1 Theoretical Background

1.1 Simplices

A collection of k + 1 points is affinely independent if the members lie on an affine hyperplane of dimension k. The convex hull of a set $X \subseteq \mathbb{R}^n$ is the smallest convex set containing X. For a finite set $\{x_0, \ldots, x_k\}$ this is the set of combinations $\sum_{i=0}^{k} \lambda_i x_k$ with $\sum_{i=0}^{k} \lambda_i = 1$.

Definition 1.1. Given an affinely independent set $X = \{x_0, \ldots, x_k\} \subseteq \mathbb{R}^n$, the k-dimensional simplex (sometimes called a geometric simplex) $\sigma = [x_0, \ldots, x_n]$ spanned by X is the convex hull of X. The points of X are called the vertices of σ, and the simplices spanned by subsets of X (which are necessarily affinely independent) are called the faces of σ.

Definition 1.2. A simplicial complex K is a finite collection of geometric simplices such that

(i) for any simplex $\sigma \in K$, every face of σ is in K;

(ii) for any two simplices $\sigma, \tau \in K$, $\sigma \cap \tau$ is either empty, or a face of both σ and τ.

The dimension of K is the largest dimension of any simplex in K. A subcomplex of K is a subset of K which is a simplicial complex.

1.2 Simplicial Homology

Definition 1.3. Given a simplicial complex K, we define a p-chain as a subset of the p-simplices in K.

We can write a p-chain c as the formal sum $c = \sum a_i \sigma_i$ where the sum is over all p-simplices and the coefficients are in $\mathbb{Z}_2$. This gives rise to an abelian group $(C_p, +)$, where $c + c' = \sum (a_i + b_i)\sigma_i$ and the coefficients are reduced mod 2. It can be further extended to a vector space by defining scalar multiplication as $a \cdot c = \sum (a \cdot a_i)\sigma_i$.
The boundary of a p-simplex is the set of (p-1)-faces. The boundary of a p-chain is the sum of the boundaries of its p-simplices: $\partial_p c = \sum a_i \partial_p \sigma_i$.

Definition 1.4. This can be formalised as an operation between vector spaces:

$$\partial_p : C_p \to C_{p-1}$$

called the boundary homomorphism.

These vector spaces and maps can be lined up into a sequence

$$\cdots \to \partial_{p+2} C_{p+1} \to \partial_{p+1} C_p \to \partial_p C_{p-1} \to \partial_{p-1} \ldots$$

called the chain complex of K.

Proposition 1.1. $\partial\partial c = 0$.

Proof. Let σ be a p-simplex. The vertices of a (p-2)-face are a subset of size p-1 from the p+1 vertices of σ. There are two subsets of $V(p)$ with size p containing these p-2 vertices. It follows that every (p-2)-face of σ is contained in exactly two (p-1)-faces, and therefore that $\partial\partial$ vanishes on σ as coefficients are reduced mod 2. By homomorphism properties this now follows for p-chains. $\square$

Definition 1.5. A p-cycle is a p-chain without boundary, the set of all p-cycles is therefore $\ker \partial_p$. As the kernel of a homomorphism it is a subgroup of C_p, and we denote it by Z_p.
Similarly, we define a p-boundary to be the boundary of a (p+1)-chain, the set of all p-boundaries is therefore $\mathrm{im}\,\partial_{p+1}$. As the image of a homomorphism it is a subgroup of C_p, and we denote it by B_p.

Note also that $B_p \trianglelefteq Z_p$ because Z_p is abelian, we can therefore take the quotient Z_p/B_p which represents the distinct cycles up to boundary.

Definition 1.6. We say that $z, z' \in Z_p$ are homologous if they fall in the same conjugacy class in Z_p/B_p.

We denote the quotient Z_p/B_p by H_p, and call it the p^{th} homology group. Its members are referred to as homology classes.
The above definitions can be extended to incorporate the vector space structure. Then, the rank-nullity theorem can then be applied to give

$$\text{rank}H_p = \text{rank}Z_p - \text{rank}B_p,$$

which we refer to the p^{th} Betti number of K and notate by $\beta_p = rankH_p$.

Definition 1.7. Define the Euler characteristic of a simplicial complex K by

$$\chi(K) = \sum_{i=0}^{k}(-1)^i \text{rank}C_p,$$

where $k = \text{dim}K$.

The map $\partial_p : C_p \to B_{p-1}$ is surjective, so we can apply the rank-nullity theorem to get $\text{rank}B_{p-1} = \text{rank}C_p - \text{rank}Z_p$. We can use this, along with that $B_i = \mathbf{0}$ for i outside $\{0, \ldots, k-1\}$, to rewrite the Euler characteristic:

$$\begin{aligned}
\chi(K) &= \sum_{i=0}^{k}(-1)^i(\text{rank}Z_p + \text{rank}B_{i-1}) \\
&= \sum_{i=0}^{k}(-1)^i \text{rank}Z_p - \sum_{i=0}^{k}(-1)^i B_i \\
&= \sum_{i=0}^{k}(-1)^i(\text{rank}Z_p - B_i) \\
&= \sum_{i=0}^{k}(-1)^i \beta_i.
\end{aligned}$$

1.3 Homology

Definition 1.8. The underlying space of a simplicial complex K is defined as the union

$$|K| = \bigcup_{\sigma \in K} \sigma,$$

and equipt with the subspace topology.

Definition 1.9. A triangulation of a topological space X is a simplicial complex K, whose underlying space is homeomorphic to X.

The previous definition of the Euler characteristic can be extended to be a well-defined topological invariant by defining it on a triangulation of a space. It is independent of the specific choice of triangulation. We note that having the same homology groups is weaker than having the same homotopy type, which is again weaker than being homeomorphic:

$$X \approx Y \implies X \simeq Y \implies H_p(X) \cong H_p(Y) \text{ for all p..}$$

The implications of this are that to compute the Betti numbers of X, we may find a space Y with the same homotopy type and compute its Betti numbers.