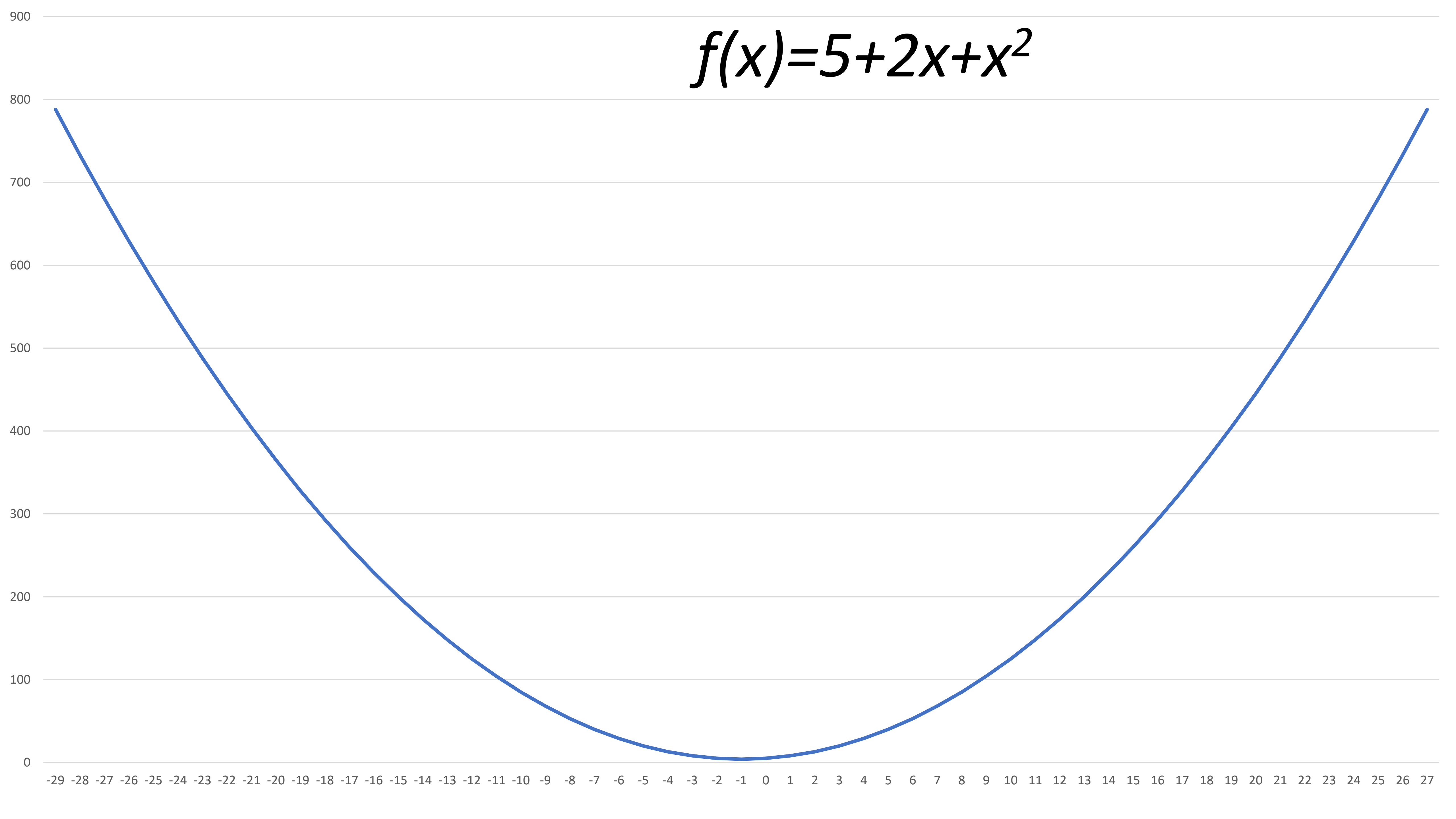
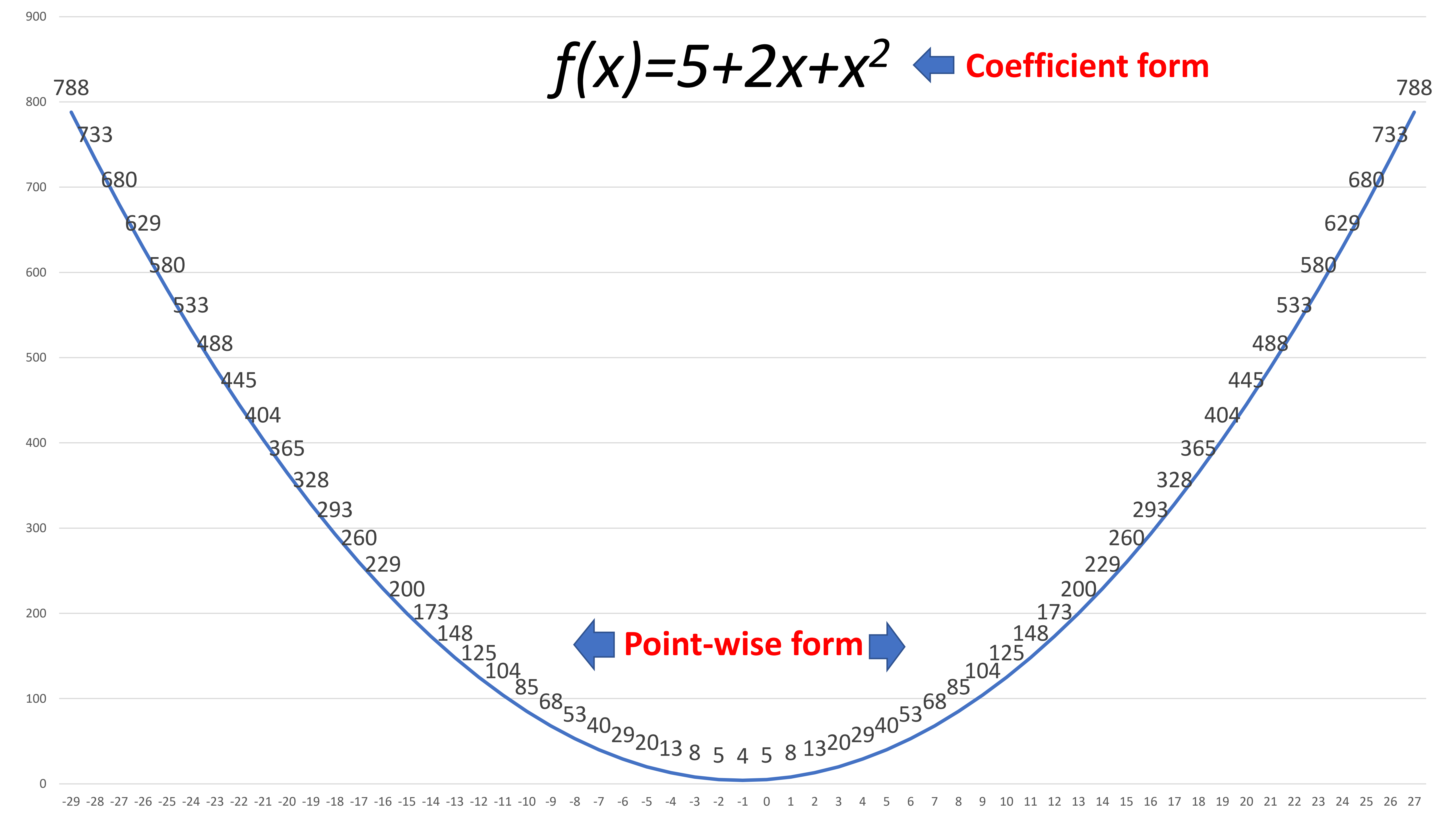
FFT

Big Idea

Change of representation

- **Change of representation of a polynomial:**
  - **From coefficient form → point-wise form**
- **Divide and Conquer**


$f(x)=5+2x+x^2$
900
800
700
600
500
400
300
200
100
0
-29 -28 -27 -26 -25 -24 -23 -22 -21 -20 -19 -18 -17 -16 -15 -14 -13 -12 -11 -10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27



$f(x)=5+2x+x^2$
Coefficient form
Point-wise form

$$A(x)=a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots + a_{n-1}x^{n-1}$$

How efficient is the transformation?

Brute force:

A(1)=

A(2)=

A(3)=

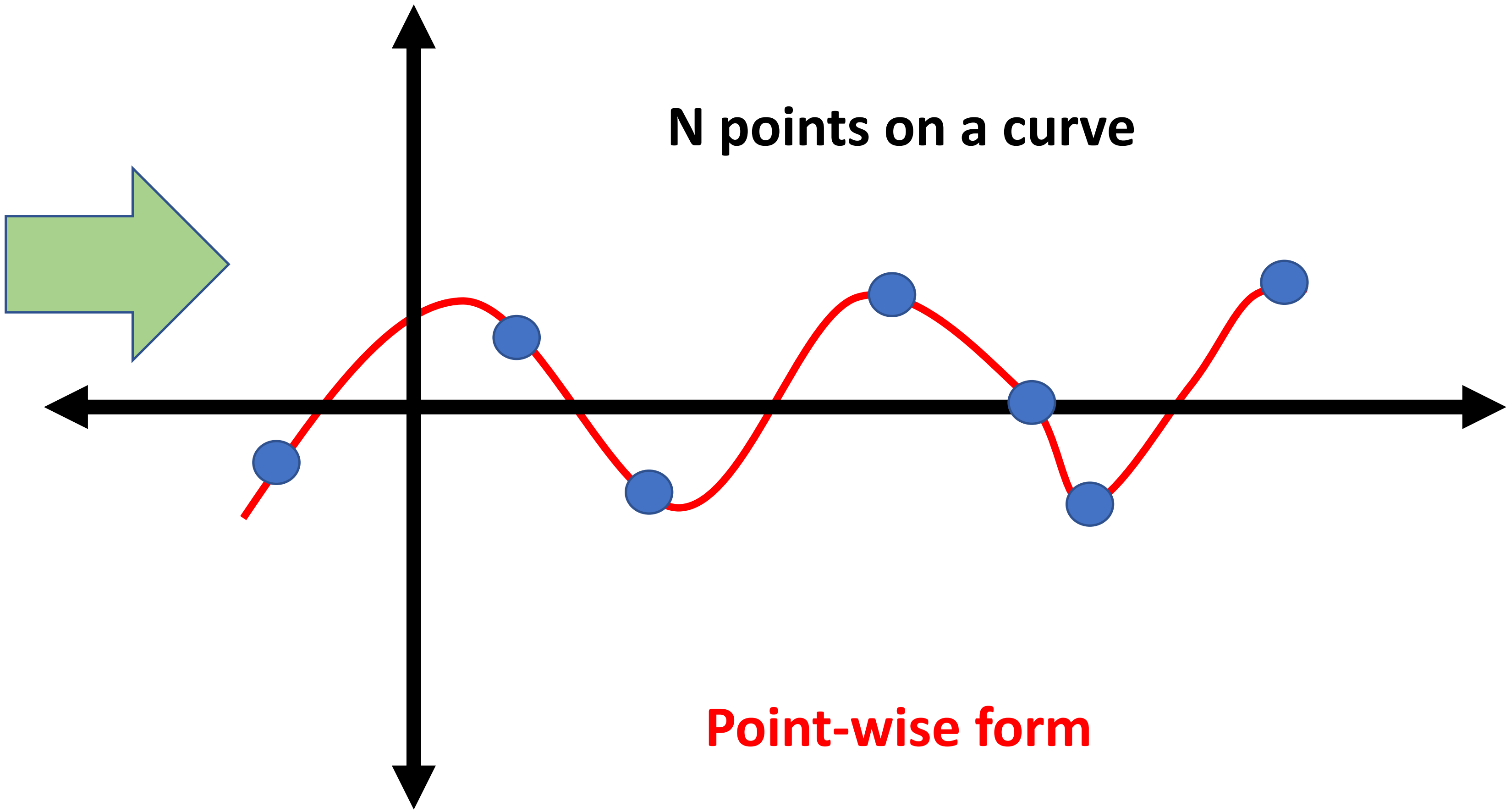
$$A(x)=a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots + a_{n-1}x^{n-1}$$

How efficient is the transformation?

Problem Statement

A(x)

Degree n-1 polynomial



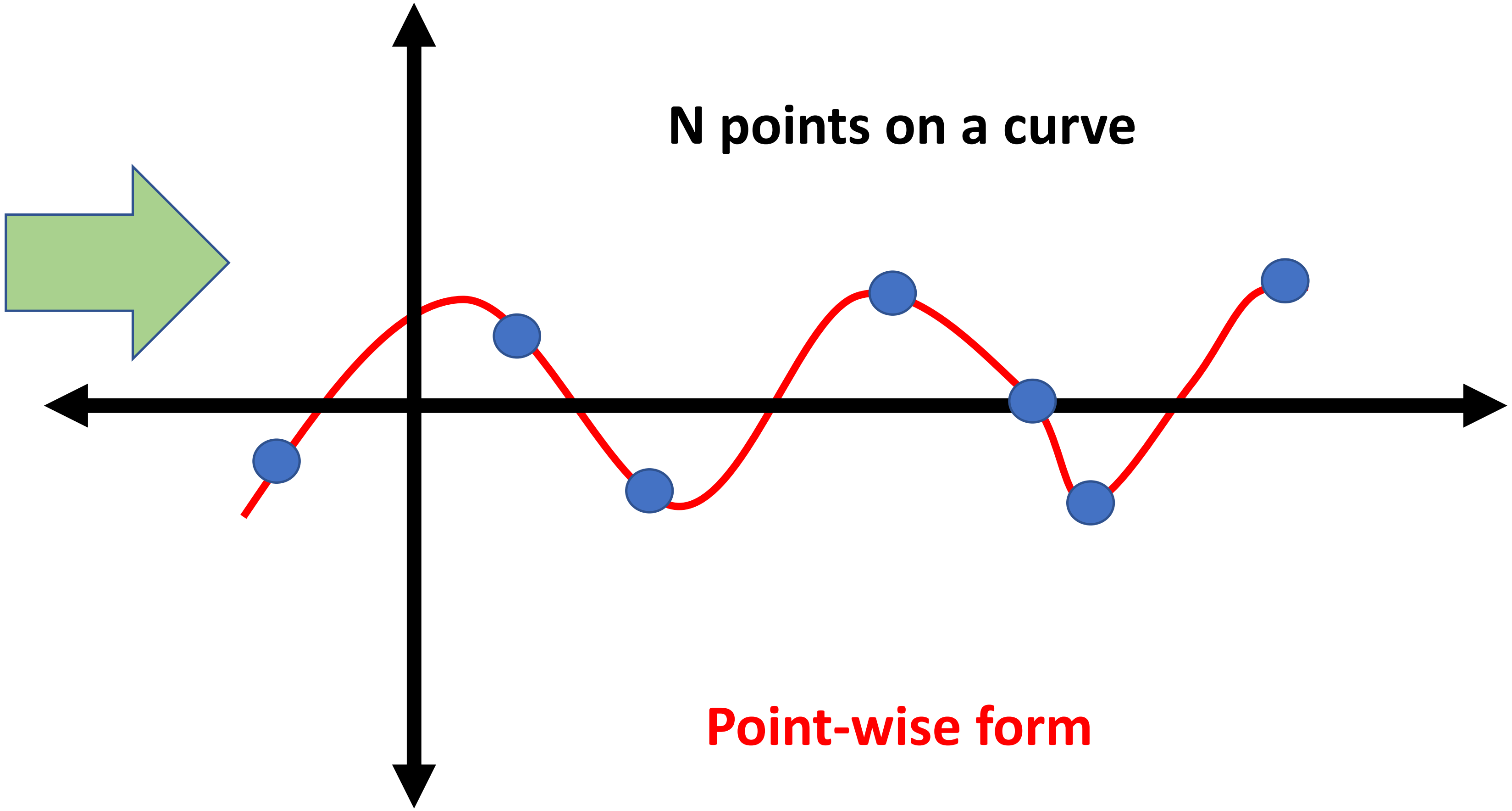
Coefficient form

Point-wise form

Problem Statement

A(x)

Degree n-1 polynomial



Coefficient form

Point-wise form

$$A(x)=a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots + a_{n-1}x^{n-1}$$

How efficient our algorithm will be?

$$T(n)=2*T(\frac{n}{2})+\theta(n)$$

Recurrence

$$
\begin{aligned}
A(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_{n-1}x^{n-1} \\
&= a_0 + a_2x^2 + a_4x^4 + \ldots\ldots + a_{n-2}x^{n-2} \\
&\quad + a_1x + a_3x^3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_{n-1}x^{n-1}
\end{aligned}
$$

$$A(x)=A_e(x^2)+x*A_o(x^2)$$

$$\mathbf{A_e(x) = a_0 + a_2x + a_4x^2 + \ldots\ldots\ \ldots\ldots\ldots + a_{n-2}x^{(n-2)/2}}$$
$$\mathbf{A_o(x) = a_1 + a_3\,x + a_5x^2 + \ldots\ldots\ldots\ldots\ldots + a_{n-1}x^{(n-2)/2}}$$

$$
\left.\begin{aligned}
&\mathbf{A_e(y) = a_0 + a_2y + a_4y^2 + \ldots\ldots\ \ldots\ldots\ldots + a_{n-2}y^{(n-2)/2}} \\
&\mathbf{A_o(z) = a_1 + a_3\,z + a_5z^2 + \ldots\ldots\ldots\ldots\ldots + a_{n-1}z^{(n-2)/2}}
\end{aligned}\right\}
$$

Both polynomials with degree $\boldsymbol{\frac{n}{2} - 1}$

$$A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots\ldots\ldots\ldots\ldots + a_{n-1}x^{n-1}$$
$$= a_0 + a_2x^2 + a_4x^4 + \ldots\ldots + a_{n-2}x^{n-2}$$
$$+ a_1x + a_3x^3 + \ldots\ldots\ldots\ldots\ldots\ldots + a_{n-1}x^{n-1}$$

$$\boldsymbol{A(x)=A_e(x^2)+x*A_o(x^2)}$$

$$\mathbf{A_e(x) = a_0 + a_2x + a_4x^2 + \ldots\ldots\ \ldots\ldots\ldots + a_{n-2}x^{(n-2)/2}}$$
$$\mathbf{A_o(x) = a_1 + a_3\,x + a_5x^2 + \ldots\ldots\ldots\ldots\ldots + a_{n-1}x^{(n-2)/2}}$$

$A(x)=A_e(x^2)+x*A_o(x^2)$

Suppose we have already evaluated A_e and A_o on {4,9,16,25}

We can compute 8 terms

$A(2)=A_e(4) + 2A_o(4)$
$A(-2)=A_e(4) + (-2)A_o(4)$

$A(3)=A_e(9) + 2A_o(9)$
$A(-3)=A_e(9) + (-3)A_o(9)$

$A(4)=$
$A(-4)=$

$A(5)=$
$A(-5)=$

$A_e(4)$ $A_o(4)$

$A_e(9)$ $A_o(9)$

$A_e(16)$ $A_o(16)$

$A_e(25)$ $A_o(25)$

$A(x)=A_e(x^2)+x*A_o(x^2)$

Suppose we have already evaluated A_e and A_o on {4,9,16,25}

We can compute 8 terms

$A(2)=A_e(4) + 2A_o(4)$
$A(-2)=A_e(4) + (-2)A_o(4)$

$A(3)=A_e(9) + 2A_o(9)$
$A(-3)=A_e(9) + (-3)A_o(9)$

$A(4)=A_e(16) + 4A_o(16)$
$A(-4)= A_e(16) + (-4)A_o(16)$

$A(5)=A_e(25) + 5A_o(25)$
$A(-5)= A_e(25) + (-5)A_o(25)$

$A_e(4)$ $A_o(4)$

$A_e(9)$ $A_o(9)$

$A_e(16)$ $A_o(16)$

$A_e(25)$ $A_o(25)$

FFT (f=a[1,2,….n])

E $\leftarrow$ FFT(A_e) *// E[1,2….,n/2]*

O $\leftarrow$ FFT(A_o) *// O[1,2….,n/2]*

Then compute:

$A(x)=A_e(x^2)+xA_o(x^2)$

T(n)=

$A(x)=A_e(x^2)+x*A_o(x^2)$

$A_e(4)$ $A_o(4)$
$A_e(9)$ $A_o(9)$
$A_e(16)$ $A_o(16)$
$A_e(25)$ $A_o(25)$

Suppose we have already evaluated A_e and A_o on {4,9,16,25}

$A(2)=A_e(4) + 2A_o(4)$
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$A(4)=A_e(16) + 4A_o(16)$
$A(-4)= A_e(16) + (-4)A_o(16)$

$A(5)=A_e(25) + 5A_o(25)$
$A(-5)= A_e(25) + (-5)A_o(25)$

$A(x)=A_e(x^2)+x*A_o(x^2)$

$A_e(4)$ $A_o(4)$
$A_e(9)$ $A_o(9)$
$A_e(16)$ $A_o(16)$
$A_e(25)$ $A_o(25)$

Suppose we have already evaluated A_e and A_o on {4,9,16,25}

We are going to need points that have logn square-roots

$A(2)=A_e(4) + 2A_o(4)$
$A(-2)=A_e(4) + (-2)A_o(4)$

$A(3)=A_e(9) + 2A_o(9)$
$A(-3)=A_e(9) + (-3)A_o(9)$

$A(4)=A_e(16) + 4A_o(16)$
$A(-4)= A_e(16) + (-4)A_o(16)$

$A(5)=A_e(25) + 5A_o(25)$
$A(-5)= A_e(25) + (-5)A_o(25)$

Complex numbers

Roots of Unity

$X^n=1$

Should have n solutions.

Euler identity:

$e^{2\pi i}=1$

Roots of Unity

$X^n=1$

$\{e^{2\pi i(j/n)}\}_{j=0,1,..,n-1}$

$\{1, e^{2\pi i/n}, e^{2\pi i2/n}, e^{2\pi i3/n}, \ldots.\ e^{2\pi i(n-1)/n}\}$

Euler identity:

$[e^{2\pi i(j/n)}]^n=$

$e^{2\pi i}=1$

Roots of Unity

$$X^n=1$$

$$\{1,e^{2\pi i/n}, e^{2\pi i2/n}, e^{2\pi i3/n},\dots e^{2\pi i(n-1)/n}\}$$

$$\{e^{2\pi i(j/n)}\}_{j=0,1,..,n-1}$$

Euler identity:

$$e^{2\pi i}=1$$

- All of these numbers are unique
- There are n of them
- They are all solutions to this equation
- Squaring these numbers gives (n/2)th roots of unity

If we take any of these numbers, we will have correct number of square roots

Taylor series

$$e^{2\pi ij/n} = \omega_{j,n}$$

$$e^{ix} = \cos(x) + i * \sin(x)$$

$$e^{2\pi ij/n} = \cos(2\pi ij/n) + i * \sin(2\pi ij/n)$$

$$e^{2\pi ij/n} = \omega_{j,n}$$

$$e^{2\pi ij/n} = \cos(2\pi ij/n) + i*\sin(2\pi ij/n)$$

Let’s compute:

$$\omega_{1,8} = \omega_{j=1,n=8} = \cos(2\pi 1/8) + i*\sin(2\pi 1/8)$$

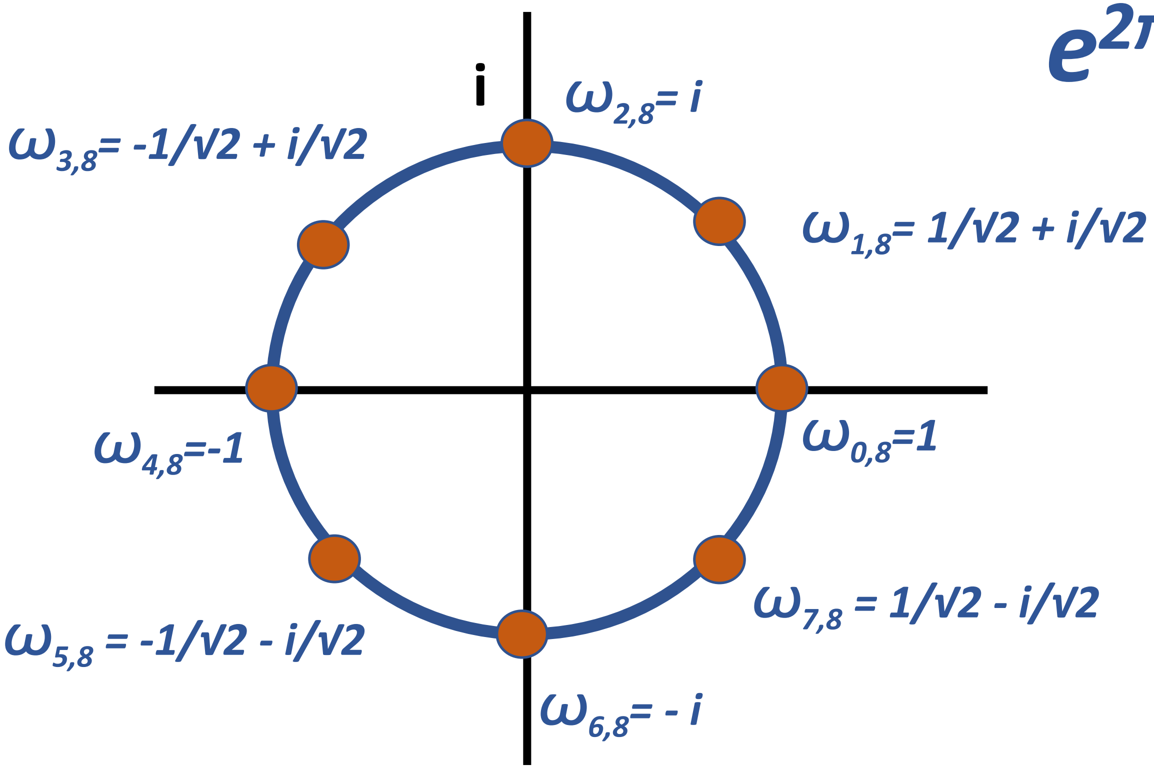
$$= \cos(\pi/4) + i*\sin(\pi/4)$$

$$= 1/\sqrt{2} + i/\sqrt{2}$$

Compute 8 roots of unity

$$e^{2\pi ij/n} = \omega_{j,n}$$

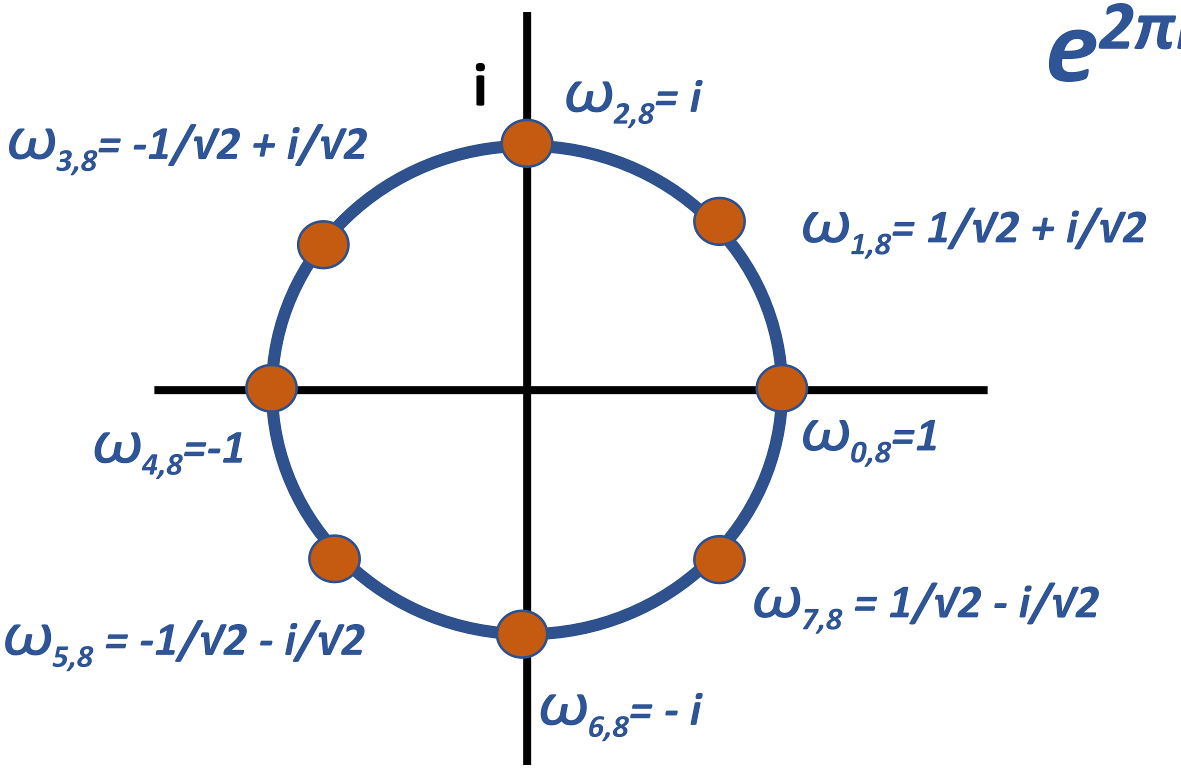
$$e^{2\pi ij/n} = \cos(2\pi ij/n) + i*\sin(2\pi ij/n)$$



Compute 8 roots of unity

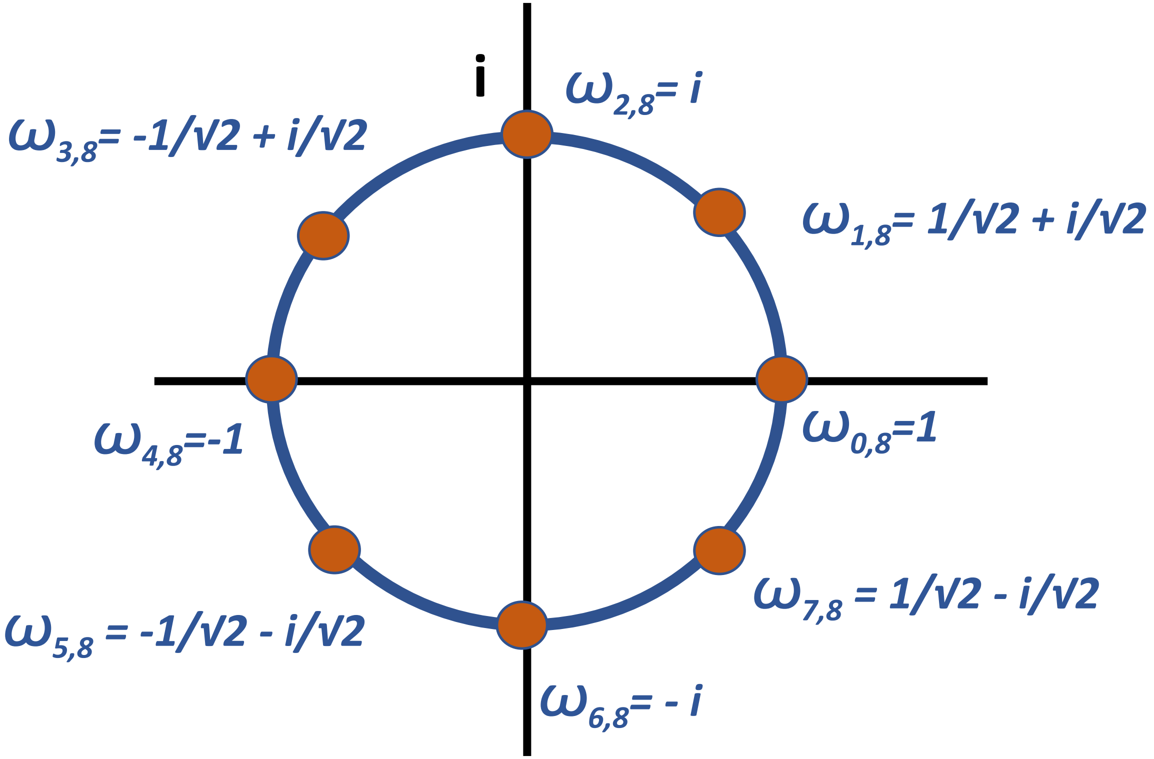
$$e^{2\pi ij/n} = \omega_{j,n}$$

$$e^{2\pi ij/n} = \cos(2\pi ij/n) + i*\sin(2\pi ij/n)$$



Squaring the nth roots of unity :

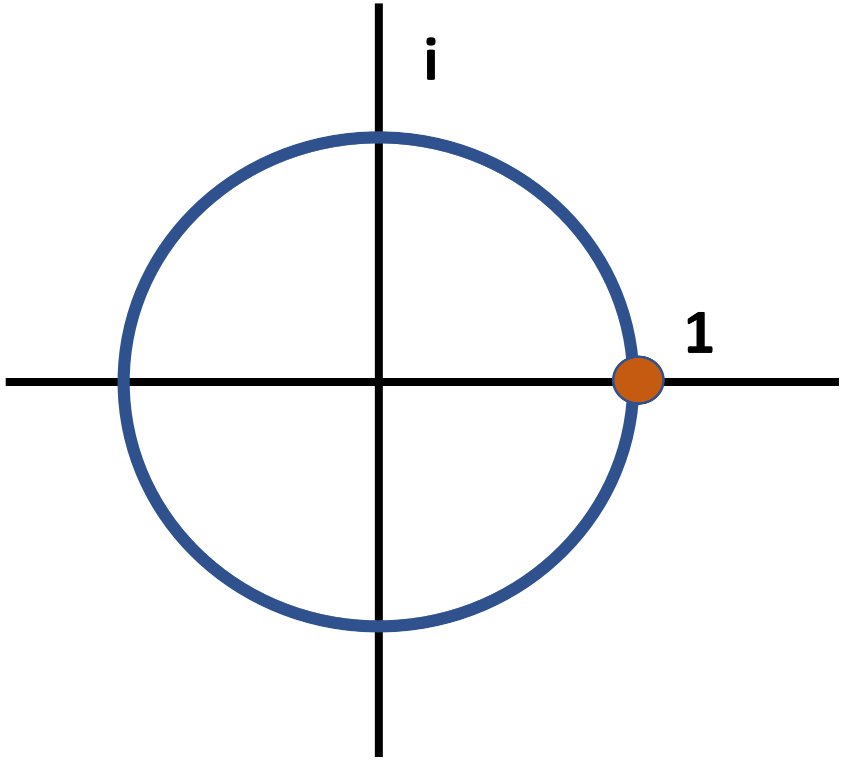
$$X^n=1$$



$$(\omega_{0,8})^2= (1)^2 = 1$$

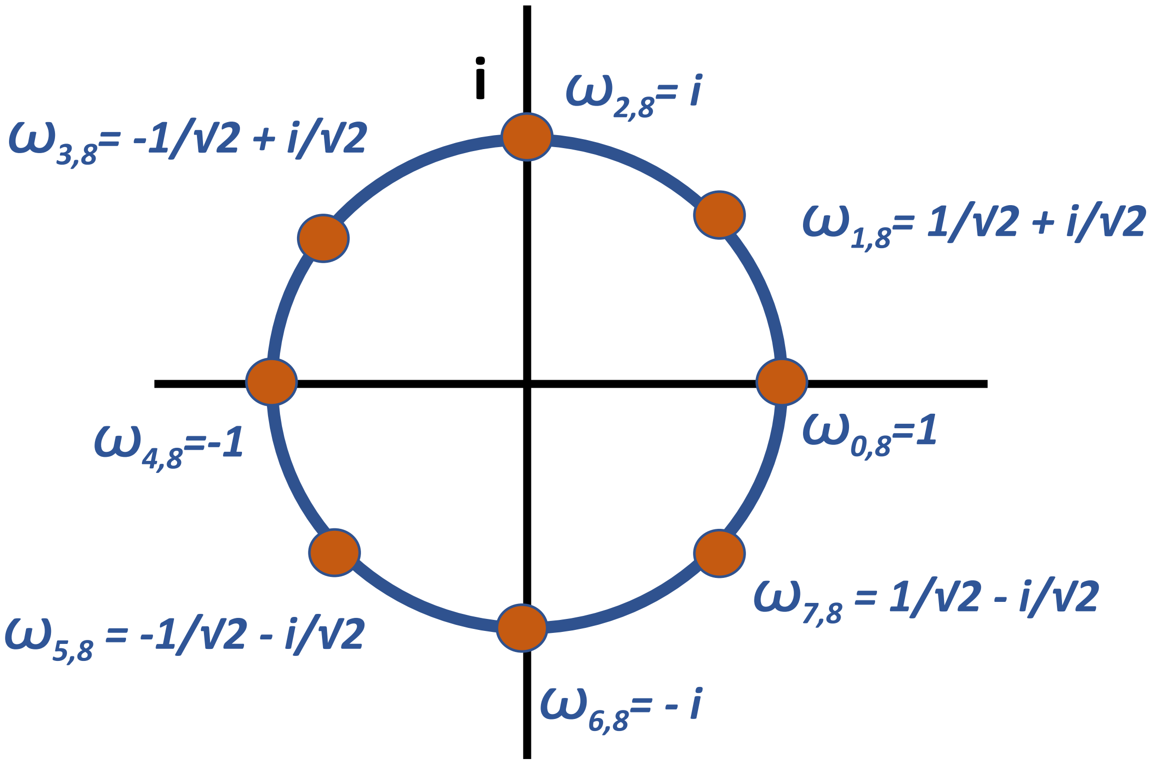
(n/2)th root on unity

$$X^{n/2}=1$$

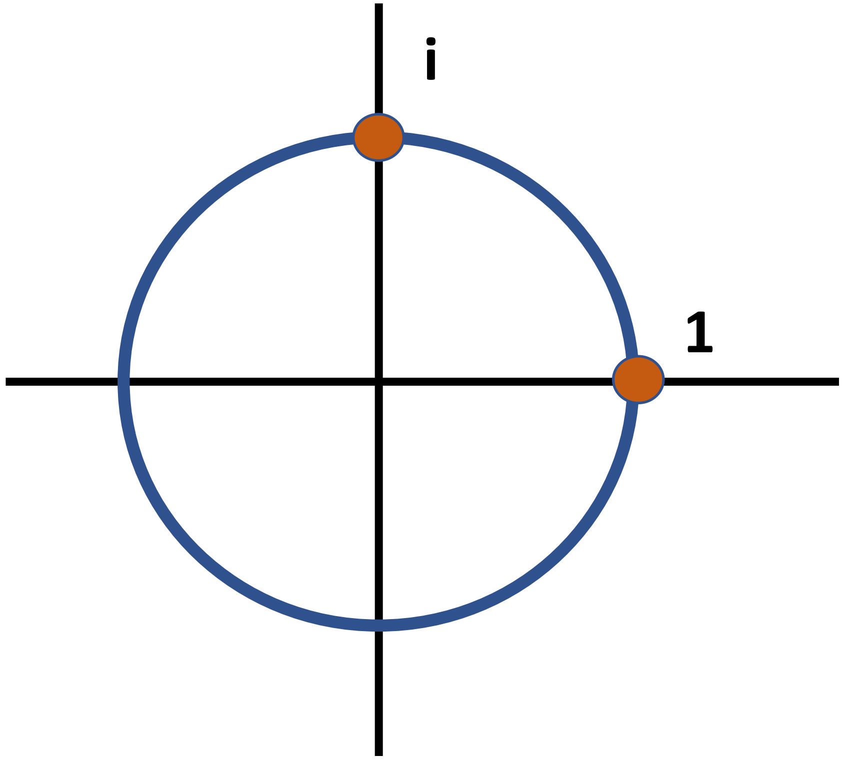


Squaring the nth roots of unity :

$$X^n=1$$



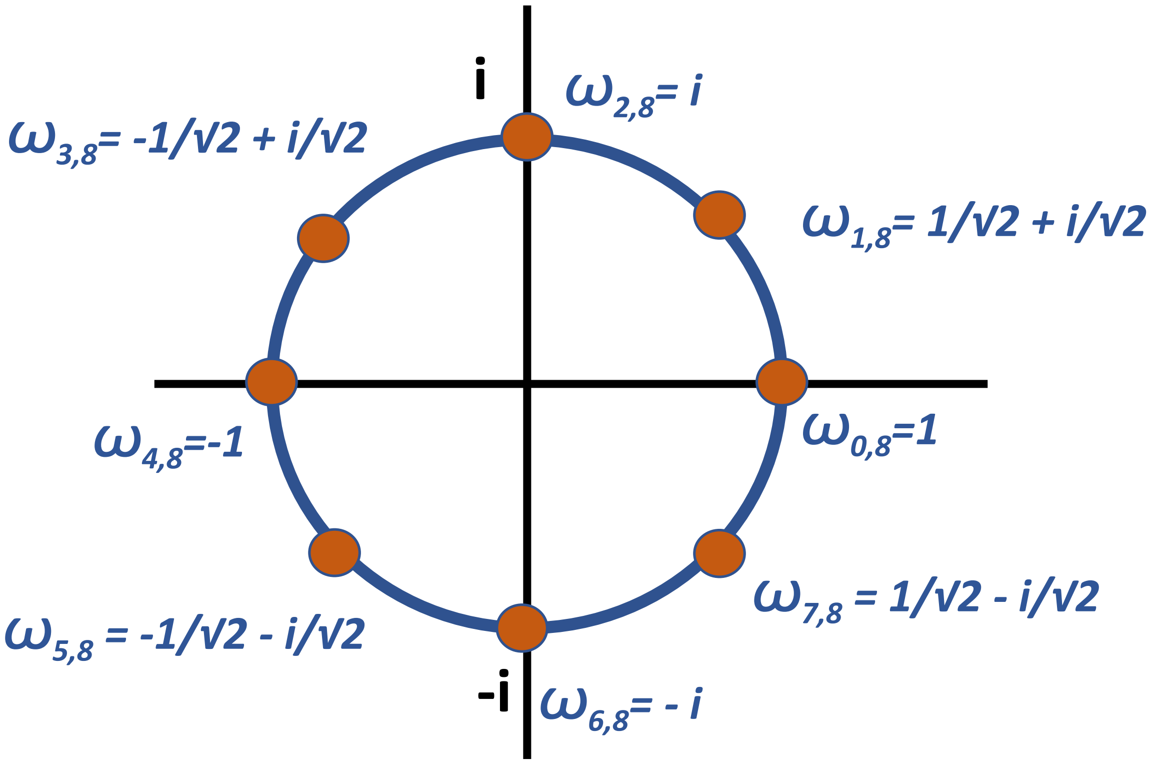
(n/2)th root on unity



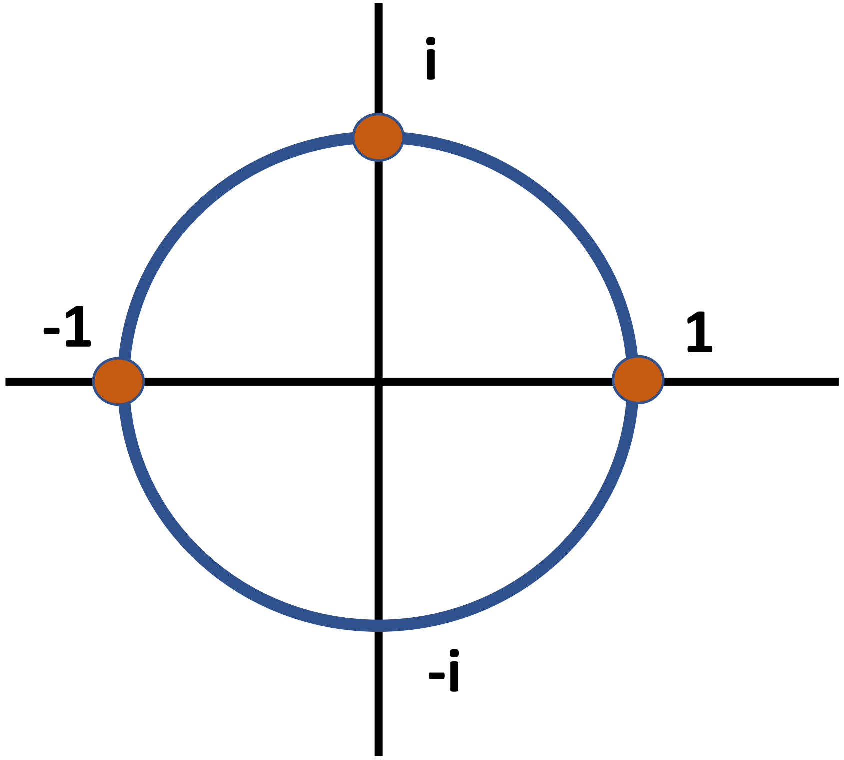
$$(\omega_{1,8})^2= (1/\surd 2 + i/\surd 2)^2 \quad = (1/\surd 2)^2 + 2(1/\surd 2 * i/\surd 2)+(i/\surd 2)^2 = ½ + i – ½$$
$$=i$$

Squaring the nth roots of unity :

$$X^n=1$$



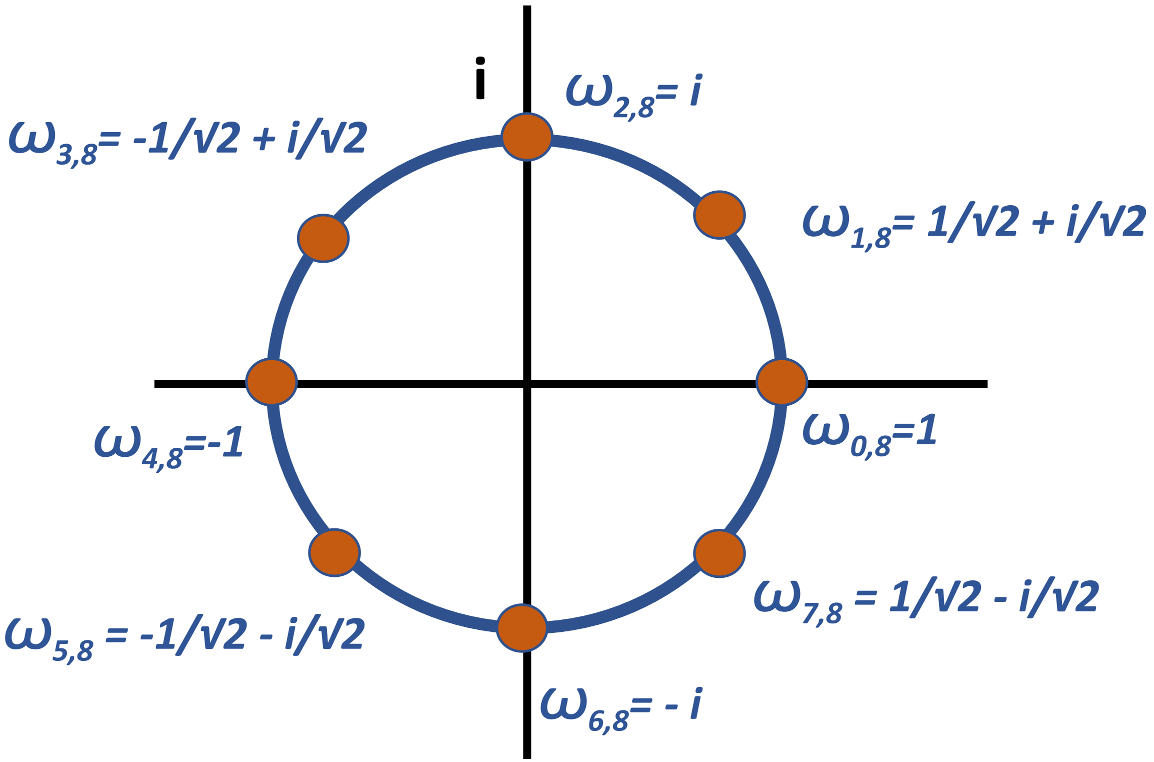
(n/2)th root on unity



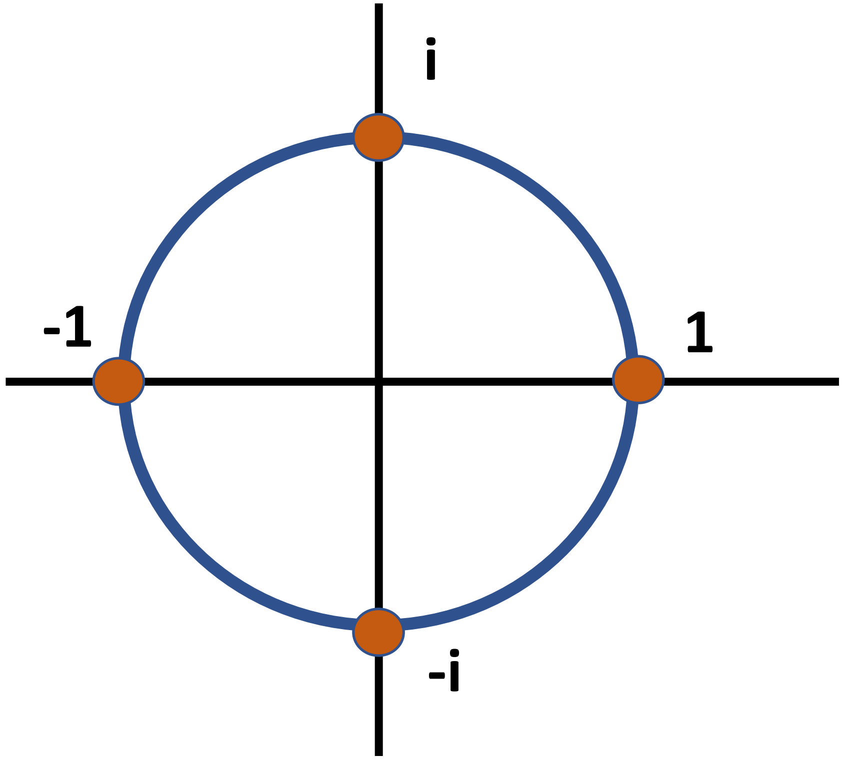
$$(\omega_{2,8})^2= (i)^2 = -1$$

Squaring the nth roots of unity :

$$X^n=1$$



(n/2)th root on unity



$$(\omega_{3,8})^2= (-1/\sqrt{2} + i/\sqrt{2})^2 \quad = (-1/\sqrt{2})^2 - 2(1/\sqrt{2}*i/\sqrt{2})+(i/\sqrt{2})^2 = ½ - i – ½$$

$$=-i$$

Squaring an nth root produces an n/2 th root/

$\{1, e^{2\pi i/n}, e^{2\pi i2/n}, e^{2\pi i3/n}, \ldots, e^{2\pi i(n/2)/n}, e^{2\pi i(n/2+1)/n}, \quad e^{2\pi i(n-1)/n}\}$

Squaring an nth root produces an n/2 th root of unity

$\{1, e^{2\pi i/n}, e^{2\pi i2/n}, e^{2\pi i3/n}, \ldots., e^{2\pi i(n/2)/n}, e^{2\pi i(n/2+1)/n}, \quad e^{2\pi i(n-1)/n}\}$

$$(e^{2\pi i(n/2)/n})^2 = e^{2\pi i} = 1$$

Squaring an nth root produces an n/2 th root of unity

$\{1, e^{2\pi i/n}, e^{2\pi i2/n}, e^{2\pi i3/n}, \ldots, e^{2\pi i(n/2)/n}, e^{2\pi i(n/2+1)/n}, \qquad e^{2\pi i(n-1)/n}\}$

$$(e^{2\pi i(n/2)/n})^2 = e^{2\pi i} = 1$$

$$(e^{2\pi i(n/2+1)/n})^2 = e^{2\pi i\,(n+2)/n}$$

$$= e^{2\pi i}\, e^{2\pi i(1/(n/2))}$$

$$= e^{2\pi i(1/(n/2))}$$

FFT (f=a[1,2,….n])

E ← FFT(A_e) *// E[1,2….,n/2]*

O ← FFT(A_o) *// O[1,2….,n/2]*

Then compute:

$$A(x)=A_e(x^2)+xA_o(x^2)$$

$$T(n)=2T(n/2)+\theta(n)$$

If n=16

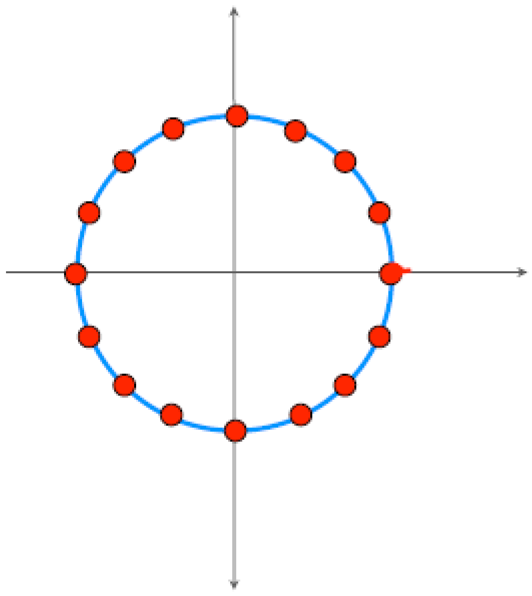
n=8

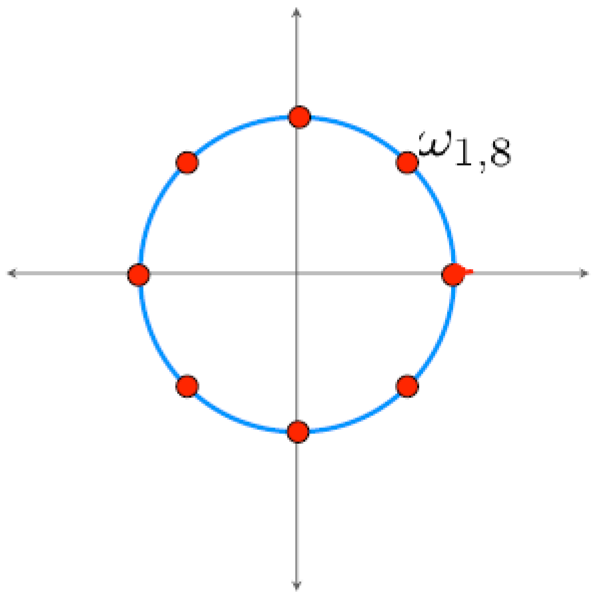
A_e, A_o

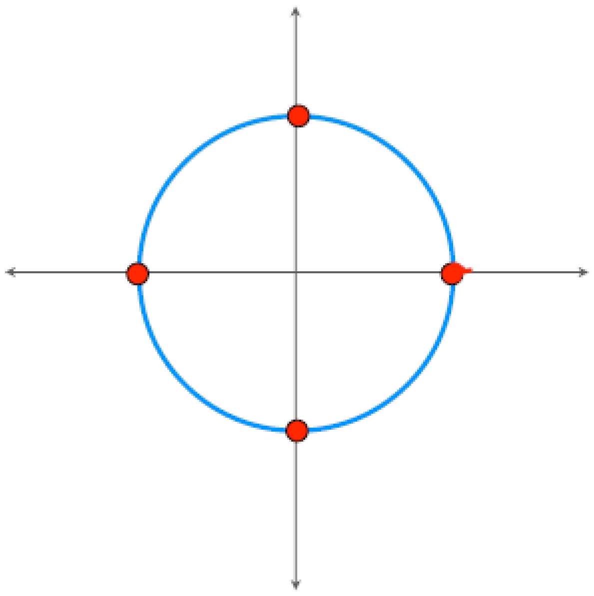
n=4

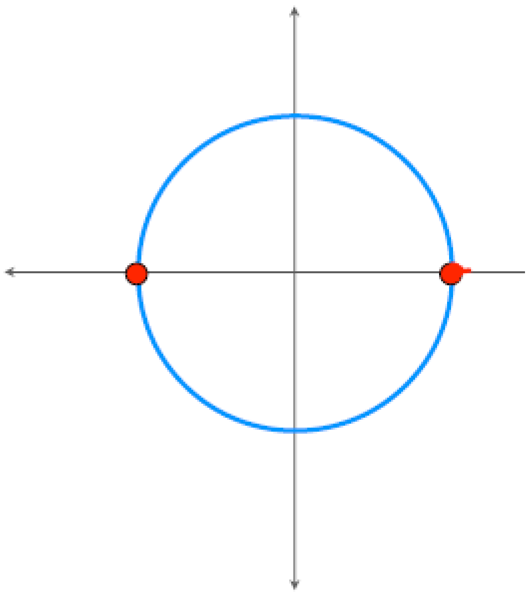
A_{eo}, A_{ee},

A_{oo}, A_{oe}









We evaluate A at
The 16^{th} roots of unity

Divide A into A_e & A_o

$n/2 = 8^{th}$
roots of
unity

$n/4 = 4^{th}$
roots of
unity

$n/8 = 2^{th}$
roots of
unity

(Base case)

Then compute:

$$A(x)=A_e(x^2)+xA_o(x^2)$$

Evaluate at a root of unity:

$$A(\omega_{i,n})=A_e(\omega_{i,n}{}^2)+\omega_{i,n}A_o(\omega_{i,n}{}^2)$$

Nth root Of unity *N/2 th root Of unity* *N/2 th root Of unity*

FFT (f=a[1,2,….n])

Base case if: n<=2

E ← FFT(A_e) *// E[1,2….,n/2]*

O ← FFT(A_o) *// O[1,2….,n/2]*

Combine result using:

$$A(\omega_{i,n}) = A_e(\omega_{i,n}^{\,2}) + \omega_{i,n} A_o(\omega_{i,n}^{\,2})$$

$$T(n) = 2T(n/2) + \theta(n)$$

Compute the FFT on values: 4,3,2,8,2,0,0,0

$A(x) = 4 + 3x + 2x^2 + 8x^3 + 2x^4 + 0x^5 + 0x^6 + 0x^7$

ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
1	$\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$	i	$\frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$	-1	$\frac{-1}{\sqrt{2}} + \frac{-i}{\sqrt{2}}$	$-i$	$\frac{1}{\sqrt{2}} + \frac{-i}{\sqrt{2}}$

$$A_e(x) = 4 + 2x + 2x^2 + 0x^3$$

$$A_o(x) = 3 + 8x$$

4 roots of unity $\omega_1, \omega_3, \omega_5, \omega_7$ which are $\{1, i, -1, -i\}$

		1	1	-1	-1	
FFT on A_e:	$\{$	$8,$	$2+2i,$	$4,$	$2-2i$	$\}$
FFT on A_o:	$\{$	$11,$	$3+8i,$	$-5,$	$3-8i$	$\}$

Compute the FFT on values: 4,3,2,8,2,0,0,0

$A(x) = 4 + 3x + 2x^2 + 8x^3 + 2x^4 + 0x^5 + 0x^6 + 0x^7$

ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
1	$\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$	i	$\frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$	-1	$\frac{-1}{\sqrt{2}} + \frac{-i}{\sqrt{2}}$	$-i$	$\frac{1}{\sqrt{2}} + \frac{-i}{\sqrt{2}}$

$$A_e(x) = 4 + 2x + 2x^2 + 0x^3$$

$$A_o(x) = 3 + 8x$$

4 roots of unity $\omega_1, \omega_3, \omega_5, \omega_7$ which are $\{1, i, -1, -i\}$

$$\begin{array}{lccccc}
 & & 1 & 1 & -1 & -1 \\
\text{FFT on } A_e: & \{ & 8, & 2+2i, & 4, & 2-2i \quad \} \\
\text{FFT on } A_o: & \{ & 11, & 3+8i, & -5, & 3-8i \quad \}
\end{array}$$

Compute the FFT on values: 4,3,2,8,2,0,0,0

A(x) = 4 + 3x + 2x^2 + 8x^3 + 2x^4 + 0x^5 + 0x^6 + 0x^7

$$A(x) = A_e(x^2) + xA_o(x^2)$$

$$A(\omega_1) = A_e(\omega_1^2) + \omega_1 A_o(\omega_1^2) = A_e(1) + 1A_o(1) = 8 + 1 \cdot 11 = \underline{19}$$

$$A(\omega_2) = A_e(\omega_2^2) + \omega_2 A_o(\omega_2^2) = A_e(i) + \omega_2 A_o(i) = (2+2i) + \omega_2(3+8i)$$

$$A(\omega_3) = A_e(i^2) + iA_o(i^2) = 4 - 5i$$

$$A(\omega_4) = A_e(-i) + \omega_4 A_o(-i) = (2-2i) + \omega_4(3-8i)$$

$$A(\omega_5) = A_e(1) - A_o(1) = 8 - 11 = -3$$

$$A(\omega_6) =$$

$$A(\omega_7) =$$

$$A(\omega_8) =$$

$$A_e(x) = 4 + 2x + 2x^2 + 0x^3$$

$$A_o(x) = 3 + 8x$$

$$\begin{array}{lccccc} & 1 & i & -1 & -i & \\ \text{FFT on } A_e: \{ & 8, & 2+2i, & 4, & 2-2i & \} \\ \text{FFT on } A_o: \{ & 11, & 3+8i, & -5, & 3-8i & \} \end{array}$$

Compute the FFT on values: 4,3,2,8,2,0,0,0

$A(x) = 4 + 3x + 2x^2 + 8x^3 + 2x^4 + 0x^5 + 0x^6 + 0x^7$

$$A(x) = A_e(x^2) + xA_o(x^2)$$

$$A(\omega_1) = A_e(\omega_1^2) + \omega_1 A_o(\omega_1^2) = A_e(1) + 1A_o(1) = 8 + 1 \cdot 11 = \underline{19}$$

$$A(\omega_2) = A_e(\omega_2^2) + \omega_2 A_o(\omega_2^2) = A_e(i) + \omega_2 A_o(i) = (2+2i) + \omega_2(3+8i)$$

$$A(\omega_3) = A_e(i^2) + iA_o(i^2) = 4 - 5i$$

$$A(\omega_4) = A_e(-i) + \omega_4 A_o(-i) = (2-2i) + \omega_4(3-8i)$$

$$A(\omega_5) = A_e(1) - A_o(1) = 8 - 11 = -3$$

$$\begin{array}{lccccc} & 1 & i & -1 & -i & \\ \text{FFT on } A_e\text{:} & \{ \; 8, & 2+2i, & 4, & 2-2i & \} \\ \text{FFT on } A_o\text{:} & \{ \; 11, & 3+8i, & -5, & 3-8i & \} \end{array}$$

$$A(\omega_6) = A_e(i) + \omega_6 A_o(i) = (2+2i) + \omega_6(3+8i)$$

$$A(\omega_7) = A_e(-1) - iA_o(-1) = 4 + 5i$$

$$A(\omega_8) = A_e(-i) + \omega_8 A_o(-i) = (2-2i) + \omega_8(3-8i)$$