Simulation of ELMs using BOUT++

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Contents

1 Introduction

This manual describes the simulation of ELMs using the `elm_pb` BOUT++ model. For a more general introduction to BOUT++ see the users' guide and papers, which also describe the internal workings of the code.

Setting up input files and running a simulation are described in sections 2 and **??**. Analysing the resulting data using IDL is then described in section **??**. These sections, together with relevant sections of the users manual, should enable a new user to begin running useful ELM simulations using BOUT++.

The remainder of this manual describes the physics model (section 3) which evolves pressure P, vorticity U and magnetic potential ψ:

$$\rho\frac{dU}{dt} = B^2\mathbf{b}\cdot\nabla\left(\frac{J_{\|}}{B}\right) + 2\mathbf{b}\times\kappa\cdot\nabla P \tag{1}$$

$$\frac{\partial\psi}{\partial t} = -\frac{1}{B_0}\nabla_{\|}\phi \tag{2}$$

$$\frac{\partial P}{\partial t} = -\frac{1}{B_0}\mathbf{b}_0\times\nabla\phi\cdot\nabla P \tag{3}$$

$$U = \frac{1}{B_0}\nabla_{\perp}^2\phi \qquad J_{\|} = J_{\|0} - \frac{1}{\mu_0}B_0\nabla_{\perp}^2\psi \tag{4}$$

The coordinate system used is described in the BOUT++ coordinates manual. The normalisation used in this BOUT++ model is the MHD one, given in section 3.5.3.

> **NOTE**: All quantities here are in SI units except where otherwise stated

2 Inputs

The general options for BOUT++ are described in the user manual. The following options control the behavior of the elm_pb physics model, and are put under the [highbeta] section of BOUT.inp.

Currently, the model used has constant mass density:

```
density = 1.0e19      # number density of deuterium [m^-3]
```

This doesn't affect the result, but scales the time normalisation (through the Alfvén speed).

```
evolve_jpar = false   # If true, evolve J raher than Psi
```

By default, the magnetic potential ψ is evolved. Setting this option to true evolves the parallel current $J_{\|}$ instead using equation 37. This does suppress currents in the vacuum region, but results in significantly higher growth-rates than evolving ψ. For now, setting this option to true is not recommended.

There are also a lot of switches to change which effects are included. The first two:

```
include_jpar0 = true     # determines whether to include jpar0 terms
include_curvature = true # include curvature drive term?
```

determine which drive terms are included in the vorticity equation (eq. 2).

```
compress = false      # set compressible (evolve Vpar)
```

If this is set to true, parallel velocity is also evolved (see section 3.8).

```
nonlinear  = true
```

This determines whether the nonlinear terms are included (v.important!)

```
diamag = false

relax_j_vac = true     # Relax to zero-current in the vacuum (EXPERIMENTAL)

filter_z = false  # remove all except one mode
filter_z_mode = 1 # Leave only fundamental

simple_rmp = false   # Enable/disable
rmp_n = 3            # Toroidal mode number
rmp_m = 6            # Poloidal mode number
rmp_factor = 1.e-4  # Amplitude of Apar [Tm]
rmp_ramp = 1.e-4    # Timescale [s] of ramp

vacuum_pressure = 0.02 # the pressure below which it is considered vacuum
                       # fraction of peak pressure
vacuum_trans = 0.01    # transition width (fraction of P)

vac_lund = -1   # Lundquist number in vacuum
core_lund = 1e4   # Lundquist number in core

viscos_par = 0.1   # Parallel viscosity

phi_flags = 202  # inversion flags for phi

apar_flags = 384
```

3 Flute-Reduced MHD

Some notation: Unit magnetic field vector

$$\mathbf{b} = \frac{\mathbf{B}}{B}$$

Perpendicular Del operator

$$\nabla f = \nabla_{\parallel} f + \nabla_{\perp} f \Rightarrow \nabla_{\perp} f = \nabla f - \nabla_{\parallel} f \Rightarrow$$

$$\nabla_{\perp} f = \nabla f - \mathbf{b} \left(\mathbf{b} \cdot \nabla f \right)$$

Note: The dot product of the unit vector $\mathbf{b}$ and ∇f, namely, $\mathbf{b} \cdot \nabla f$ gives the (scalar) parallel value of ∇f. Multiplied by $\mathbf{b}$ gives the vector $\nabla_{\parallel} f$.

3.1 Curvature vector

The field-line curvature vector κ is defined as

$$\kappa \equiv \mathbf{b} \cdot \nabla \mathbf{b} = (\nabla \times \mathbf{b}) \times \mathbf{b} \tag{5}$$

This can be re-written as

$$\kappa = \left[\frac{1}{B} \nabla \times \mathbf{B} - \mathbf{B} \times \nabla \left(\frac{1}{B} \right) \right] \times \mathbf{b}$$

using the identity $\nabla \times (\varphi \mathbf{F}) = \nabla \varphi \times \mathbf{F} + \varphi \nabla \times \mathbf{F}$ namely,

$$\nabla \times \mathbf{b} = \nabla \times \frac{\mathbf{B}}{B} = \frac{1}{B} (\nabla \times \mathbf{B}) - \mathbf{B} \times \left(\nabla \frac{1}{B} \right)$$

Using $\nabla \left(\frac{1}{B} \right) = -\nabla (B) / B^2$, this becomes

$$
\begin{aligned}
\kappa &= \left[\frac{1}{B} \nabla \times \mathbf{B} - \mathbf{B} \times \nabla \left(\frac{1}{B} \right) \right] \times \mathbf{b} \\
&= \left[\frac{1}{B} \nabla \times \mathbf{B} + \frac{1}{B^2} (\mathbf{B} \times \nabla B) \right] \times \mathbf{b} \\
&= \frac{1}{B} \mu_0 \mathbf{J} \times \mathbf{b} + \frac{1}{B^2} (\mathbf{B} \times \nabla B \times \mathbf{b}) \\
&= \frac{\mu_0}{B} \mathbf{J} \times \frac{\mathbf{B}}{B} + \frac{1}{B} \left(\frac{\mathbf{B}}{B} \times \nabla B \times \mathbf{b} \right) \\
&= \frac{\mu_0}{B^2} \mathbf{J} \times \mathbf{B} + \frac{1}{B} (\mathbf{b} \times \nabla B) \times \mathbf{b} \\
&= \frac{\mu_0}{B^2} \mathbf{J} \times \mathbf{B} + \frac{1}{B} [\nabla B (\mathbf{b} \cdot \mathbf{b}) - \mathbf{b} (\mathbf{b} \cdot \nabla B)] \tag{6} \\
&= \frac{\mu_0}{B^2} \mathbf{J} \times \mathbf{B} + \frac{1}{B} \nabla_\perp B \tag{7}
\end{aligned}
$$

where in 6 we used the identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$. Note that the second term is the perpendicular Del operator since the dot product of the unit vectors is $\mathbf{b} \cdot \mathbf{b} = 1$.

3.2 Shear-Alfvén Law

Following the derivation in Hazeltine& Meiss, we start with a fairly general fluid momentum equation:

$$\rho \frac{d\mathbf{V}}{dt} + \nabla \cdot \Pi = -\nabla P + \mathbf{J} \times \mathbf{B} \tag{8}$$

where the usual convective derivative is used:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla$$

Taking the parallel component of the curl of this equation (i.e. apply $\mathbf{B} \cdot \nabla \times$ to each side)

$$\mathbf{B} \cdot \nabla \times \left[\rho \frac{d\mathbf{V}}{dt} + \nabla \cdot \Pi \right] = \underbrace{-\mathbf{B} \cdot \nabla \times \nabla P}_{\Rightarrow 0} + \mathbf{B} \cdot \nabla \times (\mathbf{J} \times \mathbf{B}) \tag{9}$$

Note the identity $\nabla \times (\nabla \phi) = \vec{0}$ for any scalar ϕ.

The $\mathbf{J} \times \mathbf{B}$ term can be written as:

$$\mathbf{B} \cdot \nabla \times (\mathbf{J} \times \mathbf{B}) = \nabla \cdot [(\mathbf{J} \times \mathbf{B}) \times \mathbf{B}] + (\mathbf{J} \times \mathbf{B}) \cdot (\nabla \times \mathbf{B}) \tag{10}$$

based on the identity $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$.

Note that $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ and thus

$$(\mathbf{J} \times \mathbf{B}) \cdot (\nabla \times \mathbf{B}) = \frac{1}{\mu_0} (\mathbf{J} \times \mathbf{B}) \cdot \mathbf{J} = 0$$

due to the identity $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{a}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{b}) = \mathbf{a} \cdot (\mathbf{a} \times \mathbf{a}) = 0$.

Thus,

$$\mathbf{B} \cdot \nabla \times (\mathbf{J} \times \mathbf{B}) = \nabla \cdot [(\mathbf{J} \times \mathbf{B}) \times \mathbf{B}] \tag{11}$$
$$= \nabla \cdot [-\mathbf{B} \times (\mathbf{J} \times \mathbf{B})] = \nabla \cdot [-\mathbf{J}(\mathbf{B} \cdot \mathbf{B}) + \mathbf{B}(\mathbf{J} \cdot \mathbf{B})] \tag{12}$$
$$= \nabla \cdot [-B^2 \mathbf{J}(\mathbf{b} \cdot \mathbf{b}) + B^2 \mathbf{b} J_{\parallel} \mathbf{b}] \tag{13}$$
$$= \nabla \cdot [-B^2 \mathbf{J}(\mathbf{bb}) + B^2 \mathbf{b} J_{\parallel} \mathbf{b}] \tag{14}$$
$$= -\nabla \cdot [B^2 \mathbf{J}_{\perp}] \tag{15}$$
$$\tag{16}$$

using $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = -\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = -(\mathbf{c} \cdot \mathbf{b})\mathbf{a} + (\mathbf{c} \cdot \mathbf{a})\mathbf{b}$.

From Hazeltine& Meiss (eq. 3.74) we have $\mathbf{B} \cdot \nabla \left(\frac{J_{\parallel}}{B} \right) = -\nabla \cdot \mathbf{J}_{\perp}$, thus

$$-\nabla \cdot [B^2 \mathbf{J}_{\perp}] = -B^2 \nabla \cdot \mathbf{J}_{\perp} - \mathbf{J}_{\perp} \cdot \nabla B^2 \tag{17}$$
$$= B^2 \mathbf{B} \cdot \nabla \left(\frac{J_{\parallel}}{B} \right) - \frac{1}{B^2} \mathbf{B} \times (\mathbf{J} \times \mathbf{B}) \cdot \nabla_{\perp} B^2 \tag{18}$$

where we have used the above result $(\mathbf{J} \times \mathbf{B}) \times \mathbf{B} = -B^2 \mathbf{J}_{\perp}$ and ∇B^2 becomes $\nabla_{\perp} B^2$ on account that it multiplies $\mathbf{J}_{\perp}$. Now from 7 we get

$$\mathbf{J} \times \mathbf{B} = \frac{1}{\mu_0} [B^2 \kappa - B \nabla_{\perp} B] \tag{19}$$

So 9 becomes

$$\mathbf{B} \cdot \nabla \times \left[\rho \frac{d\mathbf{V}}{dt} + \nabla \cdot \Pi\right] = B^2 \mathbf{B} \cdot \nabla \left(\frac{J_{||}}{B}\right) - \frac{1}{B^2}\mathbf{B} \times \left(\frac{1}{\mu_0} B^2 \kappa - B\nabla_\perp B\right) \cdot \nabla_\perp B^2$$

$$= B^2 \mathbf{B} \cdot \nabla \left(\frac{J_{||}}{B}\right) - \frac{1}{\mu_0}\mathbf{B} \times \kappa \cdot \nabla_\perp B^2 + \cancel{\frac{\mathbf{B}}{B^2} \times \frac{1}{2}\nabla_\perp B^2 \cdot \nabla_\perp B^2}^{\ 0}$$

$$\Rightarrow \mathbf{b} \cdot \nabla \times \left[\rho \frac{d\mathbf{V}}{dt} + \nabla \cdot \Pi\right] = B^2 \mathbf{b} \cdot \nabla \left(\frac{J_{||}}{B}\right) - \frac{1}{\mu_0}\mathbf{b} \times \kappa \cdot \nabla_\perp B^2 \tag{20}$$

Denoting the LHS of equation 8 as $\mathbf{f}$, and using the expression for the curvature in equation 7 gives:

$$\kappa = \frac{\mu_0}{B^2}\left(\mathbf{f} + \nabla P\right) + \frac{1}{B}\nabla_\perp B$$

$$\Rightarrow \nabla_\perp B^2 = 2B^2 \kappa - 2\mu_0 \left(\mathbf{f} + \nabla P\right)$$

Substituting this into equation 20 gives:

$$\mathbf{b} \cdot \nabla \times \mathbf{f} = B^2 \mathbf{b} \cdot \nabla \left(\frac{J_{||}}{B}\right) - \frac{1}{\mu_0}\mathbf{b} \times \kappa \cdot \left[-2\mu_0 \left(\mathbf{f} + \nabla P\right) + 2B^2 \kappa\right]$$

$$= B^2 \mathbf{b} \cdot \nabla \left(\frac{J_{||}}{B}\right) + 2\mathbf{b} \times \kappa \cdot \mathbf{f} + 2\mathbf{b} \times \kappa \cdot \nabla P + \mathcal{O}(\kappa^2) \Rightarrow$$

$$\mathbf{b} \cdot \left[\nabla \times \mathbf{f} - 2\kappa \times \mathbf{f}\right] = B^2 \mathbf{b} \cdot \nabla \left(\frac{J_{||}}{B}\right) + 2\mathbf{b} \times \kappa \cdot \nabla P \tag{21}$$

The LHS side of this is primarily the parallel component of the vorticity $\nabla \times \mathbf{V}$, so this equation is often called the vorticity equation.

3.3 Flute reduction

This is a reduction for perturbations which are aligned to the magnetic field. Formally, this is done by setting

$$k_{||} \sim \epsilon k_\perp$$

with a small parameter ϵ. The perturbed magnetic field is written

$$\mathbf{B}_1 = \nabla \psi \times \mathbf{B}_0 + B_{||}\mathbf{b}_0 \tag{22}$$

where ψ is a normalised $A_{||}$ with units of length. Hence the parallel divergence can be written:

$$\nabla_{||} \equiv \mathbf{b} \cdot \nabla = \mathbf{b}_0 \cdot \nabla - \mathbf{b}_0 \cdot \nabla \psi \times \nabla \tag{23}$$

where the $B_{||}$ term is dropped because it is of higher order in terms of ϵ.

3.4 High-β MHD closure

This is an assumption of small β, but is called "High-β" because it assumes $\beta \sim \epsilon$, rather than the Low-β model which assumes $\beta \sim \epsilon^2$

This assumption means that the parallel velocity can be dropped from the above equations. This means the plasma is incompressible, and hence in the pressure can never go outside the initial bounds.

> **NOTE**: This is not a rigorous derivation, but is intended to make sense of the equations in terms I can understand. For a more rigorous but highly confusing derivation see the Hazeltine textbook.

3.4.1 Vorticity

Taking equation 21 and neglecting anisotropic pressure Π, variation in mass density ρ and the curvature term gives:

$$b \cdot [\nabla \times \mathbf{f} - 2\kappa \times \mathbf{f}] \simeq \rho \frac{dU}{dt}$$

where U is the vorticity:

$$U \;\equiv\; \mathbf{b} \cdot \nabla \times \mathbf{V} \tag{24}$$

$$\rho \frac{dU}{dt} \;=\; B^2 \mathbf{b} \cdot \nabla \left(\frac{J_{\parallel}}{B} \right) + 2\mathbf{b} \times \kappa \cdot \nabla P \tag{25}$$

To first order, the perpendicular velocity is the $\mathbf{E} \times \mathbf{B}$ velocity:

$$\mathbf{V} \simeq \mathbf{V}_E = \frac{1}{B^2} \mathbf{B} \times \nabla \phi$$

Taking the curl of this gives:

$$\nabla \times \mathbf{V}_E \;=\; \frac{\mathbf{b}}{B} \nabla \cdot \nabla \phi - \nabla \phi \left(\nabla \cdot \frac{\mathbf{b}}{B} \right) + (\nabla \phi \cdot \nabla) \frac{\mathbf{b}}{B} - \left(\frac{\mathbf{b}}{B} \cdot \nabla \right) \nabla \phi$$

$$\simeq \; \frac{\mathbf{b}}{B} \nabla_{\perp}^2 \phi$$

where only the first term has been kept, and parallel derivatives neglected (flute assumption). Electrostatic potential can therefore be related to vorticity by:

$$U = \frac{1}{B} \nabla_{\perp}^2 \phi \tag{26}$$

3.4.2 Magnetic field

In perturbed magnetic field $\mathbf{B}$ can be written as $\mathbf{B} = \nabla \times \mathbf{A}$. Here only the parallel component component is evolved, so the perturbed field is

$$
\begin{aligned}
\mathbf{B}_1 &= \nabla \times \left(b_0 A_{||}\right) = A_{||}\nabla \times b_0 + \nabla A_{||} \times b_0 \\
&\simeq \nabla A_{||} \times \mathbf{b}_0
\end{aligned}
$$

In Hazeltine, $\psi = A_{||}/B_0$ is used instead:

$$
\begin{aligned}
\mathbf{B}_1 &= \nabla \times \left(\mathbf{B}_0 \psi\right) = A_{||}\nabla \times \mathbf{B}_0 + \nabla \psi \times \mathbf{B}_0 \\
&\simeq \nabla \psi \times \mathbf{B}_0
\end{aligned}
\tag{27}
$$

The total current is then:

$$
\begin{aligned}
\mathbf{J} &= \mathbf{J}_0 + \frac{1}{\mu_0}\nabla \times \mathbf{B}_1 = \mathbf{J}_0 + \frac{1}{\mu_0}\nabla \times \nabla \psi \times \mathbf{B}_0 \\
\nabla \times \nabla \psi \times \mathbf{B}_0 &= \nabla \psi \left(\nabla \cdot \mathbf{B}_0\right) - \mathbf{B}_0 \nabla^2 \psi \\
&+ \left(\mathbf{B}_0 \cdot \nabla\right) \nabla \psi - \left(\nabla \psi \cdot \nabla\right) \mathbf{B}_0
\end{aligned}
$$

The first term of this is identically zero, and only the second term is kept. As before, parallel derivatives are neglected, and taking the parallel component of this equation gives

$$
J_{||} = J_{||0} - \frac{1}{\mu_0}B_0 \nabla_\perp^2 \psi
\tag{28}
$$

The evolution equation for ψ comes from the parallel electric field (Ohm's law). Here resistivity is neglected, so

$$
E_{||} = \mathbf{b}_0 \cdot \left(-\nabla \phi - \frac{\partial A_{||}}{\partial t}\right) = 0
$$

and since $A_{||} = \psi B_0$,

$$
\frac{\partial \psi}{\partial t} = -\frac{1}{B_0}\nabla_{||}\phi
\tag{29}
$$

The total $\mathbf{b}$ unit vector (equilibrium + perturbation) can be written

$$
\mathbf{b} = \frac{\mathbf{B}_0 + \mathbf{B}_1}{|\mathbf{B}_0 + \mathbf{B}_1|} \simeq \mathbf{b}_0 + \frac{\mathbf{B}_1}{B_0}
$$

for small perturbations. Hence the parallel derivative becomes

$$
\mathbf{b} \cdot \nabla \simeq \mathbf{b}_0 \cdot \nabla - \mathbf{b}_0 \times \nabla \psi \cdot \nabla
$$

3.4.3 Pressure

The only remaining equation is that for the pressure. We have used that the perpendicular velocity is just the $\mathbf{E} \times \mathbf{B}$ velocity $\mathbf{V}_E = (\mathbf{b}_0 \times \nabla\phi)/B_0$, so

$$\frac{\partial P}{\partial t} = -\frac{1}{B_0}\mathbf{b}_0 \times \nabla\phi \cdot \nabla P \tag{30}$$

Where P is the total pressure.

3.4.4 Summary

The equations derived above for vorticity U, pressure P, and vector potential (ish) $\psi = A_{||}/B_0$ are

$$\begin{aligned}
\rho\frac{dU}{dt} &= B^2\mathbf{b}\cdot\nabla\left(\frac{J_{||}}{B}\right) + 2\mathbf{b}\times\kappa\cdot\nabla P \\
\frac{\partial\psi}{\partial t} &= -\frac{1}{B_0}\nabla_{||}\phi \\
\frac{\partial P}{\partial t} &= -\frac{1}{B_0}\mathbf{b}_0\times\nabla\phi\cdot\nabla P \\
U &= \frac{1}{B_0}\nabla_\perp^2\phi \\
J_{||} &= J_{||0} - \frac{1}{\mu_0}B_0\nabla_\perp^2\psi
\end{aligned}$$

with everything in SI units. Splitting into equilibrium (subscript 0) and evolving (subscript 1), making explicit which terms are non-linear gives:

$$\begin{aligned}
\rho_0\frac{\partial U_1}{\partial t} &= -\frac{\rho_0}{B_0}\mathbf{b}_0\times\nabla\phi_1\cdot\nabla U_1 \\
&+ B_0^2\left[\mathbf{b}_0\cdot\nabla\left(\frac{J_{||1}}{B_0}\right) - \mathbf{b}_0\times\nabla\psi_1\cdot\nabla\left(\frac{J_{||0}+J_{||1}}{B_0}\right)\right] \\
&+ 2\mathbf{b}\times\kappa\cdot\nabla P_1 \tag{31} \\
\frac{\partial\psi_1}{\partial t} &= -\frac{1}{B_0}\mathbf{b}_0\cdot\nabla\phi_1 \tag{32} \\
\frac{\partial P_1}{\partial t} &= -\frac{1}{B_0}\mathbf{b}_0\times\nabla\phi_1\cdot\nabla(P_0+P_1) \tag{33} \\
U_1 &= \frac{1}{B_0}\nabla_\perp^2\phi_1 \tag{34} \\
J_{||1} &= -\frac{1}{\mu_0}B_0\nabla_\perp^2\psi_1 \tag{35}
\end{aligned}$$

3.5 Normalisation

3.5.1 Hazeltine-Meiss

Relating the above set of equations to that given in Hazeltine, define:

$$\Im = -\frac{\mu_0}{B_0} J_{\|1}$$

$$\Phi = \frac{\phi}{B_0}$$

$$\beta = \frac{2\mu_0 P_0}{\overline{B}^2}$$

$$p = \frac{2\mu_0 P_1}{\overline{B}^2}$$

where $\overline{B}$ is a constant normalisation factor. Note the minus sign on the perturbed current. Substituting into the vorticity equation this gives:

$$\rho \frac{dU}{dt} = -\frac{B^2}{\mu_0}\mathbf{b} \cdot \nabla\Im + \frac{\overline{B}^2}{\mu_0}\mathbf{b} \times \kappa \cdot \nabla p$$

Multiplying through by $\mu_0/\overline{B}^2$ and noting that the Alfén speed $\overline{V_A}^2 = \overline{B}^2/\mu_0\rho$ gives the Hazeltine-Meiss vorticity equation

$$\frac{1}{\overline{V_A}^2}\frac{dU}{dt} = -\frac{B^2}{\overline{B}^2}\mathbf{b} \cdot \nabla\Im + \mathbf{b} \times \kappa \cdot \nabla p$$

The pressure equation can be re-written as

$$\frac{\partial p}{\partial t} + \mathbf{b}_0 \times \nabla\Phi \cdot \nabla p = -\mathbf{b}_0 \times \nabla\Phi \cdot \nabla\beta$$

which becomes

$$\frac{dp}{dt} = -\mathbf{b}_0 \times \nabla\Phi \cdot \nabla\beta$$

because the convective derivative can be written as

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{b}_0 \times \nabla\Phi \cdot \nabla$$

The other equations become

$$\frac{\partial\psi}{\partial t} = -\frac{1}{B_0}\nabla_{\|}B_0\Phi$$

$$U = \nabla_{\perp}^2\Phi$$

$$\Im = \nabla_{\perp}^2\psi$$

where terms like $\nabla_{\perp}^2(1/B_0)$ are neglected.

3.5.2 BOUT

Choose typical magnetic field B_x, density N_{ix} and temperature T_{ex} to get the normalisation factors

$$\omega_{cix} = \frac{eB_x}{m_i} \qquad C_{sx} = \sqrt{\frac{eT_{ex}}{m_i}} \qquad \rho_{sx} = C_{sx}/\omega_{cix} \qquad \beta_x = \frac{2\mu_0 N_{ix} T_{ex}}{B_x^2}$$

and the normalisations

$$\hat{t} = \omega_{cix}t \qquad \hat{\nabla} = \rho_{sx}\nabla \qquad \hat{B}_0 = B_0/B_x \qquad \hat{N}_i = \frac{\rho}{N_{ix}m_i} \qquad \hat{\kappa} = \rho_{sx}\kappa$$

Evolving variables normalised as (note that vorticity includes density evolution)

$$\hat{A}_{||} = \frac{m_i}{m_e}\frac{A_{||}}{B_x\rho_{sx}} \qquad \hat{\omega} = \frac{\rho U B_0}{B_x N_{ix} m_i \omega_{cix}} \qquad \hat{P} = \frac{P}{eN_{ix}T_{ex}}$$

and auxilliary variables

$$\hat{J}_{||} = \frac{J_{||}}{eN_{ix}C_{sx}} \qquad \hat{\phi} = \frac{\phi}{T_{ex}}$$

so the equations become:

$$\frac{d\hat{\omega}}{d\hat{t}} = \hat{B}_0^3 \mathbf{b}\cdot\hat{\nabla}\left(\frac{\hat{J}_{||}}{\hat{B}_0}\right) + 2\hat{B}_0 \mathbf{b}_0 \times \hat{\kappa}\cdot\hat{\nabla}\hat{P}$$

$$\frac{\partial\hat{A}_{||}}{\partial\hat{t}} = -\frac{m_i}{m_e}\mathbf{b}_0\cdot\hat{\nabla}\hat{\phi}$$

$$\frac{\partial\hat{P}}{\partial\hat{t}} = -\frac{1}{\hat{B}_0}\mathbf{b}_0\times\hat{\nabla}\hat{\phi}\cdot\hat{\nabla}\hat{P}$$

$$\hat{\nabla}_\perp^2\hat{A}_{||} = -\frac{1}{2}\beta_x\frac{m_i}{m_e}\hat{J}_{||}$$

$$\hat{\nabla}_\perp^2\hat{\phi} = \frac{\hat{\omega}}{\hat{N}}$$

with the full parallel derivative

$$\mathbf{b}\cdot\hat{\nabla} = \mathbf{b}_0\cdot\hat{\nabla} - \frac{m_e}{m_i}\frac{1}{\hat{B}_0}\mathbf{b}_0\times\hat{\nabla}\hat{A}_{||}\cdot\hat{\nabla}$$

3.5.3 MHD

NOTE: This is the normalisation used in the elm_pb code

Define typical length scale $\overline{L}$, timescale $\overline{T}$ and magnetic field $\overline{B}$, related by $\overline{V_A}^2 = \overline{B}^2/\mu_0\rho m = \overline{L}^2/\overline{T}^2$.

$$\hat{t} = \frac{t}{\overline{T}} \qquad \hat{B} = \frac{B}{\overline{B}} \qquad \hat{\nabla} = \overline{L}\nabla \qquad \hat{\kappa} = \overline{L}\kappa$$

Evolving variables are then normalised as:

$$\hat{U} = \overline{T}U \qquad \hat{\psi} = \frac{\psi}{\overline{L}} \qquad \hat{P} = \frac{2\mu_0 P}{\overline{B}^2}$$

and auxillary variables

$$\hat{J}_{||} = -\frac{\mu_0 \overline{L}}{B_0} J_{||} \qquad \hat{\phi} = \frac{\phi}{V_A L B_0}$$

The normalised high-β reduced MHD equations are

$$\frac{d\hat{U}}{d\hat{t}} = -\hat{B}_0^2 \mathbf{b} \cdot \hat{\nabla}\hat{J}_{||} + \mathbf{b} \times \hat{\kappa} \cdot \hat{\nabla}\hat{P}$$

$$\frac{\partial\hat{\psi}}{\partial\hat{t}} = -\frac{1}{\hat{B}_0}\hat{\nabla}_{||}\left(\hat{B}_0\hat{\phi}\right)$$

$$\frac{d\hat{P}_1}{d\hat{t}} = -\mathbf{b}_0 \times \hat{\nabla}\hat{\phi} \cdot \hat{\nabla}\hat{P}_0$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{b}_0 \times \hat{\nabla}\hat{\phi} \cdot \hat{\nabla}$$

$$\hat{U} = \hat{\nabla}_\perp^2\hat{\phi} \qquad \hat{J}_{||} = \hat{\nabla}_\perp^2\hat{\psi}$$

3.6 Normalisation example

Using cbm18dens8 grid we have $\overline{B} = 1.94129908[T]$ ($Bmag$ from grid file), $\overline{L} = 3.49717236[m]$ ($Rmag$ from grid file), $\mu_0 = 4\pi 1e^{-7}[N \cdot A^{-2}]$ and number density $n = 1e^{19}[m^{-3}]$. Using as mass that of deuteron $m = 2. * 1.6726e^{-27}[kg]$ we get

$$\overline{V_A}^2 = \overline{B}^2/\mu_0 nm = 1.94129908^2/(4\pi 1e^{-7}1e^{19}2.0*1.6726e^{-27}) = 8.96505469e^{13} \Rightarrow \overline{V_A} = 9468397.[m/s]$$

Since $V_A = \overline{L}/\overline{T}$ we get $\overline{T} = 3.69352108e^{-7}[s]$ or .369 μs. Note that the Alfvén frequency is $1/\overline{T}$. Also there is a dependency of frequency and the ratio of magnetic field and major radius. From above we have,

$$V_A = \overline{L}/\overline{T} \Rightarrow \overline{B} \sim Rmag/\overline{T} \Rightarrow \overline{B}/R \sim 1/\overline{T}$$

where we $\overline{L} = Rmag$ (major radius). So to compare frequencies between two configurations with different $Bmag$ and $Rmag$ we need to scale the results appropriately. i.e between ELITE results with $Bmag = 2[T]$ and $Rmag = 3[m]$ and BOUT++ results with $Bmag = 1.94129908[T]$ and $Rmag = 3.49717236[m]$ we have

$$\omega_{BOUT}/\omega_{ELITE} = B_{bout}/R_{bout} \div B_{ELITE}/R_{ELITE} = \frac{3.49717236}{1.94129908} \div \frac{3}{2} = 1.80146 \div 1.5 = 1.201.$$

So in order to compare, we multiply BOUT's results with $1/1.201 = .8326$.

3.7   Dimensionless quantities

$$\beta_t = \frac{p[Pa]}{B_0[T]/2\mu_0[N \cdot A^{-2}]} \sim \frac{Pa \cdot N \cdot A^{-2}}{T^2} \sim \frac{(Kg \cdot m^{-1} \cdot s^{-2}) \cdot (kg \cdot m \cdot s^{-2} \cdot A^{-2})}{kg^2 \cdot s^{-4} \cdot A^{-2}} \checkmark$$

3.8   Parallel velocity

In SI units (eq. 7.51-52 from Hazeltine & Meiss) we have

$$\rho \frac{dV_{||}}{dt} = -\nabla_{||} P_1 + \mathbf{b}_0 \times \nabla\psi \cdot \nabla P_0$$

$$\frac{dP_1}{dt} + \frac{1}{B_0} \mathbf{b}_0 \times \nabla\phi \cdot \nabla P_0 = \frac{B_0^2}{\mu_0} \frac{v_s^2}{v_A^2 + v_s^2} \left[-\frac{2}{B_0} \mathbf{b}_0 \times \kappa_0 \cdot \nabla\phi - \nabla_{||} V_{||} + \frac{V_{||}}{B_0} \mathbf{b}_0 \cdot \nabla B_0 \right]$$

Transforming the above eqs into MHD units the first equation gives

$$\rho \frac{dV_{||}}{dt\frac{\overline{T}}{T}} = -\nabla_{||} \frac{\overline{L}}{L} P_1 \frac{2\mu_0 \overline{B}^2}{2\mu_0 \overline{B}^2} + \mathbf{b}_0 \times \nabla \frac{\overline{L}}{L} \psi \frac{\overline{L}}{L} \cdot \nabla \frac{\overline{L}}{L} P_0 \frac{2\mu_0 \overline{B}^2}{2\mu_0 \overline{B}^2} \Rightarrow$$

$$\rho \frac{dV_{||}}{d\hat{t}\overline{T}} = -\hat{\nabla}_{||} \frac{1}{\overline{L}} \hat{P}_1 \frac{\overline{B}^2}{2\mu_0} + \mathbf{b}_0 \times \hat{\nabla} \frac{1}{\overline{L}} \hat{\psi} \overline{L} \cdot \hat{\nabla} \frac{1}{\overline{L}} \hat{P}_0 \frac{\overline{B}^2}{2\mu_0} \Rightarrow$$

$$\rho \frac{dV_{||}}{d\hat{t}} = \frac{\overline{B}^2 \overline{T}}{2\mu_0 \overline{L}} \left[-\hat{\nabla}_{||} \hat{P}_1 + \mathbf{b}_0 \times \hat{\nabla}\hat{\psi} \cdot \hat{\nabla}\hat{P}_0 \right] \Rightarrow$$

$$\rho \frac{dV_{||}}{d\hat{t}} = \frac{\rho \overline{V_A}^2}{2\overline{V_A}} \left[-\hat{\nabla}_{||} \hat{P}_1 + \mathbf{b}_0 \times \hat{\nabla}\hat{\psi} \cdot \hat{\nabla}\hat{P}_0 \right] \Rightarrow$$

Note that $\overline{V_A} = \overline{L}/\overline{T}$. Then using $\hat{V}_{||} = V_{||}/\overline{V_A}$ we get

$$2 \frac{d\hat{V}_{||}}{d\hat{t}} = -\hat{\nabla}_{||} \hat{P}_1 + \mathbf{b}_0 \times \hat{\nabla}\hat{\psi} \cdot \hat{\nabla}\hat{P}_0.$$

The second...

$$\frac{dP_1}{dt\frac{\overline{T}}{T}} \frac{2\mu_0 \overline{B}^2}{2\mu_0 \overline{B}^2} + \frac{1}{B_0} \mathbf{b}_0 \times \nabla \frac{\overline{L}}{L} \phi \frac{\overline{V_A L B_0}}{\overline{V_A L B_0}} \cdot \nabla \frac{\overline{L}}{L} P_0 \frac{2\mu_0 \overline{B}^2}{2\mu_0 \overline{B}^2} =$$

$$\frac{B_0^2}{\mu_0} \frac{v_s^2}{v_A^2 + v_s^2} \left[-\frac{2}{B_0} \mathbf{b}_0 \times \kappa_0 \frac{\overline{L}}{L} \cdot \nabla \frac{\overline{L}}{L} \phi \frac{\overline{V_A L B_0}}{\overline{V_A L B_0}} - \nabla_{||} \frac{\overline{L}}{L} V_{||} \frac{\overline{V_A}}{\overline{V_A}} + \frac{V_{||} \frac{\overline{V_A}}{\overline{V_A}}}{B_0 \frac{\overline{B}}{\overline{B}}} \mathbf{b}_0 \cdot \nabla \frac{\overline{L}}{L} B_0 \frac{\overline{B}}{\overline{B}} \right] \Rightarrow$$

$$\frac{d\hat{P}_1}{d\hat{t}}\frac{\overline{B}^2}{2\mu_0\overline{T}} + \frac{1}{\cancel{B_0}}\mathbf{b}_0 \times \hat{\nabla}\frac{1}{\cancel{L}}\hat{\phi}\overline{V_A}\cancel{L}\cancel{B_0}\cdot\hat{\nabla}\frac{1}{\cancel{L}}\hat{P}_0\frac{\overline{B}^2}{2\mu_0} =$$

$$\frac{B_0^2}{\mu_0}\frac{v_s^2}{v_A^2 + v_s^2}\left[-\frac{2}{\cancel{B_0}}\mathbf{b}_0 \times \hat{\kappa}_0\frac{1}{L}\cdot\hat{\nabla}\frac{1}{\cancel{L}}\hat{\phi}\overline{V_A}\cancel{L}\cancel{B_0} - \hat{\nabla}_{||}\frac{1}{L}\hat{V}_{||}\overline{V_A} + \frac{\hat{V}_{||}\overline{V_A}}{\hat{B}_0\cancel{\overline{B}}}\mathbf{b}_0\cdot\hat{\nabla}\frac{1}{L}\hat{B}_0\cancel{\overline{B}}\right] \Rightarrow$$

using $\overline{V_A} = \overline{L}/\overline{T}$ we get

$$\left[\frac{d\hat{P}_1}{dt} + \mathbf{b}_0 \times \hat{\nabla}\hat{\phi}\cdot\hat{\nabla}\hat{P}_0\right]\frac{\overline{B}^2}{2\mu_0\cancel{\overline{T}}} =$$

$$\frac{B_0^2}{\mu_0\cancel{\overline{T}}}\frac{v_s^2}{v_A^2 + v_s^2}\left[-2\mathbf{b}_0 \times \hat{\kappa}_0\cdot\hat{\nabla}\hat{\phi} - \hat{\nabla}_{||}\hat{V}_{||} + \frac{\hat{V}_{||}}{\hat{B}_0}\mathbf{b}_0\cdot\hat{\nabla}\hat{B}_0\right] \Rightarrow$$

$$\left[\frac{d\hat{P}_1}{dt} + \mathbf{b}_0 \times \hat{\nabla}\hat{\phi}\cdot\hat{\nabla}\hat{P}_0\right] =$$

$$\frac{2B_0^2}{\overline{B}^2}\frac{v_s^2}{v_A^2 + v_s^2}\left[-2\mathbf{b}_0 \times \hat{\kappa}_0\cdot\hat{\nabla}\hat{\phi} - \hat{\nabla}_{||}\hat{V}_{||} + \frac{\hat{V}_{||}}{\hat{B}_0}\mathbf{b}_0\cdot\hat{\nabla}\hat{B}_0\right] \Rightarrow$$

and thus

$$\frac{d\hat{P}_1}{dt} + \mathbf{b}_0 \times \hat{\nabla}\hat{\phi}\cdot\hat{\nabla}\hat{P}_0 = 2\hat{B}_0^2\frac{v_s^2}{v_A^2 + v_s^2}\left[-2\mathbf{b}_0 \times \hat{\kappa}_0\cdot\hat{\nabla}\hat{\phi} - \hat{\nabla}_{||}\hat{V}_{||} + \frac{\hat{V}_{||}}{\hat{B}_0}\mathbf{b}_0\cdot\hat{\nabla}\hat{B}_0\right]$$

In **elm_pb.cxx**, the factor in front of the square brackets is `beta`:

$$\beta \equiv 2\hat{B}_0^2\frac{v_s^2}{v_A^2 + v_s^2} = \frac{2\hat{B}_0^2}{v_A^2/v_s^2 + 1} = \hat{B}_0^2/\left(\frac{\hat{B}_0^2}{\gamma\hat{P}_0} + \frac{1}{2}\right)$$

with

$$\frac{v_A^2}{2v_s^2} = \frac{B_0^2\cancel{\rho}}{2\cancel{\rho}\mu_0\gamma P_0} = \frac{1}{\gamma}\frac{\overline{B}^2 B_0^2}{2\mu_0 P \overline{B}^2} = \frac{\hat{B}_0^2}{\gamma\hat{P}_0}.......\textbf{CHECK}$$

and $v_A^2 = B_0^2/\rho\mu_0$ and $v_s^2 = \gamma P_0/\rho$.

3.9 Diamagnetic effects

When including the diamagnetic drift, the vorticity U (eq. 24) now includes the diamagnetic drift:

$$\mathbf{V} \simeq \mathbf{V}_E + \mathbf{V}_D = \frac{1}{B^2}\mathbf{B} \times \nabla\phi + \frac{1}{enB^2}\mathbf{B} \times \nabla P$$

NOTE: Using the ion drift direction. For **constant density** n, this gives (see 3.4.1 above)

$$U = \frac{1}{B}\nabla_\perp^2\left(\phi + \frac{P}{en}\right) \qquad (36)$$

Another term is the modification to the parallel electric field **CHECK**:

$$\frac{\partial A_{||}}{\partial t} = -\nabla_{||}\phi + 1.71\nabla_{||}T_e + \frac{Te}{n}\nabla_{||}n$$

For highbeta_reduced (MHD units), n is constant (so $\nabla_{||}n = 0$). The modified expression for vorticity (eq. 36) can be normalized using the expressions from 3.5.3 as:

$$U\frac{\overline{T}}{\overline{\overline{T}}} = \nabla_\perp^2\frac{\overline{L}^2}{\overline{L}^2}\frac{1}{B_0}\left[\phi\frac{\overline{V_A L}}{\overline{V_A L}} + \left(\frac{P}{en}\right)\right] \Rightarrow$$

$$\frac{\hat{U}}{\overline{\overline{T}}} = \frac{\hat{\nabla}_\perp^2}{\overline{L}^2}\left[\hat{\phi}\overline{V_A L} + \left(\frac{P}{B_0 en}\right)\right] \Rightarrow$$

$$\hat{U} = \hat{\nabla}_\perp^2\left[\hat{\phi}\frac{\overline{V_A L T}}{\overline{L}^2}^{\,1} + \left(\frac{P\overline{T}}{\overline{L}^2 B_0 en}\right)\right] \Rightarrow$$

$$\hat{U} = \hat{\nabla}_\perp^2\left[\hat{\phi} + \left(\frac{P\overline{TT}}{\overline{T}\overline{L}^2 B_0 en}\right)\right] \Rightarrow$$

$$\hat{U} = \hat{\nabla}_\perp^2\left[\hat{\phi} + \left(\frac{P}{\overline{T}\overline{V_A}^2 B_0 en}\right)\right] \Rightarrow$$

$$\hat{U} = \hat{\nabla}_\perp^2\left[\hat{\phi} + \left(\frac{P}{\overline{T}\frac{\overline{B}^2}{\mu_0 nm_i}B_0 en}\right)\right] \Rightarrow$$

$$\hat{U} = \hat{\nabla}_\perp^2 \left[\hat{\phi} + \left(\frac{2\mu_0 P}{\overline{B}^2} \frac{m_i}{2\overline{T} B_0 e} \right) \right] \Rightarrow$$

$$\hat{U} = \hat{\nabla}_\perp^2 \left[\hat{\phi} + \left(\hat{P} \frac{m_i}{2\overline{T} \frac{B_0}{\overline{B}} \overline{B} e} \right) \right] \Rightarrow$$

$$\hat{U} = \hat{\nabla}_\perp^2 \left[\hat{\phi} + \left(\frac{m_i}{2e\overline{T}\overline{B}} \right) \hat{P} / \hat{B}_0 \right]$$

where the term in round brackets is called `dnorm`. Note also that `dnorm` is also the drift scale δ over 2 , since it is the ratio of the transit frequency (in our problem) $1/\overline{T}$ over the gyrofrequency $\Omega = e\overline{B}/m_i$.

Since $P = 2enT$ for $Te = Ti$,

$$\frac{\partial \hat{\psi}}{\partial \hat{t}} = -\frac{1}{\hat{B}_0} \hat{\nabla}_\parallel \left(\hat{B}_0 \hat{\phi} \right) + 1.71 \left(\frac{m_i}{2e\overline{T}\overline{B}} \right) \frac{1}{\hat{B}_0} \nabla_\parallel \hat{P} / 2$$

3.10   Plasma-vacuum interface

One issue in modelling of edge plasmas is the handling of the vacuum region, and the plasma-vacuum boundary. This is an active research area, so there is no single "correct" answer at the moment and it needs an introduction here.

Initial simulations using BOUT++ ignored the issue and treated the entire domain as an ideal plasma. This made life easier, but meant that the result would contain currents driven in the "vacuum" region, which is obviously unphysical. Correct handling of current-driven instabilities such as peeling modes is very sensitive to the vacuum region, so this may be a big problem for handling these modes.

The first attempt to remove these unphysical currents was to alter the evolution equations so that no currents could be driven in the vacuum. Taking $\nabla_\perp^2$ of equation 2, and assuming that $\nabla_\parallel$ commutes with $\nabla_\perp^2$ (which it doesn't) gives

$$\frac{\partial J_\parallel}{\partial t} \simeq -\nabla_\parallel \left(\frac{1}{B_0} \nabla_\perp^2 \phi \right) \simeq -\nabla_\parallel U \tag{37}$$

Since at the start of the simulation $U = 0$ and $\nabla P = 0$ in the vacuum, $J_\parallel$ is zero for all time. Unfortunately, whilst this system does suppress currents in the vacuum, it also greatly increases ELM growth-rates for unknown reasons: evidently the discarded terms are important.

Another approach, which is commonly used in other simulation codes, is to treat the vacuum as a resistive plasma. This is implemented here by specifying Lundquist numbers

for the core and vacuum regions, and then using a tanh function of pressure Θ to produce a smooth transition between regions:

$$\Theta = \frac{1}{2}\left[1 - \tanh\left(\frac{P - P_{vac}}{\Delta P_{vac}} \right) \right] \tag{38}$$

where P_{vac} is the pressure at the plasma-vacuum interface, and ΔP_{vac} is the transition width. Θ is therefore 0 in the core and 1 in vacuum. Using this, the resistivity is given by

$$\eta = \eta_{vac}\Theta + \eta_{core}\left(1 - \Theta \right) \tag{39}$$

There are two problems with this method: current tends to diffuse from the plasma into the vacuum (making the situation worse), and the timestep becomes small. The second issue is the most troublesome, since the diffusion timestep dominates the solution, making the simulation harder to run.

Modelling a vacuum region as a resistive plasma is not a particularly good approximation: in a vacuum magnetic fields propagate at the speed of light, rather than diffuse on resistive timescales. A better model for a vacuum might be to treat the propagation of $\mathbf{B}$ as instantaneous i.e. at any given time calculate the vacuum field based on only the currents in the plasma. This is done in BOUT++ using a relaxation process. At any given time, the solution $\psi\left(\mathbf{x}, t\right)$ gives a current $J_{||}^{sol}$

$$J_{||}^{sol} = -\frac{1}{\mu_0}B_0\nabla_{\perp}^2\psi$$

which may or may not include currents in the vacuum region. From this, a "target" current $J_{||}^{target}$ is calculated by setting all vacuum currents to zero

$$J_{||}^{target} = J_{||}^{sol}\left(1 - \Theta \right)$$

This is the closest physically acceptable solution to the current result. From this, a target ψ can be calculated

$$\psi^{target} = \nabla_{\perp}^{-2}\left(\mu_0 J_{||}^{target}/B_0 \right)$$

In the vacuum region, this is the solution which would give zero current, but simply setting ψ to this value in the vacuum would result in an inconsistency because ψ in the core is affected by currents in the vacuum. Instead, by making ψ converge on the target value with a small time-constant τ_{jvac}, it can evolve to a self-consistent state with zero current in the vacuum region:

$$\frac{\partial\psi}{\partial t} = -\left(1 - \Theta \right)\frac{1}{B_0}\nabla_{||}\phi + \Theta\left(\psi^{target} - \psi \right)/\tau_{jvac}$$