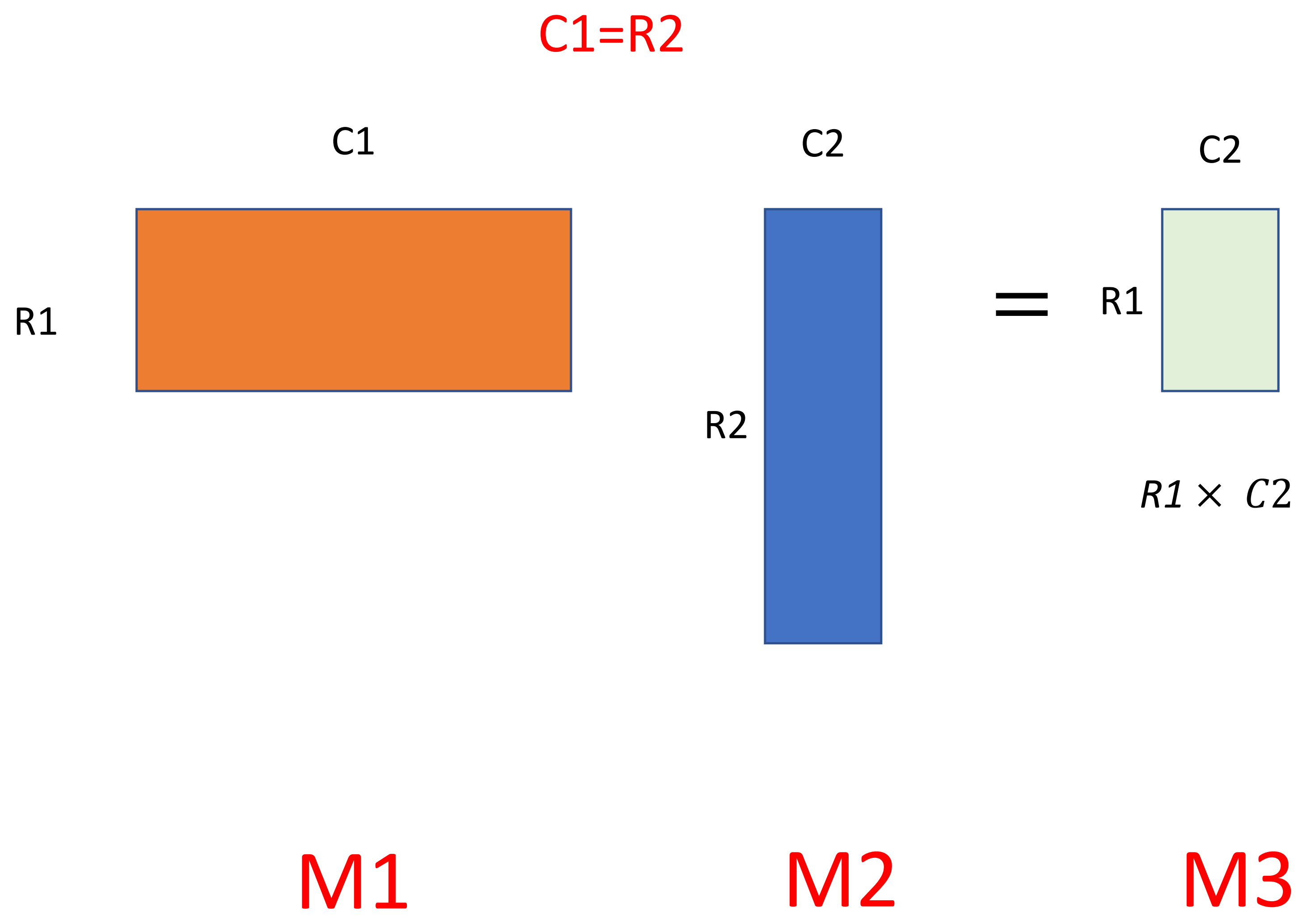
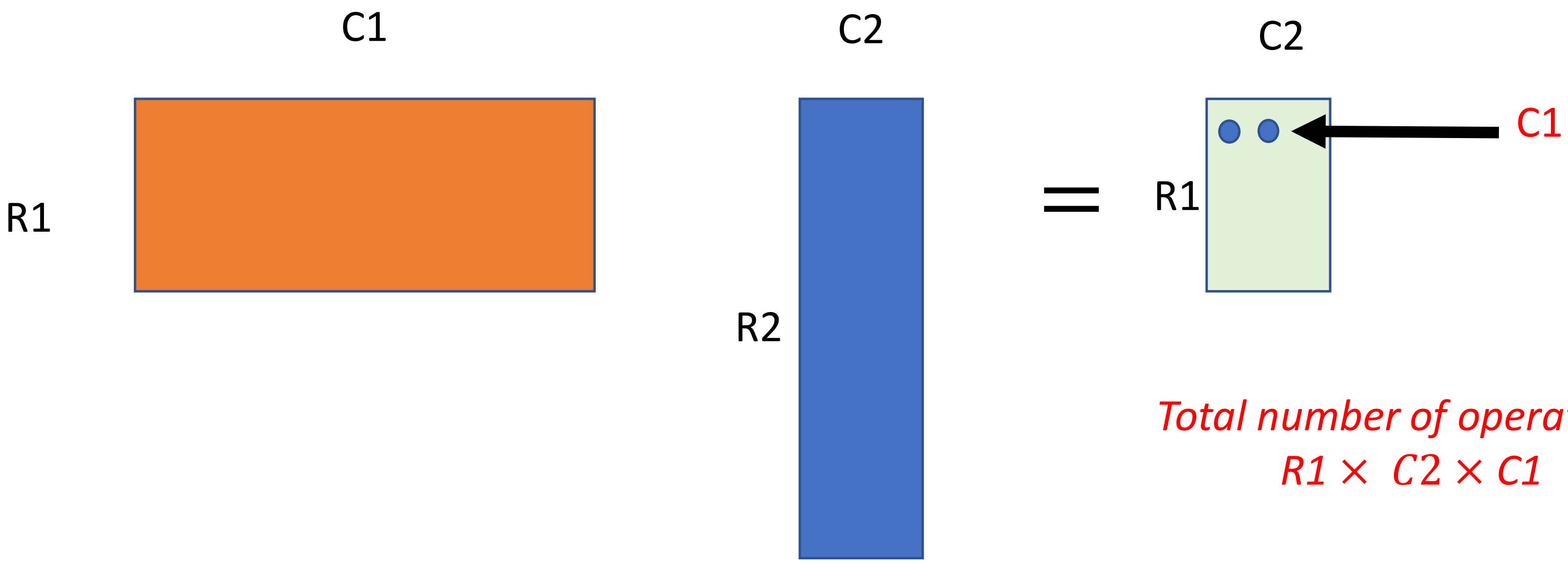
Dynamic Programming

lecture 2


C1=R2
C1
C2
C2
R1
=
R1
R2
$R1 \times C2$
M1
M2
M3

$C1=R2$



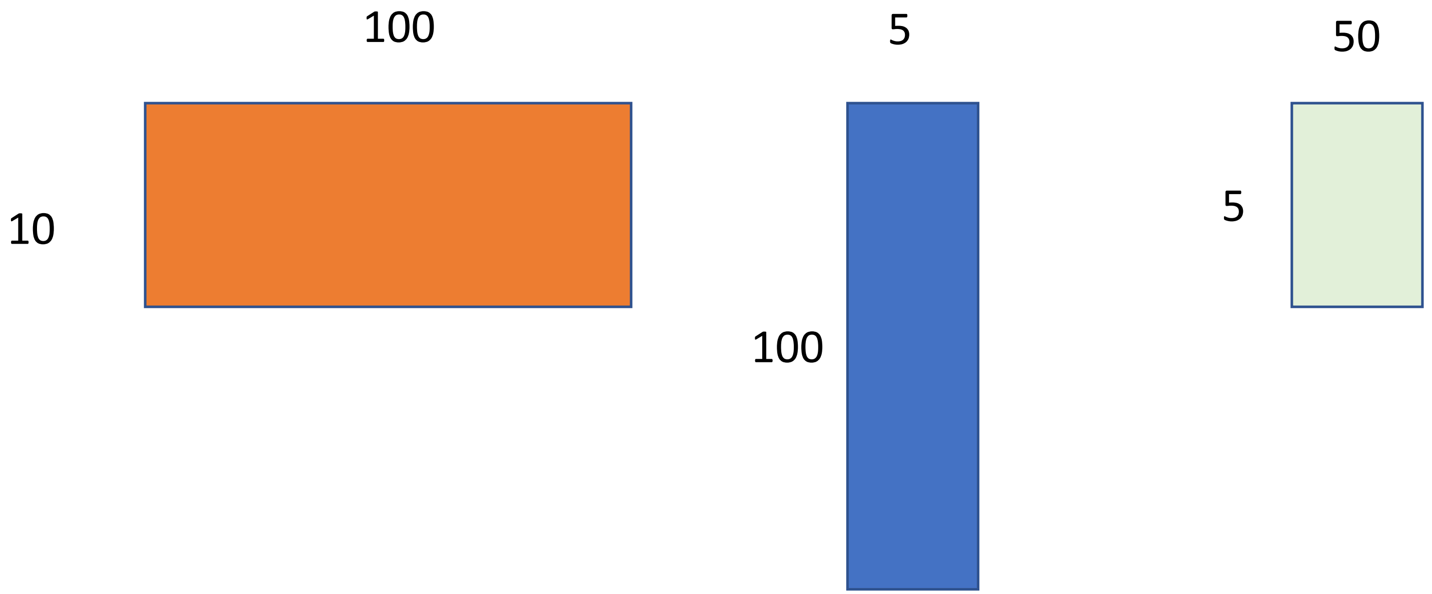
Total number of operations:
$R1 \times C2 \times C1$

$$A_1 \cdot A_2 \cdot A_3$$

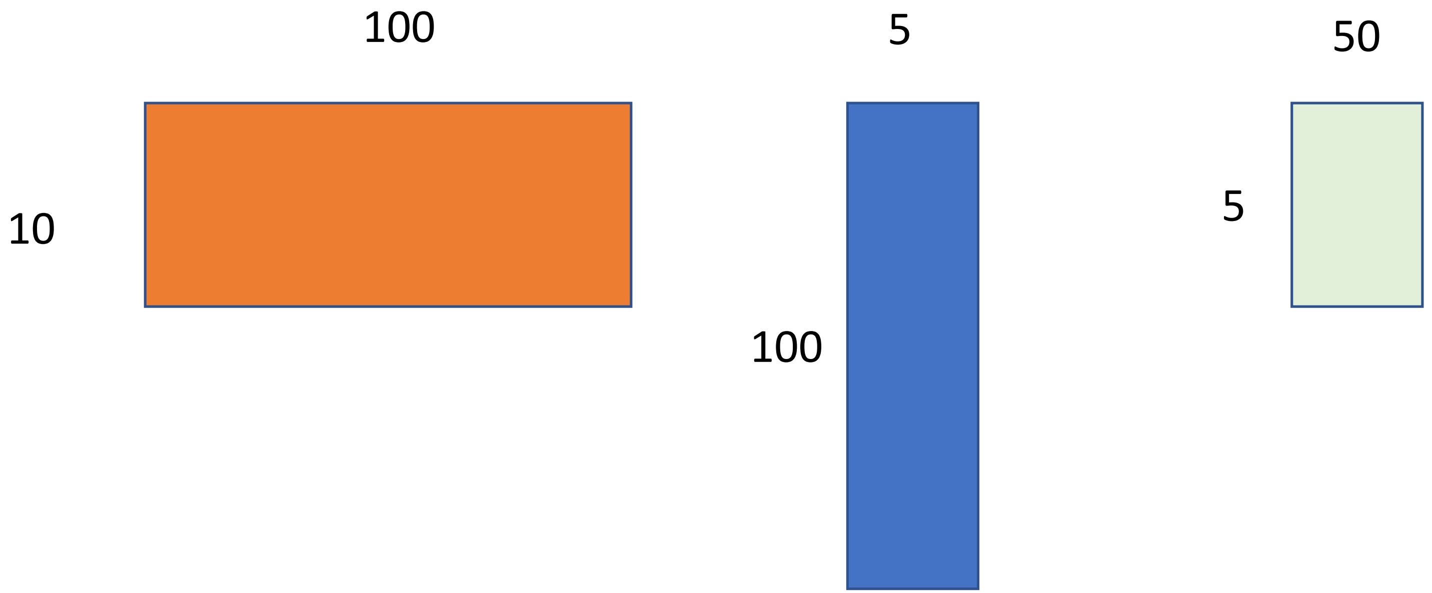
Associative

$$(A_1 \cdot A_2) \cdot A_3 \qquad A_1 \cdot (A_2 \cdot A_3)$$

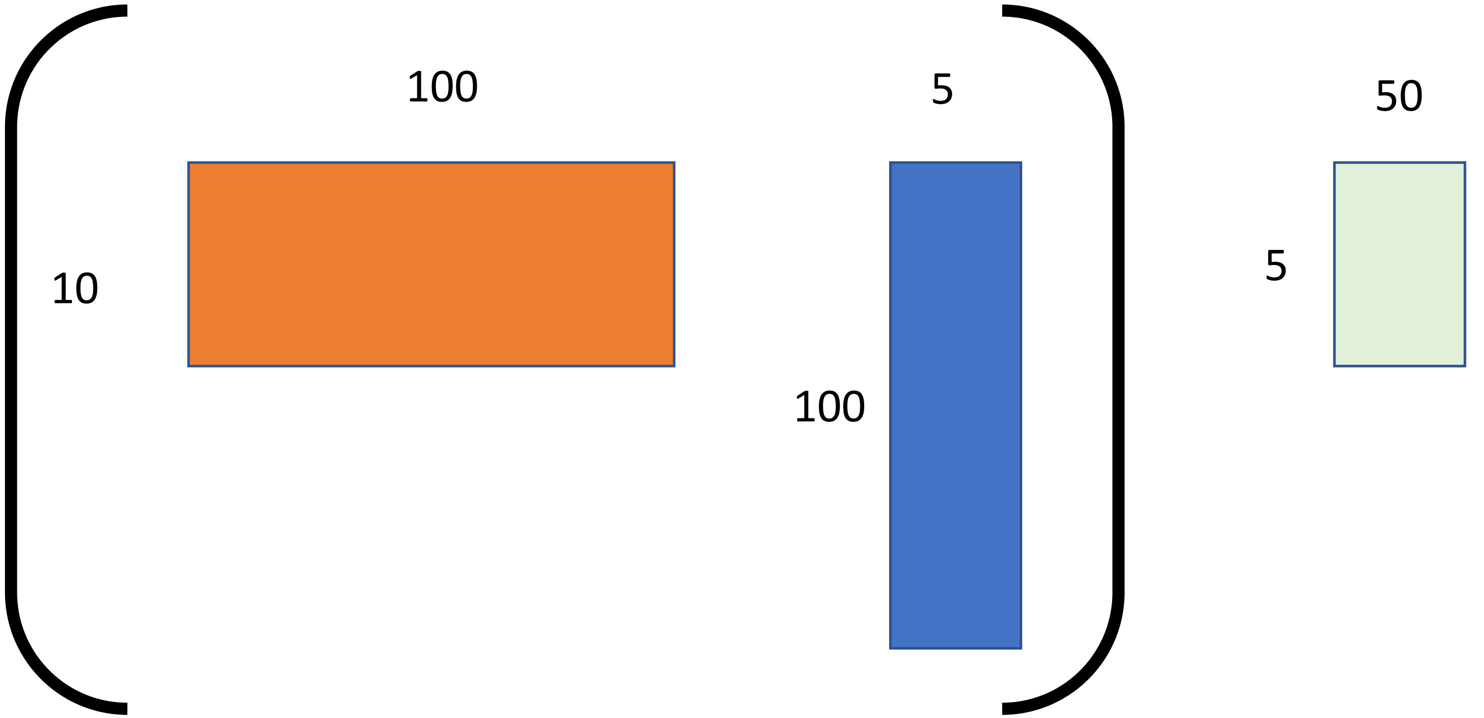
$$A_1 \cdot A_2 \cdot A_3$$



$$(A_1 \cdot A_2) \cdot A_3$$



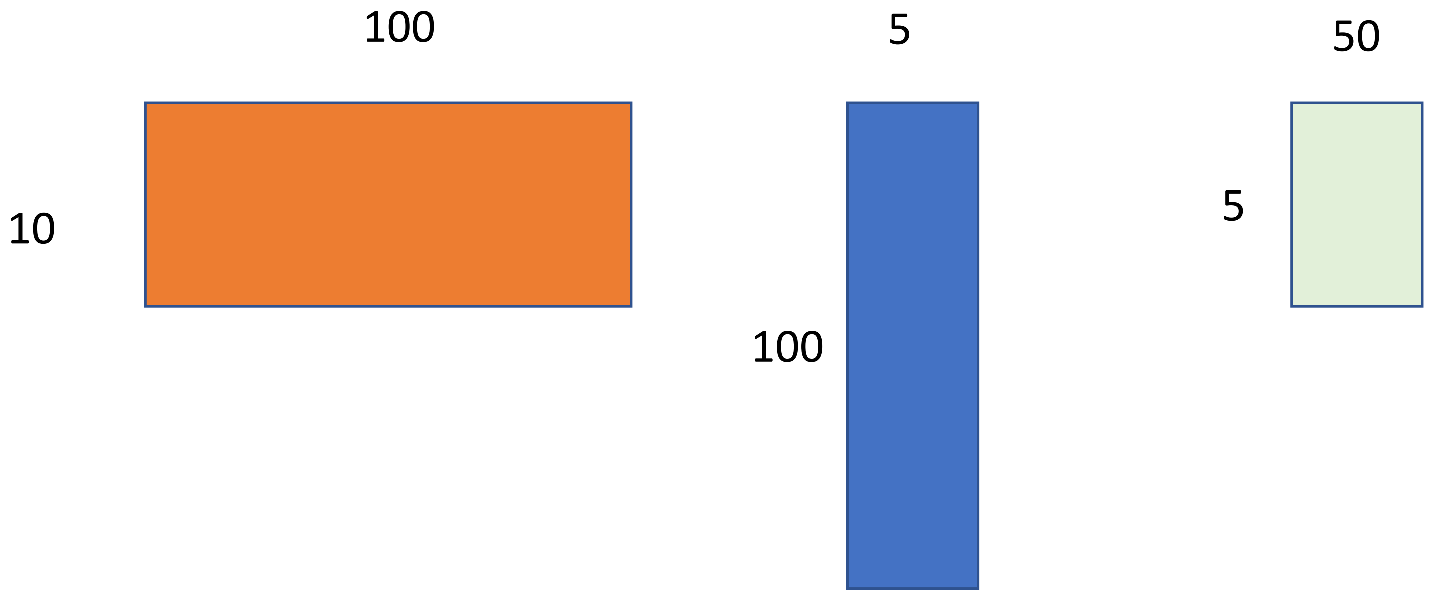
$$(A_1 \cdot A_2) \cdot A_3$$



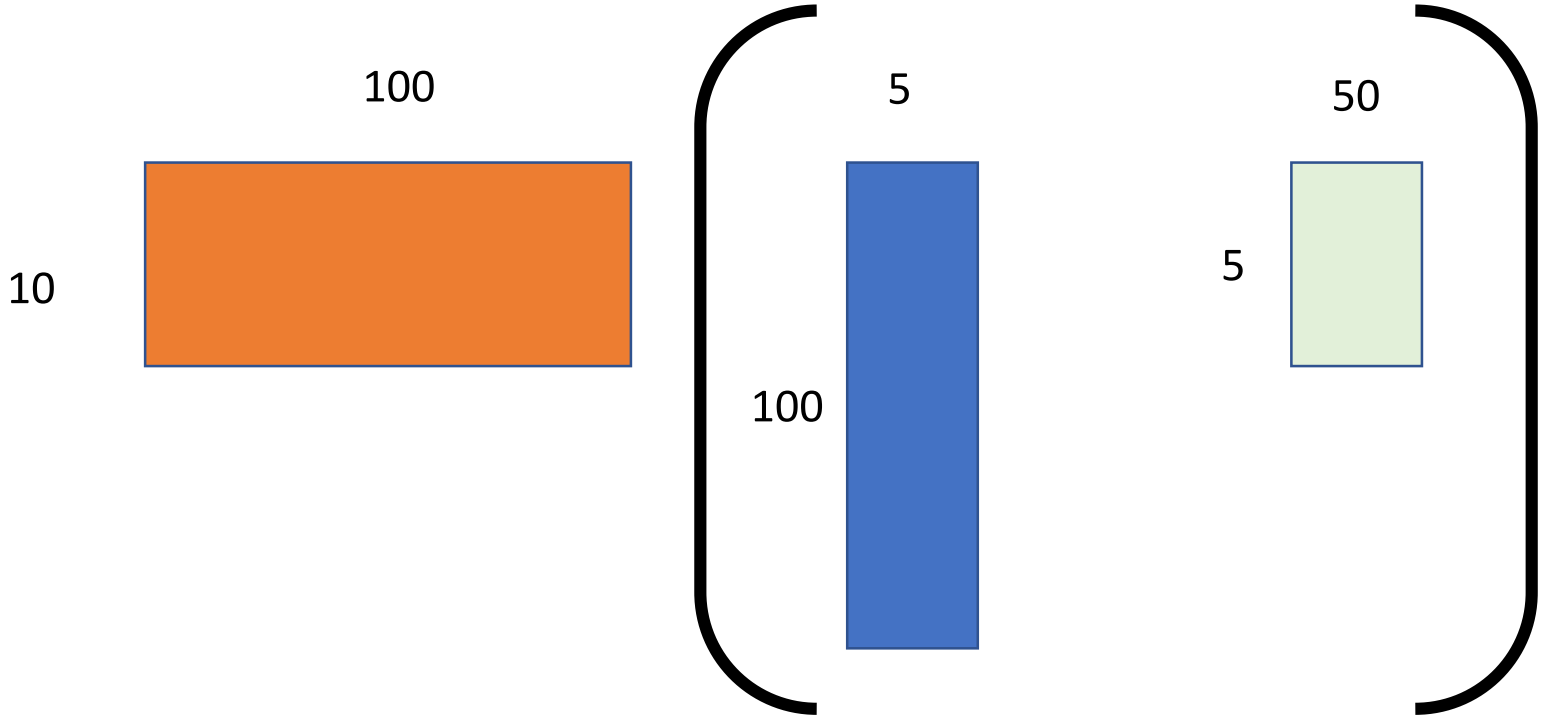
7500

$$10.100.5 + 10.5.50$$

$$A_1 \cdot (A_2 \cdot A_3)$$



$$A_1 \cdot (A_2 \cdot A_3)$$



75000

$$100.5.50 + 10.100.50$$

Order Matters

How many ways to multiply:

$$A_1 \cdot A_2 \cdot A_3 \cdot \cdot \cdot \cdot \quad A_n$$

N-1 multiplication

P(n): number of ways to multiply the n matrices

How many ways to multiply:

$$A_1 \cdot A_2 \cdot A_3 \cdot \cdot \cdot \cdot \; A_n$$

P(n): number of ways to multiply the n matrices

$P(n) = P(1).P(n-1) +$

$$\boxed{A_1} \cdot \boxed{A_2 \cdot A_3 \cdot \cdot \cdot \cdot \; A_n}$$

How many ways to multiply:

$$A_1 \cdot A_2 \cdot A_3 \cdot \cdot \cdot \cdot \; A_n$$

P(n): number of ways to multiply the n matrices

P(n) = P(1).P(n-1) + P(2).P(n-2) +

$$\boxed{A_1 \cdot A_2} \cdot \boxed{A_3 \cdot \cdot \cdot \cdot \; A_n}$$

How many ways to multiply:

$$A_1 \cdot A_2 \cdot A_3 \cdot \cdot \cdot \cdot \quad A_n$$

P(n): number of ways to multiply the n matrices

P(n) = P(1).P(n-1) + P(2).P(n-2) + P(3).P(n-3) +

$$\boxed{A_1 \cdot A_2 \cdot A_3} \cdot \boxed{A_4 \cdot \cdot \cdot \quad A_n}$$

How many ways to multiply:

$$A_1 \cdot A_2 \cdot A_3 \cdot \cdot \cdot \cdot \; A_n$$

P(n): number of ways to multiply the n matrices

$P(n) = P(1).P(n-1) + P(2).P(n-2) + P(3).P(n-3) + \ldots\ldots\ldots + P(n-1)P(1)$

$$\boxed{A_1 \cdot A_2 \cdot A_3 \cdots\cdots A_{n-1}} \cdot \boxed{A_n}$$

How many ways to multiply:

$$A_1 \cdot A_2 \cdot A_3 \cdot \cdot \cdot \cdot \; A_n$$

P(n): number of ways to multiply the n matrices

P(n) = P(1).P(n-1) + P(2).P(n-2) + P(3).P(n-3) + ……… + P(n-1)P(1)

$$= \sum_{i=1}^{n-1} P(i).P(n-i) \approx 4^n$$

Optimal Way to Compute

$$A_1 \cdot A_2 \cdot A_3 \ldots\ldots A_l \cdot A_{l+1} \cdot \; \cdot \; \cdot \; A_n$$

Optimal Way to Compute

$$\overset{C1}{{}^{R1}A_1} \cdot \overset{C2}{{}^{R2}A_2} \cdot \overset{C3}{{}^{R3}A_3} \cdots\cdots \overset{Cl}{{}^{Rl}A_l} \cdot \overset{C_{l+1}}{{}^{R_{l+1}}A_{l+1}} \cdot \cdot \cdot \overset{Cn}{{}^{Rn}A_n}$$

B[1,n]= smallest number of operations needed to multiply the chain

Optimal Way to Compute

$$\begin{array}{c} \\ R1\ A_1 \end{array} \cdot \begin{array}{c} C1 \end{array}$$

$$\overset{C1}{{}^{R1}A_1} \cdot \overset{C2}{{}^{R2}A_2} \cdot \overset{C3}{{}^{R3}A_3} \cdots\cdots \overset{Cl}{{}^{Rl}A_l} \cdot \overset{C_{l+1}}{{}^{R_{l+1}}A_{l+1}} \cdot\ \cdot\ \cdot \overset{Cn}{{}^{Rn}A_n}$$

B[1,n]= smallest number of operations needed to multiply the chain

Optimal Way to Compute

$$\underset{R1}{\overset{C1}{A_1}} \cdot \overset{C2}{\underset{R2}{A_2}} \cdot \overset{C3}{\underset{R3}{A_3}} \cdots\cdots \overset{Cl}{\underset{Rl}{A_l}} \cdot \overset{C_{l+1}}{\underset{R_{l+1}}{A_{l+1}}} \cdot\ \cdot\ \cdot \overset{Cn}{\underset{Rn}{A_n}}$$

Optimal last step: A[1…l] . A[l+1,….n]

B[1,n]= smallest number of operations needed to multiply the chain

$$\mathbf{B[1,n] = B[1,l] + B[l+1,n] + R_1.C_l.C_{l+1}}$$

Optimal Way to Compute

$$\overset{R1}{} \boxed{\overset{C1}{A_1} \cdot^{R2} \overset{C2}{A_2} \cdot^{R3} \overset{C3}{A_3} \ldots\ldots^{Rl} \overset{Cl}{A_l}} \cdot^{R_{l+1}} \boxed{\overset{C_{l+1}}{A_{l+1}} \cdot \cdot \cdot^{Rn} \overset{Cn}{A_n}}$$

Optimal last step: A[1…l] . A[l+1,….n]

B[1,n]= smallest number of operations needed to multiply the chain

$$\mathbf{B[1,n] = B[1,l] + B[l+1,n] + R_1.C_l.C_{l+1}}$$

How many choices we have for l?

$l \in [1,n-1]$

Optimal Way to Compute

$$\overset{R_1 \quad C_1}{A_1} \cdot \overset{R_2 \quad C_2}{A_2} \cdot \overset{R_3 \quad C_3}{A_3} \cdots\cdots \overset{R_l \quad C_l}{A_l} \cdot \overset{R_{l+1} \quad C_{l+1}}{A_{l+1}} \cdot \; \cdot \; \cdot \overset{R_n \quad C_n}{A_n}$$

B[1,n]= smallest number of operations needed to multiply the chain

B[1,1]	**B[1,2]**		**B[1,n-2]**	**B[1,n-1]**
B[2,n]	**B[3,n]**	**.... ...**	**B[n-1,n]**	**B[n,n]**
$R_1C_1C_n$	$R_1C_2C_n$		$R_1C_{n-2}C_n$	$R_1C_{n-1}C_n$

Which Order to Solve?

$$A_1 \cdot A_2 \cdot A_3 \cdot \ldots \cdot A_{n-1} \cdot A_n$$

$$B(i,i)=0$$

$$B(i,j)= \min_{k=i}^{j-1} B(i,k) + B(k+1\,,\,j) + R_i C_k C_j$$

$${}^{R_1}\overset{C_1}{A_1} \cdot {}^{R_2}\overset{C_2}{A_2} \cdot {}^{R_3}\overset{C_3}{A_3} \cdots\cdots {}^{R_k}\overset{C_k}{A_k} \cdot {}^{R_{k+1}}\overset{C_{k+1}}{A_{k+1}} \cdots {}^{R_{n-1}}\overset{C_{n-1}}{A_{n-1}} \cdot {}^{R_n}\overset{C_n}{A_n}$$

Which Order to Solve?

$$A_1 \cdot A_2 \cdot A_3 \cdot \ldots \cdot A_{n-1} \cdot A_n$$

$B(i,i)=0$

$$B(i,j) = \min_{k=i}^{j-1} B(i,k) + B(k+1\,,\,j) + R_i C_k C_j$$

$i=2$

$j=n-1$

C_1 C_2 C_3 C_k C_{k+1} C_{n-1} C_n

$$R_1\; A_1 \cdot \boxed{R_2\; A_2 \cdot R_3\; A_3 \cdots\cdots R_k\; A_k} \cdot R_{k+1}\; \boxed{A_{k+1} \cdots R_{n-1}\; A_{n-1}} \cdot R_n\; A_n$$

Which Order to Solve?

$A_1 \cdot A_2 \cdot A_3 \cdot \ldots \cdot A_{n-1} \cdot A_n$

$B(i,i)=0$

$B(i,j) = \min_{k=i}^{j-1} B(i,k) + B(k+1, j) + R_i C_k C_j$

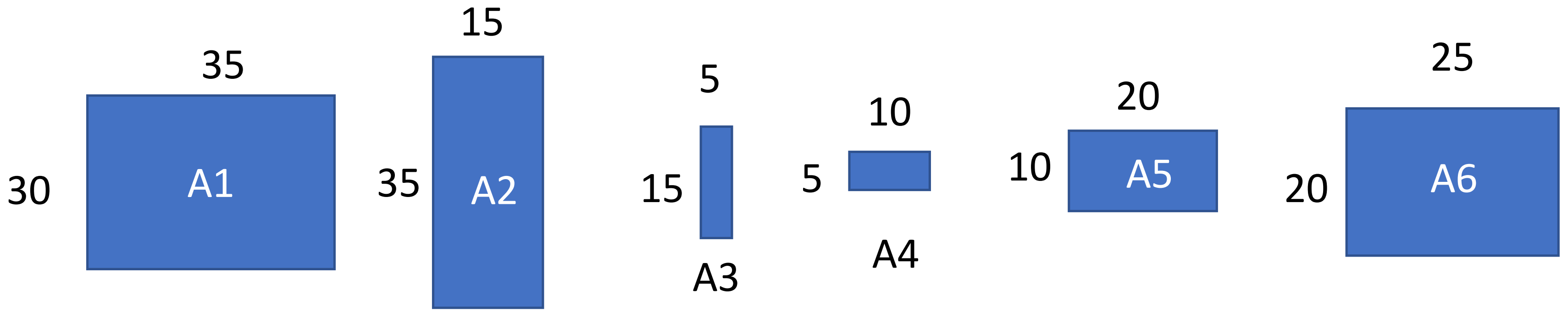
$i=2$

$j=n-1$

$K=3$

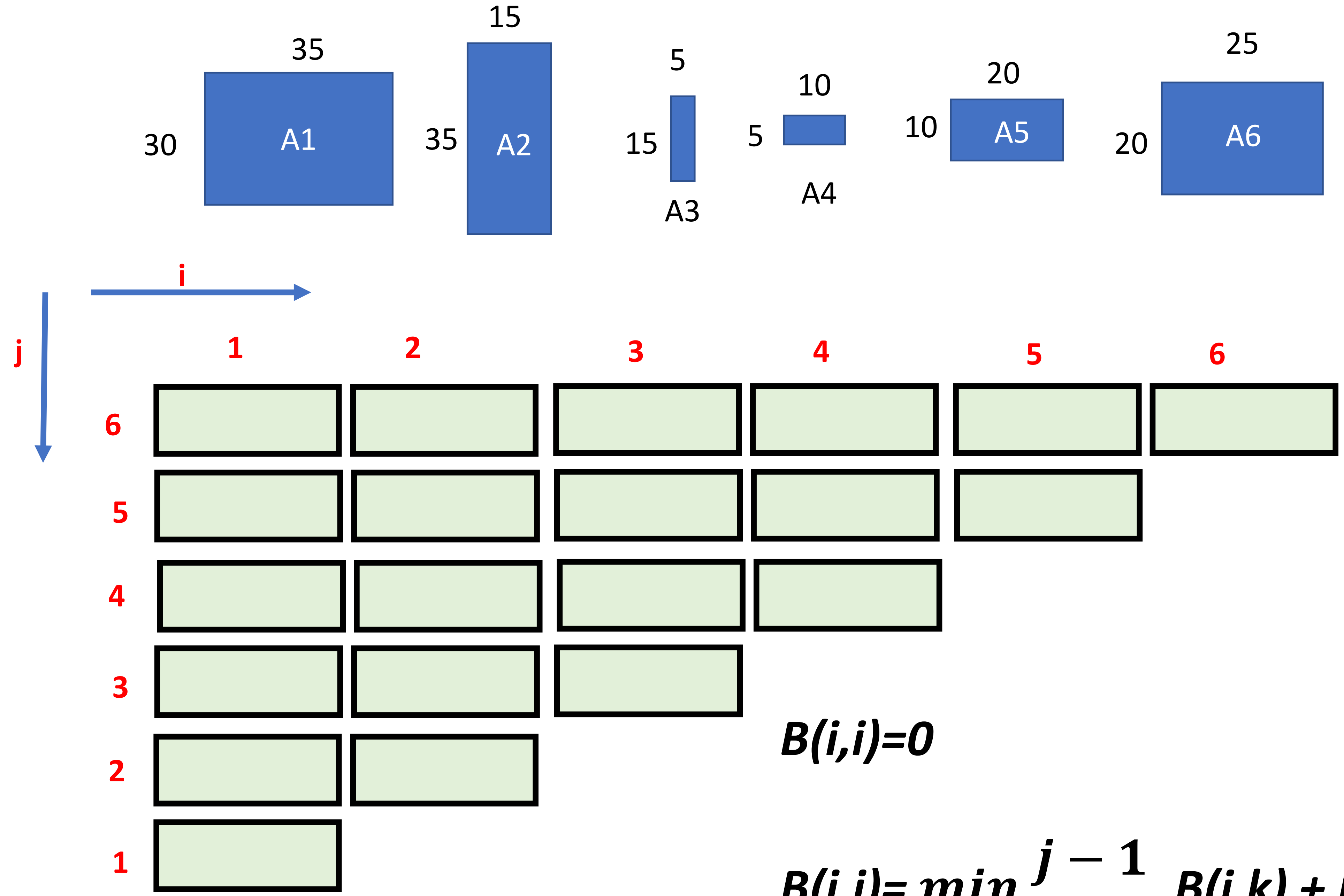
C_1 C_2 C_3 C_4 C_{n-1} C_n

$R_1\ A_1 \cdot \boxed{R_2\ A_2 \cdot R_3\ A_3} \cdot \boxed{R_4\ A_4 \ldots\ldots\ldots\ldots\ldots \ldots R_{n-1}\ A_{n-1}} \cdot R_n\ A_n$



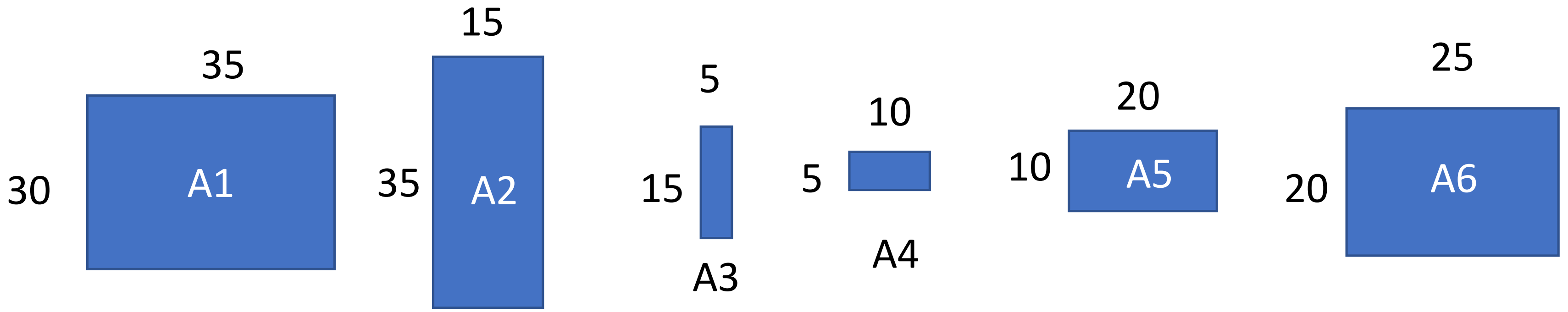
$$B(i,i)=0$$

$$B(i,j)= \min_{k=i}^{j-1} B(i,k) + B(k+1\ ,\ j) + R_i C_k C_j$$



$$B(i,i)=0$$

$$B(i,j) = \min_{k=i}^{j-1} B(i,k) + B(k+1\ ,\ j) + R_i C_k C_j$$



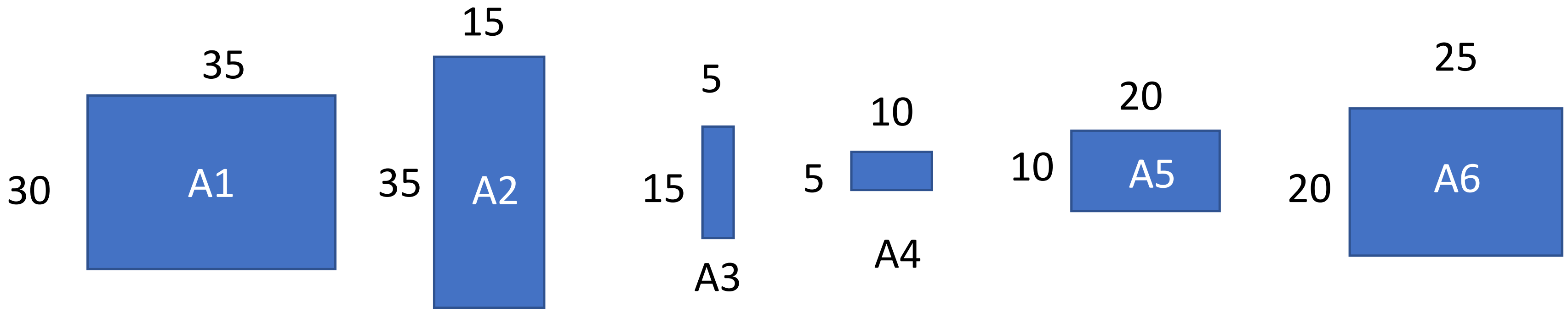
i

j

	1	2	3	4	5	6
6						0
5					0	
4				0		
3			0			
2		0				
1	0					

$$B(i,i)=0$$

$$B(i,j)= \min_{k=i}^{j-1} B(i,k) + B(k+1\,,\,j) + R_iC_kC_j$$



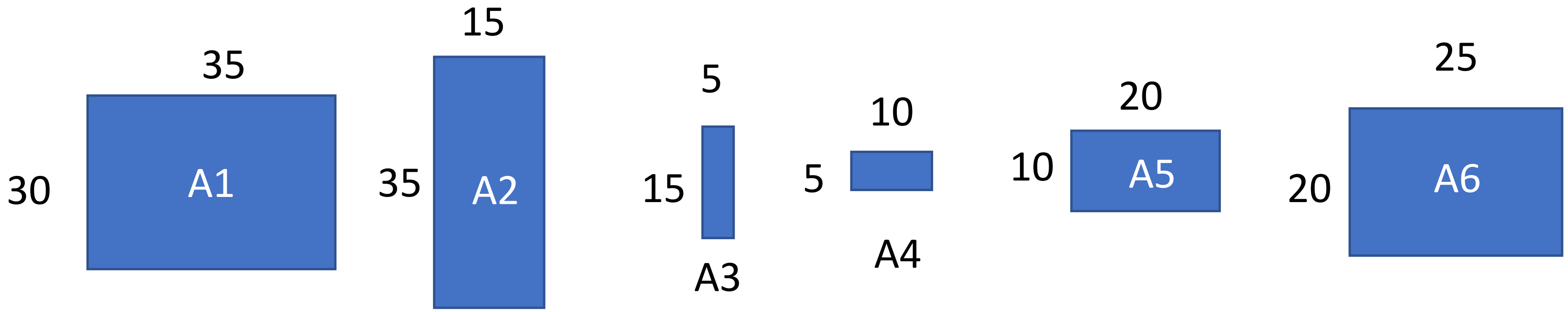
i

j

	1	2	3	4	5	6
6	*B(1,6)*					0
5					0	
4				0		
3			0			
2		0				
1	0					

$$B(i,i)=0$$

$$B(i,j)= \min_{k=i}^{j-1} B(i,k) + B(k+1\,,\,j) + R_i C_k C_j$$

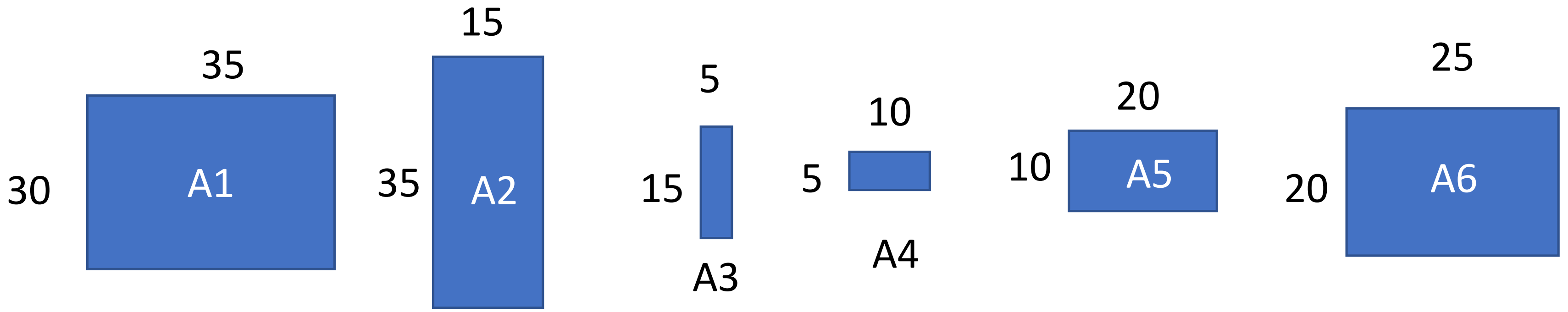


j \ i	1	2	3	4	5	6
6						0
5					0	
4				0		
3			0			
2		0				
1	0					

$$B(1,2)=B(1,1)+B(2,2)+30.35.15$$

$$B(i,i)=0$$

$$B(i,j)= \min_{k=i}^{j-1} B(i,k) + B(k+1\ ,\ j) + R_i C_k C_j$$



i

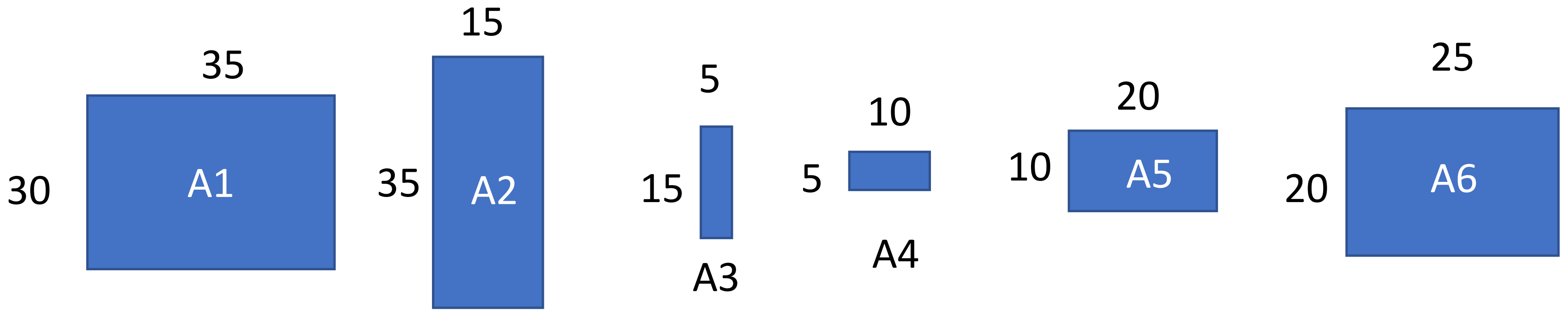
j

	1	2	3	4	5	6
6						0
5					0	
4				0		
3			0			
2	15750	0				
1	0					

$$B(1,2)=B(1,1)+B(2,2)+30.35.15$$

$$B(i,i)=0$$

$$B(i,j)= \min_{k=i}^{j-1} B(i,k) + B(k+1\ ,\ j) + R_i C_k C_j$$



i →

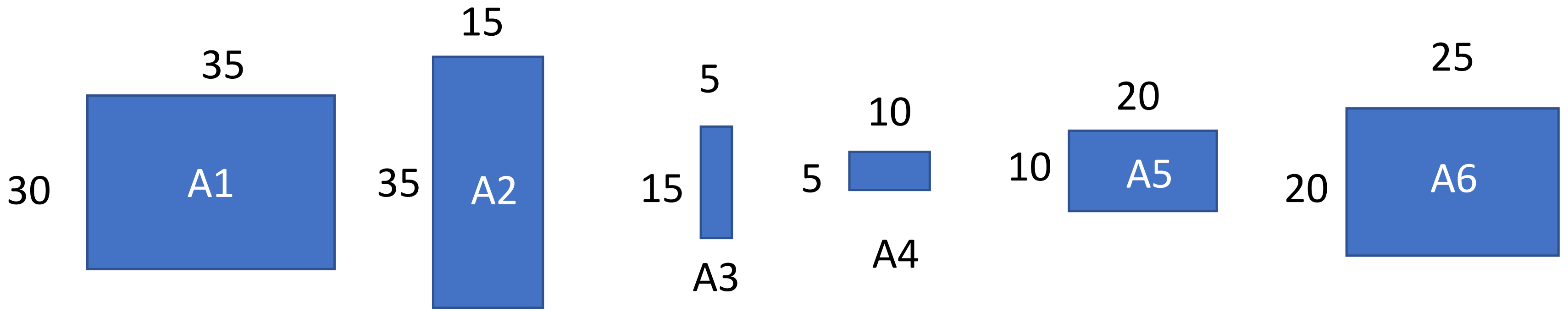
j ↓

j \ i	1	2	3	4	5	6
6						0
5					0	
4				0		
3			0			
2	15750	0				
1	0					

$$B(2,3)=B(2,2)+B(3,3)+35.15.5$$

$$B(i,i)=0$$

$$B(i,j)= \min_{k=i}^{j-1} B(i,k) + B(k+1\,,\,j) + R_i C_k C_j$$

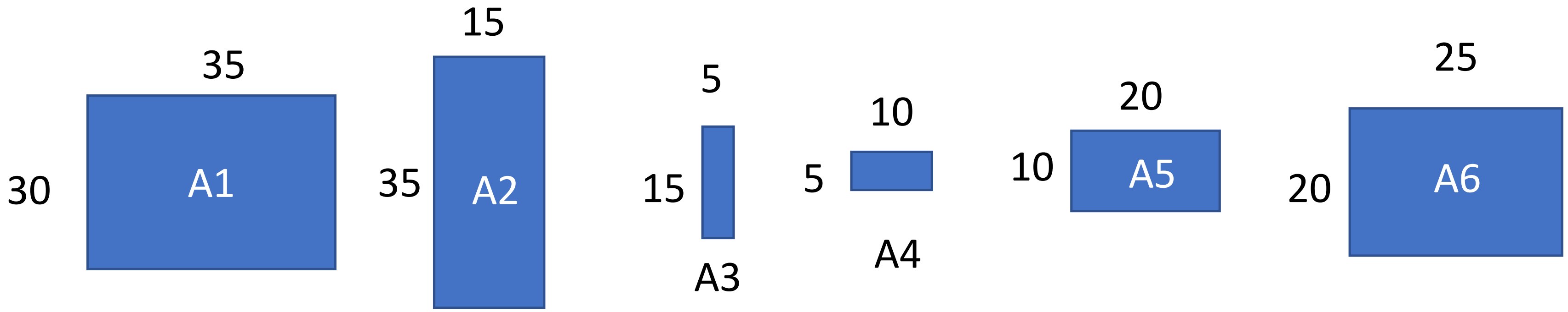


j \ i	1	2	3	4	5	6
6						0
5					0	
4				0		
3		2625	0			
2	15750	0				
1	0					

$$B(2,3)=B(2,2)+B(3,3)+35.15.5$$

$$B(i,i)=0$$

$$B(i,j)= \min_{k=i}^{j-1} B(i,k) + B(k+1\ ,\ j) + R_iC_kC_j$$

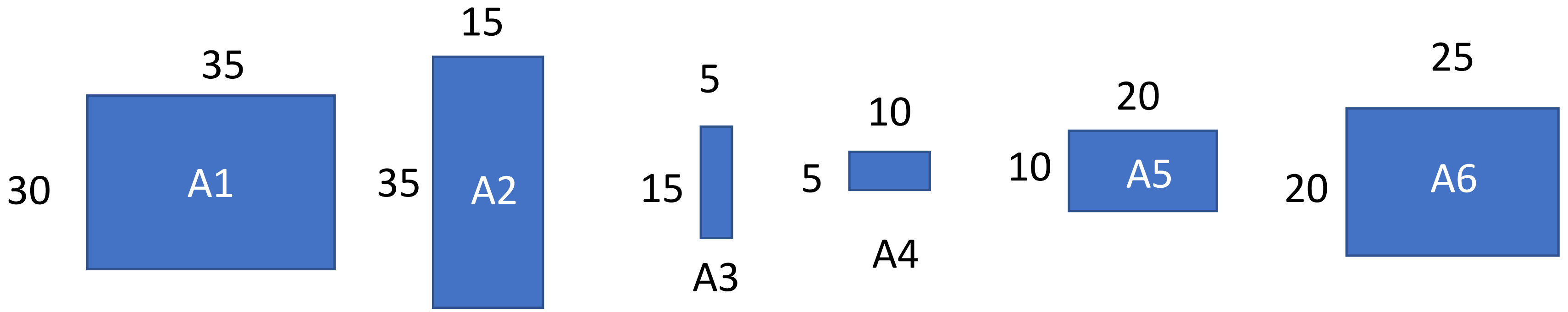


j \ i	1	2	3	4	5	6
6						0
5					0	
4				0		
3		2625	0			
2	15750	0				
1	0					

$$B(3,4)=B(3,3)+B(4,4)+15.5.10$$

$$B(i,i)=0$$

$$B(i,j)= \min_{k=i}^{j-1} B(i,k) + B(k+1\ ,\ j) + R_i C_k C_j$$



i

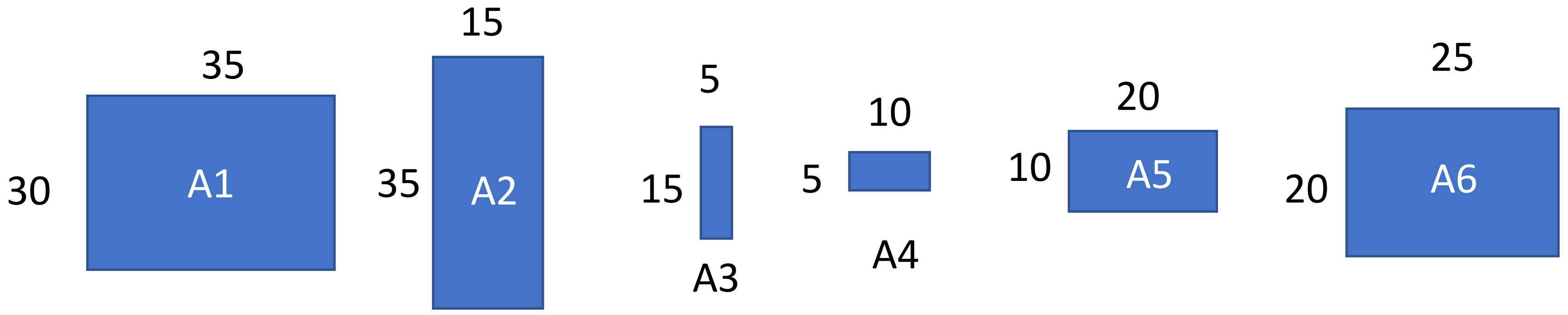
j

	1	2	3	4	5	6
6						0
5					0	
4			750	0		
3		2625	0			
2	15750	0				
1	0					

$$B(3,4)=B(3,3)+B(4,4)+15.5.10$$

$$B(i,i)=0$$

$$B(i,j)= \min_{k=i}^{j-1} B(i,k) + B(k+1\ ,\ j) + R_i C_k C_j$$



i →

j ↓

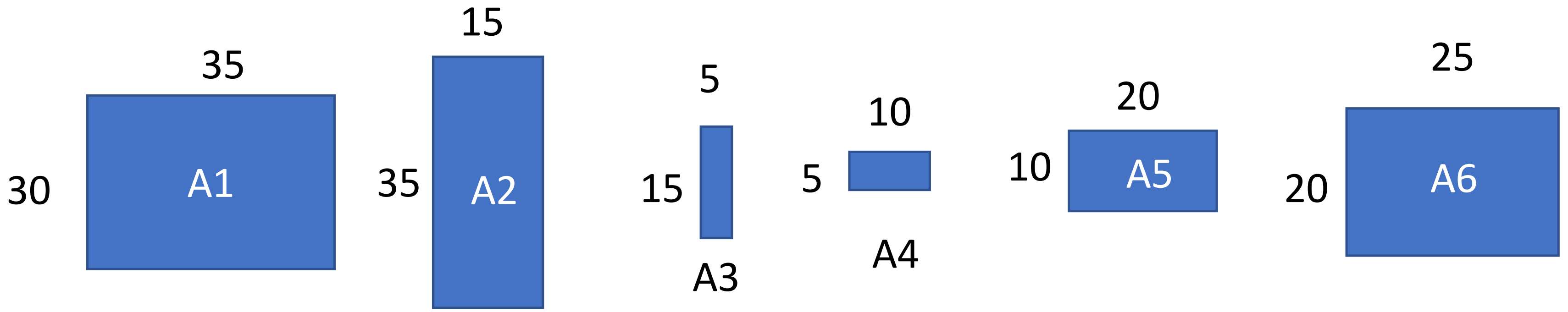
j \ i	1	2	3	4	5	6
6					5000	0
5				1000	0	
4			750	0		
3		2625	0			
2	15750	0				
1	0					

$$B(4,5)=B(4,4)+B(5,5)+5.10.20$$

$$B(5,6)=B(5,5)+B(6,6)+10.20.25$$

$$B(i,i)=0$$

$$B(i,j)= \min_{k=i}^{j-1} B(i,k) + B(k+1\ , j) + R_i C_k C_j$$



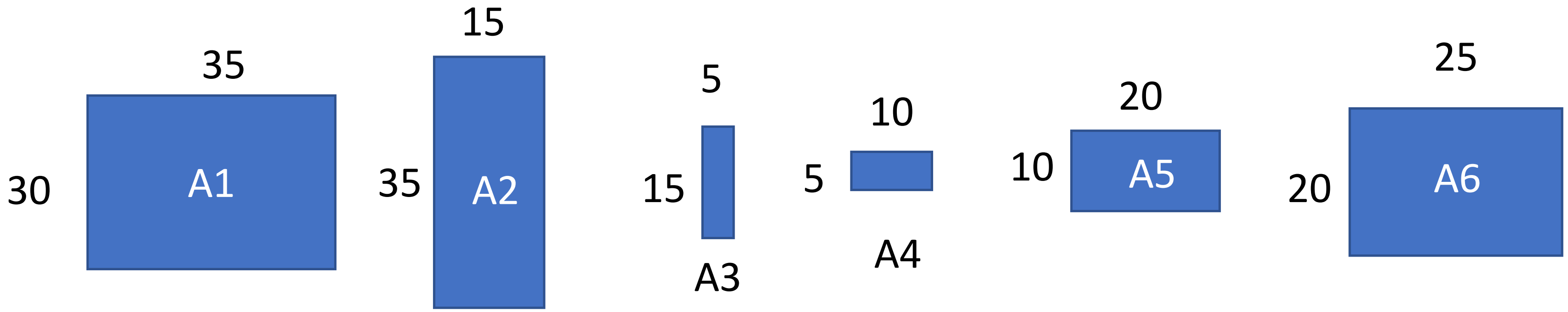
i

j	1	2	3	4	5	6
6					5000	0
5				1000	0	
4			750	0		
3		2625	0			
2	15750	0				
1	0					

B(1,3)

$$B(i,i)=0$$

$$B(i,j)= \min_{k=i}^{j-1} B(i,k) + B(k+1\ ,\ j) + R_i C_k C_j$$

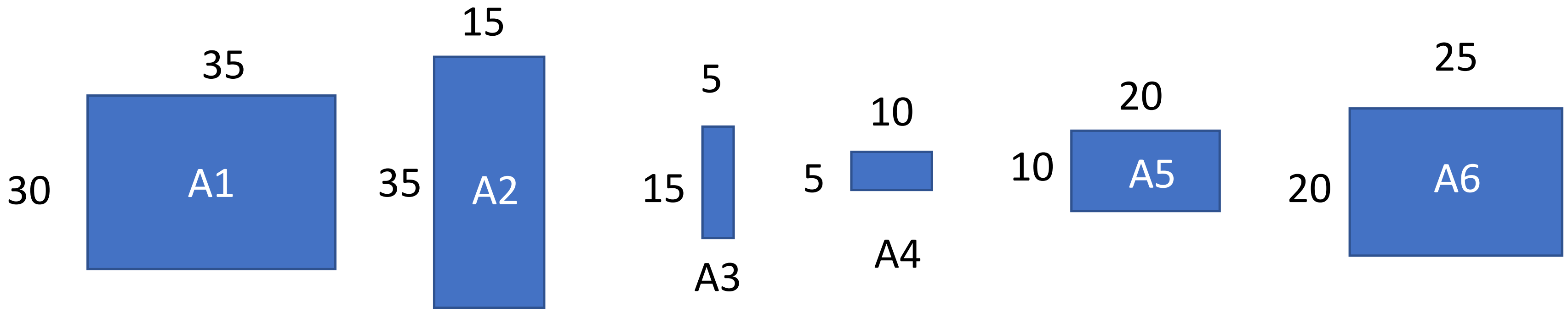


j \ i	1	2	3	4	5	6
6					5000	0
5				1000	0	
4			750	0		
3		2625	0			
2	15750	0				
1	0					

$$B(1,3)=\min\begin{cases} K=2 \\ B(1,2)+B(3,3)+30.15.5 \\ B(1,1)+B(2,3)+30.35.5 \\ K=1 \end{cases}$$

$$B(i,i)=0$$

$$B(i,j)=\min_{k=i}^{j-1} B(i,k)+B(k+1,j)+R_iC_kC_j$$

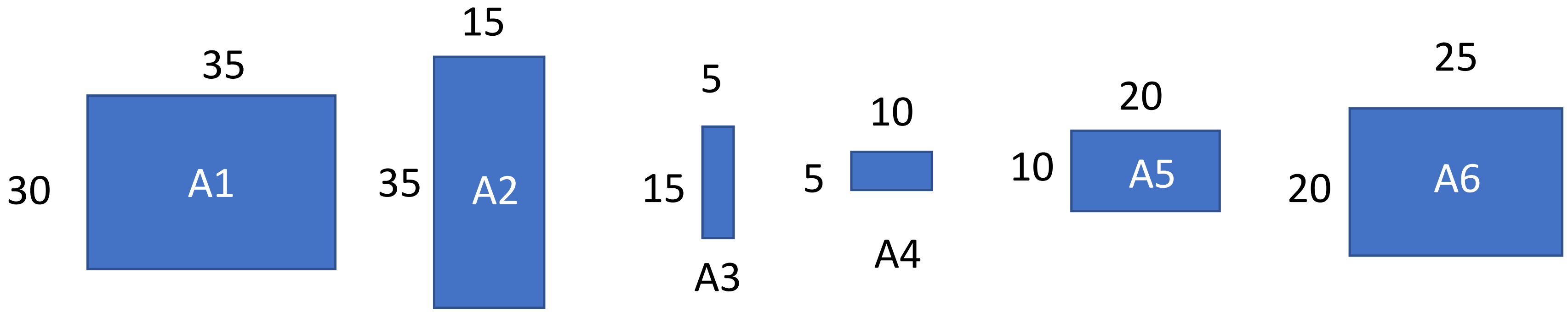


j \ i	1	2	3	4	5	6
6					5000	0
5				1000	0	
4			750	0		
3		2625	0			
2	15750	0				
1	0					

$$B(1,3)=\min\begin{cases}\overset{15750}{B(1,2)}+B(3,3)+\overset{2250}{30.15.5}\\[2ex] B(1,1)+\underset{2625}{B(2,3)}+\underset{5250}{30.35.5}\end{cases}$$

$$B(i,i)=0$$

$$B(i,j)=\min_{k=i}^{j-1} B(i,k)+B(k+1,j)+R_iC_kC_j$$

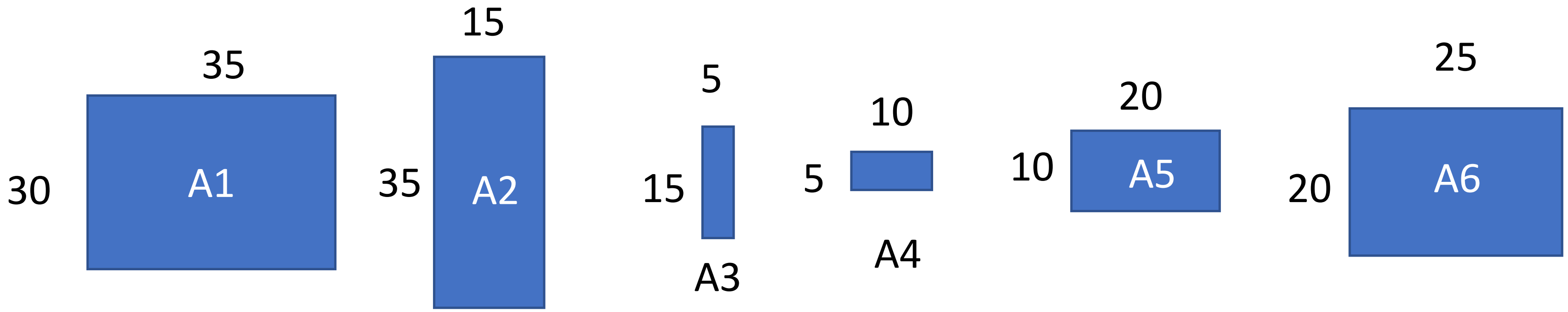


j \ i	1	2	3	4	5	6
6					5000	0
5				1000	0	
4			750	0		
3	7875	2625	0			
2	15750	0				
1	0					

$$B(1,3)=\min\begin{cases} \overset{15750}{B(1,2)}+\overset{2250}{B(3,3)+30.15.5} \\ B(1,1)+\underset{2625}{B(2,3)}+\underset{5250}{30.35.5} \end{cases}$$

$$B(i,i)=0$$

$$B(i,j)=\min_{k=i}^{j-1} B(i,k)+B(k+1,j)+R_iC_kC_j$$



i

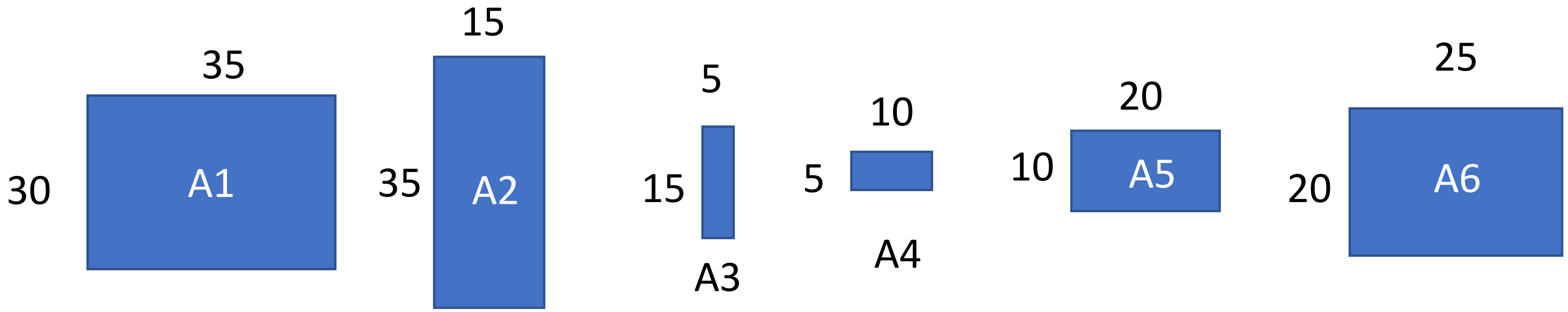
j

j \ i	1	2	3	4	5	6
6					5000	0
5				1000	0	
4			750	0		
3	7875	2625	0			
2	15750	0				
1	0					

$$B(2,4)=\min\begin{cases} K=3 \\ B(2,3)+B(4,4)+35.5.10 \\ B(2,2)+B(3,4)+35.15.10 \\ K=2 \end{cases}$$

$$B(i,i)=0$$

$$B(i,j)= \min_{k=i}^{j-1} B(i,k) + B(k+1\,,\,j) + R_i C_k C_j$$

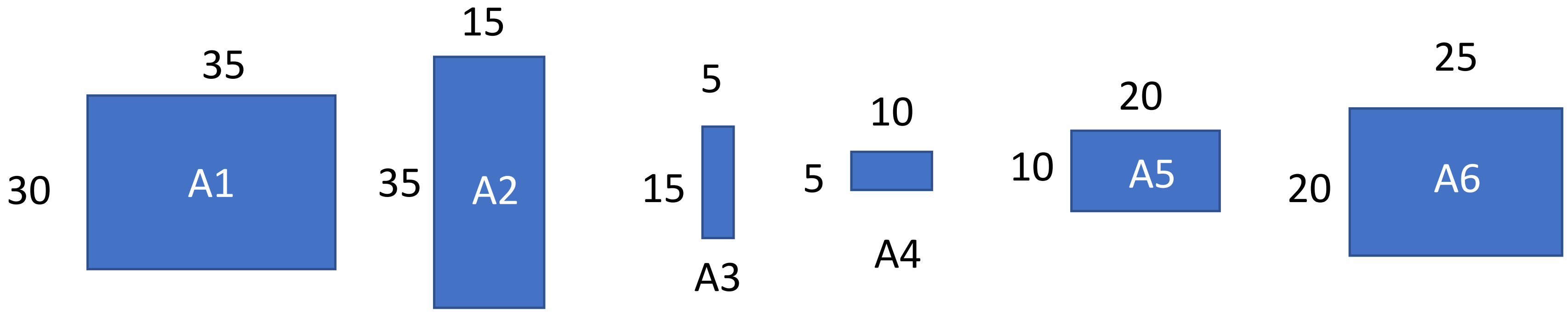


j \ i	1	2	3	4	5	6
6					5000	0
5				1000	0	
4			750	0		
3	7875	2625	0			
2	15750	0				
1	0					

$$B(2,4)=\min\begin{cases} B(2,3)+B(4,4)+35.5.10 \\ B(2,2)+B(3,4)+35.15.10 \end{cases}$$

$$B(i,i)=0$$

$$B(i,j)=\min_{k=i}^{j-1} B(i,k)+B(k+1\,,\,j)+R_iC_kC_j$$

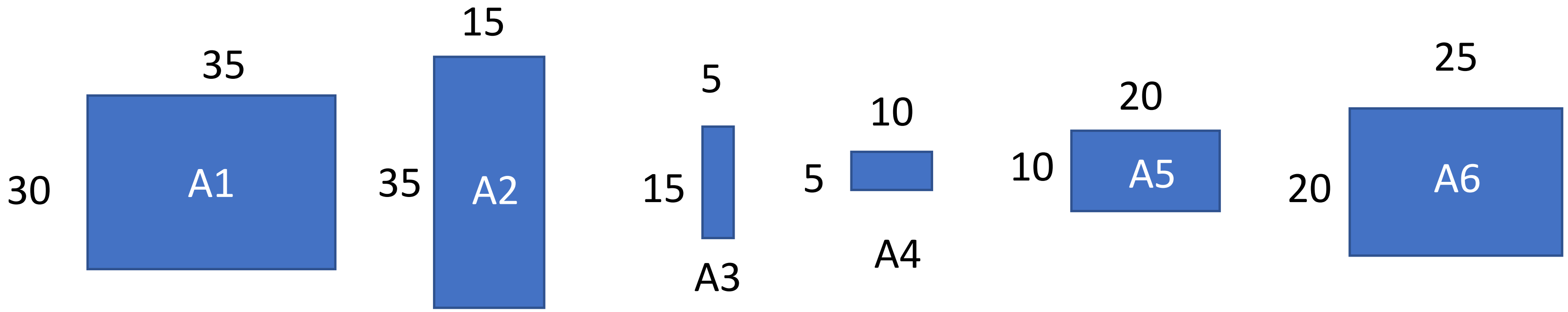


j \ i	1	2	3	4	5	6
6					5000	0
5				1000	0	
4		4375	750	0		
3	7875	2625	0			
2	15750	0				
1	0					

$$B(2,4)=\min\begin{cases} B(2,3)+B(4,4)+35.5.10 \\ B(2,2)+B(3,4)+35.15.10 \end{cases}$$

$$B(i,i)=0$$

$$B(i,j)=\min_{k=i}^{j-1} B(i,k)+B(k+1,j)+R_iC_kC_j$$

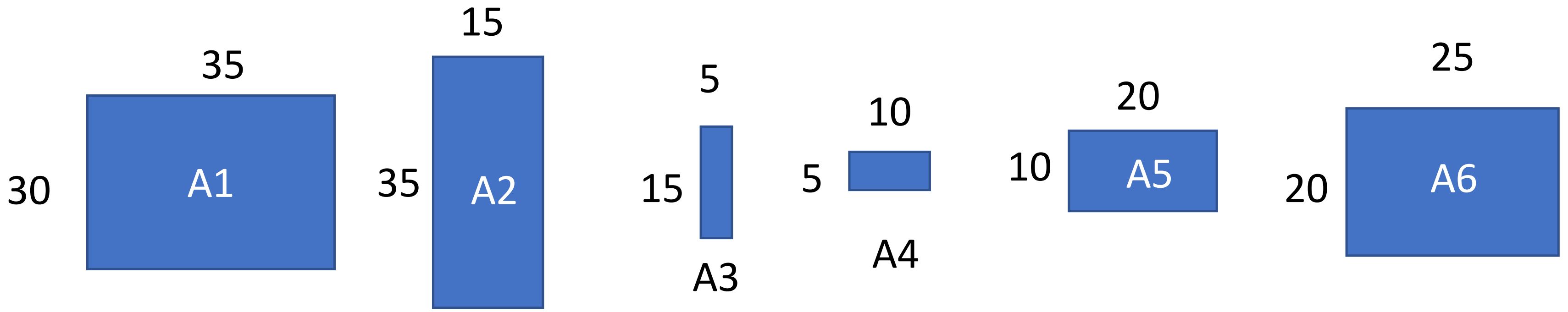


j \ i	1	2	3	4	5	6
6					5000	0
5				1000	0	
4		4375	750	0		
3	7875	2625	0			
2	15750	0				
1	0					

$B(3,5)=$

$B(i,i)=0$

$$B(i,j)= \min_{k=i}^{j-1} B(i,k) + B(k+1\,,\, j) + R_i C_k C_j$$

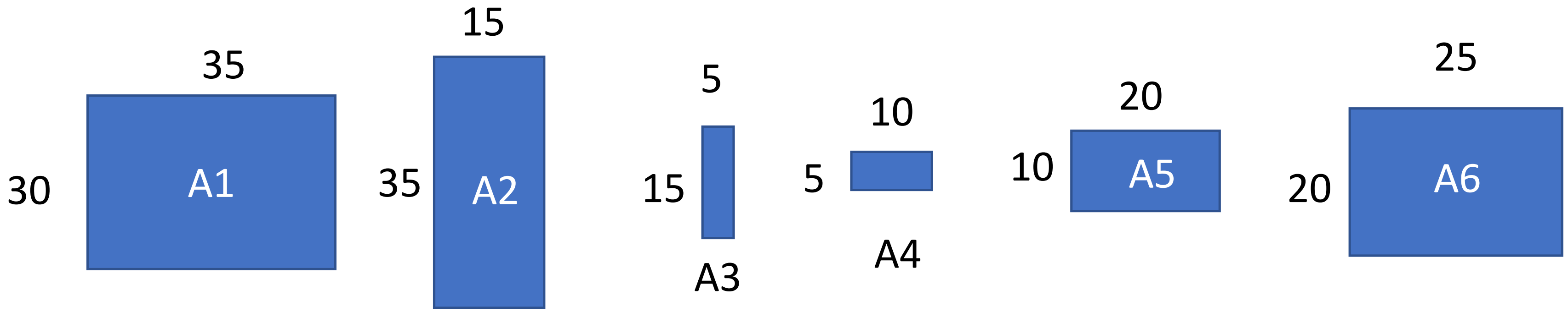


j \ i	1	2	3	4	5	6
6					5000	0
5			2500	1000	0	
4		4375	750	0		
3	7875	2625	0			
2	15750	0				
1	0					

B(3,5)=

$$B(i,i)=0$$

$$B(i,j)= \min_{k=i}^{j-1} B(i,k) + B(k+1\,,\,j) + R_i C_k C_j$$



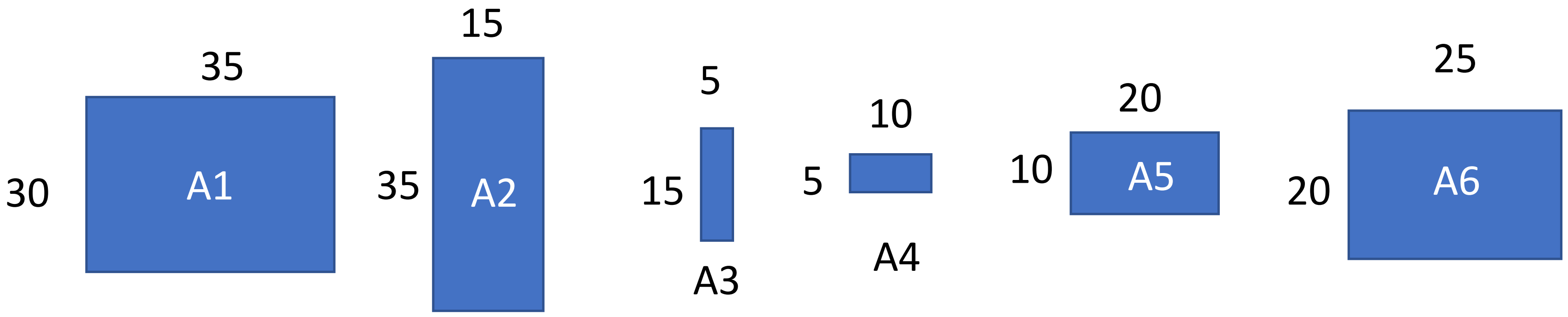
i

j

	1	2	3	4	5	6
6		10500	5375	3500	5000	0
5	11875	7125	2500	1000	0	
4	9375	4375	750	0		
3	7875	2625	0			
2	15750	0				
1	0					

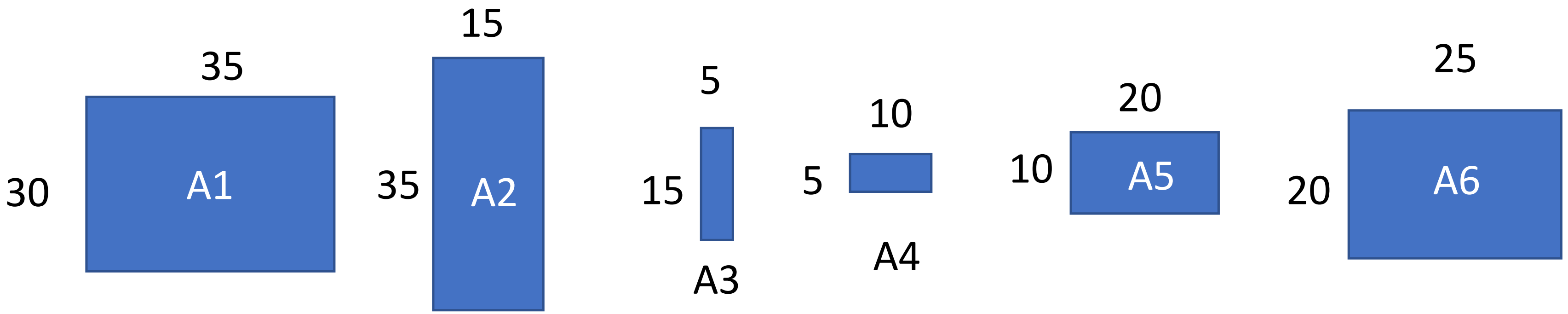
$$B(i,i)=0$$

$$B(i,j)= \min_{k=i}^{j-1} B(i,k) + B(k+1\,,\,j) + R_i C_k C_j$$



j \ i	1	2	3	4	5	6
6		10500	5375	3500	5000	0
5	11875	7125	2500	1000	0	
4	9375	4375	750	0		
3	7875	2625	0			
2	15750	0				
1	0					

$$B(1,6)=\min\begin{cases} K=1 & B(1,1)+B(2,6)+R1C1C6 \\ K=2 & B(1,2)+B(3,6)+R1C2C6 \\ K=3 & B(1,3)+B(4,6)+R1C3C6 \\ K=4 & B(1,4)+B(5,6)+R1C4C6 \\ K=5 & B(1,5)+B(6,6)+R1C5C6 \end{cases}$$



i

j

	1	2	3	4	5	6
6	15125	10500	5375	3500	5000	0
5	11875	7125	2500	1000	0	
4	9375	4375	750	0		
3	7875	2625	0			
2	15750	0				
1	0					

$$B(1,6)=\min\begin{cases} K=1 & B(1,1)+B(2,6)+R1C1C6 \\ K=2 & B(1,2)+B(3,6)+R1C2C6 \\ K=3 & B(1,3)+B(4,6)+R1C3C6 \\ K=4 & B(1,4)+B(5,6)+R1C4C6 \\ K=5 & B(1,5)+B(6,6)+R1C5C6 \end{cases}$$

Matrix Chain Multiplication

Initialize array m[x,y] to zero
Starting at diagonal, working toward upper left $\theta(n^2)$

Compute B[i,j] according to:

B(i,i)=0

$$B(i,j) = \min_{k=i}^{j-1} B(i,k) + B(k+1\,,\,j) + R_iC_kC_j \qquad \theta(n)$$

Runtime: $\boldsymbol{\theta(n3)}$

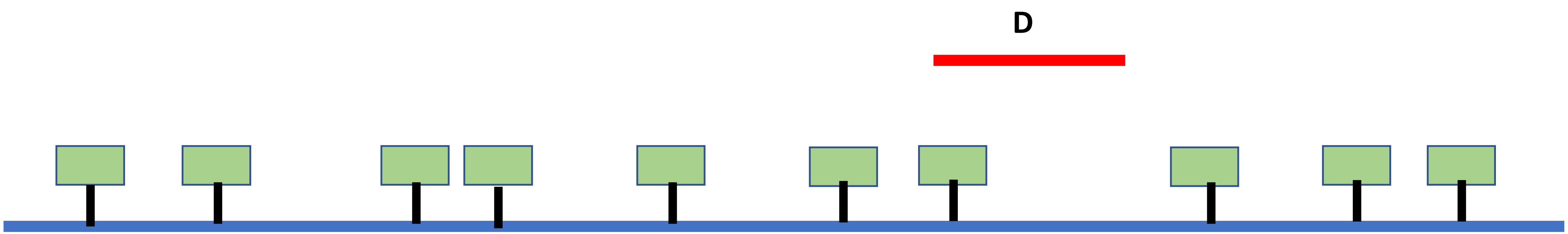

TIME FOR A NEW TRUCK?
Ford
verizon
LOWE'S


Coca-Cola
102° reasons to
refresh yourself.
CLEARCHANNEL
078706


012004
STONE HILL
WINERY
STORYBOOK VIEWS
HERMANN • MISSOURI


Coca-Cola
97° reasons to
refresh yourself.
CLEARCHANNEL
078706


Coca-Cola
100° reasons to
refresh yourself.
CLEARCHANNEL
078706

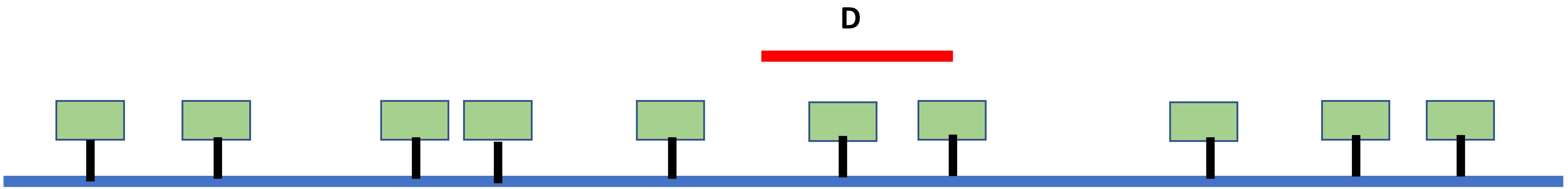


$(x_1, x_2, x_3, \ldots, X_n)$: mile markers

$(v_1, v_2, v_3, \ldots., v_n)$: Viewership, e.g., v_i = number of people that view billboard at x_i

D: distance parameters, can not place ads that are closer than D miles apart

Goal: is to maximize viewership for an acceptable campaign

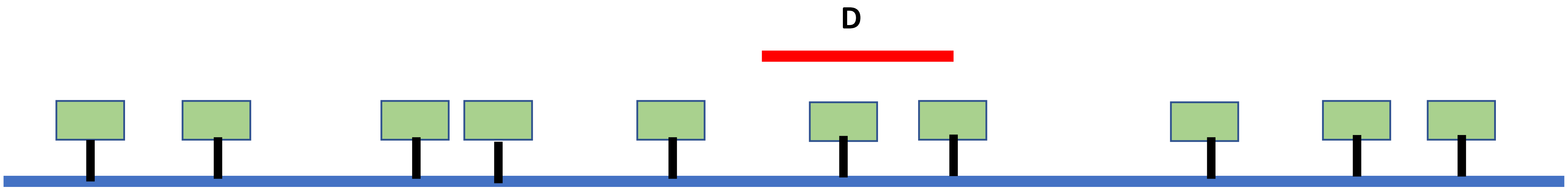


$$(x_1,x_2,x_3,\ldots,X_n),\ (v_1,v_2,v_3,\ldots.,v_n),\ D$$

$Best_n$ = max viewership for an acceptable campaign that considers the first n billboards

$Best_j$= max viewership for an acceptable campaign that considers the first j billboards

$$Best_j = \max \begin{cases} \mathbf{Best_{j-1}} \\ \mathbf{V_j + Best_{(closest\ billboard\ that\ is\ atleastD\ away)}} \end{cases}$$



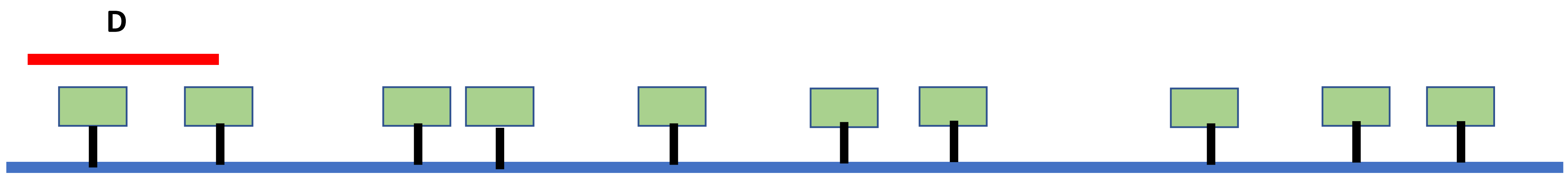
$(x_1, x_2, x_3, \ldots, X_n), (v_1, v_2, v_3, \ldots., v_n), D$

$Best_n$ = max viewership for an acceptable campaign that considers the first n billboards

$Best_j$= max viewership for an acceptable campaign that considers the first j billboards

$$Best_j = \max \begin{cases} \mathbf{Best_{j-1}} \\ \mathbf{V_j + Best_{(closest\ billboard\ that\ is\ atleastD\ away)}} \end{cases}$$

Closest-Buddy

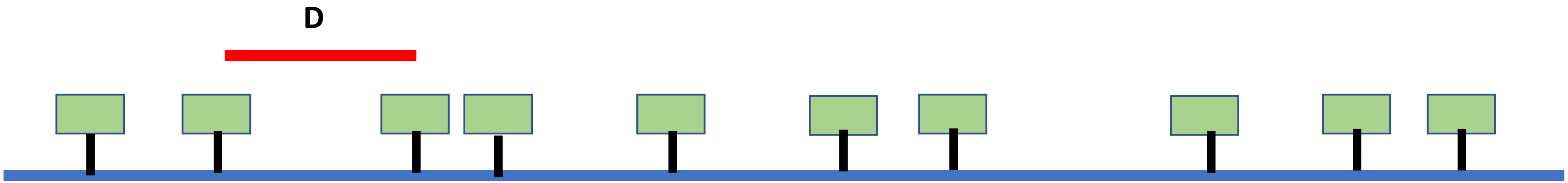


$(x_1, x_2, x_3, \ldots, X_n), (v_1, v_2, v_3, \ldots., v_n), D$

$Best_1 = v_1$

$Best_2$ = Max (Best1 , v2+$Best_{\text{Closest-Buddy}}$)

$$Best_j = \max \begin{cases} Best_{j-1} \\ V_j + Best_{\text{(closest billboard that is atleastD away)}} \end{cases}$$



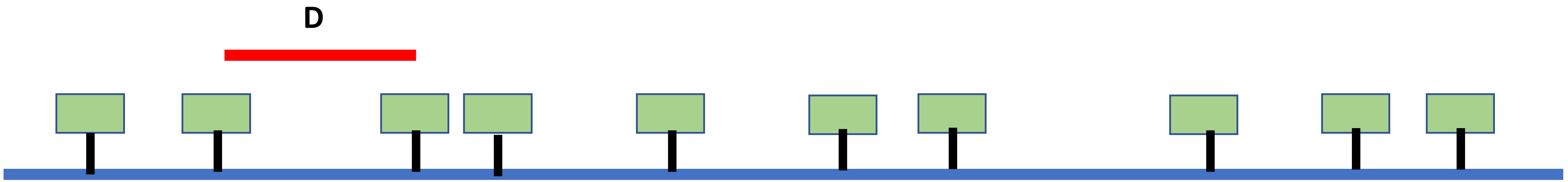
$$(x_1,x_2,x_3,\ldots,X_n),\ (v_1,v_2,v_3,\ldots.,v_n),\ D$$

$$Best_1 = v_1$$

$$Best_2 = Max\ (\ Best_1\ ,\ v_2+Best_{Closest\text{-}Buddy}\ )$$

$$Best_3 = Max\ (\ Best_2\ ,\ v_3+Best_{Closest\text{-}Buddy}\ )$$
$$= Max(\ v_2\ ,\ v_3+Best_1\ )$$

$$Best_j = \max \begin{cases} Best_{j-1} \\ V_j+Best_{(\text{closest billboard that is atleastD away})} \end{cases}$$



$(x_1, x_2, x_3, \ldots, X_n), (v_1, v_2, v_3, \ldots., v_n), D$

$Best_1 = v_1$

$Best_2 = Max\ (Best_1, v_2 + Best_{Closest\text{-}Buddy})$

$Best_3 = Max\ (Best_2, v_3 + Best_{Closest\text{-}Buddy})$
$= Max(v_2, v_3 + Best_1)$

$$Best_j = \max \begin{cases} Best_{j-1} \\ V_j + Best_{(\text{closest billboard that is atleastD away})} \end{cases}$$

$$\mathbf{Best_j = max} \begin{cases} \mathbf{Best_{j-1}} \\ \mathbf{V_j + Best_{cl(j)}} \end{cases}$$

Cl(j) := closest buddy that is at least D distance away

```
Best[0]=0
For i=1 to n

        cl=i-1
        while(dist(x[cl],x[i]<D) cl--;

        Best[i]=max {best[i-1] , v[i]+best[cl] }

Return best[n]
```

$\theta(n)$

$\theta(n)$

But, we can do better?
Next class

$\theta(n^2)$