{14}$$

where $j : \Delta \to \Delta$ is an arbitrary orientation preserving homeomorphism sending y to x_0. We have precomposed ϕ with j^{-1} to move x_0 to y, and postcomposed with a translation to ensure that the ϕ^y respects the basepoints, i.e., $\phi^y(x_0) = \mathbf{0}$. Formally, we define our basepoint changing map by $\widetilde{\mathrm{BC}}([\phi, y]) = [\phi^y]$. The following proposition proves that this map is well defined.

Proposition 49. *The class $[\phi^y] \in \tilde{\mathcal{M}}$ of is independent of the choice of j above and the choice of (ϕ, y) in the equivalence class $(P, p) \subset PLH \times \Delta/Homeo_+(\Delta, x_0)$.*

Proof. Suppose (ψ, z) is also in the class (P, p), and let $k : \Delta \to \Delta$ be an arbitrary orientation preserving homeomorphism of Δ sending z to x_0. Analogous to the definition of ϕ^y, we define

$$\psi^z(x) = \psi \circ k^{-1}(x) - \psi(z).$$

Since both (ϕ, y) and (ψ, z) lie in the equivalence class (P, p), there is an $h \in \mathrm{Homeo}_+(\Delta, x_0)$ so that $\psi = \phi \circ h^{-1}$ and $z = h(y)$. In particular, note that

$$\psi(z) = \phi \circ h^{-1}\big(h(y)\big) = \phi(y). \tag{15}$$

In addition, the composition $k \circ h \circ j^{-1}$ lies in $\mathrm{Homeo}_+(\Delta, x_0)$, because it is an orientation preserving homeomorphism of Δ which fixes x_0. We will show that ψ^z can be obtained from ϕ^y by precomposing with the inverse of this homeomorphism, proving the desired equivalence. We have

$$\begin{aligned} \phi^y \circ j \circ h^{-1} \circ k^{-1}(x) &= \phi \circ j^{-1} \circ j \circ h^{-1} \circ k^{-1}(x) - \phi(y) \\ &= \phi \circ h^{-1} \circ k^{-1}(x) - \phi(y) = \psi \circ k^{-1}(x) - \psi(z) = \psi^z(x). \end{aligned}$$

$\square$

Let $(P, p) \in \tilde{\mathcal{E}}$ and let $P^p = \widetilde{\mathrm{BC}}(P, p)$. We will now formally define the basepoint changing (translation) isomorphism $\tilde{\beta}_p : P \to P^p$ which caries p to the basepoint o_{P^p}. Choose (ϕ, y) from the equivalence class of (P, p). Define ϕ^y using an arbitrary orientation preserving homeomorphism j sending y to x_0 as in equation 14. Thus, ϕ^y represents a choice from the equivalence class of $P^p \in \tilde{\mathcal{M}}$. Consider the map $j : \Delta \to \Delta$. The map j induces a translation isomorphism from the translation structure determined by ϕ to the structure determined by ϕ^y because

$$\phi \circ j^{-1} : x \mapsto \phi \circ j^{-1}(x) \quad \text{and} \quad \phi^y : x \mapsto \phi^y(x) = \phi \circ j^{-1}(x) - \phi(y)$$

differ only by postcomposition with a translation. It follows that the basepoint changing isomorphism can be defined as

$$\tilde{\beta}_p : P \to P^p; \quad [\phi, y] \mapsto [\phi^y, j(y)],$$

where j is taken from the definition of ϕ^y. This map is well defined, and satisfies $\tilde{\beta}_p(p) = o_{P^p}$ and

$$Dev|_{P^p} \circ \tilde{\beta}_p(q) = Dev|_P(q) - Dev|_P(p) \quad \text{for all } q \in P.$$

That is, $\tilde{\beta}_p$ is a translation isomorphism from P to P^p which moves p to the basepoint of P^p. As a remark, we note that all translation isomorphisms of planar surfaces are basepoint changing homeomorphisms.

Proposition 50. *Let $i : P \to Q$ be a translation isomorphism. Let $p = i^{-1}(o_Q)$. Then, $i = \tilde{\beta}_p$.*

Proof. We note that $i \circ \tilde{\beta}_p^{-1} : Q \to Q$ is an immersion from Q to itself. It is the identity by the uniqueness of immersions. □

10.2. **Restated results.** In this subsection, we break Theorem 17 into pieces, and give a more careful statement of the results. We will devote the remainder of the section to proving these results.

Theorem 51. *The basepoint changing map $\widetilde{BC} : \tilde{\mathcal{E}} \to \tilde{\mathcal{M}}$ given by $\widetilde{BC}(P, p) = P^p$ is continuous in the immersive topology.*

The domain of the basepoint changing isomorphism $\tilde{\beta}_p : P \to P^p$ is most broadly

$$\tilde{\mathcal{D}} = \big\{\big((P, p), (Q, q)\big) \in \tilde{\mathcal{E}}^2 \ : \ P = Q\big\}.$$

We will write points in $\tilde{\mathcal{D}}$ as a triple, (P, p, q). Then we can define

$$\tilde{\beta} : \tilde{\mathcal{D}} \to \tilde{\mathcal{E}}; \quad (P, p, q) \mapsto \big(P^p, \tilde{\beta}_p(q)\big).$$

Theorem 52. *The map $\tilde{\beta}$ is continuous.*

Recall that $\tilde{\beta}_p : P \to P^p$ is a translation isomorphism. So, it is also natural to consider the inverse $\tilde{\beta}_p^{-1} : P^p \to P$. The most general domain for this "inverse map" is

$$\tilde{\mathcal{D}}^- = \big\{\big((P, p), (Q, q)\big) \in \tilde{\mathcal{E}}^2 \ : \ Q = P^p\big\}.$$

Then the map $\tilde{\beta}_p^{-1}$ becomes:

$$\tilde{\beta}^- : \tilde{\mathcal{D}}^- \to \tilde{\mathcal{E}}; \quad \big((P, p), (P^p, q)\big) \mapsto \big(P, \tilde{\beta}_p^{-1}(q)\big).$$

Corollary 53. *The map $\tilde{\beta}^-$ is continuous.*

Proof. There is a natural continuous map from $\tilde{\mathcal{E}} \to \tilde{\mathcal{D}}$ given by $i : (P, p) \mapsto (P, p, o_P)$. This map is continuous by Proposition 12. For brevity, let $o = o_P$ denote the basepoint of P. Then $\tilde{\beta} \circ i(P, p) = \big(P^p, \tilde{\beta}_p(o)\big)$. Observe that the map on $\tilde{\mathcal{D}}^-$ which does this in the first coordinate sends $\tilde{\mathcal{D}}^-$ into $\tilde{\mathcal{D}}$:

$$\big((P, p), (P^p, q)\big) \mapsto \Big(\big(P^p, \tilde{\beta}_p(o)\big), (P^p, q)\Big) = \big(P^p, \tilde{\beta}_p(o), q\big).$$

Furthermore, another application of $\tilde{\beta}$ yields:

$$\tilde{\beta}\big(P^p, \tilde{\beta}_p(o), q\big) = \big((P^p)^{\tilde{\beta}_p(o)}, \tilde{\beta}_{\tilde{\beta}_p(o)}(q)\big) = \big(P, \tilde{\beta}_p^{-1}(q)\big).$$

The last equality above follows from the observation that we are changing basepoints twice, first to p and then to the image $\tilde{\beta}_p(o)$ of the original basepoint. In particular, $\tilde{\beta}^- = \tilde{\beta}\circ(i\times id)$. Thus, it is continuous. $\square$

10.3. **Proofs.** In this subsection, we prove Theorems 51 and 52. We will use the observation that basepoint changes interact in a natural way with embeddings and immersions:

Proposition 54. *Let P be a planar surface, let $A \in PC(P)$, and choose $a \in A$. Suppose Q is another planar surface and there is an immersion (resp. embedding) $\iota_1 : A \to Q$. Let $q = \iota_1(a)$. Then, there is an immersion (resp. embedding) $\iota_2 : \tilde{\beta}_a(A) \to Q^{i_1(a)}$, and the following diagram commutes.*

$$\begin{array}{ccc} A & \xrightarrow{\iota_1} & Q \\ \Big\downarrow{\scriptstyle \tilde{\beta}_a} & & \Big\downarrow{\scriptstyle \tilde{\beta}_q} \\ \tilde{\beta}_a(A) & \xrightarrow{\iota_2} & Q^q. \end{array}$$

Proof. To make the diagram commute, we define ι_2 as a composition

$$\iota_2 : \tilde{\beta}_a(A) \xrightarrow{\tilde{\beta}_a^{-1}} A \xrightarrow{\iota_1} Q \xrightarrow{\tilde{\beta}_q} Q^q.$$

The individual maps respect the translation surface structure, and send the basepoint $\tilde{\beta}_a(a)$ of P^a to the basepoint $\tilde{\beta}_q(q)$ of Q^q. Thus the composition is an immersion. Furthermore, since $\tilde{\beta}_a^{-1}$ and $\tilde{\beta}_q$ are translation isomorphisms, ι_2 is an embedding if ι_1 is. $\square$

Before we give the proof of the two theorems, we recall that the sets of the form $\tilde{\mathcal{M}}_{\rightsquigarrow}(K)$, where $K \in \mathrm{PC}$ is a closed disk, and sets of the form $\tilde{\mathcal{M}}_{\not\rightsquigarrow}(U)$, where $U \in \mathrm{PC}$ is an open disk generate the immersive topology on $\tilde{\mathcal{M}}$. See Corollary 9. Our proof provides a direct proof of continuity using this subbasis.

Proof of Theorems 51 and 52. To prove that $\widetilde{\mathrm{BC}}$ and $\tilde{\beta}$ are continuous, it suffices to prove that the inverse images of a subbasis of open sets in the ranges are open. It suffices to prove the following statements for an arbitrary planar surface Q.

(1) If $K \in \overline{\mathrm{Disk}}(Q)$, then $\widetilde{\mathrm{BC}}^{-1}\big(\tilde{\mathcal{M}}_{\rightsquigarrow}(K)\big)$ is open.
(2) If K is as above and $U \subset K^\circ$ is open, then $\tilde{\beta}^{-1}\big(\tilde{\mathcal{E}}_+(K,U)\big)$ is open in the domain $\tilde{\mathcal{D}}$.
(3) The set $\widetilde{\mathrm{BC}}^{-1}\big(\tilde{\mathcal{M}}_{\not\rightsquigarrow}(Q)\big)$ is open.

We will simultaneously prove the related statements (1) and (2). Let $K \in \overline{\mathrm{Disk}}(Q)$ and let $U \subset K^\circ$ be open. Let P be a planar surface and $p, q \in P$. We will assume that $P^p \in \tilde{\mathcal{M}}_{\rightsquigarrow}(K)$ and that $\tilde{\beta}_p(q) \in \tilde{\mathcal{E}}_+(K,U)$. Let $q' = \tilde{\beta}_p(q) \in P^p$ and $o' = \tilde{\beta}_p(o_P) \in P^p$. Then, $\exists \iota : K \rightsquigarrow P^p$ and $q' \in \iota(U)$. Let $K_1 = \iota(K) \subset P^p$. Using the closed disk family provided by Proposition 26, we can find a $K_2 \in \overline{\mathrm{Disk}}(P^p)$ so that $K_1 \subset K_2^\circ$ and $o' \in K_2^\circ$. Define

$$\epsilon = \min\big\{ER(K_1 \subset K_2^\circ), \frac{1}{2}ER(q' \in \iota(U)\big\}.$$

Let $V \subset P^p$ be the open ϵ-ball about the basepoint, o_{P^p}, of P^p, and let $W \subset P^p$ be the open ϵ-ball about q'. Since both o_{P^p} and q' lie in K_1, we know that $V, W \subset K_2^\circ$.

We will now prove the following statement:

$$(R,r,s) \in \tilde{\mathcal{E}}_+(K_2,V) \times \tilde{\mathcal{E}}_+(K_2,W) \quad \text{implies } \exists \iota_R : K \rightsquigarrow R^r \text{ and } \tilde{\beta}_r(s) \in \iota_R(U). \tag{16}$$

We note that the hypothesis is satisfied for the quadruple (P^p, o_{P^p}, q'), and the conclusion is the desired conclusion that $(R,r) \in \widetilde{\mathrm{BC}}^{-1}\big(\tilde{\mathcal{M}}_{\rightsquigarrow}(K)\big)$ and $(R,r,s) \in \tilde{\beta}^{-1}\big(\tilde{\mathcal{E}}_+(K,U)\big)$. We will later modify this implication to hold on an open neighborhood of (P,p,q). Now we will prove the implication. Given such (R,r,s), we get an immersion $\iota_1 : K_2 \rightsquigarrow R$ so that $r \in \iota_1(V)$ and $s \in \iota_1(W)$. In particular r is within ϵ of o_R. So, by Corollary 58, there is an immersion $j_1 : K_1 \rightsquigarrow R^r$ which satisfies $d\big(\tilde{\beta}_r \circ \iota_1(q'), j_1(q')\big) < \epsilon$. In particular, we get our immersion $\iota_R = j_1 \circ \iota$, where $\iota : K \rightsquigarrow K_1$ is as in the prior paragraph. Because W was a ball of radius ϵ about q', we see that s is within ϵ of $\iota_1(q')$. Then by the triangle inequality

$$d\big(\tilde{\beta}_r(s), j_1(q')\big) \leq d\big(\tilde{\beta}_r(s), \tilde{\beta}_r \circ \iota_1(q')\big) + d\big(\tilde{\beta}_r \circ \iota_1(q'), j_1(q')\big) \leq 2\epsilon.$$

By definition of ϵ, $\iota(U)$ contains the 2ϵ ball about q'. So, we conclude that $\iota_R(U) = j_1 \circ \iota(U)$ contains the 2ϵ ball about $j_1(q')$ and this includes $\tilde{\beta}_r(s)$.

We will now use the translation isomorphism $\tilde{\beta}_p : P \to P^p$ to pull these sets back into P. We define:

$$K_1' = \tilde{\beta}_p^{-1}(K_1), \quad K_2' = \tilde{\beta}_p^{-1}(K_2), \quad V' = \tilde{\beta}_p^{-1}(V), \quad \text{and} \quad W' = \tilde{\beta}_p^{-1}(W).$$

Note that $K_2' \in \overline{\mathrm{Disk}}(P)$ since $o' = \tilde{\beta}_p(o_p) \in K_2^\circ$. We claim that for any (T,t,u) in the domain $\tilde{\mathcal{D}}$ of $\tilde{\beta}$, we have

$$(T,t,u) \in \tilde{\mathcal{E}}_+(K_2', V') \times \tilde{\mathcal{E}}_+(K_2', W') \quad \text{implies } \exists \iota_T : K \rightsquigarrow T^t \text{ and } \tilde{\beta}_t(u) \in \iota_T(U). \tag{17}$$

This will prove that $\widetilde{\mathrm{BC}}^{-1}\big(\tilde{\mathcal{M}}_{\rightsquigarrow}(K)\big)$ and $\tilde{\beta}^{-1}\big(\tilde{\mathcal{E}}_+(K,U)\big)$ are open, since (P,p,q) lies in this set. To prove our claim, suppose that there is an immersion $\iota_2 : K_2' \rightsquigarrow T$ so that $t \in \iota_2(V')$ and $u \in \iota_2(W')$. Let $v = \iota_2(p)$. By applying Proposition 54, we see there is an immersion $\iota_3 : K_2 \to T^v$ satisfying $t \in \tilde{\beta}_v \circ \iota_3(V)$ and $u \in \tilde{\beta}_v \circ \iota_3(W)$. This proves that

$$\big(T^v, \tilde{\beta}_v(t), \tilde{\beta}_v(u)\big) \in \tilde{\mathcal{E}}_+(K_2, V) \times \tilde{\mathcal{E}}_+(K_2, W),$$

which via equation 16 implies that

$$\exists \iota_R : K \rightsquigarrow (T^v)^{\tilde{\beta}_v(t)} \quad \text{so that} \quad \tilde{\beta}_{\tilde{\beta}_v(t)} \circ \tilde{\beta}_v(u) \in \iota_R(U). \tag{18}$$

Note we are changing basepoints of T twice: first we relocate the basepoint to v and then to the updated location of t. We can do this in one step by just relocating the basepoint to t. Thus,

$$(T^v)^{\tilde{\beta}_v(t)} = T^t \quad \text{and} \quad \tilde{\beta}_{\tilde{\beta}_v(t)} \circ \tilde{\beta}_v(u) = \tilde{\beta}_t(u).$$

This combines with equation 18 to prove equation 17, and finishes the proof of statements (1) and (2).

Now consider statement (3). Suppose $Q \not\rightsquigarrow P^p$. We will find an open set containing (P,p) so that if (R,r) is in this set, then $Q \not\rightsquigarrow R^r$.

Using a closed disk family, we can choose a compact set $K_1 \in \overline{\mathrm{Disk}}(Q)$ so that $K_1^\circ \not\rightsquigarrow P^p$. Choose another $K_2 \in \overline{\mathrm{Disk}}(Q)$ so that $K_1 \subset K_2^\circ$. Choose

$$\epsilon < \min\big\{ER(K_1 \subset K_2^\circ), ER(p)\big\}.$$

Let $\bar{B}_\epsilon$ denote the closed ball of radius ϵ centered at the origin in $\mathbb{R}^2$. We claim that

$$(R,r) \in \tilde{\mathcal{E}}_+(\bar{B}_\epsilon, B_\epsilon) \cap \pi^{-1}\big(\tilde{\mathcal{M}}_{\not\rightsquigarrow}(K_1^\circ)\big) \quad \text{implies} \quad Q \not\rightsquigarrow R^r.$$

Suppose not. Then there is an (R,r) so that $\bar{B}_\epsilon \hookrightarrow R$, $d(r,o_R) < \epsilon$ and $\exists \iota : Q \rightsquigarrow R^r$ but $K_1^\circ \not\rightsquigarrow R$. We will derive a contradiction by producing an embedding $K_1^\circ \rightsquigarrow R$. Note that restriction of ι produces an immersion of K_2 into R^r. By Proposition 29, we have

$$ER\big(\iota(K_1) \subset \iota(K_2^\circ)\big) \geq \epsilon.$$

Let $o = \tilde{\beta}_r(o_R)$. Then $\tilde{\beta}_o = \tilde{\beta}_r^{-1}$. Furthermore, we have $d(o_{R^r}, o) = d(r, o_R) < \epsilon$. Therefore, Corollary 58 implies that $K_1 \rightsquigarrow (R^r)^o$. But $(R^r)^o = R$, so $K_1 \rightsquigarrow R$ giving the desired contradiction.

Now choose $D \in \overline{\text{Disk}}(P)$ to be large enough so that it contains the closed ϵ ball about p. Let $U \subset D^\circ$ be the open ϵ ball about p. Let F be the fusion $\tilde{\beta}_p(D^\circ) \curlyvee K_1^\circ$. Then there is a canonical immersion $\iota : D^\circ \to F$. Let, $f = \iota \circ \tilde{\beta}_p(o_P) \in F$. We claim that

$$(S,s) \in \tilde{\mathcal{E}}_+(D,U) \cap \pi^{-1}\big(\tilde{\mathcal{M}}_{\not\rightsquigarrow}(F^f)\big) \quad \text{implies} \quad Q \not\rightsquigarrow S^s.$$

The left hand side gives the open set which will prove the second case. First observe that (P,p) lies in this set. By construction, $D \subset P$ and $p \in U$. We also need to show that $F^f \not\rightsquigarrow P$. Suppose $\exists \iota_2 : F^f \rightsquigarrow P$. Then we have the following commutative diagram of immersions (horizontal) and basepoint changes (vertical) given by Proposition 54:

$$\begin{tikzcd}
D^\circ \arrow[r] \arrow[d, "\tilde{\beta}_p"] & F^f \arrow[r, "\iota_2"] & P \arrow[d, "\tilde{\beta}_p"] \\
\tilde{\beta}_p(D^\circ) \arrow[r, "\iota"] & F \arrow[r] \arrow[u, "\tilde{\beta}_f"] & P^p.
\end{tikzcd}$$

In particular, $F \rightsquigarrow P^p$. But, by the Fusion Theorem, we have $K_1^\circ \rightsquigarrow F$, so by composition, $K_1^\circ \rightsquigarrow P^p$. But, we constructed K_1 so that this does not hold.

Now suppose that $(S,s) \in \tilde{\mathcal{E}}_+(D,U) \cap \pi^{-1}\big(\tilde{\mathcal{M}}_{\not\rightsquigarrow}(F^f)\big)$. We will prove that $Q \not\rightsquigarrow S^s$. By assumption, $\exists \iota_2 : D \to S$ and $s \in \iota_2(U)$, and $\tilde{\beta}_f(F) \not\rightsquigarrow S$. Let $t = \iota_2(p)$. Note that $\iota_2(U)$ is the ball of radius ϵ about t. By applying Proposition 54, there is an immersion $\iota_3 : \tilde{\beta}_p(D) \to S^t$ so that $\iota_3(s)$ lies within the (embedded) open ball of radius ϵ about the basepoint of S^t. By drawing a diagram similar to the above, we can conclude that $F \not\rightsquigarrow S^t$. By restriction ι_3 yields an immersion of $\tilde{\beta}_p(D^\circ)$ into S^t, and since $F \not\rightsquigarrow S^t$, we conclude by the Fusion Theorem that $K_1^\circ \not\rightsquigarrow S^t$. We conclude that

$$\big(S^t, \tilde{\beta}_t(s)\big) \in \tilde{\mathcal{E}}\big(\tilde{\beta}_p(D), \tilde{\beta}_p(U)\big) \cap \pi^{-1}\big(\tilde{\mathcal{M}}_{\not\rightsquigarrow}(K_1^\circ)\big).$$

We note that $\tilde{\mathcal{E}}\big(\tilde{\beta}_p(D), \tilde{\beta}_p(U)\big) \subset \tilde{\mathcal{E}}_+(\bar{B}_\epsilon, B_\epsilon)$, so by equation 17, we know that $Q \not\rightsquigarrow (S^t)^{\tilde{\beta}_t(s)}$. We conclude the case by observing that $(S^t)^{\tilde{\beta}_t(s)} = S^s$. □

11. The homeomorphism group action

This section investigates the continuity of the $\text{Homeo}(\mathbb{R}^2, \mathbf{0})$-action on $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{E}}$. This action was defined in §3.5.

11.1. Continuity of a homeomorphism's action.

It is fairly evident that the action of $\text{Homeo}(\mathbb{R}^2, \mathbf{0})$ behaves nicely with respect to immersions and embeddings. Indeed:

Proposition 55. *Let P and Q be planar surfaces, let $A \in PC(P)$ and $B \in PC(Q)$. Let $H \in Homeo(\mathbb{R}^2, \mathbf{0})$. Then, $A \rightsquigarrow B$ if and only if $H(A) \rightsquigarrow H(B)$. Also, $A \hookrightarrow B$ if and only* *if $H(A) \hookrightarrow H(B)$. Furthermore, the following diagram commutes in either case:*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \big\downarrow{\scriptstyle H} & & \big\downarrow{\scriptstyle H} \\ H(A) & \longrightarrow & H(B). \end{array}$$

Proof. Suppose $\exists \iota : A \rightsquigarrow B$. We will show that $\iota' = H \circ \iota \circ H^{-1}|_{H(A)}$ is an immersion from $H(A)$ to $H(B)$. It clearly $\iota'(o_P) = o_Q$. Furthermore, by equation 2 and using the fact that ι is an immersion, for any $p \in H(A)$, we have

$$\begin{aligned} Dev|_{H(Q)} \circ \iota'(p) &= Dev|_{H(Q)} \circ H \circ \iota \circ H^{-1}(p) = H \circ Dev|_Q \circ \iota \circ H^{-1}(p) \\ &= H \circ Dev|_P \circ H^{-1}(p) = H \circ H^{-1} \circ Dev|_{H(P)}(p) = Dev|_{H(P)}(p). \end{aligned}$$

This proves that ι' satisfies the developing map condition to be an immersion. Furthermore, since H and H^{-1} are bijections, ι is an embedding if and only if ι' is. The same argument proves (by acting by H^{-1}) that $H(A) \rightsquigarrow H(B)$ implies $A \rightsquigarrow B$ and that $H(A) \hookrightarrow H(B)$ implies $A \hookrightarrow B$. $\square$

As a consequence, we will see that the actions of any $H \in \mathrm{Homeo}(\mathbb{R}^2, \mathbf{0})$ on moduli and total spaces are continuous.

Lemma 56. *Fix any $H \in Homeo(\mathbb{R}^2, \mathbf{0})$. The actions of H on $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{E}}$ are continuous. That is, the maps $P \mapsto H(P)$ and $(P, p) \mapsto H(P, p)$ are continuous.*

Proof. By the previous proposition, it follows that

$$H^{-1}\big(\tilde{\mathcal{M}}_{\rightsquigarrow}(K)\big) = \tilde{\mathcal{M}}_{\rightsquigarrow}\big(H^{-1}(K)\big) \quad \text{and} \quad H^{-1}\big(\tilde{\mathcal{M}}_{\not\hookrightarrow}(Q)\big) = \tilde{\mathcal{M}}_{\not\hookrightarrow}\big(H^{-1}(Q)\big).$$

This proves that H acts continuously on $\tilde{\mathcal{M}}$. The fact that H acts continuously on $\tilde{\mathcal{E}}$ follows from equation 3 and the observation that

$$H^{-1}\big(\tilde{\mathcal{E}}_+(K, U)\big) = \tilde{\mathcal{E}}_+\big(H^{-1}(K), H^{-1}(U)\big).$$

$\square$

11.2. **Perturbations.** We use the ball embeddings introduced in §7.4 to perturb embeddings and immersions to planar surfaces. Recall that B_ϵ denotes the open ball of radius ϵ in $\mathbb{R}^2$. If the embedding radius, $ER(P, p)$, at $p \in \mathbb{P}$ is greater than ϵ, then the ball embedding $\widetilde{BE} : B_\epsilon \to P$ is the immersion $B_\epsilon \rightsquigarrow P^p$ post-composed with the translation isomorphism $\tilde{\beta}_p^{-1} : P^p \to P$, which moves the basepoint of P^p to $p \in P$. The following lemma and corollary describe the mentioned perturbations.

Lemma 57 (Perturbing subsets)**.** *Let P be a planar surface and let $K \subset P$ be a compact set containing the basepoint o_P. Let $\epsilon = ER(K)$ and choose any $p \in P$ so that $d(o_P, p) < \epsilon$. It follows that there is a $\mathbf{v} \in B_\epsilon$ so that $p = \widetilde{BE}_{o_P}(\mathbf{v})$. Then, there is an embedding of K into P^p, and it is given by*

$$e : K \hookrightarrow P^p; \quad k \mapsto \tilde{\beta}_p \circ \widetilde{BE}_k(\mathbf{v}),$$

where $\tilde{\beta}_p : P \to P^p$ is the basepoint changing isomorphism.

Proof. We will check that e is an embedding. The map $k \mapsto \widetilde{BE}_k(\mathbf{v})$ is continuous on K because each point of K has an embedding radius larger than $|\mathbf{v}|$. Since $\tilde{\beta}_p$ is also continuous, e is continuous. By definition, we have $e(o_P) = \tilde{\beta}_p(p) = o_{P^p}$. We observe that $\mathbf{v} = Dev|_P(p)$. Therefore, for all $k \in K$ we have

$$\begin{aligned} Dev|_{P^p}\big(e(k)\big) &= Dev|_{P^p}\big(\tilde{\beta}_p \circ \widetilde{BE}_k(\mathbf{v})\big) \\ &= Dev|_P\big(\widetilde{BE}_k(\mathbf{v})\big) - Dev|_P(p) = Dev|_P(k) + \mathbf{v} - \mathbf{v} = Dev|_P(k). \end{aligned}$$

This proves that e is an embedding. □

Corollary 58 (Perturbing immersions)**.** *Let K_1 and K_2 be closed disks in the planar surface Q, and suppose that $K_1 \subset K_2^\circ$. Let $\epsilon = ER(K_1 \subset K_2^\circ)$. Let $U \subset Q$ be the open ball of radius ϵ about the basepoint, and note that $U \subset K_2^\circ$. Let $(P,p) \in \tilde{\mathcal{E}}_+(K_2, U)$ so that there is an immersion $\iota : K_2 \rightsquigarrow P$ and $p \in \iota(U)$. Then, there is an immersion $j : K_1 \rightsquigarrow P^p$, and this immersion satisfies*

$$d\big(\tilde{\beta}_p \circ \iota(q), j(q)\big) < \epsilon \quad \textit{for all } q \in K_1.$$

Proof. Assume that $(P,p) \in \tilde{\mathcal{E}}_+(K_2, U)$ so that $\exists \iota : K_2 \rightsquigarrow P$ and $p \in \iota(U)$ as stated. By definition of U, there is a preimage $q \in \iota^{-1}(p)$ which is within ϵ of o_Q. So $q \in K_2^\circ$, and Lemma 57 guarantees that $\exists e : K_1 \hookrightarrow (K_2^\circ)^q$. The desired immersion, j, is given by the composition:

$$K_1 \xrightarrow{e} (K_2^\circ)^q \xrightarrow{\tilde{\beta}_q^{-1}} K_2^\circ \xrightarrow{\iota} P \xrightarrow{\tilde{\beta}_p} P^p.$$

Finally, we note that by definition (in the Lemma) the composition $\tilde{\beta}_q^{-1} \circ e$ moves points by distance less than ϵ, and the motion is along a straight line in K_2°. The inequality follows. □

11.3. **Continuity of the homeomorphism group action.** The group $\mathrm{Homeo}(\mathbb{R}^2, \mathbf{0})$ of all homeomorphisms of the plane fixing the origin is naturally endowed with the compact-open topology (which is the same as the topology of uniform convergence on compact sets). In this topology, a subbasis for the open sets is given by sets of the form $CO(K,U)$ where $K \subset \mathbb{R}^2$ is compact and $U \subset \mathbb{R}^2$ is open. We define

$$CO(K,U) = \{H \in \mathrm{Homeo}(\mathbb{R}^2, \mathbf{0}) \ : \ H(K) \subset U\}.$$

This topology makes $\mathrm{Homeo}(\mathbb{R}^2, \mathbf{0})$ into a topological group.

We will use the tools developed in the prior subsections to prove that the $\mathrm{Homeo}(\mathbb{R}^2, \mathbf{0})$ actions are continuous. The proofs of continuity of the $\mathrm{Homeo}(\mathbb{R}^2, \mathbf{0})$ actions essentially reduce to an understanding of the action of elements of $\mathrm{Homeo}(\mathbb{R}^n, \mathbf{0})$ in a small neighborhoods of the origin in the group.

Lemma 59. *Let P be a planar surface and $K \subset P$ be compact. Let $U \subset P$ be an open set containing K. Then, there is a neighborhood N of the identity in $Homeo(\mathbb{R}^2, \mathbf{0})$ so that for all $H \in N$, there is an immersion $\iota : H(K) \rightsquigarrow U$ whenever $H \in N$. Moreover, for any $\epsilon > 0$, there is a neighborhood $N' \subset N$ of the identity so that $H \in N'$ implies that*

$$d\big(p, \iota \circ H(p)\big) < \epsilon \quad \textit{for all } p \in K.$$

Proof. We can assume without loss of generality that $\epsilon \leq ER(K \subset U)$. Since our topology on $\mathrm{Homeo}(\mathbb{R}^2, \mathbf{0})$ is equivalent to the topology of uniform convergence on compact sets, the set

$$N = \big\{H \in \mathrm{Homeo}(\mathbb{R}^2, \mathbf{0}) \ : \ |H(\mathbf{v}) - \mathbf{v}| < \epsilon \text{ for all } v \in Dev|_P(K)\big\}$$

is a neighborhood of the identity. Fix an $H \in N$.

We will now specify a natural map $\phi : K \to U$. Pick $p \in K$. We define $\phi(p) \in U$ to be the unique point within distance ϵ of p so that

$$H \circ Dev|_P(p) = Dev|_P \circ \phi(p).$$

Observe that ϕ is continuous, because the images of points nearby p also must land within the ϵ ball about p, and $\phi(p)$ is uniquely and continuously determined.

We claim that the composition $\phi \circ H^{-1} : H(K) \to U$ is the desired immersion. Indeed, it can be seen to be an immersion because it respects the basepoints and because of the following commutative diagram:

$$\begin{array}{ccccc}
H(K) & \xrightarrow{H^{-1}} & K & \xrightarrow{\phi} & U \\
\big\downarrow Dev|_{H(P)} & & \big\downarrow Dev|_P & & \big\downarrow Dev|_P \\
\mathbb{R}^2 & \xrightarrow{H^{-1}} & \mathbb{R}^2 & \xrightarrow{H} & \mathbb{R}^2.
\end{array}$$

Furthermore, by construction it satisfies the inequality given in the lemma. □

Proof of Theorem 18. Suppose $\langle H_n \rangle$ is a sequence in $\text{Homeo}(\mathbb{R}^2, \mathbf{0})$ converging to H, $\langle P_n \rangle$ is a sequence in $\tilde{\mathcal{M}}$ converging to P, and $p_n \in P_n$ is a sequence in $\tilde{\mathcal{E}}$ converging to $p \in P$. We must prove that $H_n(P_n) \to H(P)$ and $H_n(P_n, p_n) \to H(P, p)$. We note that by Lemma 56, it suffices to consider the case when H is the identity, because we can apply H^{-1} uniformly to the whole sequence. So assume H is the identity.

We will now prove that $H_n(P_n) \to P$. We will verify statements (A) and (B) of Proposition 7 hold.

We will first consider statement (B). Let Q be a planar surface and suppose that $Q \hookrightarrow H_n(P_n)$ for infinitely many n. We must show that $Q \rightsquigarrow P$. By taking a subsequence, we can assume that there is an embedding $e_n : Q \hookrightarrow H_n(P_n)$ for every n. To prove that $Q \rightsquigarrow P$, it suffices to show that the interior of every closed disk in Q immerses in P. See Proposition 23. Let $K \in \overline{\text{Disk}}(Q)$. By the lemma above, there is an N so that $n > N$ implies that there is an immersion $\iota_n : H_n(K) \rightsquigarrow Q$. So, the composition $e_n \circ \iota_n$ immerses $H_n(K)$ into $H_n(P_n)$ for every n. Then by Proposition 55, $K \rightsquigarrow P_n$ for every $n > N$. Therefore $K^\circ \rightsquigarrow P_n$. Then, each P_n lies in the closed set $\tilde{\mathcal{M}} \smallsetminus \tilde{\mathcal{M}}_{\not\rightsquigarrow}(K^\circ)$. Since $P_n \to P$, we know that $P \in \tilde{\mathcal{M}} \smallsetminus \tilde{\mathcal{M}}_{\not\rightsquigarrow}(K^\circ)$ as well. Thus, $K^\circ \rightsquigarrow P$ as desired, which proves statement (B) holds.

Let $K_1 \in \overline{\text{Disk}}(P)$. To prove statement (A) of Proposition 7 holds, we must show that $K_1 \rightsquigarrow H_n(P_n)$ for all but finitely many n. Choose $K_2 \in \overline{\text{Disk}}(P)$ so that $K_1 \subset K_2^\circ$. Since $P_n \to P$, there is an N_1 so that $K_2 \hookrightarrow P_n$ for $n > N_1$. It then follows that $H_n(K_2) \hookrightarrow H_n(P_n)$. By the above lemma applied to H^{-1}, there is an N_2 so that for $n > N_2$, $H_n^{-1}(K_1) \rightsquigarrow K_2$. Equivalently, we have $n > N_2$ implies $K_1 \rightsquigarrow H_n(K_2)$. Then the existence of the desired immersion follows from the composition $K_1 \rightsquigarrow H_n(K_2) \hookrightarrow H_n(P_n)$, which exists for $n > \max\{N_1, N_2\}$. This proves statement (A) of the proposition and concludes the proof that the action on $\tilde{\mathcal{M}}$ is continuous.

We now consider the action on $\tilde{\mathcal{E}}$. Since $\pi \circ H(P, p) = H(P)$, for any $K_1 \in \overline{\text{Disk}}(P)$ and any open $U \subset K_1^\circ$ such that $p \in U$, there is an N so that for $n > N$, the immersion $\iota_n : K_1 \rightsquigarrow H_n(P_n)$ exists and satisfies $H_n(p_n) \in \iota_n(U)$. Existence of ι_n was provided above, but we need to look closer to verify that $H(p_n) \in \iota_n(U)$. Once we verify this, we have continuity of the action on $\tilde{\mathcal{E}}$ by Proposition 10. To verify this, we repeat the construction above. We construct a $K_2 \in \overline{\mathrm{Disk}}(P)$ so that $K_1 \subset K_2^\circ$. We also choose an open $V \subset U$ so that the closure $\bar{V} \subset U$. We choose

$$\epsilon \leq \min\big\{ER(K_1 \subset K_2^\circ), ER(\bar{V} \subset U)\big\}.$$

As above, since $P_n \to P$, there is an N_1 so that for $n > N_1$, $j_n : \exists K_2 \rightsquigarrow P_n$. By the above lemma, there is an N_2 so that for $n > N_2$, $\exists k_n : H_n^{-1}(K_1) \rightsquigarrow K_2$ so that

$$d\big(q, k_n \circ H_n^{-1}(q)\big) < \epsilon \quad \text{for all } q \in H_n^{-1}(K_1).$$

Since $p_n \to p$, there is an $N_3 > \max\{N_1, N_2\}$ so that $p_n \in j_n(V)$. Since the ϵ neighborhood of V is contained in U, and $k_n \circ H_n^{-1}$ moves points by no more than ϵ, we know that $V \subset k_n \circ H_n^{-1}(U)$. (The map $k_n \circ H_n^{-1}$ is a local homeomorphism, and the image of ∂U encloses V.) We conclude that $p_n \in j_n \circ k_n \circ H_n^{-1}(U)$. Then $H_n(p_n) \in H_n \circ j_n \circ k_n \circ H_n^{-1}(U)$. This proves our statement after we observe that $H_n \circ j_n \circ k_n \circ H_n^{-1}$ is a representation of the immersion $\iota_n : K_1 \rightsquigarrow H_n(P_n)$. □

12. Limits and the Compactness Theorem

In this section, we prove the Compactness Theorem. We also prove the Direct Limit and Inverse Limit Propositions. The Inverse Limit Proposition is easiest, so we prove it first.

Proof of Proposition 22. Let $\langle P_n \rangle$ be a sequence of planar surfaces as in the proposition. Then for each $m \leq n$, there is an immersion $\iota_{m,n} : P_n \rightsquigarrow P_m$. Let $P_\infty = \bigwedge\{P_n\}$. Then by the Core Corollary (Corollary 20), there are immersion $j_n : P_\infty \rightsquigarrow P_n$ for all n.

We will use the convergence criterion of Proposition 7. Suppose that $K \in P_\infty$ is compact. The restriction $j_n|_K : K \rightsquigarrow P_n$ gives the needed immersions into the surfaces P_n. Now suppose U immerses inside P_n for infinitely many n. Then for each m, there is an $n \geq m$ so that there is an immersion $k : U \rightsquigarrow P_n$. The composition $\iota_{m,n} \circ k$ gives an immersion $U \rightsquigarrow P_m$. Then by statement (II') of the Core Corollary, we know that $U \rightsquigarrow P_\infty$. □

Now we will prove the Direct Limit Proposition.

Proof of Proposition 21. Let $\langle P_n \rangle_{n \geq 1}$ be a sequence of planar surfaces as stated in the proposition. Then for each $m \leq n$, there is an immersion $\iota_{m,n} : P_m \rightsquigarrow P_n$. Let $P_\infty = \bigvee\{P_n\}$. By the Fusion Theorem, there are immersions $j_n : P_n \to P_\infty$. To prove that the sequence $\langle P_n \rangle$ converges to P_∞. To prove this, we apply the convergence criterion of Proposition 7.

Let $K \in \overline{\mathrm{Disk}}(P_\infty)$. The set K is compact, and we will apply a compactness argument to say that there is an M and a lift $\tilde{K} \subset P_M$ containing the basepoint of P_M so that $j_M|_{\tilde{K}}$ is a homeomorphism from $\tilde{K}$ onto K which respects the basepoints. Then, the inverse of this restriction $(j_M|_{\tilde{K}})^{-1}$ is the needed immersion of K into P_M. We can then immerse K into all P_N with $N > M$ by composing with $\iota_{M,N}$. This will prove the first statement needed from Proposition 7.

We will now construct $\tilde{K}$. Observe that by definition of the fusion, $P_\infty = \bigcup_n j_n(P_n)$. Otherwise, the union would be a smaller trivial surface with the properties of the fusion, violating the definition of P_∞ as the fusion. See Theorem 30.

Therefore, for each $x \in K$, there is an $n(x) \geq 1$ and a $p(x) \in P_{n(x)}$ so that $j_{n(x)}\big(p(x)\big) = x$. For the basepoint of P_∞ in K, we take $n(x) = 1$ and $p(x)$ to be the basepoint of P_1. Let B_x be the open metric ball about $p(x)$ of radius equal to the embedding radius of $p(x)$ in $P_{n(x)}$. Then the collection of images $\{j_{n(x)}(B_x) \ : \ x \in K\}$ is an open cover of K. So there is a finite subcover indexed by the subset $\{x_1, \ldots, x_k\} \subset K$. We add the basepoint of P_∞ to this set and call it x_0. Consider the collection

$$\mathcal{I} = \{(i,j) \in \{0, \ldots, k\}^2 \ : \ j_{n(x_i)}(B_{x_i}) \cap j_{n(x_j)}(B_{x_j}) \neq \emptyset\}.$$

Then for each $(i,j) \in \mathcal{I}$, we can choose points $y \in B_{x_i}$ and $z \in B_{x_j}$ so that $j_{n(x_i)}(y) = j_{n(x_j)}(z)$. By Corollary 34, there is a finite subset $\mathcal{F} \subset \{P_n\}$ containing $P_{n(x_i)}$ and $P_{n(x_j)}$ so that the immersions $P_{n(x_i)} \rightsquigarrow \bigvee \mathcal{F}$ and $P_{n(x_j)} \rightsquigarrow \bigvee \mathcal{F}$ send y and z to the same point. Because we are working with a directed sequence, we just have $\mathcal{F} = P_{N(i,j)}$ where $N(i,j)$ is the maximal index of a planar surface in $\mathcal{F}$. So, there is an $N = N(i,j)$ so that $\iota_{m(x_i),N}(y) = \iota_{m(x_j),N}(z)$. Then, because $j_{n(x_i)}(B_{x_i}) \cap j_{n(x_j)}(B_{x_j})$ is path connected, the map $j_{N(i,j)}$ restricted to

$$\iota_{n(x_i),N(i,j)}(B_{x_i}) \cup \iota_{n(x_j),N(i,j)}(B_{x_j})$$

is injective. We have defined $N(i,j)$ for all $(i,j) \in \mathcal{I}$. Let $M = \max_{(i,j)\in\mathcal{I}} N(i,j)$. Then j_M restricted to

$$\bigcup_{i=0}^{k} \iota_{n(x_i),M}(B_{x_i})$$

is injective. Then, because j_M is a a local homeomorphism, this restriction is a homeomorphism onto its image. The image contains K. We conclude that we can set $\tilde{K}$ equal to the preimage of K under this restriction of j_M. This verifies the existence of $\tilde{K}$ and proves that the first statement of Proposition 7 holds.

Now we consider the second statement of Proposition 7. Suppose that Q is a planar surface, and there is an immersion $k : Q \rightsquigarrow P_n$ for some n. Then, $j_n \circ k : Q \rightsquigarrow P_\infty$. This proves the second statement needed from Proposition 7, and concludes the proof that $P_n \to P_\infty$. □

We now prove the Compactness Theorem.

Proof of Theorem 16. Let P be a planar surface. We will prove $\tilde{\mathcal{M}} \smallsetminus \tilde{\mathcal{M}}_{\not\rightsquigarrow}(P)$ is compact. Since $\tilde{\mathcal{M}}$ is second countable, it suffices to prove that $\tilde{\mathcal{M}} \smallsetminus \tilde{\mathcal{M}}_{\not\rightsquigarrow}(P)$ is sequentially compact. So, let $\langle Q_n \rangle_{n\geq 0}$ be a sequence in $\tilde{\mathcal{M}} \smallsetminus \tilde{\mathcal{M}}_{\not\rightsquigarrow}(P)$. We will provide an algorithm which produces a convergent subsequence $\langle Q_{n_k} \rangle_{k\geq 0}$ converging to some limit $R \in \tilde{\mathcal{M}} \smallsetminus \tilde{\mathcal{M}}_{\not\rightsquigarrow}(P)$.

Recall that there are only countably many open rational rectangular unions which are homeomorphic to open disks. See Corollary 40. Let $\langle P_m \in \tilde{\mathcal{M}} \rangle_{m\geq 1}$ be a sequence which enumerates all of these rectangular unions. We will construct a subsequence $\langle P_{m_k} \rangle$ of $\langle P_m \rangle$ while simultaneously producing $\langle Q_{n_k} \rangle$.

Our algorithm is really an inductive sequence of definitions:

(1) Set $R_0 = P$.
(2) Set m_0, $n_0 = 0$ and $k = 1$.
(3) Set $\mathcal{I}_0 = \{n \ : \ n \geq 1\}$.
(4) For each successive integer $m \geq 1$, if $P_m \rightsquigarrow Q_n$ for infinitely many $n \in \mathcal{I}_k$, then perform the following steps:
  (a) Set $m_k = m$.
  (b) Set $R_k = R_{k-1} \curlyvee P_{m_k}$.
  (c) Set $n_k = \min\{n \in \mathcal{I}_{k-1} \ : \ P_{m_k} \rightsquigarrow Q_n\}$.
  (d) Set $\mathcal{I}_k = \{n \in \mathcal{I}_{k-1} \ : \ P_{m_k} \rightsquigarrow Q_n \text{ and } n > n_k\}$.
  (e) Increment k. (Reassign k to be $k+1$.)

Observe that by definition of the fusion, $R_{k-1} \rightsquigarrow R_k$ for all $k \geq 1$. So by taking a direct limit, we can define $R = \lim_{k\to\infty} R_k$. We make several further remarks about this construction:

(R1) For each k and each $l \geq k$, $P_{m_k} \rightsquigarrow Q_{n_l}$. (*Proof:* This holds when $k = l$ by definition of m_k and n_k. It holds when $l > k$, because each such n_l lies in $\mathcal{I}_k$.)
(R2) We have $R_k = P \curlyvee P_{m_1} \curlyvee \ldots \curlyvee P_{m_k}$.
(R3) For each k and each $l \geq k$, $R_k \rightsquigarrow Q_{n_l}$. (*Proof:* By (R1), each $P_{m_j} \rightsquigarrow Q_{n_l}$ for $j \leq k$. So by the Fusion Theorem and (R2), $R_k \rightsquigarrow Q_{n_l}$.)

We claim that the subsequence $\langle Q_{n_k}\rangle$ also converges to R. To prove this, we will use the convergence criterion of Proposition 7. First suppose that $K \in \overline{\text{Disk}}(R)$. We will prove that there is an l so that $K \rightsquigarrow Q_{n_k}$ for $k > l$. Since $\langle R_k\rangle$ converges to R, there is a l so so that $K \rightsquigarrow R_k$ for $k > l$. So by composing these immersions with the immersions given by remark (R3), we see $K \rightsquigarrow Q_{n_k}$ for $k > l$.

Now let U be a planar surface. We will show that if U immerses in infinitely many Q_{n_k}, then $U \rightsquigarrow R$. By part (3) of Proposition 23, it suffices to prove that every compact disk $K \in \overline{\text{Disk}}(U)$ immerses in R. Fix $K \in \overline{\text{Disk}}(U)$. By Theorem 45, there is a open rational rectangular union in $\text{Disk}(U)$ which contains K. By definition of P_m, there is an m so that P_m is isomorphic to this union. In particular $P_m \rightsquigarrow U$. Since $U \rightsquigarrow Q_{n_k}$ for infinitely many k, it must be true that P_m immerses in infinitely many Q_{n_k}. It follows that $P_m = P_{m_k}$ for some k. Then, by the definition of the fusion, $P_{m_k} \rightsquigarrow R_k$. Because $R_k \rightsquigarrow R$, we know $P_{m_k} \rightsquigarrow R$. Finally because $K \rightsquigarrow P_{m_k}$, we know that $K \rightsquigarrow R$. □

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