Induction

Class Attendance

https://forms.gle/yPP7zqEHyn9iHzod8

Use the secret number:

556755

Induction

A)

S(n) = sum of all positive integers until n is $\frac{n(n+1)}{2}$

S(3)=1+2+3=6 S(4)=1+2+3+4=10

$S(5) = \frac{5 \times 6}{2} = \frac{30}{2} = 15$

$S(1) = \frac{1 \times 2}{2} = 1$

Base Case: $S(1) = \frac{1*(1+1)}{2} = 1$

- Assume true for all n, $n < n_0$, $S(n) = \boxed{\frac{n(n+1)}{2}}$
- Show it is true to $\boxed{S(n+1)}$

$$S(n+1) = 1+2+3+\ldots\ldots.+n+(n+1)$$
$$= \frac{n(n+1)}{2} + (n+1)$$
$$= \frac{n(n+1)}{2} + \frac{2*(n+1)}{2}$$
$$= \frac{(n+1)(n+2)}{2}$$

n+1+1

Induction:

1. Base Case
2. Inductive step:

   Assume if works for some positive integer k,
   Then we can prove it is going to work for the
   Next positive integer K+1

Prove for 1

Assume for S(K), and prove it is true for S(K+1)

1 2 3 4 5 6 7 8 ………

Induction

Prove that: $n! > 2^n$ for $n>=4$

Base Case:

$n=4$, $4!=24$ and $2^4=16$, so true

Induction step:

Suppose it is true for a k, k>=4, show it is true

For k+1, meaning: $(K+1)! > 2^{(K+1)}$

$$
\begin{aligned}
\mathbf{(K+1)! = k!(K+1)} \\
&> 2^K(K+1) \text{ (by induction hypothesis)} \\
&>= 2^k.2 \text{ (since } k>=4, k+1>=2) \\
&>= 2^{(K+1)}
\end{aligned}
$$

Induction:

1. Base Case

Prove for 1

2. Inductive step:

Assume if works for some positive integer k,
Then we can prove it is going to work for the
Next positive integer K+1

Assume for S(K), and prove it is true for S(K+1)

K

2

$n = 4, 5, 6, 7, 8 \ldots \infty$

Induction

Base Case: P(1)

Inductive step:
P(1)
P(2)
.....
P(K)
True

Implies P(K+1) true

Asymptotic proof difference:

Base Case: P(n*)

Inductive step:
P(n*)
P(n*+1)
.....
P(K)
True

Implies P(K+1) true

Karatsuba

$(a\times c)(100^2) + (a\times d + b\times c)\ (100) + b\times d$

$$\underbrace{\boxed{1}\ \boxed{7}}_{a\ \ n/2}\ \underbrace{\boxed{8}\ \boxed{9}}_{b\ \ n/2} \times \underbrace{\boxed{1}\ \boxed{4}}_{c\ \ n/2}\ \underbrace{\boxed{3}\ \boxed{2}}_{d\ \ n/2}$$

Recursively compute:

1. ac,bd,(a+b)(c+d)
2. ad+bc=(a+b)(c+d) –ac-bd
3. $ac\times 100^2 + (ad+bc)\times 100 + bd$

$\boxed{3T(\frac{n}{2})}$

2 addition,

4n subtraction

4n addition

Approximately, Not exactly

Karatsuba (ab,cd)

$\underbrace{1\ 7}_{a}\ \underbrace{8\ 9}_{b} \times \underbrace{1\ 4}_{c}\ \underbrace{3\ 2}_{d}$

BASE CASE:

return b × d if inputs are 1 digit

ELSE:

Compute ac=karatsuba(a,c) → $T(\frac{n}{2})$

Compute bd=karatsuba(b,d) → $T(\frac{n}{2})$

Compute t=karatsuba((a+b),(c+d)) → $T(\frac{n}{2}) + \boxed{2n}$

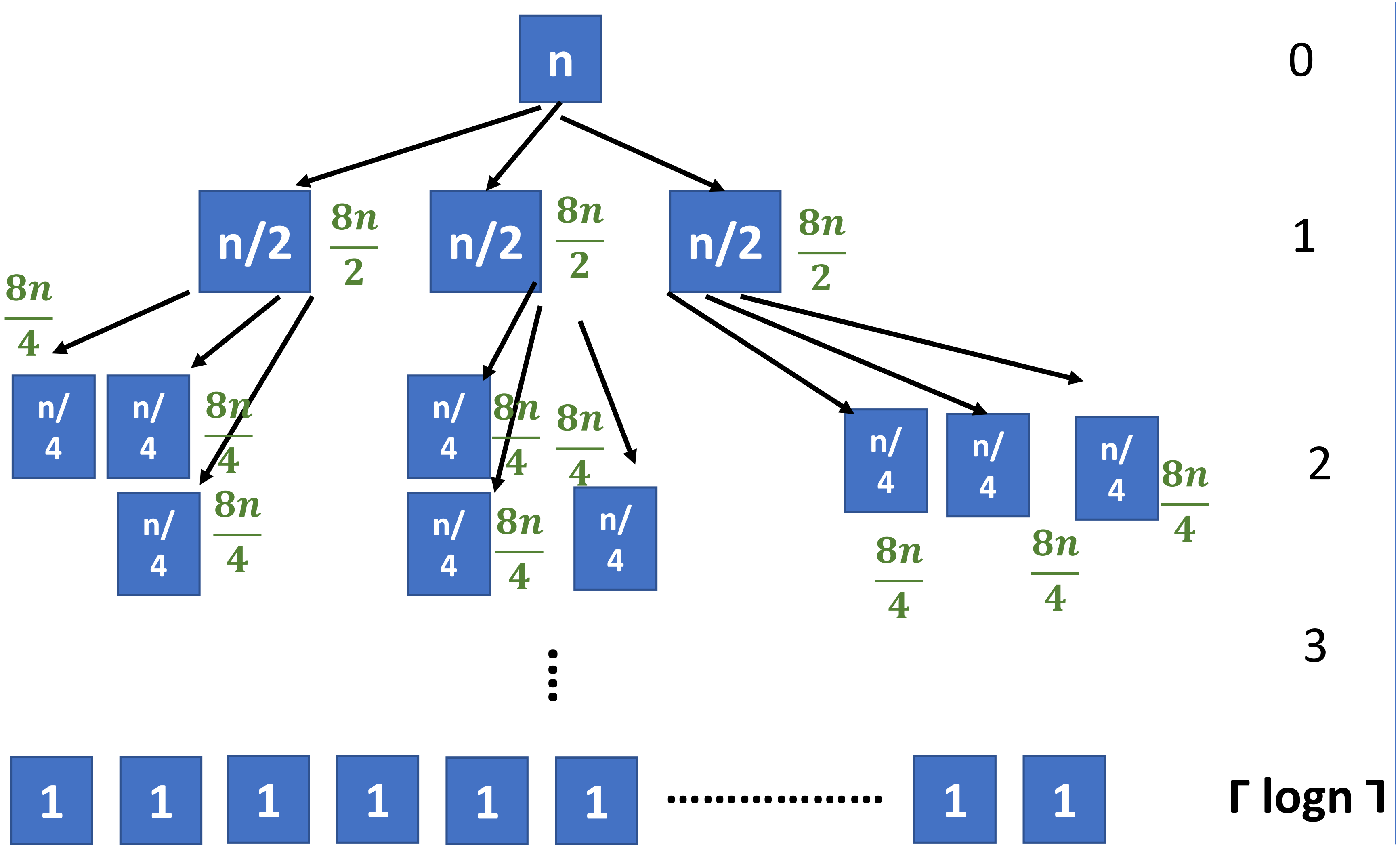
mid=t-ac-bd → $\boxed{2n}$

$T(n)=3T(\frac{n}{2}) + \boxed{8n}$

Ignoring issue of carries

Return ac × $((10)^{(\text{number of digit of a})})^2$ +mid×$(10)^{(\text{number of digit of a})}$ +bd $\boxed{\text{4n steps}}$

$$T(n)=3T\left(\frac{n}{2}\right)+8n$$



$8n$

$$3\times\frac{8n}{2} = \left(\frac{3}{2}\right)^{1}\times 8n$$

$$9\times\frac{8n}{4} = \left(\frac{3}{2}\right)^{2}\times 8n$$

$$27\times\frac{8n}{8} = \left(\frac{3}{2}\right)^{3}\times 8n$$

$$\left(\frac{3}{2}\right)^{\lceil \log n \rceil}\times 8n$$

Calculations:

$$(1+a+a^2+\ldots\ldots+a^L) = \frac{a^{L+1-1}}{a-1}$$

$$\sum_{i=0}^{L} a^i = \frac{a^{L+1-1}}{a-1}$$

$$T(n) = 8n + \frac{3}{2}\times 8n + \left(\frac{3}{2}\right)^2 \times 8n + \ldots\ldots + \left(\frac{3}{2}\right)^{\lceil \log n \rceil} \times 8n$$

$$= 8n \times \left(1 + \frac{3}{2} + \left(\frac{3}{2}\right)^2 + \ldots\ldots + \left(\frac{3}{2}\right)^{\lceil \log n \rceil}\right)$$

$$= \mathbf{8n} \times \left[\frac{\left(\frac{3}{2}\right)^{\lceil \log n \rceil + 1} - 1}{\frac{3}{2} - 1}\right] = 8n \times (2) \times \left[\left(\frac{3}{2}\right)^{\lceil \log n \rceil + 1} - 1\right]$$

$$= 2\times 8n[2^{\log_2 \frac{3}{2}}]^{\log n+1} - 16n = 16n[2^{(\log_2 3\ -1)}]^{(\log n+1)} - 16$$

$$= 16n[2^{((\log_2 n)^\wedge \log_2 3\ - \log_2 n + \log_2 3\ -1)}] - 16n = 16n\left[\frac{n^{\log_2 3} \times 2^{\log_2 3\ -1}}{n}\right] - 16n$$

$$= \boxed{16\times 2^{(\log_2 3\ -1)} \times n^{\log_2 3} - 16n} = o(n^{\log_2 3})$$

Karatsuba with Induction

$\log_2 3 = \boxed{1.6} = 1.589$

$T(n)=3T(n/2)+8n$

$T\left(\frac{n_0+1}{2}\right) < n_0$

Prove: $T(n)=O(n^{1.6})$ | So, we are going to prove: $\boxed{T(n) < 800*n^{1.6}}$ ⟹ $O(n^{1.6})$

$= 800 \times \left(\frac{n_0+1}{2}\right)^{1.6}$

Base case: Notice that for all, n=~~1~~,2, $T(n) < 800*n^{1.6}$ $= 800 \times 2^{1.6}$

Inductive step: consider that $T(n) < 800*n^{1.6}$ for all $n<n_0$, We would like to show $T(n_0+1) < 800*(n_0+1)^{1.6}$

$$T(n_0+1)=3*\boxed{T\left(\frac{n_0+1}{2}\right)}+8*(n_0+1)$$

$$< 3*\left(\frac{n_0+1}{2}\right)^{1.6}*800+8*(n_0+1)$$

$$< \frac{3}{2^{1.6}}*800*(n_0+1)^{1.6}+8*(n_0+1)$$

$$< 0.99*800*(n_0+1)^{1.6}+8*(n_0+1)$$

$$< 1*800*(n_0+1)^{1.6}-\boxed{0.01*800*}(n_0+1)^{1.6}+8*(n_0+1)$$

$$< 800*(n_0+1)^{1.6}-\cancel{8*(n_0+1)}+\cancel{8*(n_0+1)}$$

$T(n_0+1)$
$$< 800*(n_0+1)^{1.6}$$

Karatsuba with Induction

600

Guess+check

8

$T(n)=3T(n/2)+8n$

Prove: $T(n)=O(n^{1.6})$ So, we are going to prove: $T(n) < \boxed{800^*} n^{1.6}$ ➡ $O(n^{1.6})$

$0.01 \times n = 8/0.01$

Base case: Notice that for all, n=1,2, $T(n) < 800^*n^{1.6}$

Inductive step: consider that $T(n) < 800^*n^{1.6}$ for all $n<n_0$, We would like to show $T(n_0+1) < 800^*(n_0+1)^{1.6}$

$$T(n_0+1)=3^*T\left(\frac{n_0+1}{2}\right)+8^*(n_0+1)$$

$$< 3^*\left(\frac{n_0+1}{2}\right)^{1.6} {}^*800 + 8^*(n_0+1)$$

$$< \frac{3}{2^{1.6}} {}^* 800^*(n_0+1)^{1.6} + 8^*(n_0+1)$$

$$< 0.99 {}^* 800^*(n_0+1)^{1.6} + 8^*(n_0+1)$$

$$< 1^*800^*(n_0+1)^{1.6} - 0.01^*800^*(n_0+1)^{1.6} + \boxed{8^*(n_0+1)}$$

$$< 800^*(n_0+1)^{1.6} - 8^*(n_0+1) + 8^*(n_0+1)$$

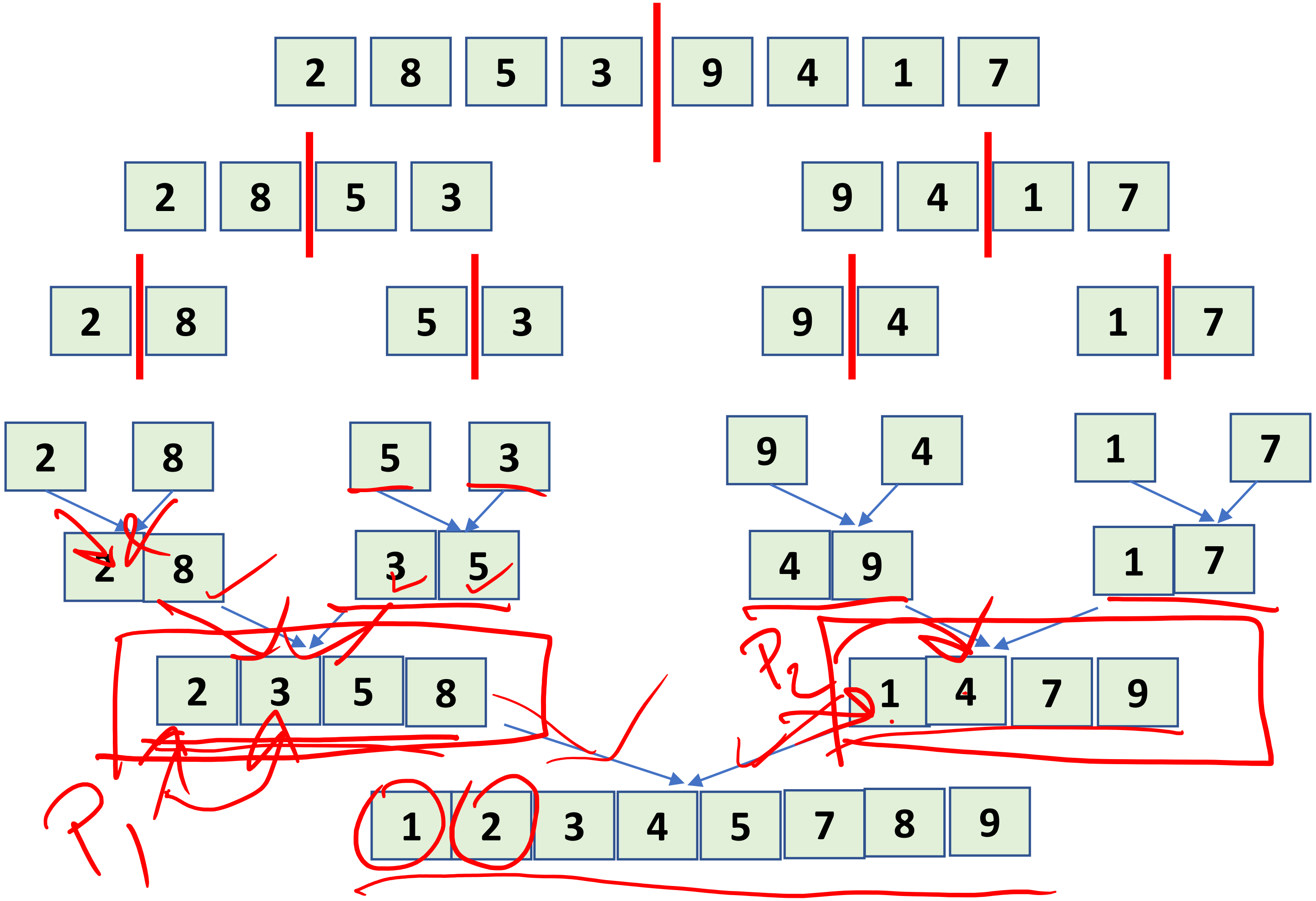
$$< 800^*(n_0+1)^{1.6}$$

$8 \times (n_0+1)^{1.6}$

$> 8 \times (n_0+1)$

1.6

Merge Sort


2 8 5 3 9 4 1 7
2 8 5 3
9 4 1 7
2 8
5 3
9 4
1 7
2 8
5 3
9 4
1 7
2 8
3 5
4 9
1 7
2 3 5 8
1 4 7 9
P1
P2
1 2 3 4 5 7 8 9

MergeSort(A,start,end)

1. **If start < end**
2. **q ← ⌊ start + end ⌋ /2**
3. Mergesort(A, start, q)
   Mergesort(A,q+1,end)
4. **Merge(A,start,q,end)**
5. **Else** *base case*

$T(n) = 2*T(n/2)+\cancel{2}n$

1

2* T(n/2)

n

MergeSort(A,start,end)

$T(n) = nlogn$

$= O(nlogn)$

$T(n)=2*T(n/2)+n$ $\boxed{O(nlogn)}$

Proof: $T(n) <= nlogn$

base case: $T(1)=0$, $T(2)=2$

- Observe that the statement holds for small n.

$< n_0$

- Now suppose $\boxed{T(n)<= nlogn}$ for all $\boxed{n<n_0}$
- Consider $T(n+1) = 2*T(\frac{n_0+1}{2}) + (n_0 +1)$

$$<= 2* (\frac{n_0+1}{2}) \log_2(\frac{n_0+1}{2}) + (n_0 +1)$$

$$<= (n_0 +1)\log(n_0 +1) - (n_0 +1)\log_2 2 + (n_0 +1)$$

$$T(n_0+1) \boxed{<= (n_0 +1)\log(n_0 +1)}$$

Karatsuba with Induction

$\frac{1}{2}$ $2^{-\log_2 3} = \frac{1}{3}$

$O\left(n^{\log_2 3}\right)$

T(n)=3T(n/2)+8n

(with appropriate base case)

Prove: $T(n) < n^{\log_2(3)} - 16n$

By inspection, the statement holds for small n. Suppose that it holds for all $n < n_0$

$< n_0$

$$T(n_0+1) = 3*T\left(\frac{n_0+1}{2}\right) + 8*(n_0+1)$$

8

$$< 3*\left[\left(\frac{n_0+1}{2}\right)^{\log_2(3)} - 16*\left(\frac{n_0+1}{2}\right)\right] + 8*(n_0+1)$$

$$< \frac{3}{3} * (n_0+1)^{\log_2(3)} - 24*(n_0+1) + 8*(n_0+1)$$

$T(n_0+1)$ $$< (n_0+1)^{\log_2(3)} - 16*(n_0+1)$$

Karatsuba

2 digits ×

Base case for Karatsuba is n=2
4 multiplication, 3 additions

Let's see how many operation is needed in assembly language?

T(2) =10 operations

4 digits ×

T(4)

3 operations on T(2) = 3*T(2) + 32

= 3*10+ 32 =62

= 62

Karatsuba

Digits	Number of operations
2	10
4	62
8	250
16	878
32	2890
64	9182
128	28570

$T(n) = 14\, n^{\log_2 3} - 1 \cancel{6n}$

c

$O(n^{\log_2 3})$

Karatsuba with Induction

T(n)=3T(n/2)+8n

Guess+check

Prove: $T(n) <= 14^* n^{\log_2(3)} -16n$

Base case

By inspection, indeed, $T(n) <= 14^* n^{\log_2(3)}$ - 16n when $n \le 128$

Inductive step

Now let's assume that $T(n) <= 14^* n^{\log_2(3)}$ -16n when $n<n_0$, We have to show it is true to $T(n_0+1)$

$< n_0$

Consider $T(n_0+1) = 3^*T\left(\frac{n_0+1}{2}\right) +8^*(n_0+1)$

$$<= 3^*\left[14^*\left(\frac{n_0+1}{2}\right)^{\log_2(3)} - 16^*\left(\frac{n_0+1}{2}\right)\right] + 8^*(n_0+1)$$

8

$$<= \frac{3}{3} * 14 * (n_0+1)^{\log_2(3)} - 24^*(n_0+1) + 8^*(n_0+1)$$

$T(n_0+1)$

$$\boxed{<= 14 * (n_0+1)^{\log_2(3)} - 16^*(n_0+1)}$$

Karatsuba with Induction

$T(n)=3T(n/2)+8n$

Guess+check

Prove: $T(n) <= 14* n^{\log_2(3)} -16n$

By inspection, indeed, $T(n) <= 14* n^{\log_2(3)} - 16n$ when $n<128$

Now let's assume that $T(n) <= 14* n^{\log_2(3)} -16n$ when $n<n_0$, We have to show it is true to $T(n_0+1)$

Consider $T(n_0+1) = 3*T\left(\frac{n_0+1}{2}\right) +8*(n_0+1)$

$$<= 3*\left[14*\left(\frac{n_0+1}{2}\right)^{\log 2(3)} - 16*\left(\frac{n_0+1}{2}\right)\right] + 8*(n_0+1)$$

$$<= \frac{3}{3} * 14 * (n_0+1)^{\log 2(3)} - 24*(n_0+1) + 8*(n_0+1)$$

$$<= \boxed{14 * (n_0+1)^{\log 2(3)} - 16*(n_0+1)}$$

Asymptotic notation:
$T(n)= O(n^{\log_2(3)})$

What happens if we skip the -16n!!!

Karatsuba with Induction

$T(n)=3T(n/2)+8n$

Guess+check

Asynptoti

Prove: $T(n) \le 14 * n^{\log_2(3)}$ $-16n$

By inspection, indeed, $T(n) \le 14 * n^{\log_2(3)}$ $-$ 16n when $n<128$

Now let's assume that $T(n) \le 14 * n^{\log_2(3)}$ -16n when $n<n_0$, We have to show it is true to $T(n_0+1)$

$T(n_0+1) \le 14*(n_0+1)^{\log_2(3)}$

no

Consider $T(n_0+1) = 3 * T\left(\frac{n_0+1}{2}\right) + 8*(n_0+1)$

$\le 3*[14*\left(\frac{n_0+1}{2}\right)^{\log_2(3)} - \cancel{16*\left(\frac{n_0+1}{2}\right)}] + 8*(n_0+1)$

$\le 14 * (n_0+1)^{\log_2(3)} + 8 * (n_0+1)$

$14 \times (n_0+1)^{\log_2 3}$

Basically we are concluding, **$T(n_0+1) \le 14 * (n_0+1)^{\log_2(3)} + 8 * (n_0+1)$**

Inductive step

prove

Question

Big-O Notation: In Practice

- Use these simplification rules:
  - Only pay attention to the dominant terms (x^3 more important than x^2)
  - Don’t include constants in your big-O expression